

Perfect matchings for the Gale-Robinson sequences

Mireille Bousquet-Mélou, CNRS, Bordeaux

Jim Propp, Brandeis University

Julian West, Malaspina University, Vancouver

A Gale-Robinson sequence

- $a(0) = a(1) = a(2) = a(3) = 1$
- $a(n)a(n-4) = a(n-1)a(n-3) + a(n-2)a(n-2)$ for $n \geq 4$.

[Somos-4]

$$a(n) = \frac{a(n - 1)a(n - 3) + a(n - 2)a(n - 2)}{a(n - 4)}$$

$a(4) = \frac{1 \times 1 + 1 \times 1}{1} = 2$	$a(8) = \frac{23 \times 3 + 7 \times 7}{2} = 59$
$a(5) = \frac{2 \times 1 + 1 \times 1}{1} = 3$	$a(9) = \frac{59 \times 7 + 23 \times 23}{3} = 314$
$a(6) = \frac{3 \times 1 + 2 \times 2}{1} = 7$	$a(10) = \frac{314 \times 23 + 59 \times 59}{7} = 1529$
$a(7) = \frac{7 \times 2 + 3 \times 3}{1} = 23$	$a(11) = \frac{1529 \times 59 + 314 \times 314}{23} = 8209$

The Gale-Robinson sequences

Initial conditions $a(0) = \dots = a(m-1) = 1$, and for $n \geq m$,

$$a(n)a(n-m) = a(n-i)a(n-j) + a(n-k)a(n-l),$$

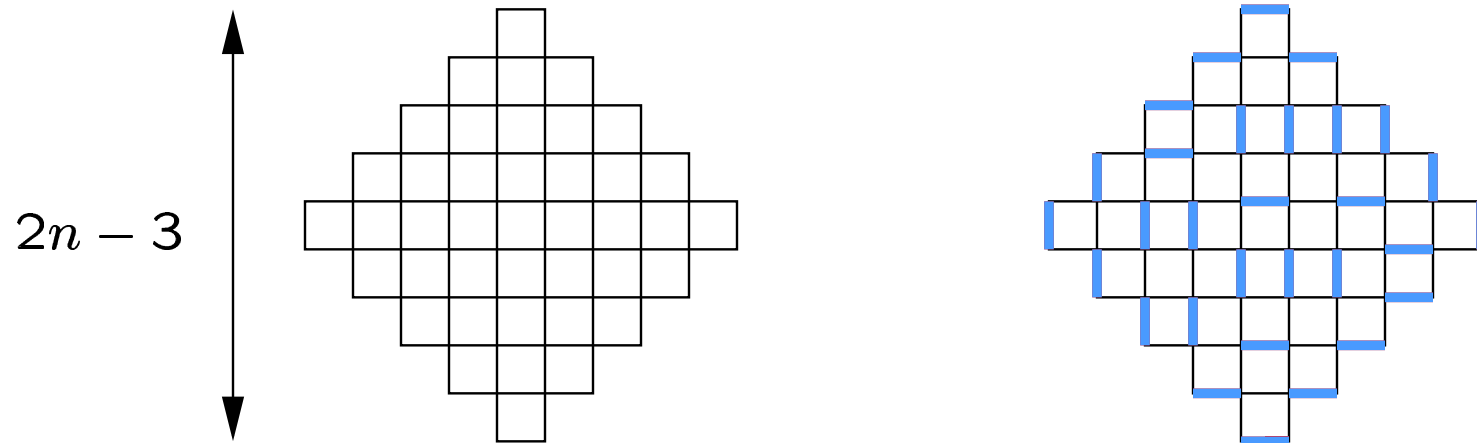
with $i + j = k + l = m$. Convention: $i \leq j, k \leq l, i \leq k$.

Ex: Somos-4

$$a(n)a(n-4) = a(n-1)a(n-3) + a(n-2)a(n-2)$$

Theorem [Fomin, Zelevinski 02]: The numbers $a(n)$ are integers.

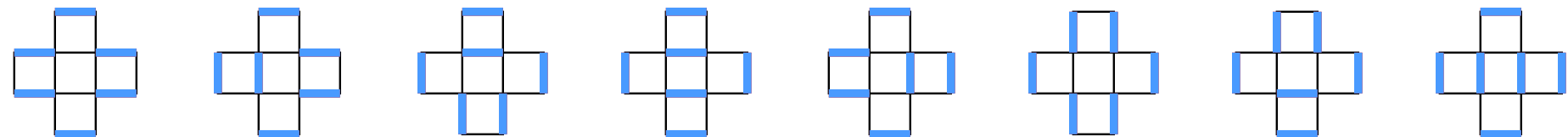
Perfect matchings of the aztec diamond



Theorem [Elkies Kuperberg Larsen Propp 92]: The aztec diamond of height $2n - 3$ has $a(n) = 2^{\binom{n}{2}}$ perfect matchings.

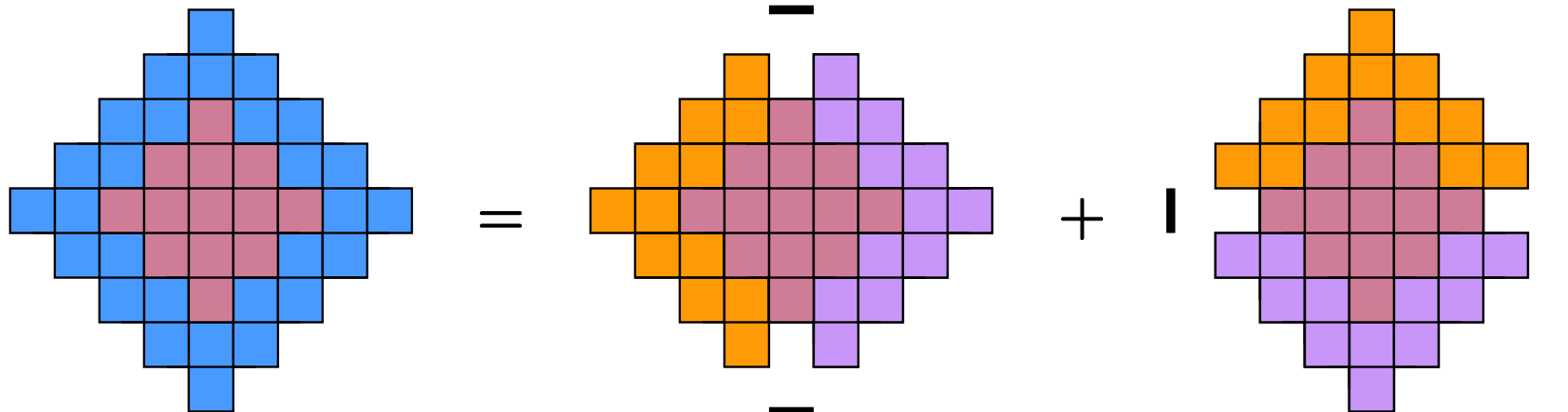
$$a(n)a(n-2) = a(n-1)a(n-1) + a(n-1)a(n-1)$$

