EXTREMES OF TOTALLY SKEWED α-STABLE PROCESSES

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We give upper and lower bounds for the probability for a local extrema of a totally skewed α-stable stochastic process. Often these bounds are sharp and coincide. The Gaussian case $\alpha=2$ is not excluded, and there our results slightly improve existing general bounds. Applications focus on moving averages and fractional stable motions.

0. Introduction. Let $\{\xi(t)\}_{t\geq 0}$ be α-stable Lévy motion, $\alpha\in(1,2)$, with skewness $\beta\in[-1,1]$. The well-known computation by Doob (1953, p. 106) implies that

$$(0.1)\qquad \mathbf{P}\{\xi(1)>u\} \leq \mathbf{P}\big\{\sup_{0\leq t\leq 1}\xi(t)>u\big\} \leq \mathbf{P}\{\xi(1)>u\}/\mathbf{P}\{\xi(1)>0\} \quad \text{for } u>0.$$

Berman (1986) showed that for a symmetric Lévy process $\eta(t)$ whose Lévy measure has regularly varying tails, and in particular for $\eta(t)=\xi(t)$ when $\beta=0$,

$$(0.2)\qquad \mathbf{P}\big\{\sup_{0\leq t\leq 1}\eta(t)>u\big\} \sim \mathbf{P}\{\eta(1)>u\} \quad \text{as } u\to\infty.$$

Willekens (1987) extended (0.2) to sub-exponentially distributed Lévy processes.

de Acosta (1977) proved that for an a.s. bounded α-stable process $\{X(t)\}_{t\in T}$

$$(0.3)\qquad \lim_{u\to\infty} u^{\alpha}\mathbf{P}\big\{\sup_{t\in T}|X(t)|>u\big\} \equiv L_1 \quad \text{exists, and} \quad L_1\in(0,\infty).$$

Samorodnitsky (1988) calculated L_1 using the spectral representation for $X(t)$.

Rosinski and Samorodnitsky (1993) proved a version of (0.3) for sub-exponential infinitely divisible processes. They also studied other functionals than $\sup_{t\in T}|X(t)|$ and developed the argument of de Acosta (1977, 1980) to allow computation of L_1.

Also for the one-sided tail of an α-stable X the limit

$$(0.4)\qquad \lim_{u\to\infty} u^{\alpha}\mathbf{P}\big\{\sup_{t\in T}X(t)>u\big\} \equiv L_2 \quad \text{exists.}$$

Further, by Rosinski and Samorodnitsky (1993), $L_2>0$ when $X(t)$ has skewness $\beta_X(t)>-1$ for some $t\in T$. But $L_2=0$ when $X(t)$ is totally skewed to the left, i.e., when $\beta_X(t)=-1$ for each $t\in T$, and then the actual tail is 'super-exponential'.

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Doob's result (0.1) is valid for α-stable Lévy motion with $\beta=-1$, and gives

$$(0.5) \quad 1 \leq L_3 \equiv \varliminf_{u\to\infty} \frac{\mathbf{P}\{\sup_{0\leq t\leq 1}\xi(t)>u\}}{\mathbf{P}\{\xi(1)>u\}} \leq \varlimsup_{u\to\infty} \frac{\mathbf{P}\{\sup_{0\leq t\leq 1}\xi(t)>u\}}{\mathbf{P}\{\xi(1)>u\}} \equiv L_4 < \infty.$$

Albin (1993) showed that $L_3=L_4>1$ and also studied stable Ornstein-Uhlenbeck processes with $\beta=-1$. These are the only results on totally skewed stable extremes.

But Braverman and Samorodnitsky (1995) extended the result of Rosinski and Samorodnitsky (1993) to 'exponential tails'. These lie between the sub-exponential setting of Rosinski and Samorodnitsky, and the super-exponential one we study.

Now consider a totally skewed α-stable process $\{X(t)\}_{t\in K}$ defined on a compact space K. In Section 3 we derive a lower bound for the tail $\mathbf{P}\{\sup_{t\in K} X(t)>u\}$ using a certain entropy function measuring the size of K 'as felt' by $X(t)$.

In Section 5 we give a sharper lower bound applicable when the entropy has bounded increase and is not 'too inhomogeneously distributed' over K, and in Section 4 we provide a useful criterion for verifying that homogenity property.

Using a second entropy we derive an upper bound for $\mathbf{P}\{\sup_{t\in K} X(t)>u\}$ in Section 8. That bound often coincide with the lower bound of Section 5.

Sections 9-10, Sections 11 and 13, and Section 12 treat moving averages, fractional stable motions, and log-fractional stable motions, respectively. These applications use tools for estimation of entropies that are developed in Section 8.

