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Harmonic Measure and "Locally Flat" Domains

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In this lecture, I will describe a series of joint works with Tatiana Toro on the relationship between regularity properties of harmonic measure and Poisson kernels, and regularity properties of the underlying domains.

Thus, consider a domain $\Omega \subseteq \mathbb{R}^{n+1}$ and the solution to the classical Dirichlet problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u\big|_{\partial\Omega} = f \in C_b(\partial\Omega), \end{cases} \tag{DP}$$

$u \in C_b(\overline{\Omega})$, where C_b is the class of bounded continuous functions.

The maximum principle and the Riesz representation theorem yield the formula

$$u(X_*) = \int_{\partial\Omega} f(Q)\, d\omega^{X_*}(Q), \quad X_* \in \Omega,$$

and the family of positive Borel probability measures $\{d\omega^{X_*}\}$ is called harmonic measure. We sometimes fix $X_* \in \Omega$ and write $d\omega = d\omega^{X_*}$.

Note that, if Ω is a smooth domain, then $d\omega^{X_*}(Q) = \frac{\partial G}{\partial \vec{n}_Q}(Q, X_*)d\sigma(Q)$, where G is the Green's function for Ω, $d\sigma$ is surface measure, and $\frac{\partial}{\partial \vec{n}_Q}$ denotes differentiation along the outward unit normal.

When Ω is unbounded and v is a minimal harmonic function in Ω with $v\big|_{\partial\Omega} \equiv 0$, we define $d\omega^\infty$, harmonic measure with pole at infinity, to be the measure satisfying

$$\int_{\partial\Omega} \varphi d\omega^\infty = \int_\Omega v\Delta\varphi, \quad \text{for } \varphi \in C_0^\infty(\Omega).$$

The existence and uniqueness of v and ω^∞ (modulo multiplicative constants) can be established, for instance, when Ω is an unbounded NTA (non-tangentially accessible) domain. For example, if $\Omega = \mathbb{R}^{n+1}_+ = \{(x,t) : t > 0\}$, then $v(x,t) = t$ and $d\omega^\infty = dx$ on $\mathbb{R}^n$.

The work I will describe originated from trying to understand, as $\alpha \to 0$, the classical theorem of Kellogg, which shows that, if Ω is of class $C^{1,\alpha}$, $0 < \alpha < 1$, then $d\omega = k\, d\sigma$ with $\log k \in C^\alpha$; and its "converse", the free boundary regularity of Alt-Caffarelli (1981), which states that, if Ω satisfies certain necessary weak conditions (to be more fully explained later) and $d\omega = k\, d\sigma$ with $\log k \in C^\alpha$, then Ω must be of class $C^{1,\alpha}$.

To motivate our results, we recall real variable characterizations of $C^{1,\alpha}$ and C^α:

$\varphi \in C^{1,\alpha}(\mathbb{R}^n)$ $(0 < \alpha < 1) \Leftrightarrow \forall r > 0, x_0 \in \mathbb{R}^n$, there exists an affine function L_{r,x_0} on $\mathbb{R}^n$ such that

$$\frac{|\varphi(x) - L_{r,x_0}(x)|}{r} \leq Cr^\alpha \text{ for } |x - x_0| < r. \tag{I$_\alpha$}$$

When $\alpha = 0$, this condition is equivalent to the Zygmund class condition $\varphi \in \Lambda_*$, i.e.,

$$\frac{|\varphi(x+h) + \varphi(x-h) - 2\varphi(x)|}{|h|} \leq C.$$

For us, when $\alpha = 0$, the λ_* class will also be relevant, where $\varphi \in \lambda_*$ if $\varphi \in \Lambda_*$ and, in addition, the ratio described above tends to 0 as $\alpha \to 0$.

$$h \in C^\alpha \Leftrightarrow \sup_{r>0} \frac{1}{r^\alpha} \mathrm{av}_{B_r} |h - h_{B_r}| \leq C, \tag{II$_\alpha$}$$