Ex: $n = 3$

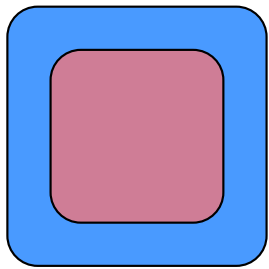


Eric Kuo's proof: a condensation theorem

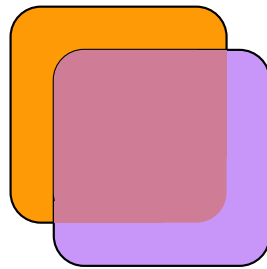
$$a(n)a(n-2) = a(n-1)a(n-1) + a(n-1)a(n-1)$$



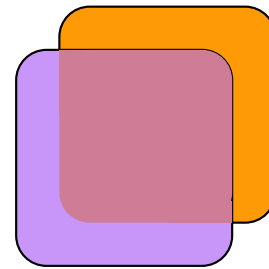
Dodgson's condensation formula



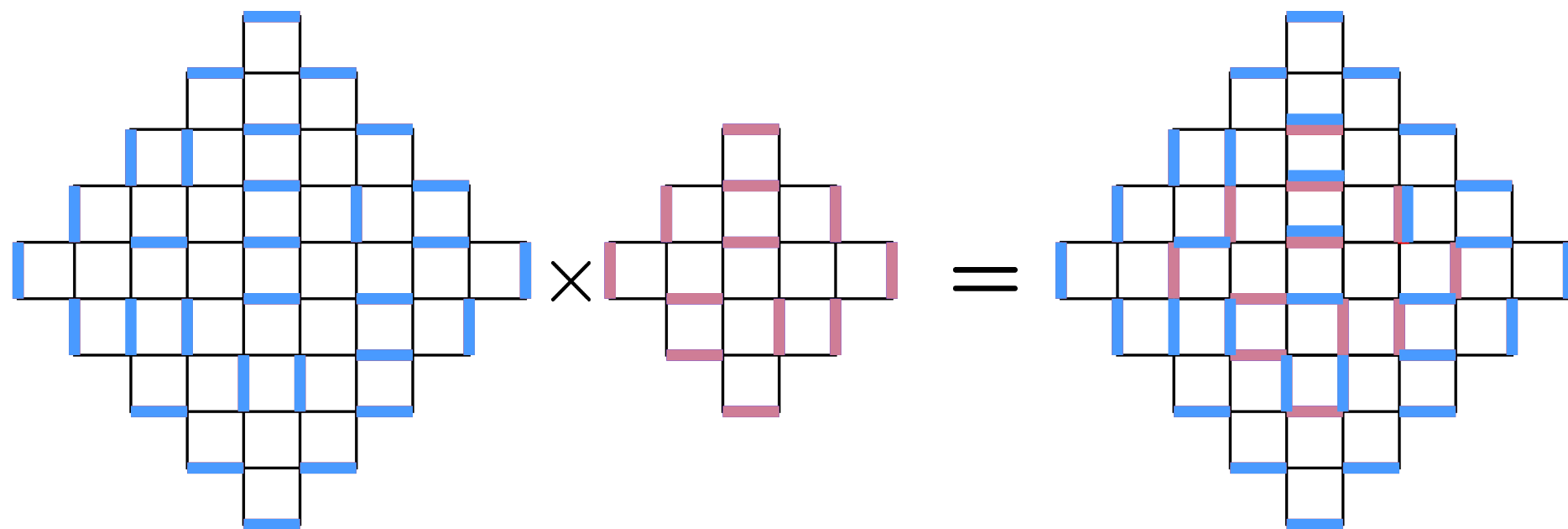
=

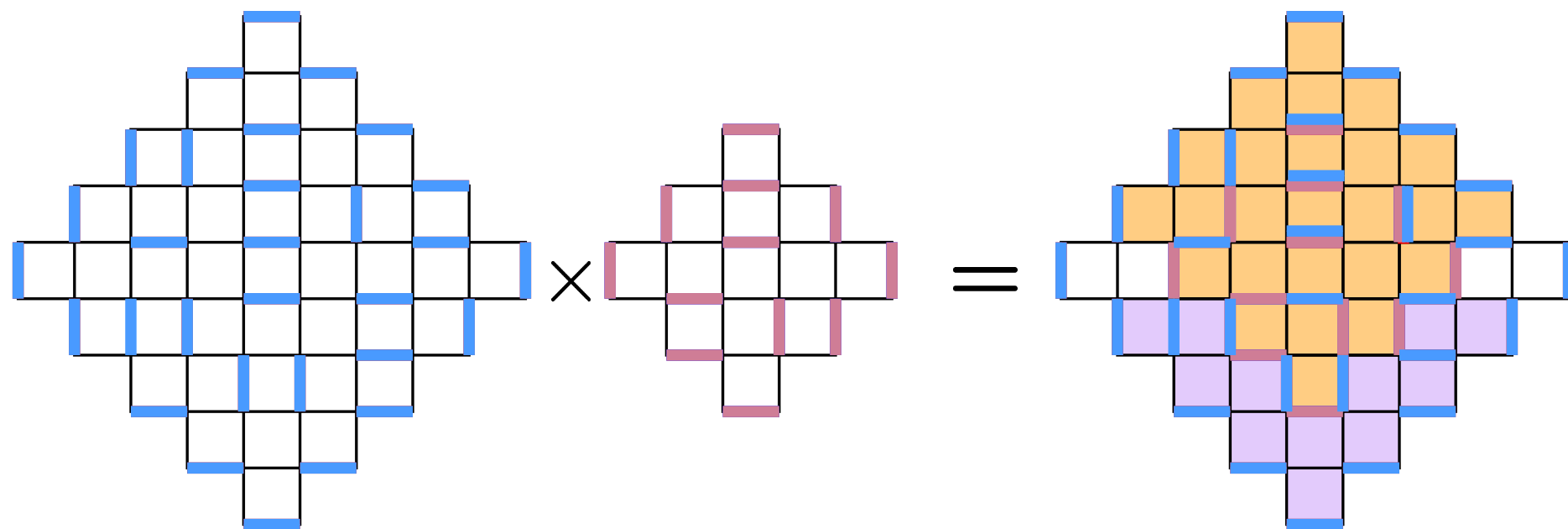