In Section 14 we specialize our results to the Gaussian case and discuss how they contribute there. We also give an application to fractional Brownian motion.

1. Stable processes. Let $(\Omega,\mathcal{F},\mathbf{P})$ be a complete probability space with random variables $\mathbb{L}^0(\Omega)$. We write $Z\in S_\alpha(\sigma,\beta)$ when $Z\in\mathbb{L}^0(\Omega)$ is an α-stable random variable with characteristic function

$$\mathbf{E}\{\exp[i\theta Z]\} = \exp\{-|\theta|^\alpha\sigma^\alpha[1-i\beta\tan(\tfrac{\pi\alpha}{2})\mathrm{sign}(\theta)]\} \quad \text{for } \theta\in\mathbb{R}.$$

Here $\alpha\in(1,2]$, $\sigma\geq 0$ and $\beta\in[-1,1]$ are parameters. Note that $\alpha=2$ is allowed.

Let (S,Σ,m) be a σ-finite measure space, $\beth:S\to[-1,1]$ a measurable map, and $M:\Sigma_0\to\mathbb{L}^0(\Omega)$ a σ-additive independently scattered α-stable set-function with control measure m and skewness $\beth$, where $\Sigma_0\equiv\{A\in\Sigma: m(A)<\infty\}$.

For $f\in\mathbb{L}^0(S)$ and $p\in[1,\infty)$ we define $\|f\|_p \equiv [\int_S|f|^p\,dm]^{1/p}$ and $\mathbb{L}^p(S)\equiv \{f\in\mathbb{L}^0(S): \|f\|_p<\infty\}$. Letting $f^{\langle\alpha\rangle}\equiv|f|^\alpha\mathrm{sign}(f)$ we further write

$$\langle f\rangle \equiv \int_S f\,dm \;\text{ for }\; f\in\mathbb{L}^1(S) \quad \text{and} \quad \beth_f \equiv \langle f^{\langle\alpha\rangle}\beth\rangle/\|f\|_\alpha^\alpha \;\text{ for }\; f\in\mathbb{L}^\alpha(S).$$

It is well-known [e.g., Samorodnitsky and Taqqu (1994, Chapter 3)] that

$$X \equiv \int_S f\,dM \;\in\; S_\alpha(\|f\|_\alpha, \beth_f) \quad \text{is well-defined for} \quad f \in \mathbb{L}^\alpha(S).$$

We shall study a separable **P**-continuous α-stable process $\{X(t)\}_{t\in T}$ on a separable topological space T. For such a process there exist S, m, $\beth$ and a continuous $F_{(\cdot)}: T \to \mathbb{L}^\alpha(S)$ such that [e.g., Samorodnitsky and Taqqu (1994, Section 3.11)]

(1.1)
the finite dimensional distributions of $\{X(t)\}_{t\in T}$ = those of $\{\int_S F_t(r)\,dM(r)\}_{t\in T}$.

In fact (1.1) always holds for $(S, m) = (\mathbb{R}$, 'Lebesgue measure'), but since natural representations often use non-Euclidian (S, Σ, m), we study the general setup.

We study extremes over a compact $K \subseteq T$ for $X(t) \in S_\alpha(\|F_t\|_\alpha, \beta_X(t))$ given by (1.1), with scale $\|F_t\|_\alpha > 0$ for some $t \in K$, and with skewness $\beta_X(t) = \beth_{F_t} = -1$ for each $t \in T$. Without loss of generality we can therefore suppose that

(1.2) $\qquad F_t(r) \geq 0$ for $(r,t) \in S \times T$ and $\beth(r) = -1$ for $r \in S$.

By continuity of $F_{(\cdot)}$, K is compact wrt. the pseudo-metric $p(s,t) \equiv \|F_s - F_t\|_\alpha$ on T. Therefore K is also p-sequentially compact, and so

(1.3) $\qquad$ there is a $\tilde{t} \in K$ satisfying $\sup\{\|F_t\|_\alpha : t \in K\} = \|F_{\tilde{t}}\|_\alpha > 0$.

2. Canonical distance and entropy I. After a general discussion of entropies we introduce a distance ρ on T that is important for our lower bounds.

