Negative definite functions. Integral representations independent of a Lévy function and related problems

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Negative definite functions

Abstract

In this paper we give a unitary method for obtaining Lévy Khinchin type formulas for negative definite functions.

We obtain integral representations, independent of a Lévy function, for negative definite functions, with real part bounded below, defined on a commutative involutive semigroup, and for continuous negative definite functions defined on the group $\mathbb{R}^n$.

We reobtain integral representations for continuous negative definite functions defined on the semigroup $\mathbb{R}^n_+$.

Key words: positive definite function, negative definite function, Radon measure, commutative involutive semigroup, quadratic form.

1 Introduction

The negative definite functions occur in probability theory and in potential theory. Their integral representation, known as Lévy-Khinchine formula, depends on a Lévy function (see [2], p. 108, Theorem 3.19 and [7], p. 316, Theorem 8).

The existence of Lévy function is proved in [8] for locally compact commutative groups and in [4] for commutative involutive semigroups.

We give, in Section 3 of this paper, integral representations, for negative definite functions with real part bounded below, defined on a commutative involutive semigroup, which characterize these functions and are independent of a Lévy function. These integral representations can also be obtained using [2], p. 108, Theorem 3.19, but the proof from this paper does not depend on a Lévy function and gives a new method for treating Lévy Khinchin type formulas.

To obtain the integral representations we give in Section 2 a result inspired of Choquet theory on adapted cones (see [5]).

We also use the result of Section 2 to reobtain, in Section 3, the quadratic forms on the semigroup $(\mathbb{N}^2, +)$ with the involution $(m,n)^* = (n,m)$ (see [2], p. 117, Lemma 4.13).

A function $f : [0,\infty[\to \mathbb{R}$ defined by

$$f(x) = C + ax + \int_{-\infty}^{0} (1 - e^{yx}) d\mu(y)$$

where $C, a \in [0,\infty[$ and μ is a positive Radon measure on $]-\infty, 0[$ such that the function $x \to \frac{x}{1+x}$ is μ integrable, is called a Bernstein function (cf. [3], p. 64, 9.8). In Section 4 of this paper, we give a generalization for Bernstein functions by completing a result of Berg ([1], p. 86, 3.2). Using the method of Section

3 we also obtain in Section 4 a new proof for the integral representation of the negative definite functions defined on the semigroup $\mathbb{R}_+^n$ ([1], p. 81, 3.1). This integral representation depends on a Lévy function.

In Section 5 we consider the continuous negative definite functions defined on the group $\mathbb{R}^n$ and we give integral representations for these functions which are also independent of a Lévy function. (see [6] for the classical Lévy-Khinchin formula on $\mathbb{R}^n$).

2 A representation theorem

Let X be a locally compact Hausdorff space. We denote by $\mathcal{C}(X)$ the set $\{f : X \to \mathbb{R} | f$ continuous and with compact support $\}$ and by $\mathcal{C}_+(X)$ the set $\{f \in \mathcal{C}(X) | f \geq 0\}$.

Theorem 1 *Let V be a linear space of real continuous functions on X, such that $V \supset \mathcal{C}(X)$, and $L : V \to \mathbb{R}$ a linear functional, such that $L(f) \geq 0$ for every $f \in V_+$, where $V_+ = \{f \in V | f \geq 0\}$. The restriction of L to $\mathcal{C}(X)$ is a positive Radon measure μ with the following properties:*

(i) *every function of V_+ is μ integrable and we have $L(f) \geq \int_X f(x) d\mu(x)$ for $f \in V_+$;*

(ii) *if we denote by M the set $\{(f,h) \in V \times V_+ |$ there is a compact $K \subset X$ such that $|f(x)| \leq h(x)$ for $x \in X - K\}$ we have $\int_X f(x) d\mu(x) = L(f)$ for every $f \in V$ which satisfies the following condition: for each $\epsilon > 0$ there is a function $h \in V_+$, with $L(h) < \epsilon$, such that $(f,h) \in M$.*

Proof. If $f \in V_+$, $g \in \mathcal{C}_+(X)$ and $f \geq g$, then

$$L(f) \geq L(g) = \mu(g).$$

It results that f is μ integrable and $L(f) \geq \mu(f)$, which proves (i).

Take $\epsilon > 0$ and $(f, h) \in M$ such that $L(h) < \epsilon$. There exists a compact set $K \subset X$ such that

$$|f(x)| \leq h(x),\ x \in X \setminus K.$$

There also exists a compact $K' \subset X$ such that $\int_{X \setminus K'} |f| d\mu \leq \epsilon$.

We choose a continuous function $\varphi : X \to [0, 1]$ with compact support such that $\varphi(x) = 1$ for $x \in K \cup K'$. We have

$$-h \leq f - f\varphi \leq h.$$

The positivity of L yields

$$|L(f) - \int f\varphi d\mu| \leq \epsilon.$$

We obtain

$$|L(f) - \int f d\mu| \leq |L(f) - \int f\varphi d\mu| + |\int f\varphi d\mu - \int f d\mu| \leq 2\epsilon,$$

which finishes the proof. ■

3 Integral representations for negative definite functions

Let $(S, +, *)$ be a commutative involutive semigroup with neutral element 0 (see [2], p. 86). We say that a function $\varphi : S \to \mathbb{C}$ is positive definite on S if for each natural number $n \geq 1$, each family $c_1, \ldots, c_n$ of complex numbers and each family $x_1, \ldots, x_n$ of elements of S, we have

$$\sum_{j,k=1}^{n} c_j \bar{c}_k \varphi(x_j + x_k^*) \geq 0.$$

A function $\varphi : S \to \mathbb{C}$ is hermitian if $\varphi(x^*) = \overline{\varphi(x)}$ for each $x \in S$.

We say that a hermitian function $\varphi : S \to \mathbb{C}$ is negative definite on S if for each natural number $n \geq 2$, each family $c_1, \ldots, c_n$ of complex numbers, such that $c_1 + \ldots + c_n = 0$, and each family $x_1, \ldots, x_n$ of elements of S, we have

$$\sum_{j,k=1}^{n} c_j \bar{c}_k \varphi(x_j + x_k^*) \leq 0.$$

We denote by Λ the set $\{\rho : S \to \mathbb{C} | \rho(0) = 1;\ \rho(x^*) = \overline{\rho(x)},\ x \in S;\ \rho(x+y) = \rho(x)\rho(y),\ x, y \in S;\ |\rho(x)| \leq 1,\ x \in S\}$ and by Ω the set $\{\rho \in \Lambda | \rho \not\equiv 1\}$.

With the product topology Λ is a compact space and Ω is a locally compact space.

Theorem 2 *For a function $\varphi : S \to \mathbb{C}$ the following conditions are equivalent:*

(i) *φ is negative definite on S and has real part bounded below ;*

(ii) *there are a real number C, a function $q : S \to [0, \infty[$, such that*

$$q(x) + q(y) = \frac{1}{2}(q(x+y) + q(x^* + y)),\ x, y \in S,$$

and a positive Radon measure μ on Ω, such that the functions $(\rho \mapsto (1 - Re\ \rho(x)))_{x \in S}$ are μ integrable, which satisfy

$$Re\ \varphi(x) = C + q(x) + \int_{\Omega} (1 - Re\ \rho(x)) d\mu(\rho),\ x \in S$$

and

$$- \,Im\ \varphi(x+y) + Im\ \varphi(x) + Im\ \varphi(y) =$$
$$\int_{\Omega} (Im\ \rho(x+y) - Im\ \rho(x) - Im\ \rho(y)) d\mu(\rho).$$

C, q and μ are uniquely determined by φ.

