Perfect matchings for the Gale-Robinson sequences

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A Gale-Robinson sequence

- $a(0) = a(1) = a(2) = a(3) = 1$

- $a(n)a(n-4) = a(n-1)a(n-3) + a(n-2)a(n-2)$ for $n \geqslant 4$.

[Somos-4]

$$a(n) = \frac{a(n-1)a(n-3) + a(n-2)a(n-2)}{a(n-4)}$$

$$a(4) = \frac{1 \times 1 + 1 \times 1}{1} = 2$$

$$a(5) = \frac{2 \times 1 + 1 \times 1}{1} = 3$$

$$a(6) = \frac{3 \times 1 + 2 \times 2}{1} = 7$$

$$a(7) = \frac{7 \times 2 + 3 \times 3}{1} = 23$$

$$a(8) = \frac{23 \times 3 + 7 \times 7}{2} = 59$$

$$a(9) = \frac{59 \times 7 + 23 \times 23}{3} = 314$$

$$a(10) = \frac{314 \times 23 + 59 \times 59}{7} = 1529$$

$$a(11) = \frac{1529 \times 59 + 314 \times 314}{23} = 8209$$

The Gale-Robinson sequences

Initial conditions $a(0) = \cdots = a(m-1) = 1$, and for $n \geqslant m$,

$$a(n)a(n-m) = a(n-i)a(n-j) + a(n-k)a(n-\ell),$$

with $i + j = k + \ell = m$. Convention: $i \leqslant j, k \leqslant \ell, i \leqslant k$.

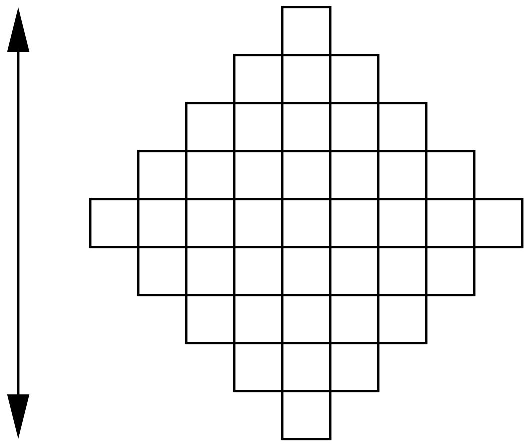
Ex: Somos-4

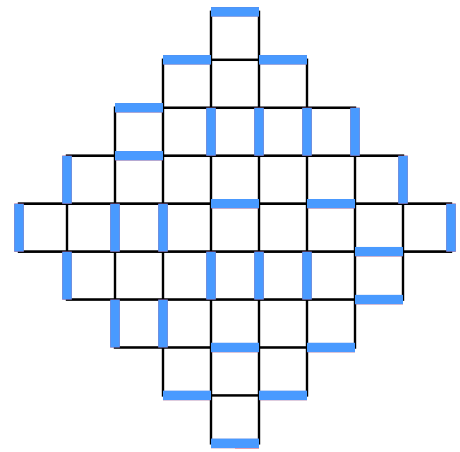
$$a(n)a(n-4) = a(n-1)a(n-3) + a(n-2)a(n-2)$$

Theorem [Fomin, Zelevinski 02]: The numbers $a(n)$ are integers.

Perfect matchings of the aztec diamond

$2n-3$

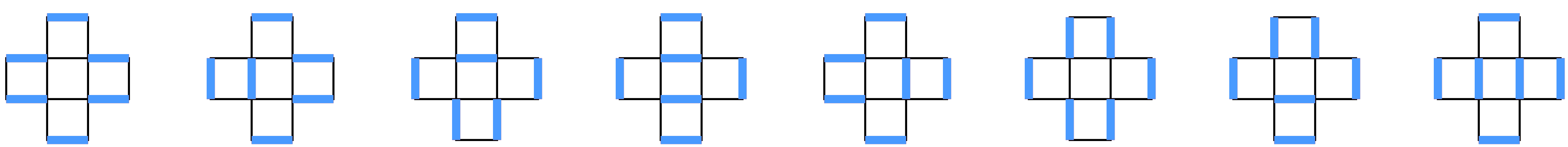




Theorem [Elkies Kuperberg Larsen Propp 92]: The aztec diamond of height $2n-3$ has $a(n) = 2^{\binom{n}{2}}$ perfect matchings.

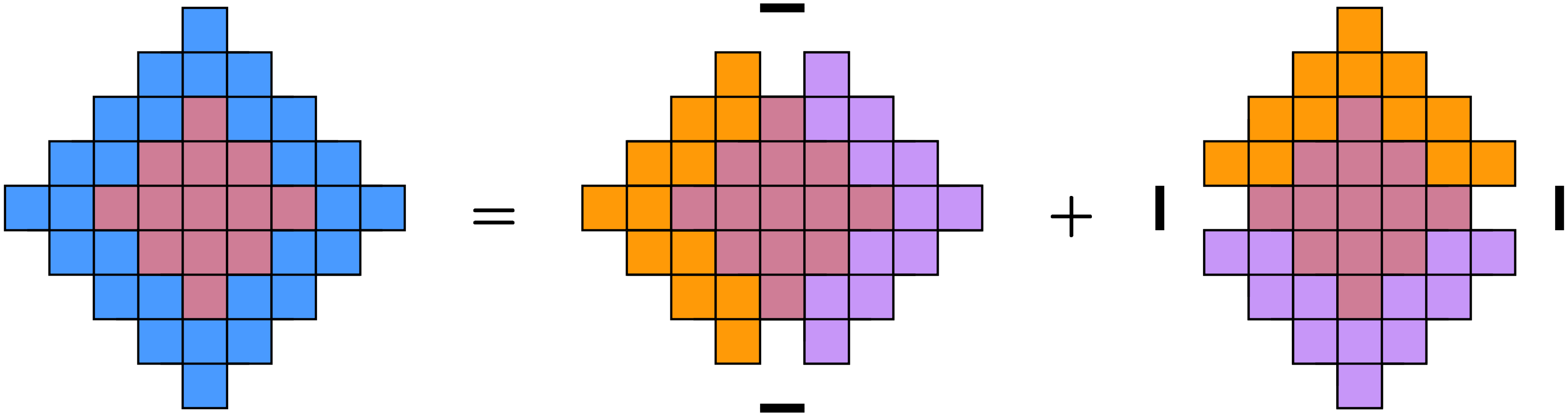
$$a(n)a(n-2) = a(n-1)a(n-1) + a(n-1)a(n-1)$$