Given a $\hat{T} \subseteq T$, an $\varepsilon > 0$, and a symmetric map $d: T^2 \to \mathbb{R}^+$, we call $B_d(t,\varepsilon) \equiv \{\tau \in T : d(\tau,t) < \varepsilon\}$ a d-ball centered at $t \in T$, and $N \subseteq \hat{T}$ an ε-net for $\hat{T}$ wrt. d if $\hat{T} \subseteq \bigcup_{t\in N} B_d(t,\varepsilon)$. Further $G \subseteq \hat{T}$ is an ε-grid in $\hat{T}$ wrt. d if $d(s,t) \geq \varepsilon$ for distinct $s, t \in G$. The d-covering number E_d and d-content number M_d are given by

$$E_d(\hat{T};\varepsilon) \equiv \inf\{\#N : N \text{ is an } \varepsilon\text{-net for } \hat{T} \text{ wrt. } d\}$$
$$M_d(\hat{T};\varepsilon) \equiv \sup\{\#G : G \text{ is an } \varepsilon\text{-grid in } \hat{T} \text{ wrt. } d\}.$$

A minimal ε-net N for $\hat{T}$ wrt. d satisfies $\#N = E_d(\hat{T};\varepsilon)$. Similarly G is a maximal ε-grid in $\hat{T}$ wrt. d if $\#G = M_d(\hat{T};\varepsilon)$. We refer to both E_d and M_d as entropies (although one usually reserves this label for E_d).

Our lower bounds rely on the assumption that $\mathbf{P}\{X(s) > u \mid X(t) > u\} \to 0$ sufficiently fast as $u \to \infty$. To prove this fact we invoke the estimate

(2.1) $\qquad \mathbf{P}\{X(s) > u \mid X(t) > u\} \leq \mathbf{P}\{X(s) + X(t) > 2u\} / \mathbf{P}\{X(t) > u\}.$

Weber [in e.g., Weber (1989)] saw that all precursors to (2.1) in the Gaussian literature are matched by (2.1). In the Gaussian case (2.1) is not crucial. But two-dimensional stable probabilities are difficult to handle, and there (2.1) is essential.

Guided by (2.1) we are led to consider the 'distance'

$$\underline{\rho(s,t)} \equiv 2\max\big\{\|F_s\|_\alpha,\, \|F_t\|_\alpha\big\} - \|F_s+F_t\|_\alpha \quad \text{between } s,t\in T.$$

Clearly $\rho: T^2 \to \mathbb{R}^+$ is continuous, but ρ need not obey the triangle inequality.

Inspired by the Gaussian treatment of Adler and Samorodnitsky (1987), we use the $\underline{\text{refined entropies } M_\rho(K_\ell(\varepsilon);\cdot)}$ to handle scale-inhomogenity, where

$$\underline{K_\ell(\varepsilon)} \equiv \big\{t\in K : \|F_t\|_\alpha \in \big[\|F_{\tilde{t}}\|_\alpha - (\ell+1)\varepsilon,\, \|F_{\tilde{t}}\|_\alpha - \ell\varepsilon\big]\big\} \quad \text{for } \varepsilon>0 \text{ and } \ell\in\mathbb{N}.$$

The lower bound in Section 3 requires that there is a $g:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$\text{(2.2)} \qquad \lim_{u\to\infty} g(u)^{-1}\ln\big[M_\rho\big(K_0(u^{-\alpha/(\alpha-1)});\, u^{-\alpha/(\alpha-1)}g(u)\big)\big] = 0.$$

Now recall that a locally bounded $\underline{f:\mathbb{R}^+\to\mathbb{R}^+ \text{ has finite order}}$ $(\underline{f\in FO})$ if

$$\underline{\text{order}(f)} \equiv \overline{\lim}_{\varepsilon\downarrow 0} \ln[f(\varepsilon)]/\ln(\varepsilon^{-1}) < \infty.$$

It is obvious that for an $f\in FO$, and given $\gamma_1\in(\text{order}(f),\infty)$ and $\varepsilon_1>0$,

(2.3) there is a $C_1 = C_1(f,\gamma_1,\varepsilon_1) > 0$ such that $f(\varepsilon) \le C_1\varepsilon^{-\gamma_1}$ for $\varepsilon\in(0,\varepsilon_1]$.

Proposition 1. **(i)** *If* $f_1(\varepsilon) \equiv M_\rho(K_0(\varepsilon);\varepsilon)$ *or* $f_2(\varepsilon) \equiv M_\rho(K;\varepsilon)$ *satisfies* $f_i \in FO$*, then* (2.2) *holds provided that* $\lim_{u\to\infty} g(u)/\ln(u) = \infty$.

(ii) *If* $f_3(\varepsilon) \equiv M_\rho(K_0(\varepsilon);\varepsilon)$ *or* $f_4(u) \equiv M_\rho(K;\varepsilon)$ *satisfies* $\ln(f_i)\in FO$ *with* $\text{order}(\ln(f_i))<\kappa$*, then* (2.2) *holds for* $g(u) = u^{\kappa\alpha/[(\kappa+1)(\alpha-1)]}$.