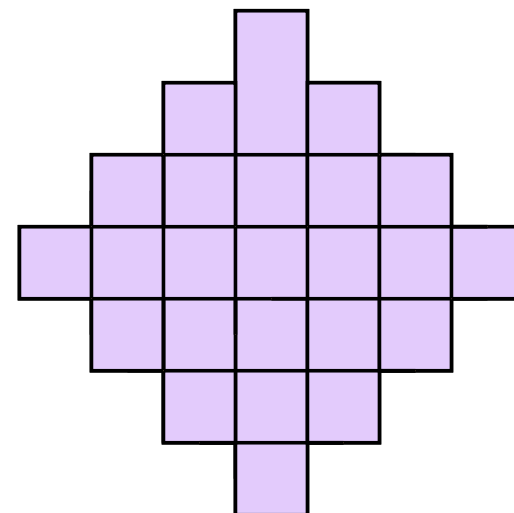
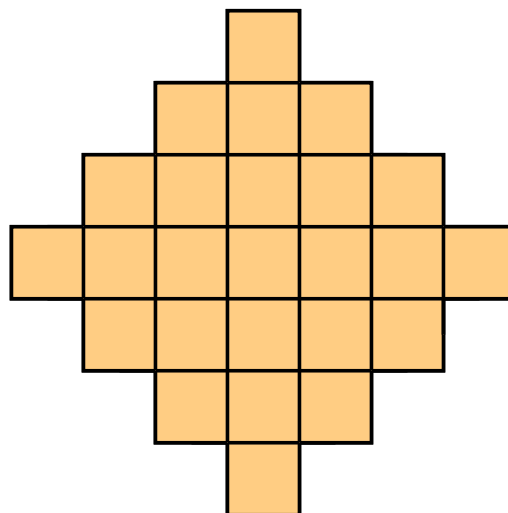
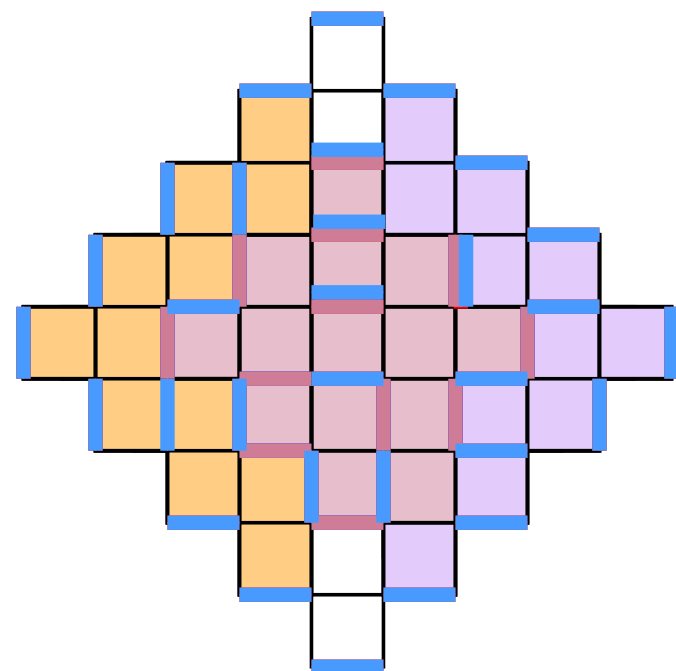
−



$$\det(M) \det(M_C) = \det(M_{NE}) \det(M_{SW}) - \det(M_{NW}) \det(M_{SE})$$

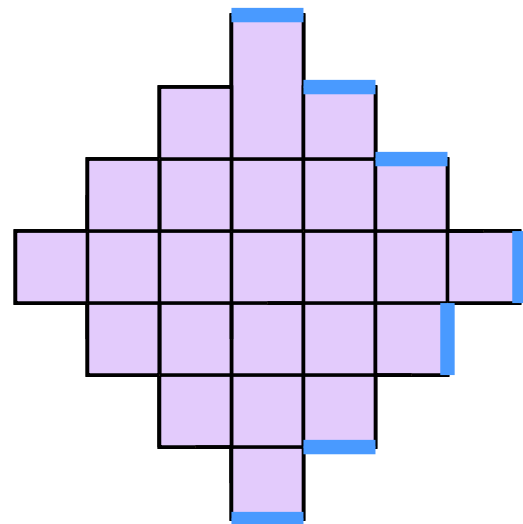
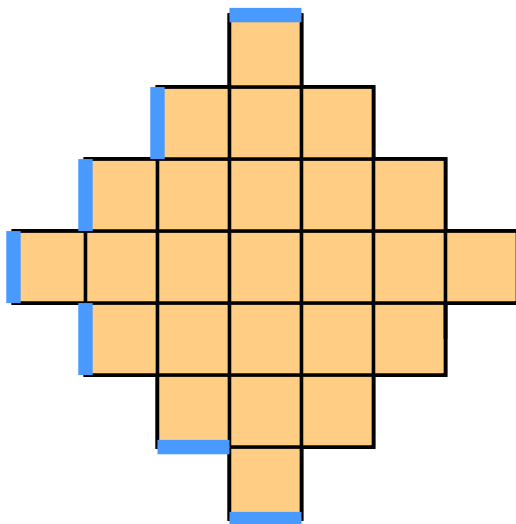
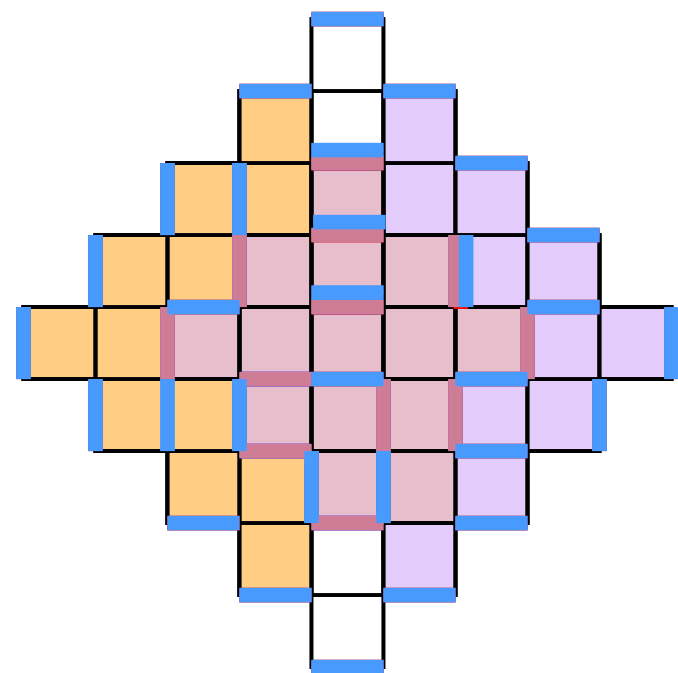