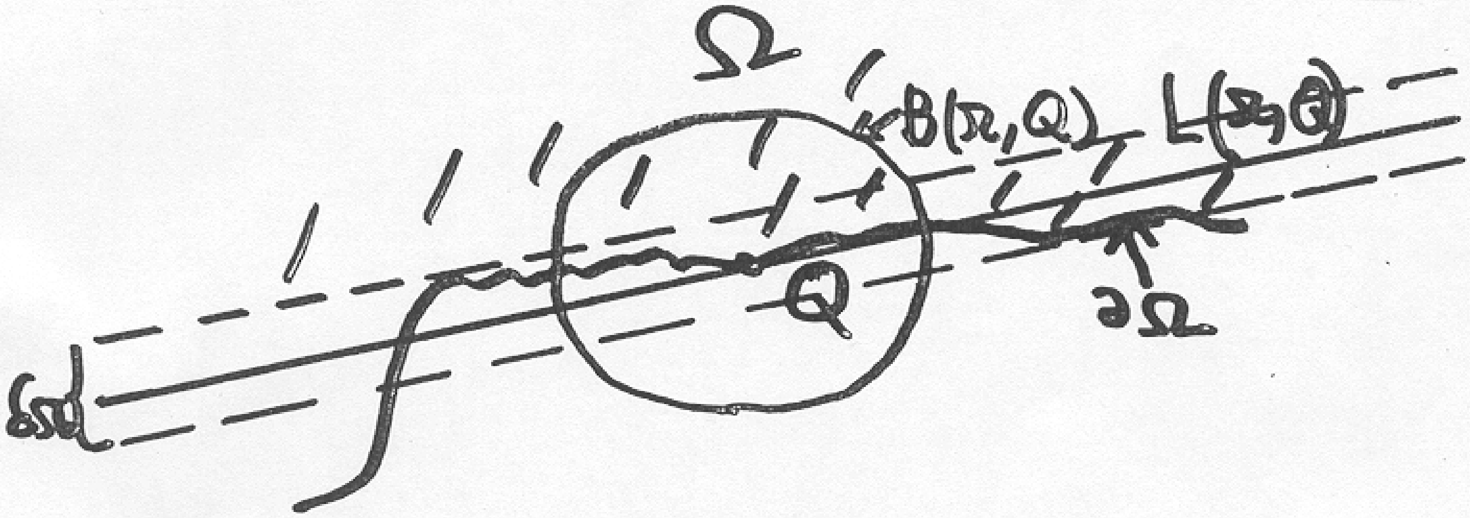
where av_A denotes the average over the set A and B_r any ball of radius r. When $\alpha = 0$, this becomes the BMO space of John-Nirenberg, but we will be more interested in VMO, where $h \in$ VMO if $h \in$ BMO and in addition $\mathrm{av}_{B_r}|h - h_{B_r}| \xrightarrow{r \to 0} 0$. Note that VMO plays the role *vis-à-vis* BMO that continuous functions play *vis-à-vis* L^∞.

We start out by giving our geometric analogue of $(\mathrm{I})_0$: We say that $\Omega \subseteq \mathbb{R}^{n+1}$ is δ-Reifenberg flat if it has the separation property (a quantitative connectivity property), and, for all compact $K \subseteq\subseteq \mathbb{R}^{n+1}$, there exists

$R_K > 0$, such that, for $0 < r < R_k$ and $Q \in \partial\Omega \cap K$, there exists an n-dimensional plane $L(r, Q)$ passing through Q such that

$$\frac{1}{r} D\big[B(r,Q) \cap \partial\Omega, B(r,Q) \cap L(r,Q)\big] \leq \delta,$$

where D denotes Hausdorff distance. Note that this is a significant condition only for $\delta < 1$. We will always assume $\delta < \frac{1}{4\sqrt{2}}$. We say that Ω is Reifenberg vanishing if, as $r \to 0$, we can take $\delta \to 0$.



For instance, the domain above the graph of a λ_* function is Reifenberg vanishing. In general, Reifenberg vanishing domains are not local graphs; they do not have tangent planes or a "surface measure". This class of domains was introduced by Reifenberg (1960) in his study of the Plateau problem for minimal surfaces in higher dimensions.

In order to state our analogue of Kellogg's theorem in this setting, we need to introduce "multiplicative" analogues of $(\mathrm{I})_0$.