Proof. If $f_1\in FO$ and $g(u)/\ln(u)\to\infty$, then (2.3) implies that

$$\begin{aligned}&\overline{\lim}_{u\to\infty} g(u)^{-1}\ln\big[M_\rho\big(K_0(u^{-\alpha/(\alpha-1)});\, u^{-\alpha/(\alpha-1)}g(u)\big)\big]\\ &\qquad\le \overline{\lim}_{u\to\infty} g(u)^{-1}\ln\big[M_\rho\big(K_0(u^{-\alpha/(\alpha-1)}g(u));\, u^{-\alpha/(\alpha-1)}g(u)\big)\big] = 0.\end{aligned}$$

When $\ln(f_3)\in FO$ with $\text{order}(\ln(f_3))<\kappa$ we similarly obtain

$$\begin{aligned}&g(u)^{-1}\ln\big[M_\rho\big(K_0(u^{-\alpha/(\alpha-1)});\, u^{-\alpha/(\alpha-1)}g(u)\big)\big]\\ &\qquad\le u^{-\kappa\alpha/[(\kappa+1)(\alpha-1)]}\ln\big[M_\rho\big(K_0(u^{-\alpha/[(\kappa+1)(\alpha-1)]});\, u^{-\alpha/[(\kappa+1)(\alpha-1)]}\big)\big] \to 0. \quad\square\end{aligned}$$

3. A general lower bound.

Theorem 1. *Let* $\{X(t)\}_{t\in T}$ *satisfy* (1.1)-(1.3)*. If* (2.2) *holds we then have*

$$\underline{\lim}_{u\to\infty} \mathbf{P}\big\{\sup_{t\in K} X(t)>u\big\}\Big/\Big[M_\rho\big(K_0(u^{-\alpha/(\alpha-1)});\, u^{-\alpha/(\alpha-1)}g(u)\big)\,\mathbf{P}\{X(\tilde{t})>u\}\Big] > 0.$$

Proof. Put $q = q(u) \equiv u^{-\alpha/(\alpha-1)} g(u)$ and let $G = G(q)$ be a maximal q-grid for $K_0\big(u^{-\alpha/(\alpha-1)}\big)$ wrt. ρ. Also note that there are constants $C_2, C_3 > 0$ such that

$$\text{(3.1)}\quad \mathbf{P}\{S_\alpha(\sigma,-1)>u\}\begin{cases} \le\ C_2\Big(\dfrac{u}{\sigma}\Big)^{-\alpha/2(\alpha-1)}\exp\Big\{-C_3\Big(\dfrac{u}{\sigma}\Big)^{\alpha/(\alpha-1)}\Big\} & \text{for } u>0\\[2mm] \ge C_2^{-1}\Big(\dfrac{u}{\sigma}\Big)^{-\alpha/2(\alpha-1)}\exp\Big\{-C_3\Big(\dfrac{u}{\sigma}\Big)^{\alpha/(\alpha-1)}\Big\} & \text{for } u\ge\sigma\end{cases}$$

[e.g., Samorodnitsky and Taqqu (1994, p. 17)], and recall the inequalities

$$\text{(3.2)}\qquad 1+\tfrac{\alpha}{\alpha-1}x \le (1+x)^{\alpha/(\alpha-1)} \le 1+2^{\alpha/(\alpha-1)}(x\vee 0) \quad \text{for } x\in[-1,1].$$

Clearly the fact that $\|F_t\|_\alpha \ge \|F_{\tilde t}\|_\alpha - u^{-\alpha/(\alpha-1)}$ for $t\in K_0(u^{-\alpha/(\alpha-1)})$ implies $u/\|F_t\|_\alpha \le u\big(1+2u^{-\alpha/(\alpha-1)}/\|F_{\tilde t}\|_\alpha\big)\,/\,\|F_{\tilde t}\|_\alpha \le 2u/\|F_{\tilde t}\|_\alpha$ for $u^{-\alpha/(\alpha-1)} \le \frac12\|F_{\tilde t}\|_\alpha$ $\big[$using the inequality $(1-x)^{-1}\le 1+2x$ for $x\in[0,\frac12]\big]$. Hence (3.1) and (3.2) yield