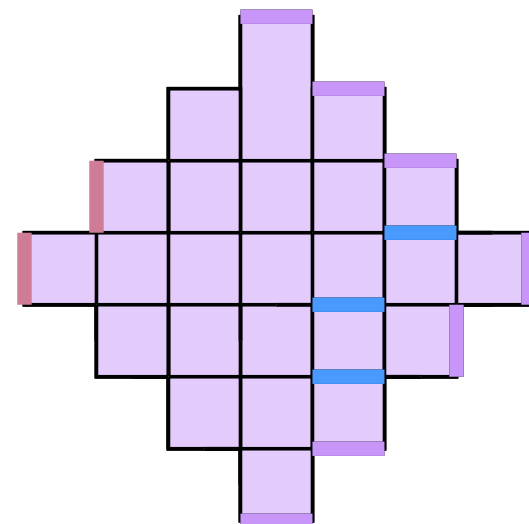
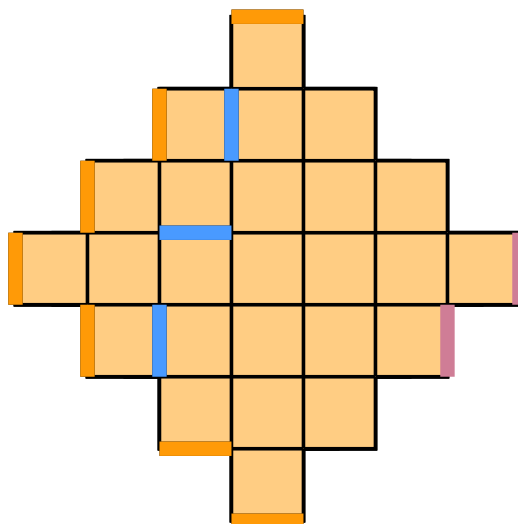
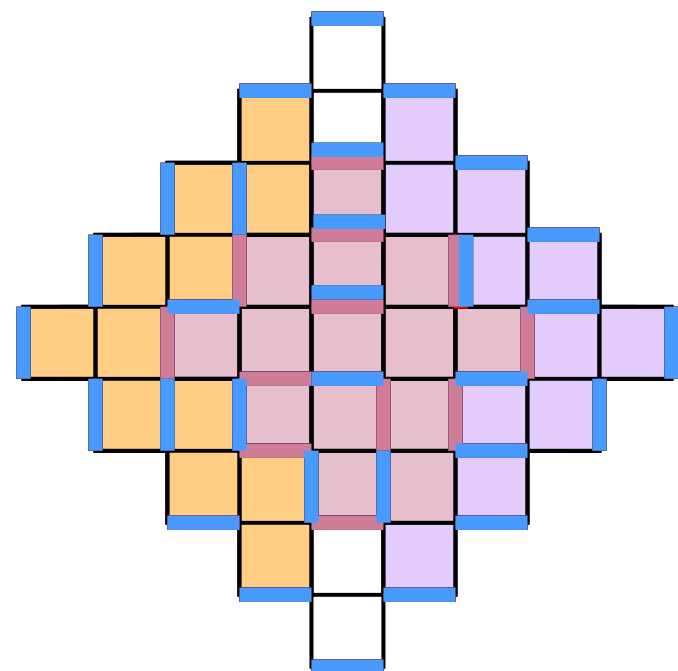
$$a(n)a(n-2)$$

$$a(n-1) \times a(n-1)$$



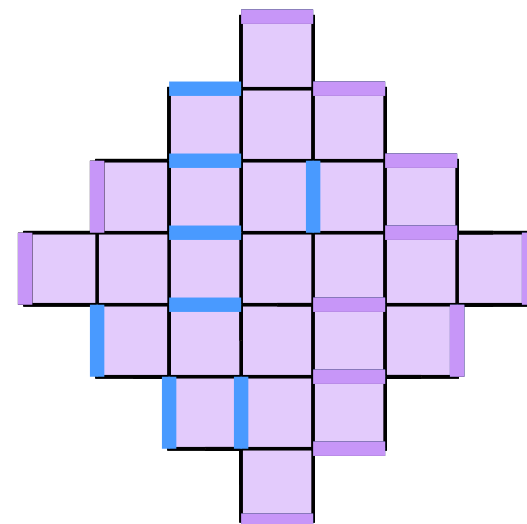
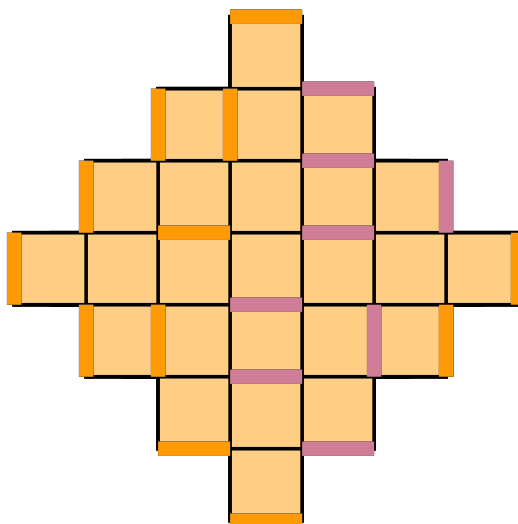
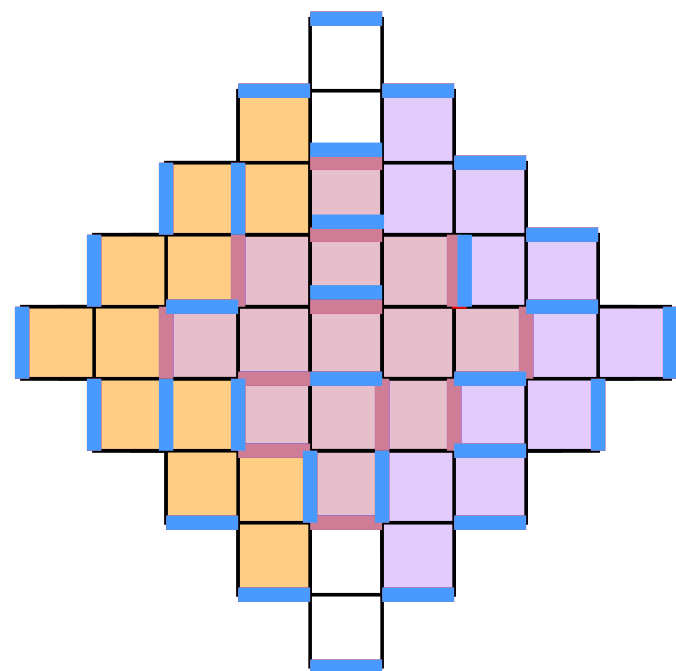
$$a(n)a(n-2)$$