Proof. (i) $\Rightarrow$ (ii). For every $t \in]0, \infty[$ the function $\psi_t : S \to \mathbb{C}$ defined by $\psi_t(x) = e^{-t\varphi(x)}$ is positive definite (cf. [2], p. 74, Theorem 2.2) and bounded.

It follows from [2], p. 93 Theorem 2.5 that for each $t \in]0, \infty[$ there is a positive Radon measure μ_t on Λ such that

$$e^{-t\varphi(x)} = \int_{\Lambda} \rho(x) d\mu_t(\rho),\ x \in S.$$

We denote by V the set

$$\{f : \Omega \to \mathbb{R} | f = F|_{\Omega},\ F : \Lambda \to \mathbb{R},\ F \text{ continuous },$$
$$F(\theta) = 0,\ \lim_{t \to 0} \frac{1}{t} \int_{\Lambda} F(\rho) d\mu_t(\rho) \text{ exists in } \mathbb{R}\},$$

where $f = F|_{\Omega}$ means that f is the restriction of F to Ω and $\theta : S \to \mathbb{C}$ is defined by $\theta(x) = 1$ for every $x \in S$.

V is a real vector space and the function $L : V \to \mathbb{R}$ defined by $L(f) = \lim_{t \to 0} \frac{1}{t} \int_{\Lambda} F(\rho) d\mu_t(\rho)$ is a linear functional on V such that $L(f) \geq 0$ for $f \in V_+$.

Let $\mathcal{F}$ denote the set of all families $(a_x)_{x \in S}$ of complex numbers such that $a_x \neq 0$ only for finite number of x and which satisfy the relation $\sum_{x \in S} a_x = 0$.

Let U denote the set

$$\{f : \Omega \to \mathbb{R} | f(\rho) = \sum_{x \in S} a_x \rho(x),\ (a_x)_{x \in S} \in \mathcal{F}\}.$$

U is a real vector space. We shall prove that U is a subspace of V. If we take $(a_x)_{x\in S} \in \mathcal{F}$ such that the function defined on Ω by

$$\rho \mapsto \sum_{x\in S} a_x \rho(x)$$

is in U we have

$$\sum_{x\in S} a_x \Big(\frac{e^{-t\varphi(x)} - 1}{t}\Big) = \frac{1}{t}\int_\Lambda \Big(\sum_{x\in S} a_x\rho(x)\Big) d\mu_t(\rho).$$

Letting t tend to 0 we obtain that the function $(\rho \mapsto \sum_{x\in S} a_x\rho(x))$ is in V and that

$$L(\rho \mapsto \sum_{x\in S} a_x\rho(x)) = -\sum_{x\in S} a_x\varphi(x).$$

Next we prove that $\mathcal{C}(\Omega) \subset V$.

Let $f \in \mathcal{C}(\Omega), f \not\equiv 0$. We suppose that the compact support of f is A.

We have $A \subset \Omega = \cup_{x\in S}\{\rho \in \Omega | |1-\rho(x)| > 0\}$ and consequently we can find a natural number $n \geq 1$ and $a_1, \ldots, a_n$ elements of S such that $\sum_{j=1}^n |1-\rho(a_j)| > 0$ on A.

The function defined on S by

$$x \mapsto \frac{1}{t}\int_\Lambda \rho(x)\Big(\sum_{j=1}^n |1-\rho(a_j)|^2\Big) d\mu_t(\rho)$$

is positive definite and it results from the inclusion $U \subset V$ that

$$\lim_{t\to 0}\frac{1}{t}\int_\Lambda \rho(x)\Big(\sum_{j=1}^n |1-\rho(a_j)|^2\Big) d\mu_t(\rho)$$

exists in $\mathbb{R}$. We denote by $u(x)$ this limit.

The function $u : S \to \mathbb{C}$ is positive definite and bounded. Using [2], p. 93, Theorem 2.5, we obtain a positive Radon measure ν on Λ such that

$$u(x) = \int_\Lambda \rho(x) d\nu(\rho), \quad x \in S$$

The Theorem 2.11 from [2], p. 97 implies that we have

$$\lim_{t\to 0}\frac{1}{t}\int_\Lambda F(\rho)d\mu_t(\rho) = \lim_{t\to 0}\frac{1}{t}\int_\Lambda G(\rho)\big(\sum_{j=1}^{n}|1-\rho(a_j)|^2\big)d\mu_t(\rho)$$
$$= \int_\Lambda G(\rho)d\nu(\rho)$$

(where $G(\rho) = \frac{F(\rho)}{\sum_{j=1}^{n}|1-\rho(a_j)|^2}$ for $\rho\in\Omega$ and $G(\theta)=0$), which means that $f\in V$.

Let x, y be elements of S and ϵ a real number such that $0<\epsilon<1$. Let $K_{\epsilon,y}$ (resp. $K'_{\epsilon,y}$) be the compact $\{\rho\in\Omega|\text{Re }\rho(y)\leq 1-\epsilon\}$ (resp. $\{\rho\in\Omega||\text{Im }\rho(y)|\geq \epsilon\}$).

If $x, y\in S$ we have

$$(1-\text{Re }\rho(x))(1-\text{Re }\rho(y))\leq \epsilon(1-\text{Re }\rho(x))$$

for $\rho\in\Omega - K_{\epsilon,y}$ and

$$|(1-\text{Re }\rho(x))\text{Im }\rho(y)|\leq \epsilon(1-\text{Re }\rho(x))$$

for $\rho\in\Omega - K'_{\epsilon,y}$.

Theorem 1 yields a positive Radon measure μ on Ω such that the elements of V_+ are μ integrable and we have

$$-\varphi(0)+\frac{1}{2}\varphi(x)+\frac{1}{2}\varphi(x^*)\geq\int_\Omega(1-\text{Re }\rho(x))d\mu(\rho). \tag{1}$$

$$\begin{aligned}&-\varphi(0)+\frac{1}{2}\varphi(x)+\frac{1}{2}\varphi(x^*)+\frac{1}{2}\varphi(y)+\frac{1}{2}\varphi(y^*)\\&-\frac{1}{4}(\varphi(x+y)+\varphi(x^*+y)+\varphi(x+y^*)+\varphi(x^*+y^*))\\&=\int_\Omega(1-\text{Re }\rho(x))(1-\text{Re }\rho(y))d\mu(\rho).\end{aligned} \tag{2}$$

$$
\begin{aligned}
&-\frac{1}{2i}(\varphi(y)-\varphi(y^*))+\frac{1}{4i}(\varphi(x+y)+\varphi(x^*+y)-\varphi(x+y^*)-\varphi(x^*+y^*)) \\
&=\int_\Omega (1-\text{Re }\rho(x))\text{Im }\rho(y)d\mu(\rho).
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
&-\frac{1}{2i}(\varphi(x)-\varphi(x^*))+\frac{1}{4i}(\varphi(x+y)+\varphi(y^*+x)-\varphi(y+x^*)-\varphi(y^*+x^*)) \\
&=\int_\Omega (1-\text{Re }\rho(y))\text{Im }\rho(x)d\mu(\rho).
\end{aligned} \tag{4}
$$