Ex: $n = 3$

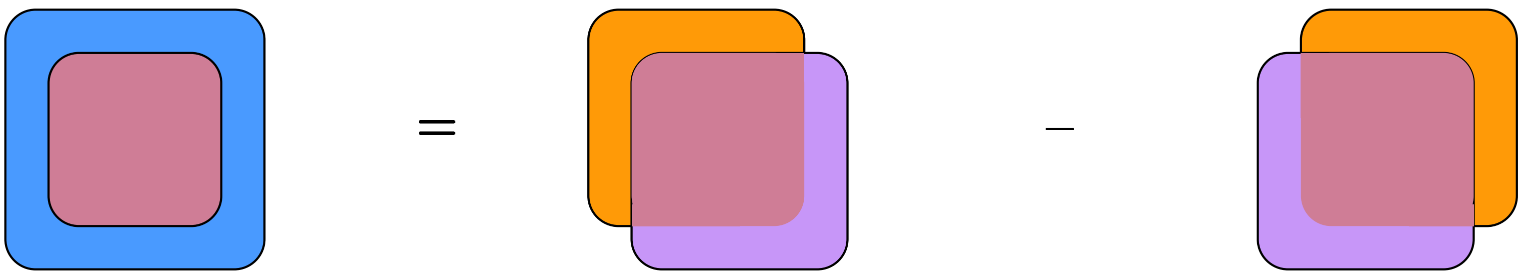


Eric Kuo's proof: a condensation theorem

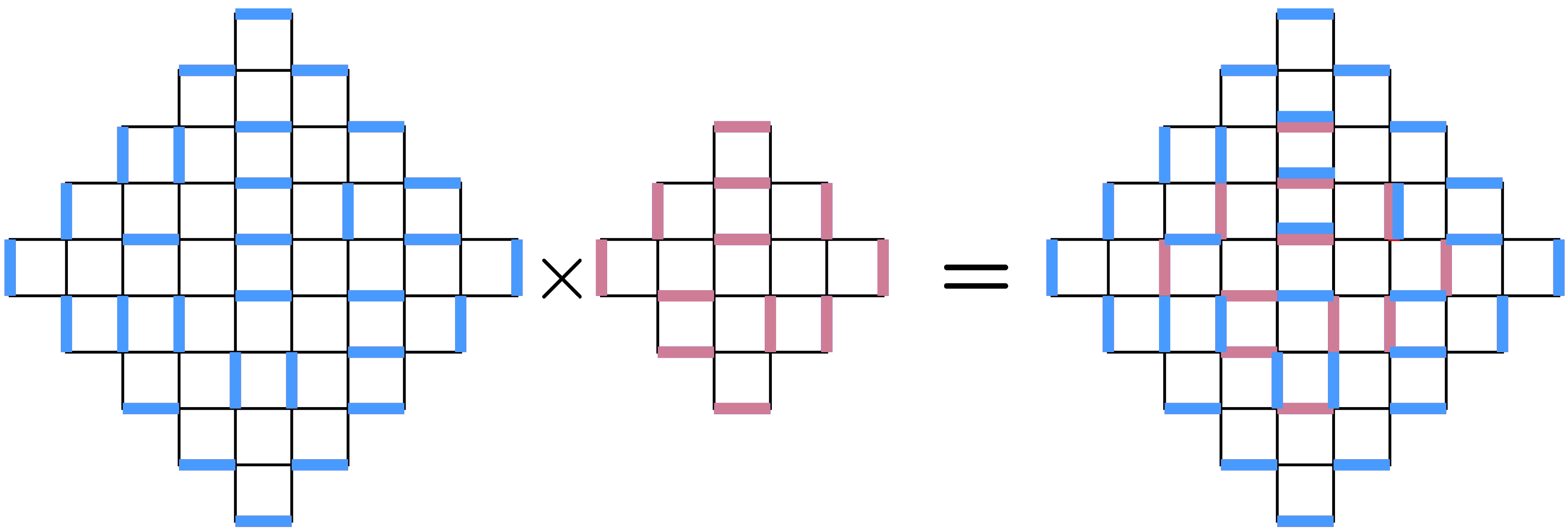
$$a(n)a(n-2) \qquad = \qquad a(n-1)a(n-1) \qquad + \qquad a(n-1)a(n-1)$$