$$a(n-1) \times a(n-1)$$



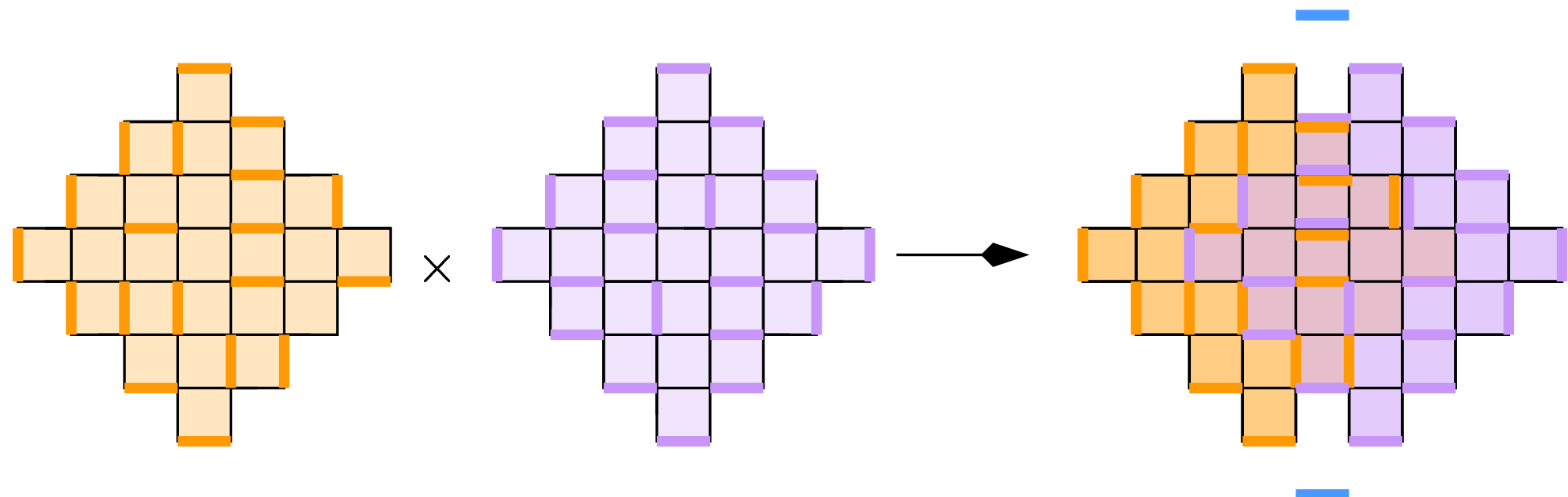
$$a(n)a(n-2)$$

$$a(n-1) \times a(n-1)$$



$$a(n)a(n-2)$$

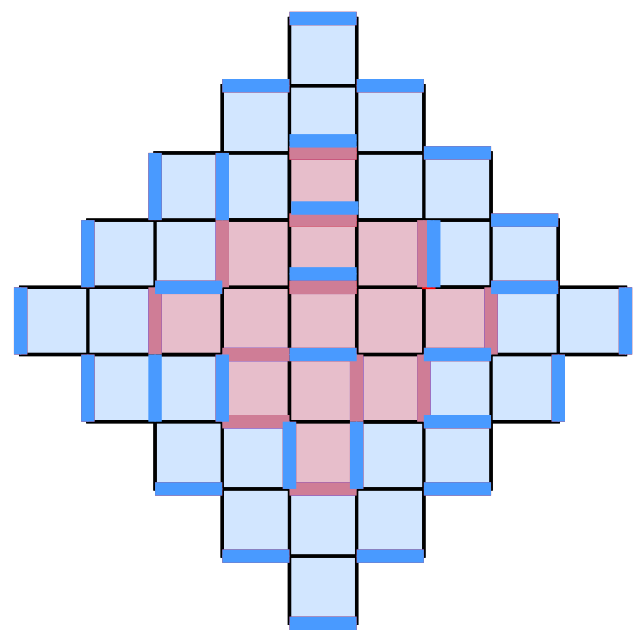
$$a(n-1) \times a(n-1)$$



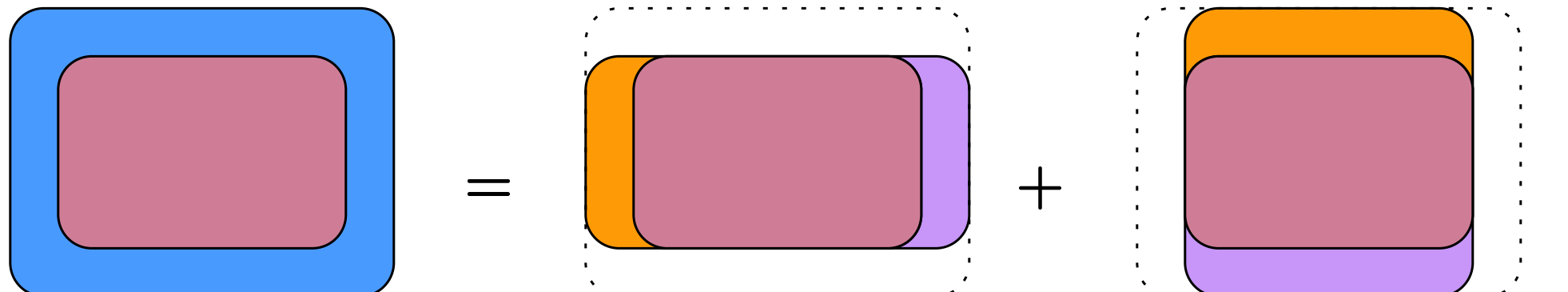
$$a(n - 1) \times a(n - 1)$$

=

$$a(n)a(n - 2) =$$



What we are aiming at...