A measure μ, supported on $\partial\Omega$, is doubling if, $\forall K \subseteq\subseteq \mathbb{R}^{n+1}$, there exists $R_K > 0$ such that, if $0 < r < R_K$, then

$$\mu\big(B(2r, Q) \cap \partial\Omega\big) \leq C\,\mu\big(B(r, Q) \cap \partial\Omega\big).$$

Such a μ is called asymptotically optimal doubling if it is doubling and

$$\lim_{r\to 0} \inf_{Q\in\partial\Omega\cap K} \frac{\mu(B(\tau r, Q) \cap \partial\Omega)}{\mu(B(r, Q) \cap \partial\Omega)} = \lim_{r\to 0} \sup_{Q\in\partial\Omega\cap K} \frac{\mu(B(\tau r, Q) \cap \partial\Omega)}{\mu(B(r, Q) \cap \partial\Omega)} = \tau^n,$$

for $0 < \tau < 1$, $K \subseteq\subseteq \mathbb{R}^{n+1}$.

For example, if Ω is of class $C^{1,\alpha}$ and $d\sigma$ denotes surface measure, then $\sigma\big(B(r, Q) \cap \partial\Omega\big) = \alpha_n r^n + O(r^{n+\alpha})$, $Q \in \partial\Omega$, and hence σ is asymptotically optimal doubling. If $\log k \in C^\alpha$, then the same is true for $d\omega = k\, d\sigma$.

Our analogue of Kellog's theorem is:

Theorem 1. *(1997) If Ω is a Reifenberg vanishing domain, then ω (ω^∞) is asymptotically optimal doubling.*

To understand a possible converse to Theorem 1, we recall a geometric measure theory (GMT) problem, first posed by Besicovitch: let μ be a positive Radon measure on $\mathbb{R}^{n+1}$ such that, for each $Q \in \Sigma$ (Σ the support of μ) and each $r > 0$, we have

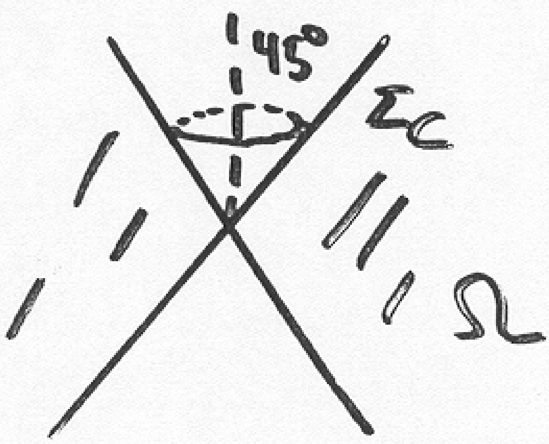
$$\mu\big(B(r, Q)\big) = \alpha r^n, \quad \alpha > 0 \text{ fixed.} \tag{B}$$

Then, what can be said about μ? Clearly, if $d\mu = dx$ on $\mathbb{R}^n \subseteq \mathbb{R}^{n+1}$, then (B) holds.

Nevertheless, in 1987, D. Preiss found the following interesting example: let Σ_C be the light cone $x_4^2 = x_1^2 + x_2^2 + x_3^2$, and $d\mu = d\sigma_{\Sigma_C}$ its surface measure. Then μ satisfies (B). Moreover, the general case of (B) is settled by the following remarkable theorem of Kowalski-Preiss (1987).

Theorem . *Let μ be a non-zero measure with property (B), and put $\Sigma = \operatorname{supp}\mu \subseteq \mathbb{R}^{n+1}$. If $n = 1, 2$, then $\Sigma = \mathbb{R}^n$. If $n \geq 3$, then either $\Sigma = \mathbb{R}^n$ or $\Sigma = \Sigma_C \otimes \mathbb{R}^{n-3}$, modulo rigid motions.*

The connection of the Preiss example to our problem comes from the fact that, if $\Omega = \{x_4^2 < x_1^2 + x_2^2 + x_3^2\}$, $\Omega \subseteq \mathbb{R}^4$, then $d\omega^\infty = d\sigma_{\Sigma_C}$ (separation of variables) and, by Preiss's result, ω^∞ is asymptotically optimal doubling, but, of course, Ω is not Reifenberg vanishing, since it is $\frac{1}{4\sqrt{2}}$-Reifenberg flat, and no better.