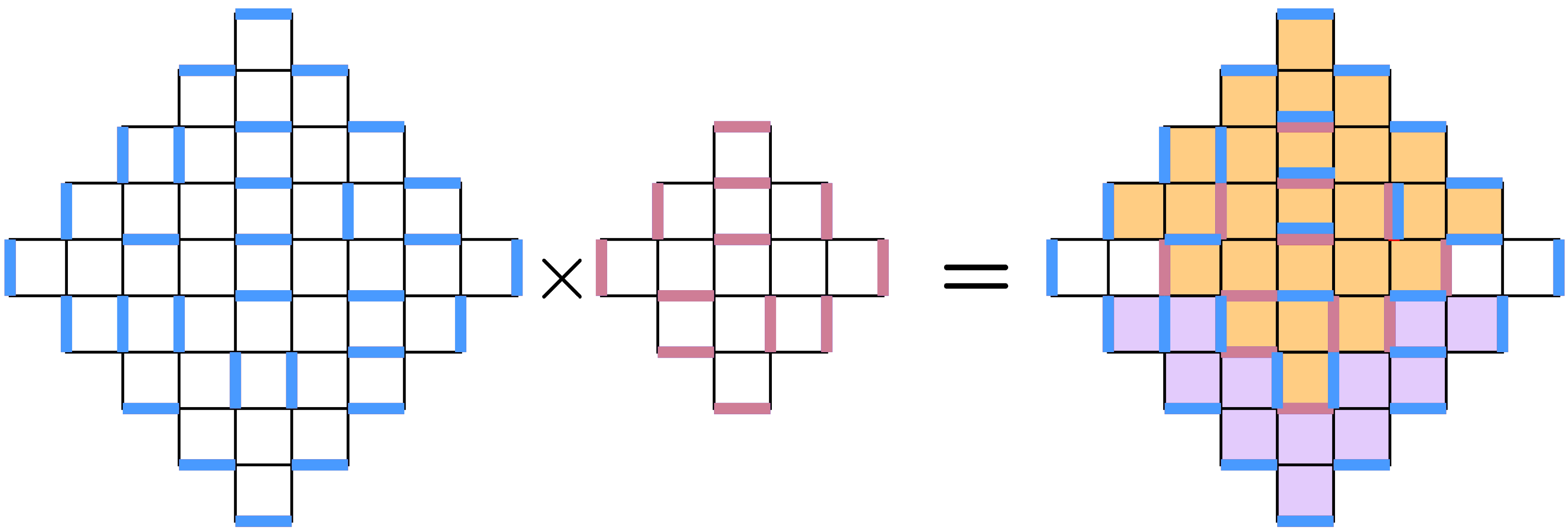


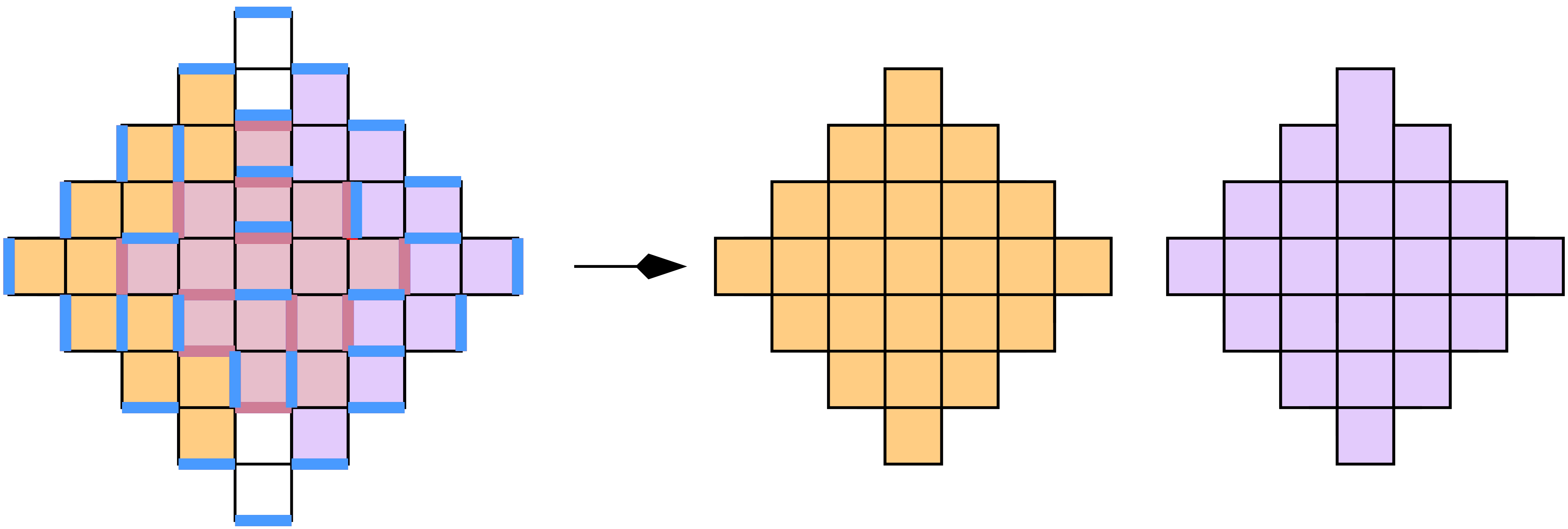
Dodgson's condensation formula



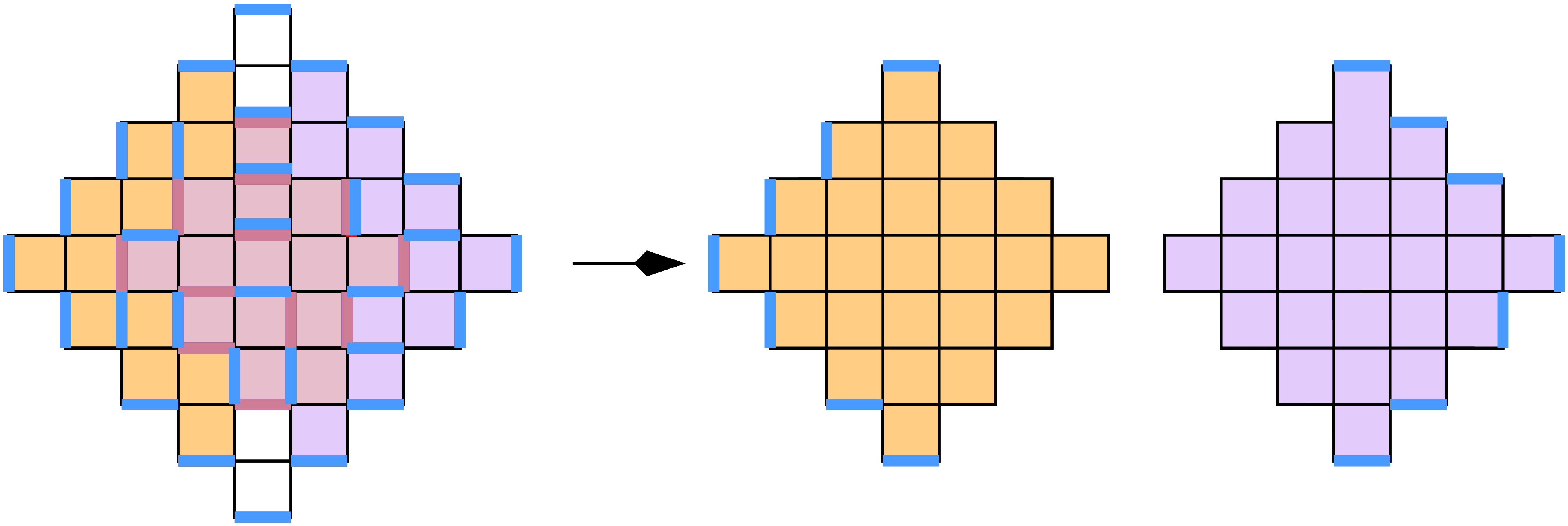
$$\det(M)\det(M_C) = \det(M_{NE})\det(M_{SW}) - \det(M_{NW})\det(M_{SE})$$