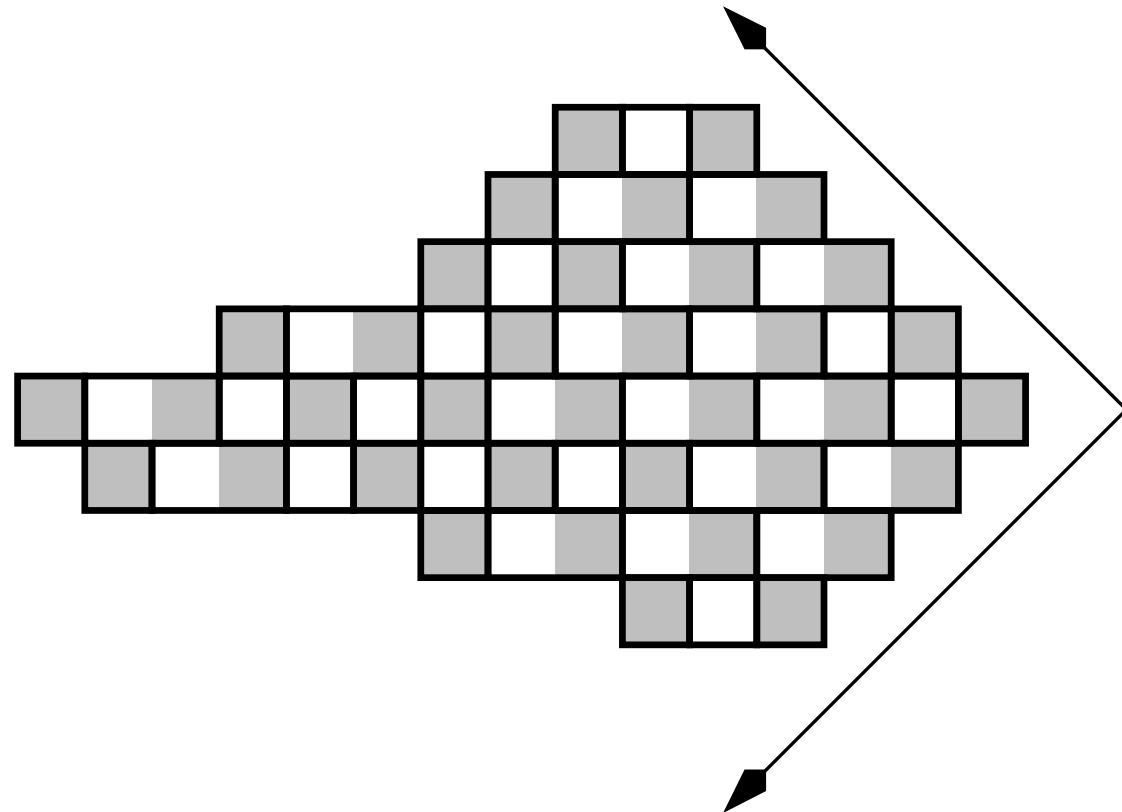


The diagram illustrates the decomposition of a product of two functions into two terms. On the left, a blue rounded rectangle contains a pink rounded rectangle. This is set equal to the sum of two terms. The first term consists of a dashed rounded rectangle containing a pink rounded rectangle with an orange rounded rectangle on its left side and a purple rounded rectangle on its right side. The second term consists of a dashed rounded rectangle containing a pink rounded rectangle with an orange rounded rectangle on its top side and a purple rounded rectangle on its bottom side.

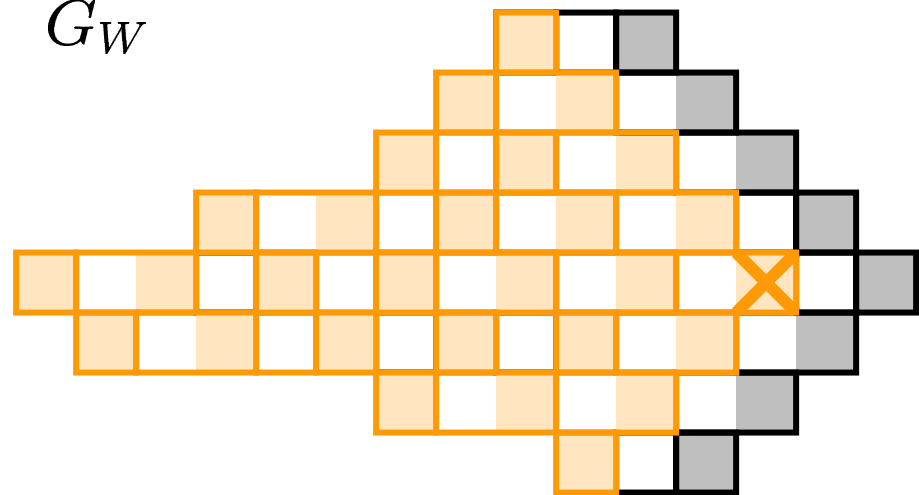
$$a(n)a(n-m) = a(n-i)a(n-j) + a(n-k)a(n-l)$$

A pinecone

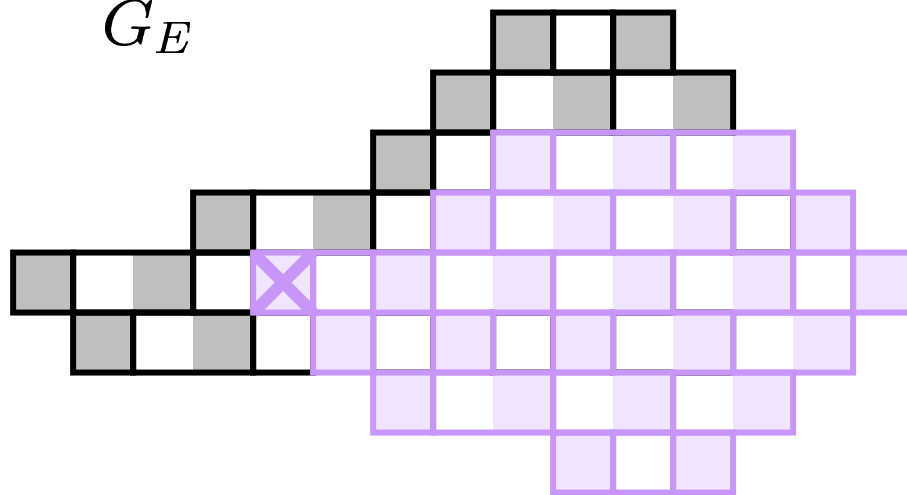
- cells  
- rows of odd length
- pinecone shape