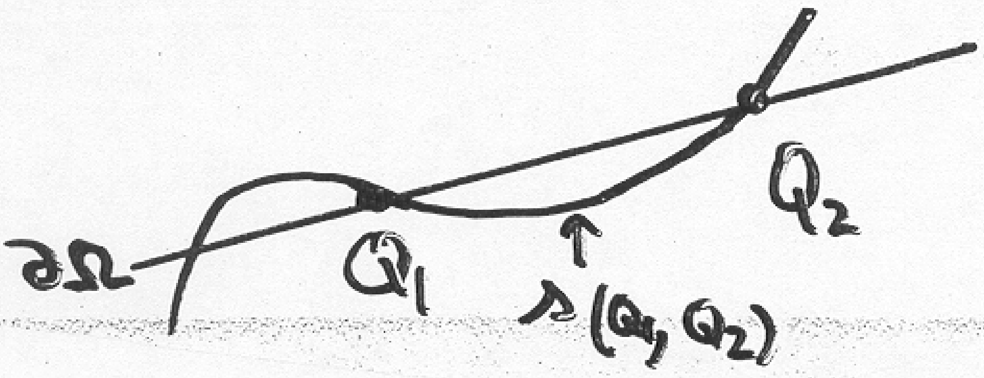
Our converse to Theorem 1 is now:

Theorem 2. *(1999) Assume that $\Omega \subseteq \mathbb{R}^{n+1}$, it verifies the separation property , and that ω (ω^∞) is asymptotically optimal doubling. If $n = 1, 2$, then Ω is Reifenberg vanishing. If $n \geq 3$ and Ω is δ-Reifenberg flat, $\delta < \frac{1}{4\sqrt{2}}$, then Ω is Reifenberg vanishing.*

This is in fact a GMT result. It remains valid if ω (ω^∞) is replaced by any asymptotically optimal doubling measure μ with support $\partial\Omega$. The idea of the proof is to use a "blow-up" argument to reduce matters to the Kowalski-Preiss theorem.

We now turn to the results motivated by $(\mathrm{II})_0$. Note that the unit normal $\vec{n}$ satisfies $|\vec{n}| = 1$, and so the BMO condition on it is automatic, but the VMO condition is not. To put our work in perspective, we recall some of the history of the subject.

A domain $\Omega \subseteq \mathbb{R}^{1+1} = \mathbb{R}^2$ is called a chord-arc domain is $\partial\Omega$ is locally rectifiable, and, whenever $Q_1, Q_2 \in \partial\Omega$, we have
$\ell(s(Q_1, Q_2)) \leq C|Q_1 - Q_2|$, where ℓ denotes length and $s(Q_1, Q_2)$ is the shortest arc between Q_1 and Q_2. Ω is called vanishing chord-arc if, in addition, as $Q_1 \to Q_2$, the ratio $\frac{\ell(s(Q_1,Q_2))}{|Q_1-Q_2|}$ tends to 1, uniformly on compact sets.



The first person to study harmonic measure on chord-arc domains in the plane was Laurentiev (1936), who proved:

Theorem . *If* $\Omega \subseteq \mathbb{R}^{1+1}$ *is chord-arc, then* $d\omega = k\,d\sigma$ *with* $\log k \in \mathrm{BMO}(d\sigma)$. *(In fact,* $\omega \in A_\infty(d\sigma)$*, the Muckenhoupt class.)*

For vanishing chord-arc domains in the plane, Pommerenke (1978) proved:

Theorem . *Suppose that* Ω *is a chord-arc domain in* $\mathbb{R}^{1+1}$. *Then* Ω *is vanishing chord-arc if and only if* $d\omega = k\,d\sigma$ *with* $\log k \in \mathrm{VMO}(d\sigma)$.

These results were obtained using function theory, so their proofs don't generalize to higher dimensions.

In higher dimensions, the first breakthrough came in the celebrated theorem of B. Dahlberg (1977), who showed that, if $\Omega \subseteq \mathbb{R}^{n+1}$ is a Lipschitz domain, then $d\omega = k\,d\sigma$ with $\log k \in \mathrm{BMO}$ (in fact, $\omega \in A_\infty(d\sigma)$).