$$\begin{aligned}\text{(3.3)}\quad &\mathbf{P}\{X(t)>u\}\\ &\ge C_2^{-1}\bigg(\frac{2\,u}{\|F_{\tilde t}\|_\alpha}\bigg)^{-\alpha/2(\alpha-1)}\exp\bigg\{-C_3\bigg(\frac{u\big(1+2u^{-\alpha/(\alpha-1)}/\|F_{\tilde t}\|_\alpha\big)}{\|F_{\tilde t}\|_\alpha}\bigg)^{\alpha/(\alpha-1)}\bigg\}\\ &\ge C_4\,\mathbf{P}\{X(\tilde t)>u\} \qquad \text{for } t\in K_0\big(u^{-\alpha/(\alpha-1)}\big) \text{ and } u \text{ sufficiently large,}\end{aligned}$$

where $C_4 = C_2^{-2}2^{-\alpha/2(\alpha-1)}\exp\big\{-C_3(2/\|F_{\tilde t}\|_\alpha)^{(2\alpha-1)/(\alpha-1)}\big\}$. Since $X(s)+X(t)\in S_\alpha\big(\|F_s+F_t\|_\alpha,-1\big)$ with $\|F_s+F_t\|_\alpha \le 2\|F_{\tilde t}\|_\alpha - \rho(s,t)$, (3.1)-(3.3) further give

$$\begin{aligned}\text{(3.4)}\quad &\mathbf{P}\{X(s)+X(t)>2u\}\\ &\le C_2\bigg(\frac{u}{\|F_{\tilde t}\|_\alpha-\rho(s,t)/2}\bigg)^{-\alpha/2(\alpha-1)}\exp\bigg\{-C_3\bigg(\frac{u}{\|F_{\tilde t}\|_\alpha-\rho(s,t)/2}\bigg)^{\alpha/(\alpha-1)}\bigg\}\\ &\le C_2\bigg(\frac{u}{\|F_{\tilde t}\|_\alpha}\bigg)^{-\alpha/2(\alpha-1)}\exp\bigg\{-C_3\bigg(\frac{u\big(1+\frac12\rho(s,t)\,/\,\|F_{\tilde t}\|_\alpha\big)}{\|F_{\tilde t}\|_\alpha}\bigg)^{\alpha/(\alpha-1)}\bigg\}\\ &\le C_2^2\,\mathbf{P}\{X(\tilde t)>u\}\,\exp\bigg\{-\frac{C_3\alpha\,u^{\alpha/(\alpha-1)}\rho(s,t)}{2(\alpha-1)\,\|F_{\tilde t}\|_\alpha^{(2\alpha-1)/(\alpha-1)}}\bigg\}\\ &\le C_4^{-1}C_2^2\,\big(\inf_{t\in G}\mathbf{P}\{X(t)>u\}\big)\,\mathrm{e}^{-2C_5 g(u)} \quad \text{for } G\ni s\ne t\in G \text{ and } u \text{ large.}\end{aligned}$$

Here $C_5 \equiv C_3\frac{\alpha}{4(\alpha-1)}\|F_{\tilde t}\|_\alpha^{-(2\alpha-1)/(\alpha-1)}$.

Since by (2.2) $\#G \le \mathrm{e}^{C_5 g(u)}$ for u large, (3.3) and (3.4) combine to show that

$$\begin{aligned}\text{(3.5)}\quad \mathbf{P}\big\{\sup_{t\in G}X(t)>u\big\} &\ge \sum_{t\in G}\mathbf{P}\{X(t)>u\} - \sum_{G\ni s\ne t\in G}\mathbf{P}\big\{X(s)>u,\,X(t)>u\big\}\\ &\ge \big(\#G - C_4^{-1}C_2^2\mathrm{e}^{-2C_5 g(u)}(\#G)^2\big)\,\big(\inf_{t\in G}\mathbf{P}\{X(t)>u\}\big)\\ &\ge \tfrac12 C_4(\#G)\,\mathbf{P}\{X(\tilde t)>u\} \qquad \text{for } u \text{ sufficiently large.} \quad \square\end{aligned}$$

4. Homogeneous entropies. In Section 5 we study processes with homogeneous entropies. Here we prove that if $M_\rho(K;\cdot)$ has bounded increase, and if ρ is equivalent with a translation invariant 'distance' $\hat\rho$, then $M_\rho(K;\cdot)$ is homogeneous.