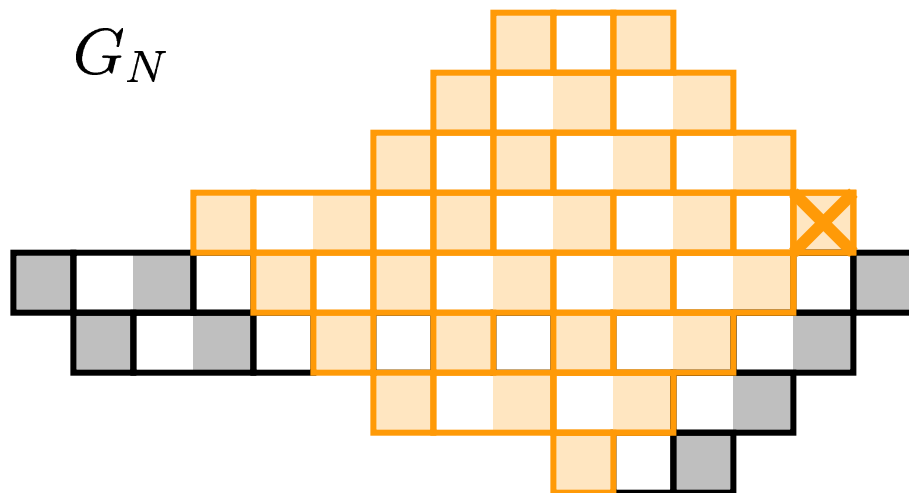
G_W



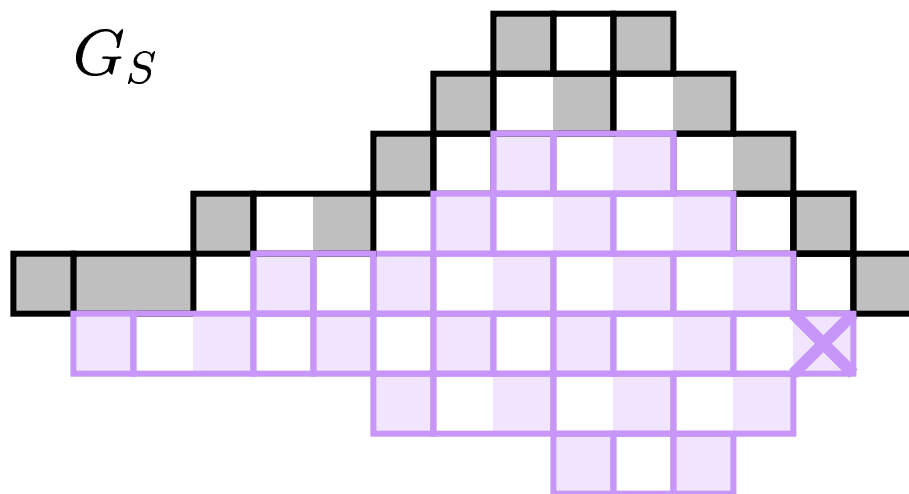
G_E



G_N



G_S

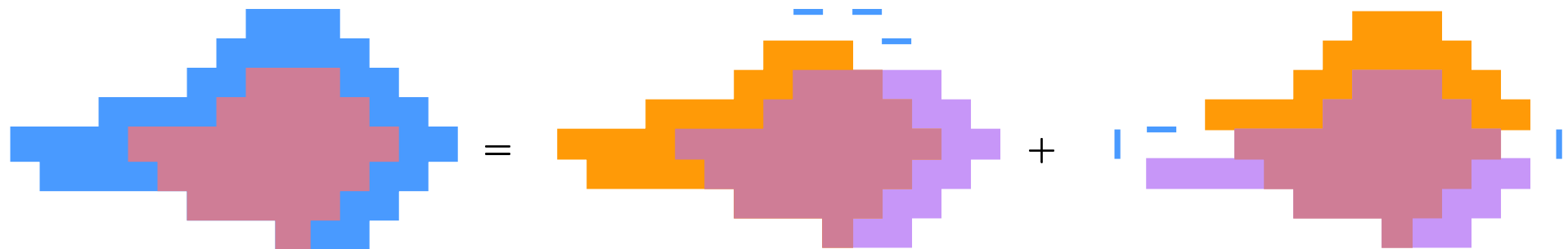


A diagram showing a 2D grid of squares. The grid is composed of white, gray, and pink squares. The pink squares form a central, roughly rectangular region. Two red 'X' marks are placed on the grid, one on the left and one on the right, both within the pink region. The label G_C is positioned to the left of the grid.

The condensation theorem

The number of perfect matchings of the pinecone G is related to the number of perfect matchings of its five sub-pinecones:

$$M(G)M(G_C) = M(G_W)M(G_E) + M(G_N)M(G_S).$$



Pinecones for the Gale-Robinson sequences

For every Gale-Robinson sequence

$$a(n)a(n - \textcolor{red}{m}) = a(n - \textcolor{brown}{i})a(n - \textcolor{violet}{j}) + a(n - \textcolor{brown}{k})a(n - \textcolor{violet}{\ell})$$

there exists a sequence $G(n)$ of pinecones such that

- $G(0), \dots, G(m-1)$ are reduced to a single edge,
- for $n \geq m$,

$$\begin{aligned} G_{\textcolor{brown}{N}}(n) &= G(n - \textcolor{brown}{k}), \\ G_{\textcolor{brown}{W}}(n) &= G(n - \textcolor{brown}{i}), \quad G_{\textcolor{red}{C}}(n) = G(n - \textcolor{red}{m}), \quad G_{\textcolor{violet}{E}}(n) = G(n - \textcolor{violet}{j}), \\ G_{\textcolor{violet}{S}}(n) &= G(n - \textcolor{violet}{\ell}). \end{aligned}$$

By the condensation theorem, $a(n)$ is the number of perfect matchings of $G(n)$, and hence

$a(n)$ is an integer!

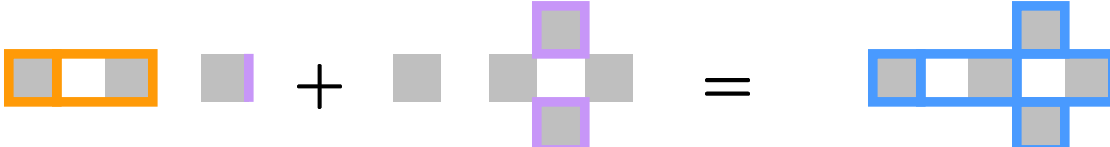
Pinecones for Somos-4

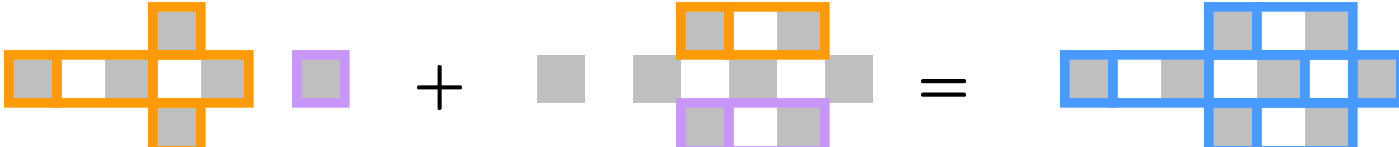
For $n \geq 4$,

$$a(n) = \frac{a(n-1)a(n-3) + a(n-2)a(n-2)}{a(n-4)}.$$

$$G(4) =$$


$$G(5) =$$


$$G(6) =$$


$$G(7) =$$


$$G(8) = \frac{\begin{array}{c} \text{Orange shape} + \text{Purple shape} \end{array}}{=} \text{Blue shape}$$