If we denote by $q : S \to \mathbb{R}$ the function defined by

$q(x) = -\varphi(0) + \text{Re }\varphi(x) - \int_\Omega (1 - \text{Re }\rho(x))d\mu(\rho)$, then the relation (1) gives $q(x) \geq 0$, $x \in S$ and the formula (2) gives

$$
q(x)+q(y)=\frac{1}{2}(q(x+y)+q(x^*+y)).
$$

From (3) and (4) we obtain

$$
\begin{aligned}
&-\frac{1}{2i}(\varphi(y)-\varphi(y^*))-\frac{1}{2i}(\varphi(x)-\varphi(x^*))+\frac{1}{2i}(\varphi(x+y)-\varphi(x^*+y^*)) \\
&=\int_\Omega (1-\text{Re }\rho(x))\text{Im }\rho(y)d\mu(\rho)+\int_\Omega (1-\text{Re }\rho(y))\text{Im }\rho(x)d\mu(\rho) \\
&=\int_\Omega (\text{Im }\rho(x)+\text{Im }\rho(y)-\text{Im }\rho(x+y))d\mu(\rho)
\end{aligned}
$$

which gives the second integral formula from the theorem.

(ii) $\Rightarrow$ (i). Let n be a natural number $\geq 2, c_1, \ldots, c_n$ complex numbers such that $c_1 + \ldots + c_n = 0$ and $x_1, \ldots, x_n$ elements of S. If we have the integral representations of (ii) it follows that

$$\sum_{j,k=1}^{n} c_j\bar{c}_k\varphi(x_j+x_k^*) =$$
$$\sum_{j,k=1}^{n} c_j\bar{c}_k\text{Re }\varphi(x_j+x_k^*) + \sum_{j,k=1}^{n} c_j\bar{c}_k\text{Im }\varphi(x_j+x_k^*) =$$
$$\sum_{j,k=1}^{n} c_j\bar{c}_k q(x_j+x_k^*) + \sum_{j,k=1}^{n} c_j\bar{c}_k(\text{Im }\varphi(x_j)+\text{Im }\varphi(x_k^*))$$
$$+\int_\Omega (-|\sum_{j=1}^{n} c_j\rho(x_j)|^2 + \sum_{j,k=1}^{n} c_j\bar{c}_k(\text{Im }\rho(x_j)+\text{Im }\rho(x_k^*))d\mu(\rho)$$
$$= \sum_{j,k=1}^{n} c_j\bar{c}_k q(x_j+x_k^*) - \int_\Omega |\sum c_j\rho(x_j)|^2 d\mu(\rho) \leq 0$$

because q is negative definite (cf. [2], p. 101, Theorem 3.9).

The unicity of μ results from the equality:

$$-\varphi(x)+\varphi(x+y)+\varphi(x+y^*)-\frac{1}{4}(\varphi(x+2y)+2\varphi(x+y+y^*)+\varphi(x+2y^*) =$$
$$\int_\Omega \rho(x)(1-\text{Re }\rho(y))^2 d\mu(\rho),\ x,y\in S.$$

Unicity of q is a consequence of the unicity of μ because $C=\varphi(0)$. ■

Remark 1 Choose a natural number $n\geq 2, c_1,\dots,c_n$ complex numbers and $x_1,\dots,x_n$ elements of S. The function defined on Ω by $\rho\mapsto|\sum_{j=1}^n c_j\rho(x_j)|^2$ is in V_+ and consequently if we have (i), we obtain

$$-\sum_{j,k=1}^{n} c_j\bar{c}_k\varphi(x_j+x_k^*) \geq \int_\Omega |\sum_{j=1}^{n} c_j\rho(x_j)|^2 d\mu(\rho).$$

This proves that q is negative definite without using [2], p. 101, Theorem 3.9.

Remark 2 It results from the proof of the theorem that we have $\mu=0$ if and only if $\varphi(x)=C+q(x)+i\ell(x)$, $x\in S$, where C and q are as in the Theorem 2 and $\ell:S\to\mathbb{R}$ is a function such that $\ell(x+y)=\ell(x)+\ell(y)$, $x,y\in S$, and $\ell(x^*)=-\ell(x)$, $x\in S$. This is Lemma 3.14 from [2], p. 105.

Remark 3 In the proof of Theorem 2 we have reobtained that $\lim_{t\to 0} \frac{1}{t}\mu_t|_\Omega = \mu$ vaguely (cf. [2], p. 103, Lemma 3.12).

Proposition. Consider the semigroup $(\mathbb{N}^2, +)$ with the involution $(m,n)^* = (n,m)$. For a function $\varphi : \mathbb{N}^2 \to \mathbb{C}$ the following conditions are equivalent:

(i) φ is negative definite and has real part bounded below;

(ii) there are real numbers C, α, β, such that $\alpha, \beta \geq 0$, and a positive Radon measure μ on $\Omega = \{\rho \in \mathbb{C} | |\rho| \leq 1, \rho \not\equiv 1\}$, such that the function $\rho \mapsto 1 - \mathrm{Re}\ \rho$ is μ integrable, which satisfy

$$\mathrm{Re}\ \varphi(m,n) = C + (m+n)\alpha + (m-n)^2\beta + \int_\Omega (1 - \mathrm{Re}\ \rho^m\bar{\rho}^n)d\mu(\rho).$$

and

$$\mathrm{Im}\ (-\varphi(m+p, n+q) + \varphi(m,n) + \varphi(p,q)) = \int_\Omega \mathrm{Im}\ (\rho^{m+p}\bar{\rho}^{n+q} - \rho^m\bar{\rho}^n - \rho^p\bar{\rho}^q)d\mu(\rho).$$

C, α, β and μ are uniquely determined by φ and we have

$$\alpha = -\varphi(0,0) + \frac{1}{2}(\varphi(1,0) + \varphi(0,1)) - \frac{1}{8}(\varphi(2,0) - 2\varphi(1,1) + \varphi(0,2)) - \int_\Omega (1 - \mathrm{Re}\ \rho - \frac{1}{2}(\mathrm{Im}\ \rho)^2)d\mu(\rho)$$

and

$$\beta = \frac{1}{4}(\varphi(2,0) + \varphi(1,1) + \varphi(0,2)) - \int_\Omega (\mathrm{Im}\ \rho)^2 d\mu(\rho).$$

Proof. We denote by D the set $\{t \in \mathbb{C} | |z| \leq 1\}$ and by Λ the set

$$\{\rho : \mathbb{N}^2 \to \mathbb{C} | \rho(0,0) = 1; \rho(m,n) = \overline{\rho(n,m)};$$
$$\rho(m+p, n+q) = \rho(m,n)\cdot\rho(p,q); |\rho(m,n)| \leq 1, m,n,p,q \in \mathbb{N}\}$$

Let $z \in D$. The function $\rho_z : \mathbb{N}^2 \to \mathbb{C}$ given by $\rho_z(m,n) = z^m \bar{z}^n$ is in Λ and the mapping $z \mapsto \rho_z$ is a topological isomorphism of D onto Λ. Using this isomorphism, the Proposition is a particular case of Theorem 2 and we only have to calculate $q(m,n)$ where

$$q(m,n) = -\varphi(0,0) + \text{Re } \varphi(m,n) - \int_\Omega (1 - \text{Re } \rho^m \bar{\rho}^n) d\mu(\rho).$$

As in the proof of the Theorem 2, we notice that the function defined on Ω by

$$\rho \mapsto (1 - \text{Re } \rho)^m (\text{Im } \rho)^n$$

is μ integrable for $m \geq 1$ or $n \geq 2$ and we have

$$L(\rho \mapsto (1 - \text{Re } \rho)^m (\text{Im } \rho)^n) = \int_\Omega (1 - \text{Re } \rho)^m (\text{Im } \rho)^n d\mu(\rho)$$

for $m \geq 2$ or $n \geq 3$ or ($m \geq 1$ and $n \geq 1$).