A locally bounded $f:\mathbb{R}^+\to\mathbb{R}^+$ is O-regularly varying ($f\in OR$) if

$$0<\underline{\lim}_{\varepsilon\downarrow 0}\inf_{\lambda\in[\Lambda,1]} f(\lambda\varepsilon)/f(\varepsilon)\le \overline{\lim}_{\varepsilon\downarrow 0}\sup_{\lambda\in[\Lambda,1]} f(\lambda\varepsilon)/f(\varepsilon)<\infty \quad \text{for } \Lambda\in(0,1].$$

Further (a locally bounded) $f:\mathbb{R}^+\to\mathbb{R}^+$ has bounded increase ($f\in BI$) if

$$\overline{\lim}_{\varepsilon\downarrow 0}\sup_{\lambda\in[\Lambda,1]} f(\lambda\varepsilon)/f(\varepsilon)<\infty \quad \text{for } \Lambda\in(0,1].$$

Clearly, to each $\varepsilon_2>0$ there are $C_6[f,\varepsilon_2]>0$ and $\gamma_2[f],\gamma_3[f]\in\mathbb{R}$ such that

$$(4.1)\qquad \begin{cases} C_6^{-1}\lambda^{-\gamma_2}f(\varepsilon)\le f(\lambda\varepsilon)\le C_6\lambda^{-\gamma_3}f(\varepsilon) & \text{for } 0<\lambda\varepsilon\le\varepsilon\le\varepsilon_2, \text{ if } f\in OR\\ f(\lambda\varepsilon)\le C_6\lambda^{-\gamma_3}f(\varepsilon) & \text{for } 0<\lambda\varepsilon\le\varepsilon\le\varepsilon_2, \text{ if } f\in BI\end{cases}.$$

Since $M_\rho(K;\varepsilon)$ is a non-increasing function of ε, we have $M_\rho(K;\cdot)\in OR$ when

$$(4.2)\qquad \overline{\lim}_{\varepsilon\downarrow 0} M_\rho(K;\varepsilon)\,/\,M_\rho(K;\lambda^{-1}\varepsilon)<\infty \quad \text{for } \lambda\in(0,1].$$

We shall assume that there is a symmetric map $\hat\rho:T^2\to\mathbb{R}^+$ such that

$$(4.3)\qquad \Re^{-1}\hat\rho(s,t)\le\rho(s,t)\le\Re\,\hat\rho(s,t) \quad \text{for } s,t\in K, \quad \text{for some } \Re\in[1,\infty),$$

and which satisfies the weak triangle inequality

$$(4.4)\qquad \hat\rho(r,t)\le\hat\Delta\big[\hat\rho(r,s)+\hat\rho(s,t)\big] \quad \text{for } r,s,t\in K, \quad \text{for some } \hat\Delta\in[1,\infty).$$

Proposition 2. *Let $(T,+)$ be a separable Abelian topological group such that K contains a ρ-ball. If (4.2) holds, and if (4.3)-(4.4) hold for some $\hat\rho$ satisfying $\hat\rho(r+s,r+t)=\hat\rho(s,t)$ for $r,s,t\in K$, then there is an $\varepsilon_3>0$ such that*

$$(4.5)\qquad \sup\Big\{M_\rho(K;\hat\varepsilon)\,M_\rho\big(K\cap B_\rho(t,\hat\varepsilon);\varepsilon\big)\,/\,M_\rho(K;\varepsilon)\;:\;t\in K,\;0<\varepsilon\le\hat\varepsilon\le\varepsilon_3\Big\}<\infty.$$

Proof. Obviously (4.3) implies that

$$(4.6)\qquad M_{\hat\rho}(\hat T;\Re\varepsilon)\le M_\rho(\hat T;\varepsilon)\le M_{\hat\rho}\big(\hat T;\varepsilon/\Re\big) \quad \text{for } \hat T\subseteq K.$$

Hence we have

$$\sup\Big\{M_\rho(K;\Re^2\hat\varepsilon)\,M_\rho\big(K\cap B_\rho(t,\hat\varepsilon);\varepsilon\big)\,/\,M_\rho(K;\varepsilon/\Re^2)\;:\;t\in K,\;0<\varepsilon\le\hat\varepsilon\le\varepsilon_4\Big\}$$

$$\le\sup\Big\{M_{\hat\rho}(K;\Re\hat\varepsilon)\,M_{\hat\rho}\big(K\cap B_{\hat\rho}(t,\Re\hat\varepsilon);\varepsilon/\Re\big)\,/\,M_{\hat\rho}(K;\varepsilon/\Re)\;:\;t\in K,\;0<\varepsilon\le\hat\varepsilon\le\varepsilon_4\Big\}$$

$$=\sup\Big\{M_{\hat\rho}(K;\hat\varepsilon)\,M_{\hat\rho}\big(K\cap B_{\hat\rho}(t,\hat\varepsilon);\varepsilon\big)\,/\,M_{\hat\rho}(K;\varepsilon)\;:\;t\in K,\;0<\Re^2\varepsilon\le\hat\varepsilon\le\Re\varepsilon_4\Big\}$$

for $\varepsilon_4 > 0$. In view of (4.2), the fact that (4.5) holds for $M_\rho(\cdot,\cdot)$ will therefore follow if we can prove (4.5) for $M_{\hat\rho}(\cdot,\cdot)$.