One direction of Pommerenke's result was extended to higher dimensions by Jerison-Kenig (1982), who showed that, if Ω is a C^1 domain, then $\log k \in$ VMO. (In general, note that Ω is of class C^1 need not imply that $\log k$ is continuous.)

In order to explain our results and to clarify the connection with condition $(\mathrm{II})_0$, we need to introduce some terminology.

We say that $\Omega \subseteq \mathbb{R}^{n+1}$ is a "δ-chord-arc domain" if Ω is δ-Reifenberg flat, Ω is of locally finite perimeter, the boundary of Ω is Ahlfors regular i.e., the surface measure σ (which is Radon measure on $\partial\Omega$, by the assumption of locally finite perimeter) satisfies the inequalities

$$C^{-1}r^n \leq \sigma\big(B(r,Q) \cap \partial\Omega\big) \leq Cr^n$$

and the BMO norm of the unit normal $\vec{n}$ is bounded by δ.

We say that Ω is "vanishing chord-arc" if, in addition, it is Reifenberg vanishing and $\vec{n} \in \mathrm{VMO}(d\sigma)$.

The two notions introduced of "vanishing chord-arc" domains in the plane are the same, and a domain is vanishing chord-arc exactly when it is of locally finite perimeter, has an Ahlfors regular boundary, it is Reifenberg vanishing and satisfies $\vec{n} \in$ VMO. (Semmes 90, KT 97,99).

Our potential-theoretic result, which extends the work of Jerison-Kenig (1982), is

Theorem 3. *(1997) If Ω is a vanishing chord-arc domain, then ω (ω^∞) has the property that $d\omega = k\,d\sigma$ ($d\omega^\infty = h\,d\sigma$) with $\log k \in \mathrm{VMO}$ ($\log h \in \mathrm{VMO}$).*

In order to understand possible converses of this, extending the work of Pommerenke to higher dimensions, we will recall precisely the Alt-Caffarelli result which we alluded to earlier. In the language that we have introduced, their local regularity theorem can be stated as follows:

Theorem . *Let Ω be a set of locally finite perimeter whose boundary is Ahlfors regular. Assume that Ω is δ-Reifenberg flat, $\delta < \delta_n$. Suppose that $d\omega = k\,d\sigma$ with $\log k \in C^\alpha(\partial\Omega)$ ($0 < \alpha < 1$). Then Ω is a $C^{1,\alpha}$ domain.*

The reason for this being a free boundary regularity result is that, in the case when Ω is unbounded and $d\omega = d\omega^\infty$, $d\omega^\infty = h\,d\sigma$, then $v > 0$ in Ω, $v\big|_{\partial\Omega} \equiv 0$, $\Delta v = 0$ in Ω and $h = \frac{\partial v}{\partial \vec{n}}$. Thus, knowledge of the regularity of the Cauchy data of v ($v\big|_{\partial\Omega}, \frac{\partial v}{\partial \vec{n}}\big|_{\partial\Omega}$) yields regularity of $\partial\Omega$ (or of $\vec{n}$, the normal).

The first connection between the above Theorem and the work of Pommerenke was made by Jerison (1990), who was also the first to formulate the higher-dimensional analogues of Pommerenke's theorem as end-point estimates as $\alpha \to 0$ in the Alt-Caffarelli theorem.

Before stating our result, it is useful to clarify the assumptions in the Alt-Caffarelli theorem. For this, we recall some examples:

Examples. When $n = 1$, Keldysh-Laurentiev (1937) constructed domains in $\mathbb{R}^{1+1}$ with locally rectifiable boundaries which can be taken to be Reifenberg vanishing and for which $d\omega = d\sigma$, i.e., $k \equiv 1$, but which are not very smooth. For instance, they fail to be chord-arc. These domains do not, of course, have Ahlfors regular boundaries.
When $n = 2$, Alt-Caffarelli constructed a double cone Γ in $\mathbb{R}^3$ such that, for Ω the domain outside the cone, $d\omega^\infty = d\sigma$, i.e., $k \equiv 1$. This is of course not smooth near the origin, the problem being that, while Ω is Ahlfors regular, Ω is not δ-Reifenberg flat for small δ.
When $n = 3$, the Preiss cone we saw before exhibits the same behavior.