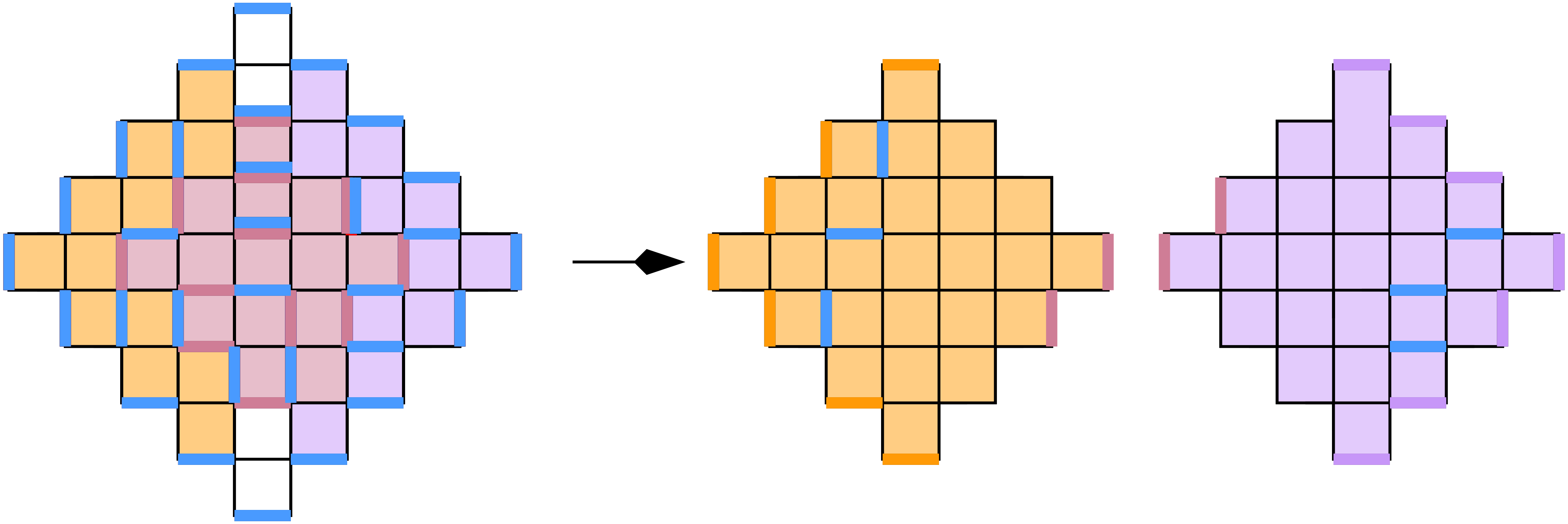




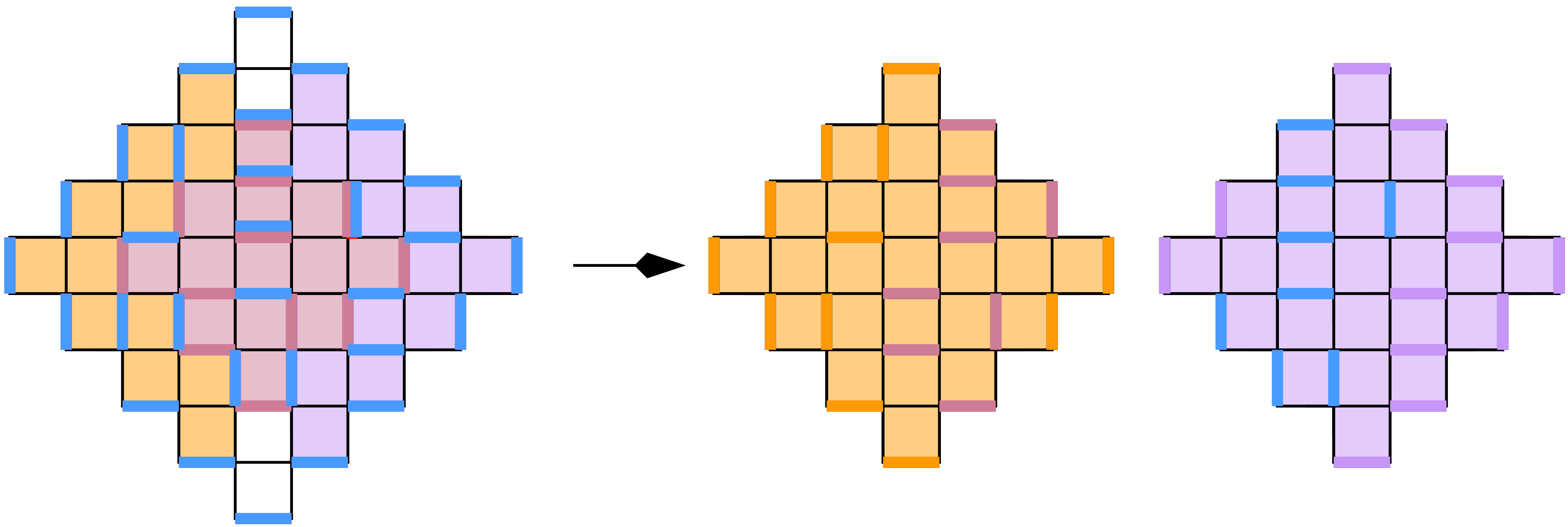
$$a(n)a(n-2) \qquad a(n-1) \times a(n-1)$$



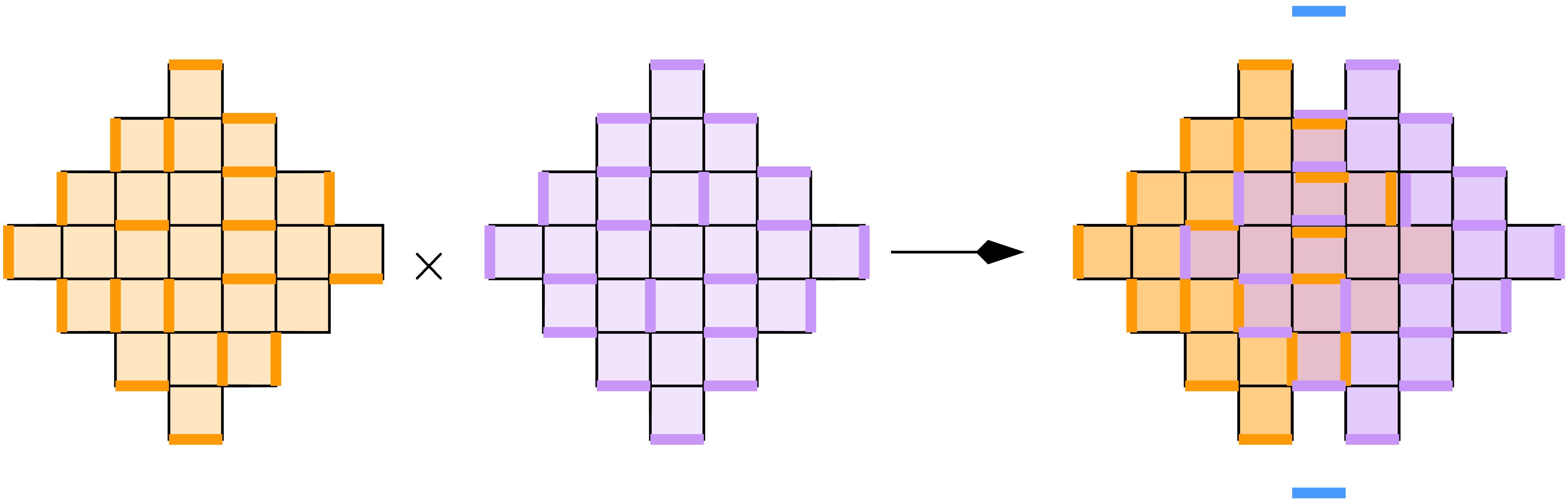
$a(n)a(n-2)$ $\qquad\qquad$ $a(n-1) \times a(n-1)$