Using this and the binomial theorem, we obtain that

$$L(\rho \mapsto 1 - \text{Re } \rho^m \bar{\rho}^n - (m+n)(1 - \text{Re } \rho - \frac{1}{2}(\text{Im } \rho)^2) - (m-n)^2 (\text{Im } \rho)^2)$$
$$= \int_\Omega (1 - \text{Re } \rho^m \bar{\rho}^n - (m+n)(1 - \text{Re } \rho - \frac{1}{2}(\text{Im } \rho)^2 - (m-n)^2 \frac{1}{2}(\text{Im } \rho)^2) d\mu(\rho)$$

This is equivalent to $q(m,n) = (m+n)\alpha + (m-n)\beta$, where

$\alpha = L(\rho \mapsto 1 - \text{Re } \rho - \frac{1}{2}(\text{Im } \rho)^2) - \int_\Omega (1 - \text{Re } \rho - \frac{1}{2}(\text{Im } \rho)^2) d\mu(\rho)$ and

$$\beta = L(\rho \mapsto (\text{Im } \rho)^2) - \int_\Omega (\text{Im } \rho)^2 d\mu(\rho).$$

We have $1 - \text{Re } \rho - \frac{1}{2}(\text{Im } \rho)^2 = \frac{1}{2}(1 - \text{Re } \rho)^2 + \frac{1}{2}(1 - (\text{Re } \rho)^2 - (\text{Im } \rho)^2) \geq 0$, which implies that $\alpha \geq 0$. That $\beta \geq 0$ is evident. This finishes the proof of the Proposition. ■

Remark 4 The integral representation of the negative definite functions considered in the Proposition, which depends on a Lévy function, is in [2], p. 119, Proposition 4.15.

4 A generalization for Bernstein functions

In this section $\mathbb{R}_+^n = ([0,\infty[)^n$ and $\langle , \rangle$ is the usual scalar product in $\mathbb{R}^n$. We will consider the semigroup $(\mathbb{R}_+^n, +)$, and assume that this semigroup has identical involution.

Theorem 3 *For a function $\varphi : \mathbb{R}_+^n \to \mathbb{R}$ the following conditions are equivalent:*

(i) *φ is positive, continuous and negative definite on $\mathbb{R}_+^n$;*

(ii) *we have*

$$\varphi(x) = C + \langle a, x\rangle + \int_\Omega (1 - e^{\langle \rho, x\rangle}) d\mu(\rho),\ x \in \mathbb{R}_+^n$$

where $C \in [0,\infty[; a = (a_1, \dots, a_n) \in \mathbb{R}_+^n,\ a_j \geq 0; \Omega = (-\infty, 0]^n \setminus (0, \dots, 0)$ and μ is a positive Radon measure on Ω such that the function $\rho \mapsto \frac{||\rho||}{1+||\rho||}$ is μ integrable.

C, a and μ are uniquely determined by φ.

Proof. (i) $\Rightarrow$ (ii) For every $t \in]0,\infty[$ the function $\psi_t : \mathbb{R}_+^n \to \mathbb{R}$ defined by $\psi_t(x) = e^{-t\varphi(x)}$ is continuous positive definite (cf. [2], p. 74, Theorem 2.2) and bounded. It follows from [2], p.115, Proposition 4.7, that for each $t \in]0,\infty[$ there is a positive Radon measure μ_t on $]-\infty, 0]^n$ such that

$$e^{-t\varphi(x)} = \int_{]-\infty,0]^n} e^{\langle \rho, x\rangle} d\mu_t(\rho).$$

We denote by V' the set

$$\{f : \Omega \to \mathbb{R} | f = F|_\Omega, F :]-\infty, 0]^n \to \mathbb{R}, F \text{ continuous},$$
$$F(0, \dots, 0) = 0, \lim_{t\to 0} \frac{1}{t} \int_{]-\infty,0]^n} F(\rho) d\mu_t(\rho) \text{ exists in } \mathbb{R}\}.$$

Let $L' : V' \to \mathbb{R}$ be the function defined by $L'(f) = \lim_{t\to 0} \frac{1}{t} \int_{]-\infty,0]^n} F(\rho) d\mu_t(\rho)$.

Let $\mathcal{F}$ denote the set of all families $(a_x)_{x\in\mathbb{R}^n_+}$ of real numbers such that $a_x \neq 0$ only for a finite number of x and which satisfy the relation

$$\sum_{x\in\mathbb{R}^n_+} a_x = 0.$$

Let U denote the set

$$\{f : \Omega \to \mathbb{R} | f(\rho) = \sum_{x\in\mathbb{R}^n_+} a_x e^{\langle\rho,x\rangle}, (a_x)_{x\in\mathbb{R}^n_+} \in \mathcal{F}\}.$$

We obtain as in the proof of Theorem 2 that U is a subspace of V and that

$$L'(\rho \mapsto \sum_{x\in\mathbb{R}^n_+} a_x e^{\langle\rho,x\rangle}) = -\sum_{x\in\mathbb{R}^n_+} a_x \varphi(x),$$

if the function $\rho \mapsto \sum_{x\in\mathbb{R}^n_+} a_x e^{\langle\rho,x\rangle}$ is an element of U.

If $\rho = (\rho_1, \ldots, \rho_n)$, we have

$$\lim_{t\to 0} \frac{1}{t} \int_{]-\infty,0]^n} \sum_{j=1}^{n} (1 - e^{\rho_j}) d\mu_t(\rho)$$
$$= -n\varphi(0, \ldots, 0) + \varphi(1, 0, \ldots, 0) + \ldots + \varphi(0, \ldots, 0, 1)$$

It results from [2], p. 52, Proposition 4.6 that there is a sequence $(t_k)_{k\in\mathbb{N}} \subset]0,1]$, with $\lim_{k\to\infty} t_k = 0$, such that the sequence

$$\Big(\frac{1}{t_k}(\rho \mapsto \sum_{j=1}^{n}(1 - e^{\rho_j}))\mu_{t_k}\Big)_{k\in\mathbb{N}}$$

converges vaguely.