Our first result was:

Theorem 4. *(1999) Assume that $\Omega \subseteq \mathbb{R}^{n+1}$ is δ-chord-arc, $\delta \leq \delta_n$, that ω (ω^∞) is asymptotically optimal doubling and that* $\log k \in \mathrm{VMO}$ *(*$\log h \in \mathrm{VMO}$*). Then* $\vec{n} \in \mathrm{VMO}$ *and Ω is vanishing chord-arc.*

Notice, however, that, when comparing the hypothesis of Theorem 4 to the Alt-Caffarelli theorem two things are apparent :

First, we are making the additional assumption that ω is asymptotically optimal doubling, and hence, in light of Theorem 2, Ω is Reifenberg vanishing.

Next, the "flatness" assumption in the Alt-Caffarelli theorem is δ-Reifenberg flatness, while in Theorem 4 we make the *a priori* assumption that, in addition, the BMO norm of $\vec{n}$ is smaller than δ.

Recently we have developed a new approach which has removed these objections. We have:

Theorem 5. *(2001) Let Ω be a set of locally finite perimeter whose boundary is Ahlfors regular. Assume that Ω is δ-Reifenberg flat, $\delta < \delta_n$. Suppose that $d\omega = k\,d\sigma$ ($d\omega^\infty = h\,d\sigma$) with $\log k \in \mathrm{VMO}(d\sigma)$ ($\log h \in \mathrm{VMO}(d\sigma)$). Then $\vec{n} \in \mathrm{VMO}(d\sigma)$ and Ω is a vanishing chord-arc domain.*

Note that Theorems 3 and 5 together give a complete characterization of the vanishing chord-arc domains in terms of their harmonic measure, in analogy with Pommerenke's 2-dimensional result, thus answering a question posed by Semmes (1990).

Our technique for the proof of Theorem 5 is to use a suitable "blow-up" to reduce matters to the following version of the "Liouville theorem" of Alt-Caffarelli:

Theorem 6. *Let Ω be a set of locally finite perimeter whose boundary is (unboundedly) Ahlfors regular. Assume that Ω is an unbounded δ-Reifenberg flat domain, $\delta < \delta_n$. Suppose that u and h satisfy:*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \ u\big|_{\partial\Omega} \equiv 0 \end{cases}$$

and

$$\int_\Omega u \Delta\varphi = \int_{\partial\Omega} \varphi h \, d\sigma, \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}^{n+1}).$$

Suppose that $\sup_{x\in\Omega} |\nabla u(x)| \leq 1$ *and* $h(Q) \geq 1$ *for* ($d\sigma$-)*a.e.* Q *on* $d\Omega$. *Then* Ω *is a half-space and* $u(x, x_{n+1}) = x_{n+1}$.

This is combined with the crucial blow-up result, which we now describe.

Let Ω be as in Theorem 5, and assume in addition that Ω is unbounded. Suppose $d\omega^\infty = h \, d\sigma$ with $\log h \in \mathrm{VMO}(d\sigma)$, and let u be the associated harmonic function. Let $Q_i \in \partial\Omega$ and assume that $Q_i \to Q_\infty \in \partial\Omega$ as $i \to \infty$ (without loss of generality, $Q_\infty = 0$). Let $\{r_i\}_{i=1}^\infty$ be a sequence of positive numbers tending to 0, and put

$$\Omega_i = \frac{1}{r_i}(\Omega - Q_i), \qquad \partial\Omega_i = \frac{1}{r_i}(\partial\Omega - Q_i),$$

$$u_i(X) = \frac{1}{r_i \mathrm{av}_{B(r_i,Q_i)} h \, d\sigma} u(r_i X + Q_i) \quad \text{and} \quad d\omega_i^\infty = h_i(Q) d\sigma_i(Q),$$

where $h_i(Q) = \frac{1}{\mathrm{av}_{B(r_i,Q_i)} h \, d\sigma} h(r_i Q + Q_i)$. Then:

Theorem 7. *There exists a subsequence of $\{\Omega_i\}$ (which we will call again $\{\Omega_i\}$) satisfying:*

$$\Omega_i \to \Omega_\infty \text{ in the Hausdorff distance sense, uniformly on compacts;} \tag{7.1}$$

$$\partial\Omega_i \to \partial\Omega_\infty \text{ in the Hausdorff distance sense, uniformly on compacts;} \tag{7.2}$$

$$u_i \to u_\infty \text{ uniformly on compacts;} \tag{7.3}$$

$$\begin{cases} \Delta u_\infty = 0 & \text{in } \Omega_\infty \\ u_\infty = 0 & \text{in } \partial\Omega_\infty \\ u_\infty > 0 & \text{in } \Omega_\infty. \end{cases} \tag{7.4}$$

Furthermore

$$\omega_i \rightharpoonup \omega_\infty \tag{7.5}$$

and

$$\sigma_i \rightharpoonup \sigma_\infty, \tag{7.6}$$

weakly as Radon measures. Here, $\sigma_\infty = \mathcal{H}^n \lfloor \partial\Omega_\infty$ and ω_∞ denotes the harmonic measure of Ω_∞ with pole at ∞ (corresponding to u_∞). Moreover, $\sup_{Z\in\Omega_\infty} |\nabla u_\infty(Z)| \le 1$ and $h_\infty(Q) = \frac{d\omega_\infty}{d\sigma_\infty}(Q) \ge 1$ for $\mathcal{H}^n$-a.e. $Q \in \partial\Omega_\infty$.

Since $\log h \in \mathrm{VMO}(\partial\Omega)$, the average $\mathrm{av}_{B(r,Q)} h\, d\sigma$ is close to the value of h in a proportionally large subset of $B(r,Q) \cap \partial\Omega$. This remark allows us to conclude that (7.6) holds, which is crucial to the application and which fails in general under just (7.1) and (7.2).

As an immediate application of Theorems 6 and 7, we obtain that Ω_∞ is a half-plane. This already establishes that Ω is Reifenberg vanishing in Theorem 5. To establish that $\vec{n}$ is in VMO, we assume otherwise, and obtain $Q_i \to Q_\infty$, $r_i \to 0$, such that $\mathrm{av}_{B(r_i,Q_i)} |\vec{n} - \vec{n}_{B(r_i,Q_i)}|^2 d\sigma \geq \ell^2$, $\ell > 0$. We consider the corresponding blow-up sequence, and let $\vec{e}_{n+1}$ be the direction perpendicular to $\partial\Omega_\infty$. By the divergence theorem and (7.1) and (7.2), we have for $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$ that

$$\lim_{i\to\infty} \int_{\partial\Omega_i} \varphi \langle \vec{n}_i, \vec{e}_{n+1} \rangle d\sigma_i = \int_{\mathbb{R}^n \times \{0\}} \varphi\, dx$$

and hence

$$\lim_{i\to\infty} \left\{ \int_{\partial\Omega_i} \varphi\, d\sigma_i - \frac{1}{2} \int_{\partial\Omega_i} \varphi |\vec{n}_i - \vec{e}_{n+1}|^2 d\sigma_i \right\} = \int_{\mathbb{R}^n \times \{0\}} \varphi\, dx,$$

so that (7.6) yields

$$\lim_{i\to\infty} \int_{\partial\Omega_i} \varphi |\vec{n}_i - \vec{e}_{n+1}|^2 d\sigma_i = 0.$$

Taking $\varphi \geq \chi_{B(1,0)}$ yields the corresponding bound for the integral on $\partial\Omega_i \cap B(1,0)$. But

$$\mathrm{av}_{B(1,0)\cap\partial\Omega_i} |\vec{n}_i - \vec{e}_{n+1}|^2 d\sigma_i = \mathrm{av}_{B(r_i,Q_i)} |\vec{n} - \vec{e}_{n+1}|^2 d\sigma,$$

and hence

$$\ell \leq \overline{\lim}_{i\to\infty}\left(\mathrm{av}_{B(r_i,Q_i)}|\vec{n} - \vec{n}_{B(r_i,Q_i)}|^2 d\sigma\right)^{1/2}$$
$$\leq 2\,\overline{\lim}_{i\to\infty}\left(\mathrm{av}_{B(r_i,Q_i)}|\vec{n} - \vec{e}_{n+1}|^2 d\sigma\right)^{1/2},$$

a contradiction. This concludes the proof.