$a(n)a(n-2)$ $\longrightarrow$ $a(n-1) \times a(n-1)$

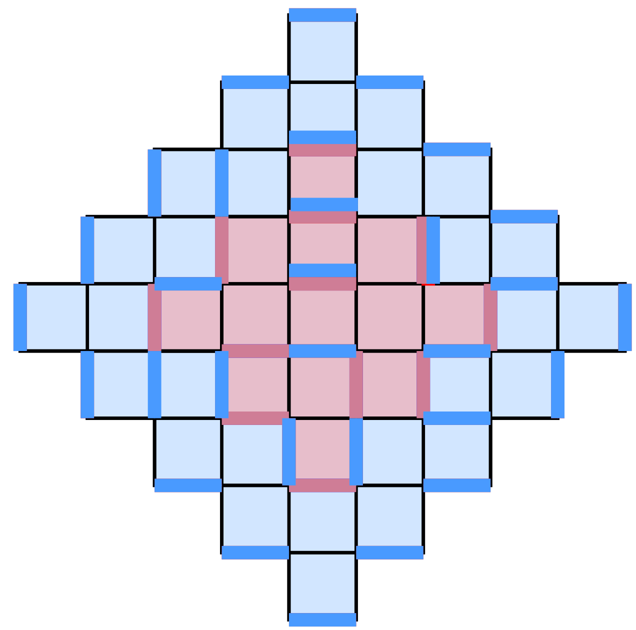


$a(n)a(n-2)$ $\qquad$ $a(n-1) \times a(n-1)$

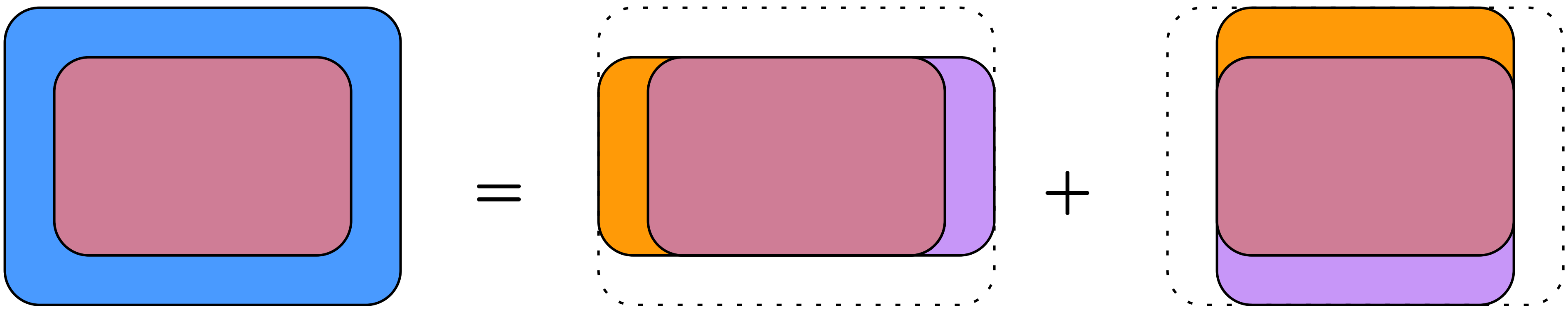


$$a(n-1) \times a(n-1) \quad =$$

$$a(n)a(n-2) \quad =$$

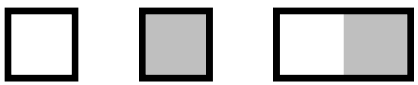


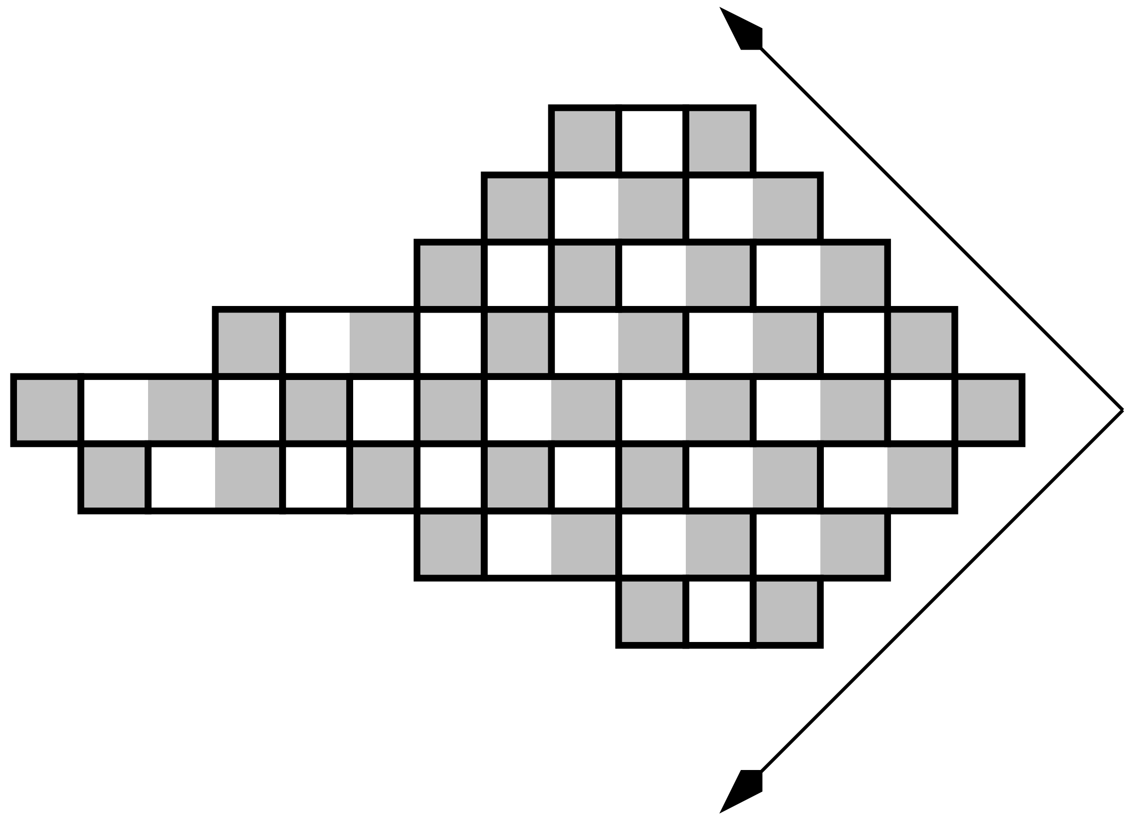
What we are aiming at...