We define

$$V = \{f : \Omega \to \mathbb{R} | f = F|_\Omega, F :]-\infty, 0]^n \to \mathbb{R},$$

F continuous, $F(0, \ldots, 0) = 0$, $\lim_{k\to\infty} \frac{1}{t_k} \int_{]-\infty,0]^n} F(\rho) d\mu_{t_k}$ exists in $\mathbb{R}\}$.

We also define $L : V \to \mathbb{R}$ by

$$L(f) = \lim_{k\to\infty} \frac{1}{t_k} \int_{]-\infty,0]^n} F(\rho) d\mu_{t_k}(\rho).$$

We have $V' \subset V$ and $L|_{V'} = L'$. If we take $f \in \mathcal{C}(\Omega)$, we obtain that

$$\begin{aligned} &\lim_{k\to\infty} \frac{1}{t_k} \int_{]-\infty,0]^n} F(\rho) d\mu_{t_k}(\rho) \\ &= \lim_{k\to\infty} \frac{1}{t_k} \int_{]-\infty,0]^n} G(\rho) \sum_{j=1}^n (1-e^{\rho_j}) d\mu_{t_k}(\rho) \end{aligned}$$

(where $G(\rho) = \frac{F(\rho)}{\sum_{j=1}^n (1-e^{\rho_j})}$ for $\rho \in \Omega$ and $G(0,\dots,0) = 0$) exists in $\mathbb{R}$. This proves that $\mathcal{C}(\Omega) \subset V$.

We have, using Taylor's formula,

$$\lim_{\rho\in\Omega, \rho\to(0,\dots,0)} \frac{1 - e^{\langle \rho, x\rangle} - \sum_{j=1}^n (1-e^{\rho_j}) x_j}{\sum_{j=1}^n (1-e^{\rho_j})} = 0,$$

where $x \in \mathbb{R}_+^n$ and $\rho = (\rho_1, \dots, \rho_n)$.

If we take $x, \alpha \in \mathbb{R}_+^n$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j > 0$, then it is easy to see that for each $\epsilon > 0$ there exists a compact set $K \subset \Omega$ such that

$$|e^{\langle \rho, \alpha\rangle}(1 - e^{\langle \rho, x\rangle} - \sum_{j=1}^n (1-e^{\rho_j}) x_j)| \le \epsilon \sum_{j=1}^n (1-e^{\rho_j}) \quad \text{for} \quad \rho \in \Omega \setminus K.$$

Theorem 1 yields a positive Radon measure on Ω such that the functions $(\rho \mapsto (1 - e^{\langle \rho, x\rangle}))_{x\in\mathbb{R}_+^n}$ are μ integrable and we have

$$-\varphi(0) + \varphi(x) \ge \int_\Omega (1 - e^{\langle \rho, x\rangle}) d\mu(\rho),\ x \in \mathbb{R}_+^n. \tag{5}$$

$$\begin{aligned} &-\varphi(\alpha) + \varphi(\alpha + x) - \sum_{j=1}^n x_j(-\varphi(\alpha) + \varphi\alpha + e_j)) \\ &= \int_\Omega e^{\langle \rho, \alpha\rangle}(1 - e^{\langle \rho, x\rangle} - \sum_{j=1}^n x_j(1-e^{\rho_j})) d\mu(\rho). \end{aligned} \tag{6}$$

($x, \alpha \in \mathbb{R}_+^n$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j > 0, (e_j)_{1 \le j \le n}$ the canonical base in $\mathbb{R}^n$).

If in (6) we let α tend to $(0, \dots, 0)$, we obtain

$$\varphi(x) = \varphi(0) + \sum_{j=1}^{n} x_j a_j + \int_\Omega (1 - e^{\langle \rho, x \rangle}) d\mu(\rho), \tag{7}$$

where $a_j = (-\varphi(0) + \varphi(e_j) - \int_\Omega (1 - e^{\rho_j}) d\mu(\rho))$. Using (5), we obtain $a_j \ge 0$.

If we show that the function, $\rho \mapsto \frac{||\rho||}{1+||\rho||}$ is μ integrable the formula (7) is the integral representation from (ii).

To this end, it is enough to prove that for a compact neighbourhood O of the origin in $\mathbb{R}^n$, we have $\mu(\Omega \setminus O) < 0$.

Take $\rho \in \Omega$. We have

$$\int_{[0,1]^n} e^{\langle \rho, x \rangle} dx = \prod_{\rho_j \neq 0} \frac{1}{\rho_j} (e^{\rho_j} - 1)$$

where dx is Lebesque measure in $\mathbb{R}^n$.

Consider the function $\psi : \Omega \to \mathbb{R}$ defined by

$$\psi(\rho) = 1 - \int_{[0,1]^n} e^{\langle \rho, x \rangle} dx = \int_{[0,1]^n} (1 - e^{\langle \rho, x \rangle}) dx$$

Using Fubini's theorem and relation (5), we obtain

$$\int_{[0,1]^n} (-\varphi(0) + \varphi(x)) dx \ge \int_{[0,1]^n} \big(\int_\Omega (1 - e^{\langle \rho, x \rangle}) d\mu(\rho) \big) dx = \int_\Omega \psi(\rho) d\rho$$

We have

$$\begin{aligned} \mu(\{\rho \in \Omega | \rho_j \ge -2, 1 \le j \le n\}) &\le \mu(\{\rho \in \Omega | \psi(\rho) \ge \frac{1}{2}\}) \\ &\le 2 \int_{[0,1]^n} (-\varphi(0) + \varphi(x)) dx < \infty. \end{aligned}$$

This finishes the proof of the implication (i) $\Rightarrow$ (ii).

The implication (ii) $\Rightarrow$ (i) is trivial.

Next we prove the assertion concerned with unicity.

We note that it is enough to prove the unicity of μ. We have the relation

$$\int_\Omega e^{\langle\rho,x+\alpha\rangle}(1-e^{\langle\rho,y\rangle})^2 d\mu(\rho)$$
$$= -\varphi(x+\alpha)+2\varphi(x+\alpha+y)-\varphi(x+\alpha+2y)$$

where $x,y,\alpha\in\mathbb{R}_+^n$, $\alpha=(\alpha_1,\dots,\alpha)$, $\alpha_j>0$. Letting α tend to $(0,\dots,0)$, we obtain

$$\int_\Omega e^{\langle\rho,x\rangle}(1-e^{\langle\rho,y\rangle})^2 d\mu(\rho)=-\varphi(x)+2\varphi(x+y)-\varphi(2y).$$

Now the unicity of μ is a consequence of the unicity of the mesure in [2], p. 115, Proposition 4.7. This completes the proof. ■

Theorem 4 *For a function $\varphi:\mathbb{R}_+^n\to\mathbb{R}$ the following conditions are equivalent*

(i) *φ is continuous and negative definite on $\mathbb{R}_+^n$;*

(ii) *we have*

$$\varphi(x)=C+\langle a,x\rangle-q(x)+\int_\Omega(1-e^{\langle\rho,x\rangle}-\sum_{j=1}^n x_j(1-e^{\rho_j}))d\mu(\rho),$$

where $C\in\mathbb{R}$, $a\in\mathbb{R}^n$, $q(x)=\sum_{j,k=1}^n p_{jk}x_jx_k$ is a positive quadratic form on $\mathbb{R}^n$, $\Omega=\mathbb{R}^n\setminus(0,\dots,0)$ and μ is a positive Radon measure on Ω such that the functions $(\rho\mapsto(1-e^{\langle\rho,x\rangle})^2)_{x\in\mathbb{R}_+^n}$ are μ integrable.