Now choose a ρ-ball $B_\rho(t_1,r_1)\subseteq K$, so that by (4.3), $B_{\hat\rho}(t_1,r)\subseteq K$ for $r\le r_2\equiv r_1/\Re$. Since maximal r-grids are r-nets, the translation invariance of $\hat\rho$ then gives

$$(4.7)\qquad M_{\hat\rho}(K;\varepsilon) \le M_{\hat\rho}(K;r)\, M_{\hat\rho}\big(B_{\hat\rho}(t_1,r);\varepsilon\big) \quad \text{for } r\le r_2.$$

Using the easy fact that (4.4) implies

$$\hat\rho(s',t') \ge \hat\Delta^{-2}\hat\rho(s,t) - \hat\rho(s,s') - \hat\rho(t,t') \quad \text{for } s,t,s',t'\in K,$$

it follows that $\hat\rho(s',t')\ge\hat\varepsilon$ when $\hat\rho(s,t)\ge 3\hat\Delta^2\hat\varepsilon$ and $\hat\rho(s,s')$, $\hat\rho(t,t')\le\hat\varepsilon$. Writing $\varepsilon_4\equiv r_2/(2\hat\Delta)$ we therefore obtain

$$(4.8)\qquad M_{\hat\rho}(K;\varepsilon) \ge M_{\hat\rho}\big(B_{\hat\rho}(t_1,\varepsilon_4);3\hat\Delta^2\hat\varepsilon\big)\, M_{\hat\rho}\big(B_{\hat\rho}(t_1,\hat\varepsilon);\varepsilon\big) \quad \text{for } 0<\varepsilon\le\hat\varepsilon\le\varepsilon_4.$$

Here we also used the fact that, by another application of (4.4),

$$\hat\rho(t_1,\tau) \le 2\hat\Delta\big[\hat\rho(t_1,s)+\hat\rho(s,\tau)\big] < r_2 \quad \text{for } \tau\in B_{\hat\rho}(s,\hat\varepsilon) \text{ with } s\in B_{\hat\rho}(t_1,\varepsilon_4).$$

But combining (4.6)-(4.8) we conclude that

$$\begin{aligned} M_{\hat\rho}(K;\hat\varepsilon)\, M_{\hat\rho}\big(K\cap B_{\hat\rho}(t,\hat\varepsilon);\varepsilon\big)\,/\,M_{\hat\rho}(K;\varepsilon) &\le M_{\hat\rho}(K;\hat\varepsilon)\,/\,M_{\hat\rho}\big(B_{\hat\rho}(t_1,\varepsilon_4);3\hat\Delta^2\hat\varepsilon\big)\\ &\le M_{\hat\rho}(K;\hat\varepsilon)\, M_{\hat\rho}(K;\varepsilon_4)\,/\,M_{\hat\rho}\big(K;3\hat\Delta^2\hat\varepsilon\big)\\ &\le M_\rho(K;\hat\varepsilon/\Re)\, M_{\hat\rho}(K;\varepsilon_4)\,/\,M_\rho\big(K;3\hat\Delta^2\Re\hat\varepsilon\big), \end{aligned}$$

where by (4.2) the right hand side is bounded for $\hat\varepsilon\in(0,\varepsilon_4]$. Hence (4.5) holds. □

5. A sharper lower bound. Homogenous ε-grids cannot have too many elements too close to any point of K. Assuming homogenity, the arguments (3.4)-(3.5) can therefore be refined to estimates that do not involve the nuisance function g.

Guided by Theorem 1 we modify the requirement (4.2) to

$$(5.1)\qquad \overline{\lim}_{\hat\varepsilon\downarrow 0}\ \sup_{\varepsilon\in(0,\hat\varepsilon]} M_\rho(K_0(\varepsilon);\hat\varepsilon)\,/\,M_\rho\big(K_0(\varepsilon);\lambda^{-1}\hat\varepsilon\big) < \infty \quad \text{for } \lambda\in(0,1].$$

Instead of (4.5) we further require that, for some $\varepsilon_5>0$, to each choice of $\hat\varepsilon\in(0,\varepsilon_5]$ and $\varepsilon\in(0,\hat\varepsilon]$, there is a maximal ε-grid G_ε in $K_0(\varepsilon)$ wrt. ρ such that

$$(5.2)\qquad \sup\Big\{M_\rho(K_0(\varepsilon);\hat\varepsilon)\ \#\big(G_\varepsilon\cap B_\rho(t,\hat\varepsilon)\big)\,/\,\#G_\varepsilon : t\in K_0(\varepsilon),\ 0<\varepsilon\le\hat\varepsilon\le\varepsilon_5\Big\} < \infty.$$

When the scale is constant, (4.2) and (4.5) imply (5.1) and (5.2). The interpretation of (5.2) [and (4.5)] is that contributions to the cardinality of grids from different parts of $K_0(\cdot)$ $[K]$ are not too inhomogeneously distributed.