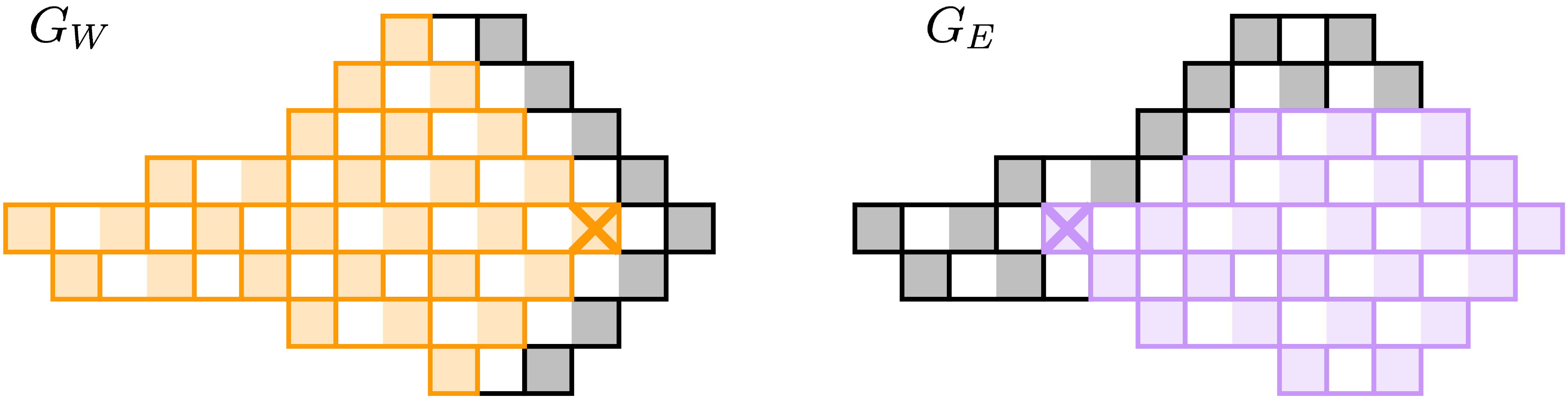


$$a(n)a(n-m) = a(n-i)a(n-j) + a(n-k)a(n-\ell)$$

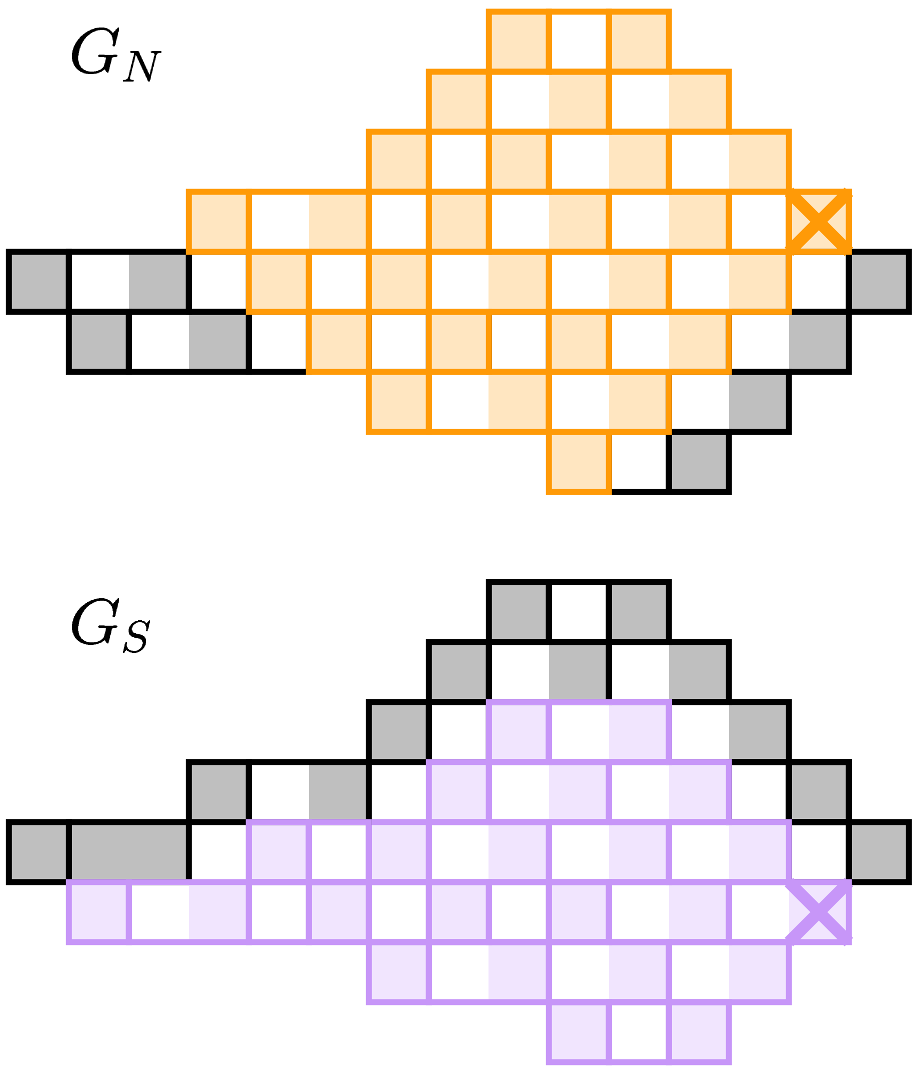
A pinecone

- cells 
- rows of odd length
- pinecone shape

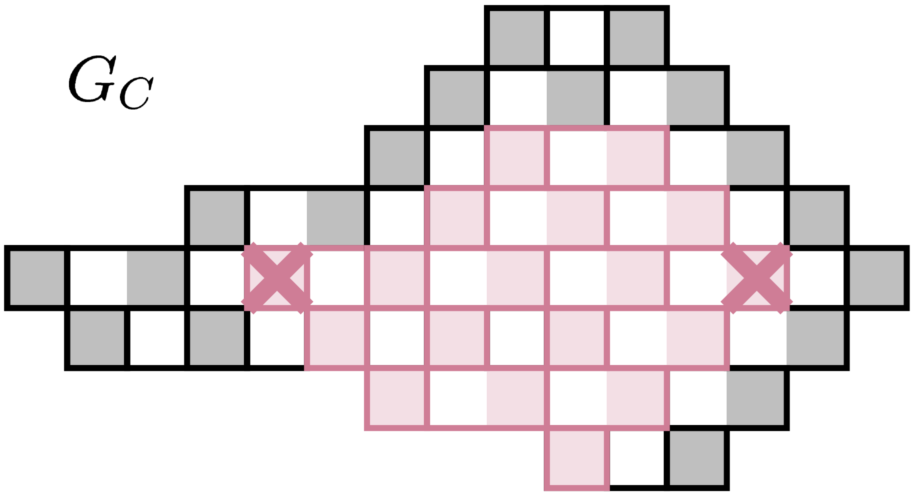




G_W
G_E



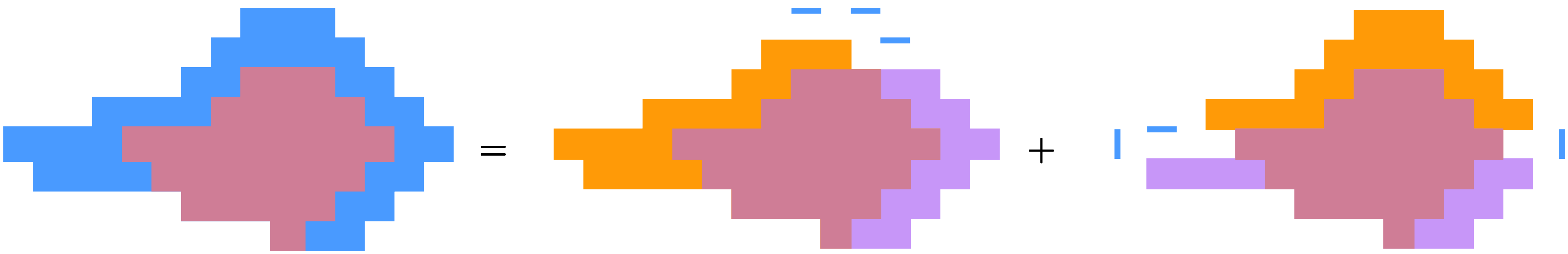
G_N
G_S


G_C

The condensation theorem

The number of perfect matchings of the pinecone G is related to the number of perfect matchings of its five sub-pinecones:

$$M(G)M(G_C) = M(G_W)M(G_E) + M(G_N)M(G_S).$$



Pinecones for the Gale-Robinson sequences

For every Gale-Robinson sequence

$$a(n)a(n-m) = a(n-i)a(n-j) + a(n-k)a(n-\ell)$$