C,a,q and μ are uniquely determined by φ.

Proof. For each $t\in]0,\infty[$ the function $\psi_t:\mathbb{R}_+^n\to\mathbb{R}$ defined by $\psi_t(x)=e^{-t\varphi(x)}$ is continuous positive definite. It follows from [2], p. 214, Theorem 5.8 that for each $t\in]0,\infty[$ there is a positive Radon measure μ_t on $\mathbb{R}^n$ such that

$$e^{-t\varphi(x)}=\int_{\mathbb{R}^n} e^{\langle\rho,x\rangle}d\mu_t(x),\ x\in\mathbb{R}^n.$$

We define $V', \mathcal{F}, U, L'$ as in the proof of Theorem 3.

We have

$$\lim_{t\to 0}\frac{1}{t}\int_{\mathbb{R}^n}\sum_{j=1}^{n}(1-e^{\rho_j})^2 d\mu_t(\rho) =$$

$$-\,n\varphi(0,\dots,0)+2(\varphi(1,0,\dots,0)+\dots+\varphi(0,\dots,0,1))$$

$$-\,(\varphi(2,0,\dots,0)+\dots+\varphi(0,\dots,0,2))$$

There is a sequence $(t_k)_{k\in\mathbb{N}}\subset]0,1]$, with $\lim_{k\to\infty} t_k = 0$, such that the sequence

$$\big(\frac{1}{t_k}(\rho\mapsto\sum_{j=1}^{n}(1-e^{\rho_j})^2)\mu_{t_k}\big)_{k\in\mathbb{N}}$$

converges vaguely.

Now we define V and L, as in the proof of Theorem 3 and obtain $U\subset V$ and $\mathcal{C}(\Omega)\subset V$.

We denote by: $T:\mathbb{R}^n_+\times\Omega\to\mathbb{R}$ the function defined by

$$T(x,\rho) = 1-e^{\langle\rho,x\rangle}-\sum_{j=1}^{n}x_j(1-e^{\rho_j})+\sum_{j=1}^{n}\frac{x_j(x_j-1)}{2}(1-e^{\rho j})^2+\sum_{\substack{j,k=1\\ j\neq k}}^{n}x_jx_k(1-e^{\rho_j})(1-e^{\rho_k}).$$

Using Taylor's formula, we have for each $x\in\mathbb{R}^n_+$

$$\lim_{\rho\in\Omega,\rho\to(0,\dots,0)}\frac{T(x,\rho)}{\sum_{j=1}^{n}(1-e^{\rho_j})^2}=0.$$

If we take $x,\alpha,\beta\in\mathbb{R}^n_+, \alpha_j>0$ and $\beta_j=1$, it is easy to see that for each $\epsilon>0$ there is a compact $K\subset\Omega$ such that

$$|e^{\langle\rho,\alpha\rangle}T(x,\rho)|\le\epsilon(\sum_{j=1}^{n}(1-e^{\rho_j})^2+(1-e^{\langle\rho,x+\alpha+\beta\rangle})^2),\ \rho\in\Omega\setminus K.$$

Theorem 1 yields a positive Radon measure μ on Ω such that

$$-\sum_{x\in\mathbb{R}^n_+}a_x\varphi(x)\ge\int_{\Omega}\sum_{x\in\mathbb{R}^n_+}a_xe^{\langle\rho,x\rangle}d\mu(\rho), \tag{8}$$

where

$$(a_x)_{x\in\mathbb{R}_+^n} \in \mathcal{F}, \text{ with } \sum_{x\in\mathbb{R}_+^n} a_x e^{\langle\rho,x\rangle} \geq 0, \text{ for } \rho \in \Omega.$$

We also have $L(\rho \mapsto e^{\langle\rho,\alpha\rangle}T(x,\rho)) = \int_\Omega (e^{\langle\rho,\alpha\rangle}T(x,\rho))d\mu(y)$, $x, \alpha \in \mathbb{R}_+^n$, $\alpha_j > 0$.

Letting α tend to $(0,\dots,0)$, we obtain

$$L(T(x,\rho)) = \int_\Omega T(x,\rho)d\mu(\rho),\ x \in \mathbb{R}_+^n$$

which can be written

$$\varphi(x) = \varphi(0,\dots,0) + \sum_{j=1}^n x_j a_j - q(x) + \int_\Omega (1 - e^{\langle\rho,x\rangle} - \sum_{j=1} x_j(1 - e^{\rho_j}))d\mu(\rho)$$

where $q(x) = \frac{1}{2}(L(\rho \mapsto (\sum_{j=1}^n x_j(1-e^{\rho_j}))^2) - \int_\Omega (\sum_{j=1}^n x_j(1-e^{\rho_j}))^2 d\mu(\rho))$, and $a_j = -\varphi(0) + \varphi(e_j) + \frac{1}{2}(L(\rho \mapsto (1-e^{\rho_j})^2) - \int_\Omega (1-e^{\rho_j})^2 d\mu(\rho))$. The function $g : \Omega \to \mathbb{R}$ defined by $g(\rho) = (\sum_{j=1}^n x_j(1-e^{\rho_j}))^2$ is in $U$and consequently the inequality (8) implies $q(x) \geq 0$, $x \in \mathbb{R}_+^n$. This completes the proof of implication (i) $\Rightarrow$ (ii). The implication (ii) $\Rightarrow$ (i) is immediate.

The unicity of μ results from the relation

$$L(\rho \mapsto e^{\langle\rho,x\rangle}(1-e^{\langle\rho,y\rangle})^4) = \int_\Omega e^{\langle\rho,x\rangle}(1-e^{\langle\rho,y\rangle})^4 d\mu(\rho),\ x, y \in \mathbb{R}_+^n.$$

If $q(x) = \sum_{j,k=1}^n p_{jk}x_jx_k$, we have

$$p_{jk} = \frac{1}{2}(-\varphi(0) + \varphi(e_j) + \varphi(e_k) - \varphi(e_j + e_k) - \int_\Omega (1-e^{\rho_j})(1-e^{\rho_k})d\mu(\rho)).$$

This proves the unicity of q and finishes the proof because $C = \varphi(0)$ and the unicity of a is also a consequence of the unicity of μ. ■

5 Integral representations for continuous negative definite function on the groupe $\mathbb{R}^n$

Theorem 5 *For a function $\varphi : \mathbb{R}^n \to \mathbb{C}$ the following conditions are equivalent:*