The diagram shows the addition of two shapes to form a third. The first shape (orange) is a cross-like structure with a central 2x2 square, a horizontal bar of 5 squares, and a vertical bar of 3 squares. The second shape (purple) is a cross-like structure with a central 2x2 square, a horizontal bar of 3 squares, and a vertical bar of 3 squares. The result (blue) is a cross-like structure with a central 2x2 square, a horizontal bar of 7 squares, and a vertical bar of 3 squares.

$$G(9) = \frac{\begin{array}{c} \text{Orange shape} + \text{Purple shape} \end{array}}{\begin{array}{c} \text{Red shape} \end{array}} = \text{Blue shape}$$

The diagram shows the addition of two shapes to form a third, with a red shape below the line. The first shape (orange) is a cross-like structure with a central 2x2 square, a horizontal bar of 5 squares, and a vertical bar of 3 squares. The second shape (purple) is a cross-like structure with a central 2x2 square, a horizontal bar of 3 squares, and a vertical bar of 3 squares. The result (blue) is a cross-like structure with a central 2x2 square, a horizontal bar of 7 squares, and a vertical bar of 3 squares. The red shape below the line is a 2x2 square.

$$G(10) = \frac{\begin{array}{c} \text{Orange shape} + \text{Purple shape} \end{array}}{\begin{array}{c} \text{Red shape} \end{array}} = \text{Blue shape}$$

The diagram shows the addition of two shapes to form a third, with a red shape below the line. The first shape (orange) is a cross-like structure with a central 2x2 square, a horizontal bar of 5 squares, and a vertical bar of 3 squares. The second shape (purple) is a cross-like structure with a central 2x2 square, a horizontal bar of 3 squares, and a vertical bar of 3 squares. The result (blue) is a cross-like structure with a central 2x2 square, a horizontal bar of 7 squares, and a vertical bar of 3 squares. The red shape below the line is a cross-like structure with a central 2x2 square, a horizontal bar of 3 squares, and a vertical bar of 3 squares.

Homework

Construct the pinecones associated with the Somos-5 sequence:

$$a(n)a(n - 5) = a(n - 1)a(n - 4) + a(n - 2)a(n - 3)$$

A non-recursive description of pinecones

Take your favourite Gale-Robinson sequence

$$a(n)a(n - m) = a(n - i)a(n - j) + a(n - k)a(n - \ell).$$

For $x, y \geq 0$, put a black square at coordinates

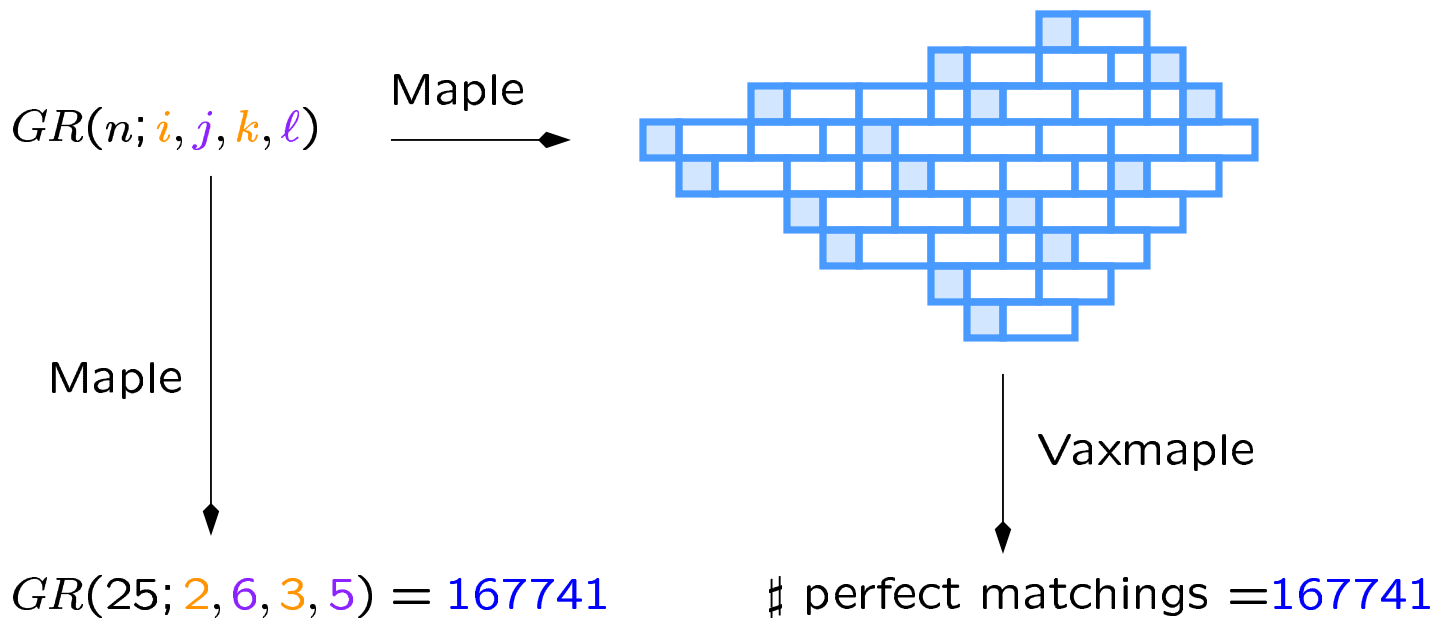
$$u = x + y - 2 \left\lfloor \frac{j - n - 1 + kx + \ell y}{i} \right\rfloor - 4, \quad v = y - x.$$

Keep those that satisfy $|v| \leq u$: they define the pinecone $G(n)$.

Let's check!

Take your favourite Gale-Robinson sequence:

$$a(n)a(n-8) = a(n-2)a(n-6) + a(n-3)a(n-5).$$



Extensions

1. Add coefficients:

$$a(n)a(n-m) = A a(n-i)a(n-j) + B a(n-k)a(n-\ell).$$

Then $a(n)$ is a polynomial in A and B with nonnegative integer coefficients [BM-P-W].

2. Generic initial conditions

$$a(0) = x_0, \dots, a(m-1) = x_{m-1}.$$

Then $a(n)$ is a Laurent polynomial in the x_i with integer coefficients [Fomin-Zelevinski] (conjecture: nonnegative coefficients).

3. Three-term Gale-Robinson sequences

$$a(n)a(n-m) = A a(n-i)a(n-j) + B a(n-k)a(n-\ell) + C a(n-p)a(n-q),$$

plus generic initial conditions. Then $a(n)$ is a Laurent polynomial in the x_i with coefficients in $\mathbb{Z}[A, B, C]$ [Fomin-Zelevinski] (conjecture: coefficients in $\mathbb{N}[A, B, C]$).