there exists a sequence $G(n)$ of pinecones such that

- $G(0), \ldots, G(m-1)$ are reduced to a single edge,
- for $n \geqslant m$,

$$G_W(n) = G(n-i), \quad \begin{array}{l} G_N(n) = G(n-k), \\ G_C(n) = G(n-m), \\ G_S(n) = G(n-\ell). \end{array} \quad G_E(n) = G(n-j),$$

By the condensation theorem, $a(n)$ is the number of perfect matchings of $G(n)$, and hence

$a(n)$ is an integer!

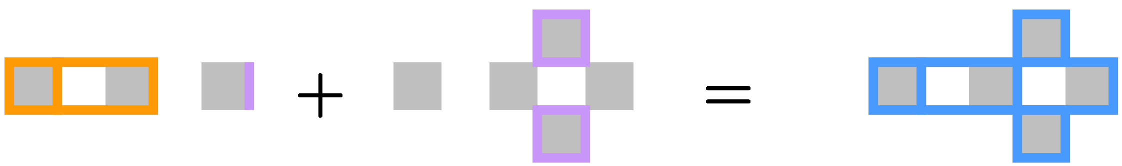
Pinecones for Somos-4

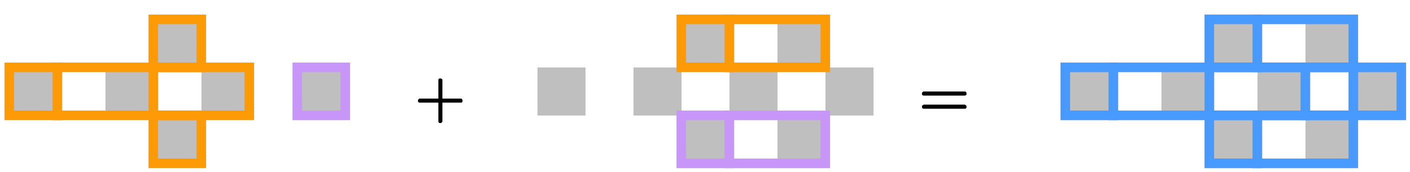
For $n \geqslant 4$,

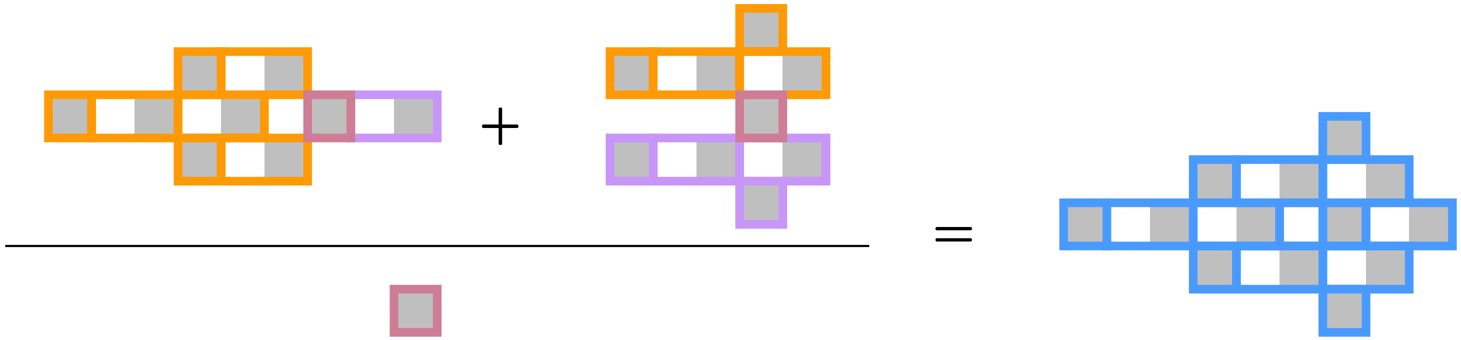
$$a(n) = \frac{a(n-1)a(n-3) + a(n-2)a(n-2)}{a(n-4)}.$$