(i) *φ is continuous and negative definite on $\mathbb{R}^n$;*

(ii) *there is a real number C, a positive quadratic form $q : \mathbb{R}^n \to \mathbb{R}$, such that $q(x) = \sum_{j,k=1}^{n} a_{jk} x_j x_k$ with $a_{jk} \in \mathbb{R}$ and $a_{jk} = a_{kj}$, and a positive Radon measure μ on $\mathbb{R}^n \setminus (0, \ldots, 0) = \Omega$, such that the function $g : \Omega \to \mathbb{R}$ defined by $g(\rho) = \frac{||\rho||^2}{1+||\rho||^2}$ is μ integrable, which satisfy*

$$Re\ \varphi(x) = C + q(x) + \int_{\Omega} (1 - \cos\langle \rho, x \rangle) d\mu(\rho)$$

and

$$\begin{aligned} &- Im\ \varphi(x+y) + Im\ \varphi(x) + Im\ \varphi(y) = \\ &\int_{\Omega} (\sin\langle \rho, x+y \rangle - \sin\langle \rho, x \rangle - \sin\langle \rho, y \rangle) d\mu(\rho) \end{aligned}$$

C, q and μ are uniquely determined by φ and we have

$$a_{jj} = -\varphi(0) + \frac{1}{2}(\varphi(e_j) + \varphi(-e_j)) - \int_{\Omega} (1 - \cos \rho_j) d\mu(\rho)$$

and

$$\begin{aligned} a_{jk} = &\frac{1}{8}(\varphi(e_j + e_k) + \varphi(-e_j - e_k) - \varphi(e_j - e_k) - \varphi(e_k - e_j)) \\ &- \frac{1}{2} \int_{\Omega} \sin \rho_j \sin \rho_k d\mu(\rho), \quad \text{for } j \neq k. \end{aligned}$$

Proof. Using Bochner's theorem in $\mathbb{R}^n$, we define the measures $(\mu_t)_{t \in]0,\infty[}$ as in Section 3. In this section $\mathcal{F}$ will be the set of all families $(a_x)_{x \in \mathbb{R}^n}$ of complex numbers such that $a_x \neq 0$ only for a finite number of x, which satisfy the relation $\sum_{x\in\mathbb{R}^n} a_x = 0$. If we denote by V the set

$$\{f : \Omega \to \mathbb{R} | f = F|_\Omega,\ F : \mathbb{R}^n \to \mathbb{R},\ F \text{ continuous},$$
$$F(0, \dots, 0) = 0, \lim_{t\to 0} \frac{1}{t} \int_{\mathbb{R}^n} F(\rho) d\mu_t(\rho) \text{ exists in } \mathbb{R}\},$$

we obtain, as in Section 3, that the set:

$$U = \{f : \Omega \to \mathbb{R} | f(\rho) = \sum_{x\in\mathbb{R}} a_x e^{i\langle\rho,x\rangle}, (a_x)_{x\in\mathbb{R}^n} \in \mathcal{F}\}$$

is included in V. Using the classical Lévy's theorem, we also obtain that $\mathcal{C}(\Omega) \subset V$. We will show that for each $\beta > 0$ the function $h_\beta : \Omega \to [0, \infty[$ defined by $h_\beta(\rho) = \frac{1}{\beta^n} \int_{[0,\beta]^n} (1 - \cos\langle\rho, x\rangle) dx$. ($dx$ Lebsegue measure in $\mathbb{R}^n$) is in V.

First it is clear that

$$\lim_{\rho\in\Omega, \rho\to(0,\dots,0)} h_\beta(\rho) = 0.$$

If we take $h_\beta(0, \dots, 0) = 0$, we have

$$\begin{aligned}
\frac{1}{t} \int_{\mathbb{R}^n} h_\beta(\rho) d\mu_t(\rho) &= \frac{1}{\beta^n t} \int_{\mathbb{R}^n} \Big(\int_{[0,\beta]^n} (1 - \cos\langle\rho, x\rangle) dx \Big) d\mu_t(\rho) \\
&= \frac{1}{\beta^n t} \int_{[0,\beta]^n} \Big(\int_{\mathbb{R}^n} (1 - \cos\langle\rho, x\rangle) d\mu_t(\rho) \Big) dx \\
&= \frac{1}{\beta^n t} \int_{[0,\beta]^n} \Big(e^{-t\varphi(0)} - \frac{1}{2} e^{-t\varphi(x)} - \frac{1}{2} e^{-t\varphi(-x)} \Big) dx
\end{aligned}$$

Consequently,

$$\lim_{t\to 0} \frac{1}{t} \int_{\mathbb{R}^n} h_\beta(\rho) d\mu_t(\rho) = \frac{1}{\beta^n} \int_{[0,\beta]^n} (\text{Re } \varphi(x) - \varphi(0)) dx$$

and therefore $h_\beta \in V$.

We define L as in Section 3. It results, from the continuity of φ in 0, that for $\varepsilon > 0$ there is a $\beta_\varepsilon > 0$ such that $L(h_{\beta_\varepsilon}) \leq \varepsilon$. An elementary calculus shows that there is a real number $M > 0$, such that $h_{\beta_\varepsilon}(\rho) \geq \frac{1}{2}$ for $\|\rho\| \geq M$.

Choose x and y in $\mathbb{R}^n$. If we take $\gamma > 0$ such that $\|\rho\| \leq \gamma$ implies $1 - \cos\langle\rho, y\rangle \leq \varepsilon$ we have

$$(1 - \cos\langle\rho, x\rangle)(1 - \cos\langle\rho, y\rangle) \leq \varepsilon(1 - \cos\langle\rho, x\rangle) + 4h_{\beta_\varepsilon}(\rho)$$

for $\|\rho\| \leq \gamma$ or $\|\rho\| \geq M$.

If we take $\delta > 0$ such that $\|\rho\| \leq \delta$ implies $|\sin\langle\rho, y\rangle| \leq \varepsilon$ we have

$$|(1 - \cos\langle\rho, x\rangle)\sin\langle\rho, y\rangle| \leq \varepsilon(1 - \cos\langle\rho, x\rangle) + 2h_{\beta_\varepsilon}(\rho)$$

for $\|\rho\| \leq \delta$ or $\|\rho\| \geq M$.

Using the preceding inequalities and Theorem 1 we can obtain as in the proof of Theorem 2, the measure μ on Ω and the integral representations of Theorem 5.

We denote by $T : \mathbb{R}^n \times \Omega \to \mathbb{R}$ the function defined by

$$T(x, \rho) = 1 - \cos\langle\rho, x\rangle - \sum_{j=1}^{n} x_j^2(1 - \cos\rho_j) - \sum_{\substack{j,k=1 \\ j \neq k}}^{n} x_j x_k \sin\rho_j \sin\rho_k,$$

and by $Q : \mathbb{R}^n \times \Omega$ the function defined by

$$Q(x, \rho) = \sum_{j=1}^{n} x_j^2(1 - \cos\rho_j) + \sum_{\substack{j,k=1 \\ j \neq k}}^{n} x_j x_k \sin\rho_j \sin\rho_k.$$

The Taylor's formula implies that

$$\lim_{\rho \in \Omega, \rho \to (0,\dots,0)} \frac{T(x, \rho)}{\sum_{j=1}^{n}(1 - \cos\rho_j)} = 0, \quad x \in \mathbb{R}^n.$$

