Sharp geometric bounds on the measures $\exp(\alpha \mid x \mid^2)dx$, $\alpha \in \mathbf{R}$

Christer Borell, November 11, 2003, at 15^{15}

Abstract The measures

$$d\mu_\alpha(x) = \exp(\alpha \mid x \mid^2)dx, \alpha \in \mathbf{R}$$

in $\mathbf{R}^n$ all satisfy isoperimetric inequalities, where either Euclidean balls ($\alpha \geq 0$) or half spaces are extremal ($\alpha < 0$). Furthermore, by Brunn, Minkowski, and Lusternik, the n:th root of μ_0 is concave (in the sense of Brunn and Minkowski) on the class of all non-empty Borel sets. For $\alpha < 0$, Ehrhard proved in 1983 that $\Phi^{-1}(\frac{\mu_\alpha}{\mu_\alpha(\mathbf{R}^n)})$ is concave restricted to convex bodies in $\mathbf{R}^n$. Here

$$\Phi(x) = \int_{-\infty}^{x} \exp(-\frac{y^2}{2})\frac{dy}{\sqrt{2\pi}}, \text{-}\infty \leq x \leq \infty.$$

In this talk we prove that $\Phi^{-1}(\frac{\mu_\alpha}{\mu_\alpha(\mathbf{R}^n)})$ is concave on the class of all Borel sets if $\alpha < 0$. This solves Problem 1 in the Ledoux and Talagrand book "Probability in Banach Spaces". At present there are no concavity results known for $\alpha > 0$.