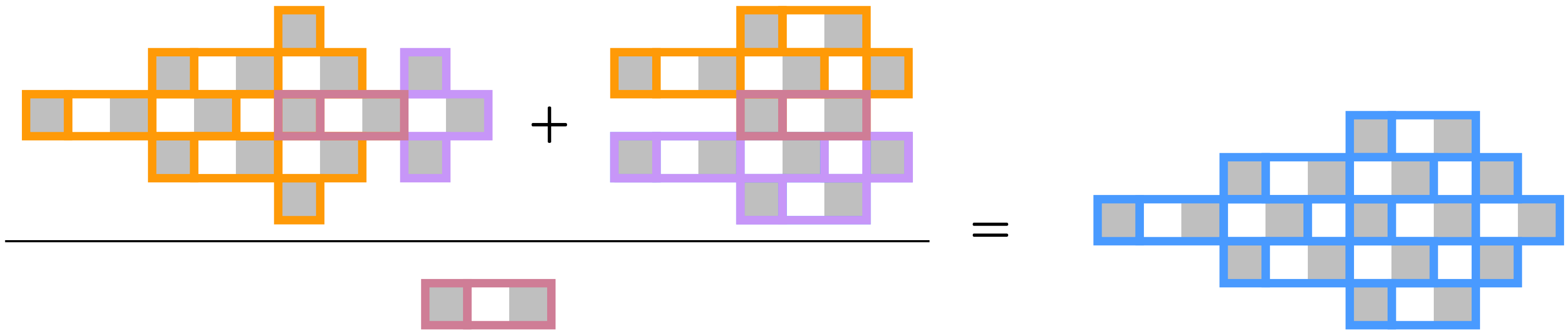
$G(4) =$ 

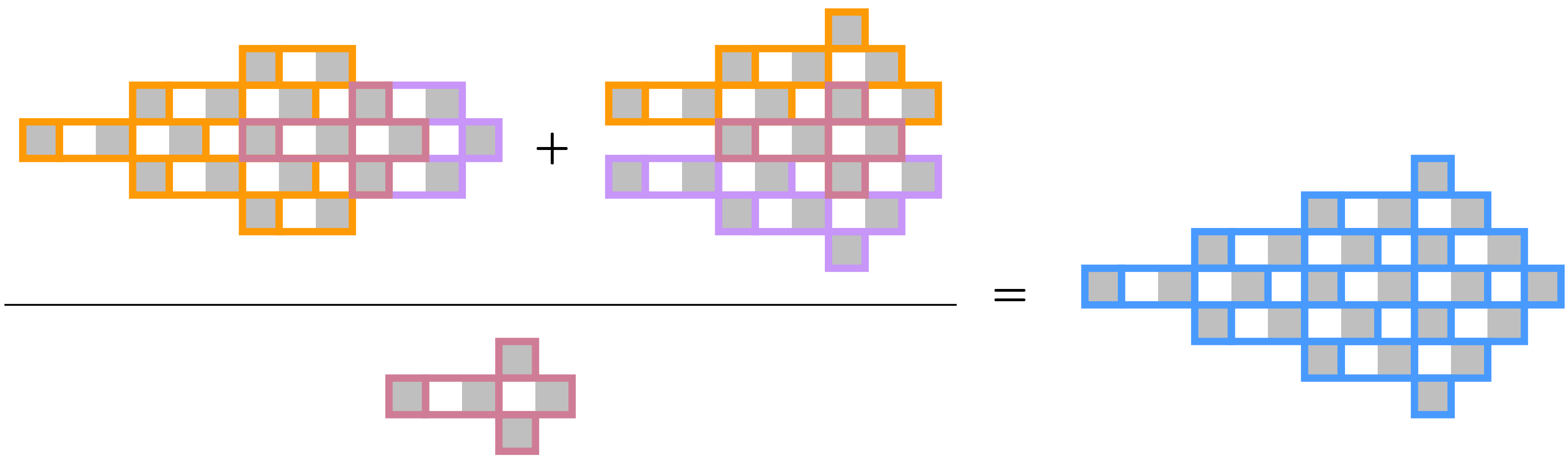
$G(5) =$ 

$G(6) =$ 

$G(7) =$ 

$G(8) =$ 

$G(9) =$ 

$G(10) =$ 

Homework

Construct the pinecones associated with the Somos-5 sequence:

$$a(n)a(n-5) = a(n-1)a(n-4) + a(n-2)a(n-3)$$

A non-recursive description of pinecones

Take your favourite Gale-Robinson sequence

$$a(n)a(n-m) = a(n-i)a(n-j) + a(n-k)a(n-\ell).$$

For $x, y \geqslant 0$, put a black square at coordinates

$$u = x + y - 2\left\lfloor \frac{j-n-1+kx+\ell y}{i} \right\rfloor - 4, \quad v = y - x.$$

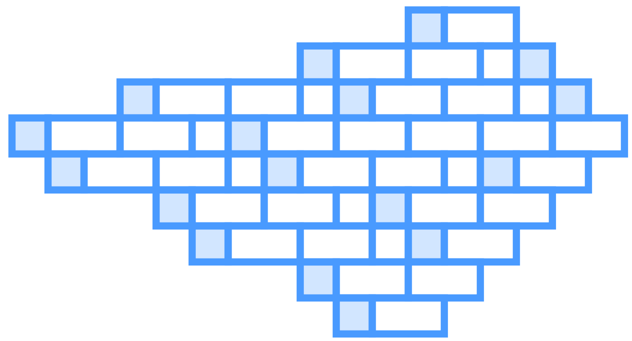
Keep those that satisfy $|v| \leqslant u$: they define the pinecone $G(n)$.

Let's check!

Take your favourite Gale-Robinson sequence:

$$a(n)a(n-8) = a(n-2)a(n-6) + a(n-3)a(n-5).$$

$GR(n; i, j, k, \ell)$ —Maple→



Maple ↓ and Vaxmaple ↓

$GR(25; 2, 6, 3, 5) = 167741$

$\sharp$ perfect matchings $= 167741$

Extensions

1. Add coefficients:

$$a(n)a(n-m) = A\ a(n-i)a(n-j) + B\ a(n-k)a(n-\ell).$$

Then $a(n)$ is a polynomial in A and B with nonnegative integer coefficients [BM-P-W].

2. Generic initial conditions

$$a(0) = x_0, \ldots, a(m-1) = x_{m-1}.$$

Then $a(n)$ is a Laurent polynomial in the x_i with integer coefficients [Fomin-Zelevinski] (conjecture: nonnegative coefficients).

3. Three-term Gale-Robinson sequences

$$a(n)a(n-m) = A\ a(n-i)a(n-j) + B\ a(n-k)a(n-\ell) + C\ a(n-p)a(n-q),$$

plus generic initial conditions. Then $a(n)$ is a Laurent polynomial in the x_i with coefficients in $\mathbb{Z}[A,B,C]$ [Fomin-Zelevinski] (conjecture: coefficients in $\mathbb{N}[A,B,C]$).