The function $\rho \mapsto T(x, \rho)$ is bounded and therefore using the preceding limit and a h_β function we obtain, as before, that

$$L(\rho \mapsto T(x, \rho)) = \int_\Omega T(x, \rho) d\mu(\rho)$$

The function $\rho \mapsto \sin \rho_j \sin \rho_k$ is μ integrable, because the functions $1 - \cos 2\rho_j$ and $1 - \cos 2\rho_k$ are μ integrable, and consequently the function $\rho \mapsto Q(x, \rho)$ is μ integrable for every $x \in \mathbb{R}^n$. We have

$$\begin{aligned} q(x) &= -\varphi(0) + \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(-x) - \int_\Omega (1 - \cos\langle \rho, x\rangle) d\mu(\rho) \\ &= L(\rho \mapsto T(x, \rho)) - \int_\Omega T(x, \rho) d\mu(\rho) + L(\rho \mapsto Q(x, \rho)) - \int_\Omega Q(x, \rho) d\mu(\rho) \\ &= \sum_{j,k=1}^n a_{jk} x_j x_k \end{aligned}$$

where $a_{jj} = L(\rho \mapsto (1 - \cos \rho_j)) - \int_\Omega (1 - \cos \rho_j) d\mu(\rho)$ and

$$a_{jk} = a_{kj} = \frac{1}{2}(L(\rho \mapsto \sin \rho_j \sin \rho_k) - \int_\Omega \sin \rho_j \sin \rho_k d\mu(\rho)) \ \text{ for } \ j \neq k.$$

Next we prove that the function $\rho \mapsto \frac{||\rho||^2}{1 + ||\rho||^2}$ is μ integrable.

We have

$$\begin{aligned} &\int_{[0,1]^n} (-\varphi(0) + \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(-x)) dx \\ &\geq \int_{[0,1]^n} \Big(\int_\Omega (1 - \cos\langle \rho, x\rangle d\mu(\rho) \Big) dx \\ &= \int_\Omega \Big(\int_{[0,1]^n} (1 - \cos\langle \rho, x\rangle) dx \Big) d\mu(\rho) \\ &= \int_\Omega h_1(\rho) d\mu(\rho). \end{aligned}$$

Choosing a real number M, such that $h_1(\rho) \geq \frac{1}{2}$ for $||\rho|| \geq M$, the preceding inequality gives

$$\mu(\{\rho \in \Omega | ||\rho|| \geq M\}) \leq 2 \int_{[0,1]^n} (-\varphi(0) + \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(-x)) dx.$$

The limits $\lim_{\rho_j \to 0} \frac{1 - \cos \rho_j}{\rho_j^2} = 1$, $j = 1, \ldots, n$ prove that the function $\rho \mapsto ||\rho||^2$ is μ integrable on a set of the form $O \setminus (0, \ldots, 0)$, where O is a neighbourhood of the origin.

We also have proved that the function $\rho \mapsto \frac{||\rho||^2}{1+||\rho||^2}$ is μ integrable. If we notice that the unicity of C, q and μ and the implication (ii) $\Rightarrow$ (i) can be proved as in Section 3 we finish the proof. ■

Remark 5 *The inclusion $\mathcal{C}(\Omega) \subset V$ results also from [3], p. 172, Proposition 18.2.*

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6 Negative definite functions on $\mathbb{N}^*$

For a function $\varphi : \mathbb{N}^* \to \mathbb{R}$ the following conditions are equivalent

1. φ is negative definite on $\mathbb{N}^*$

2. there is a positive Radon measure μ on $\mathbb{R} \setminus \{0,1\}$, such that every polynom divisible by $x^2(1-x)^2$ is μ integrable, and real number a, b, c such that $c \leq 0$, which satisfy

$$\varphi(2) \geq t(2) \quad \text{and} \quad \varphi(n) = E(n), \text{ for } n \geq 3,$$

where $E(n) = a + bn + cn^2 + \int_{\mathbb{R}\setminus\{0,1\}}(x^4 - x^n + (n-4)x^4(x-1))dx$.

Proof. It is clear that (ii) $\Rightarrow$ (i). We will prove (i) $\Rightarrow$ (ii).

The set

$$V = \{P : \mathbb{R} \setminus \{0,1\} \to \mathbb{R} | P \text{ polynomial function,}$$
$$P(0) = P'(0) = P(1) = P'(1) = 0\}$$

is an adapted space.

The function $L : V \to \mathbb{R}$ defined by

$$L_\varphi(a_2x^2 + \ldots + a_nx^n) = -a_2\varphi(2) - \ldots - a_n\varphi(n)$$

is positive on V_+ because every element of V_+ can be expressed as a sum of the form $P_1^2 + P_2^2$ where P_1 and P_2 are polynomial functions (cf. [3]).

Let P be polynomial function of degree m. We notice that for every ε there is a compact $K \subset \mathbb{R} \setminus \{0,1\}$ such that if $\varphi : \mathbb{R} \setminus \{0,1\} \to \mathbb{R}$ is a continuous function with compact support which is 1 on K the following inequality holds

$$|x^3(1-x)^3P(x)(1-\varphi(x))| \leq \varepsilon(x^2(1-x)^2 + x^{2m+2})$$

for $x \in \mathbb{R} \setminus \{0,1\} \setminus K$.

Theorem 1 and Proposition 1 yield a positive Radon measure on $\mathbb{R} \setminus \{0,1\}$ such that

$$L(x \mapsto x^2(1-x)^2 Q(x)) \geq \int_{\mathbb{R}\setminus\{0,1\}} x^2(1-x)^2 Q(x) d(x), \tag{9}$$

for every positive polynomial function Q, and

$$L(x \mapsto x^3(1-x)^3 P(x)) = \int_{\mathbb{R}\setminus\{0,1\}} x^3(1-x)^3 P(x) d\mu(x), \tag{10}$$

for every polynomial function P.

The relation (10) gives for $n \geq 3$

$$\begin{aligned} &-\varphi(4) + \varphi(n) - (n-4)(\varphi(5) - \varphi(4)) - \\ &\frac{(n-4)(n-5)}{2}(\varphi(6) - 2\varphi(5) + \varphi(4)) = \\ &\int_{\mathbb{R}\setminus\{0,1\}} (x^4 - x^n + (n-4)(x-1)x^4 + \frac{(n-4)(n-5)}{2}(x-1)^2 x^4) d\mu(x) \end{aligned}$$

Consequently, we have for $n \geq 3$

$$\varphi(n) = a + bn + cn^2 + \int_{\mathbb{R}\setminus\{0.1\}} (x-1)^2 x^4 d\mu(x)).$$

Using (9) we obtain $c \leq 0$.

For $n = 2$ the polynom

$$x^4 - x^n + (n-4)(x-1)x^4 + \frac{(n-4)(n-5)}{2}(x-1)^2 x^4$$

becomes

$$x^2(1-x)^2(-1-2x+3x^2).$$

The relation (10) implies that

$$\int_{\mathbb{R}\setminus\{0,1\}} (x^2 - x)^3 = -\varphi(6) + 3\varphi(5) - 3\varphi(4) + \varphi(3) \tag{11}$$

Using (11) and the identity

$$x^2(1-x)^2(-1-2x+3x^2)+4x^3(1-x)^3=-x^2(1-x)^4,$$

we obtain as a consequence of (9) the following relation

$$\varphi(2)-E(2)=-L(x\mapsto x^2(1-x)^4)+\int_{\mathbb{R}}(x^2(1-x)^4)d\mu(x)\leq 0$$

which completes the proof.