Theorem 2. *Let* $\{X(t)\}_{t\in T}$ *satisfy* (1.1)-(1.3)*. If* (5.1)-(5.2) *hold we then have*

$$\varliminf_{u\to\infty} \mathbf{P}\big\{\sup_{t\in K} X(t)>u\big\}\Big/\Big[M_\rho\big(K_0(u^{-\alpha/(\alpha-1)});\, u^{-\alpha/(\alpha-1)}\big)\,\mathbf{P}\{X(\tilde{t})>u\}\Big] > 0.$$

Proof. Let $G(\varepsilon;\varepsilon_5)$ be a maximal ε_5-grid in $K_0(\varepsilon)$ (wrt. ρ). Since a maximal ε_5-grid must be an ε_5-net, we then have

$$\#G_\varepsilon \;\le\; \textstyle\sum_{t\in G(\varepsilon;\varepsilon_5)} \#\big(G_\varepsilon\cap B_\rho(t,\varepsilon_5)\big) \;\le\; \#G(\varepsilon;\varepsilon_5)\;\sup_{t\in G(\varepsilon;\varepsilon_5)} \#\big(G_\varepsilon\cap B_\rho(t,\varepsilon_5)\big).$$

Consequently there is a $t_2(\varepsilon)\in G(\varepsilon;\varepsilon_5)$ such that

$$M_\rho(K_0(\varepsilon);\varepsilon) \;=\; \#G_\varepsilon \;\le\; M_\rho(K_0(\varepsilon);\varepsilon_5)\;\#\big(G_\varepsilon\cap B_\rho(t_2(\varepsilon),\varepsilon_5)\big). \tag{5.3}$$

Now write $\mathcal{G}_\varepsilon \equiv G_\varepsilon\cap B_\rho(t_2(\varepsilon),\varepsilon_5)$ and $n(\varepsilon)\equiv \#\mathcal{G}_\varepsilon$, so that [by (5.3)]

$$n(\varepsilon) \ge M_\rho(K_0(\varepsilon);\varepsilon)\,/\,M_\rho(K_0(\varepsilon);\varepsilon_5) \quad\text{and}\quad n(\varepsilon)\le M_\rho(K_0(\varepsilon);\varepsilon). \tag{5.4}$$

Then we have $n(\cdot)\in BI$, since (5.4) yields

$$\begin{aligned}
&\varlimsup_{\varepsilon\downarrow 0}\sup_{\lambda\in[\Lambda,1]} n(\lambda\varepsilon)/n(\varepsilon)\\
&\qquad\le \varlimsup_{\varepsilon\downarrow 0}\sup_{\lambda\in[\Lambda,1]} M_\rho(K_0(\lambda\varepsilon);\lambda\varepsilon)\,M_\rho(K_0(\varepsilon);\varepsilon_5)\,/\,M_\rho(K_0(\varepsilon);\varepsilon)\\
&\qquad\le \varlimsup_{\varepsilon\downarrow 0}\sup_{\lambda\in[\Lambda,1]} M_\rho(K_0(\lambda\varepsilon);\lambda\varepsilon)\,M_\rho(K;\varepsilon_5)\,/\,M_\rho(K_0(\lambda\varepsilon);\varepsilon),
\end{aligned}$$

where the right hand side is finite by (5.1). Therefore (4.1) shows that

$$n(\varepsilon) \;\le\; C_6[n(\cdot),1]\,\varepsilon^{-\gamma_3[n(\cdot)]}\,n(1) \quad\text{for}\quad \varepsilon\le 1. \tag{5.5}$$

Obviously (5.1) implies that

$$M_\rho\big(K_0(\varepsilon);\hat{\varepsilon}\big) \;\le\; R\,M_\rho\big(K_0(\varepsilon);\mathrm{e}\,\hat{\varepsilon}\big) \quad\text{for}\quad 0<\varepsilon\le\hat{\varepsilon}\le\varepsilon_6,$$

for some constants $R>0$ and $\varepsilon_6\in(0,\varepsilon_5]$. Since for each $\lambda\in(0,1]$, there is a $k\in\mathbb{N}$ such that $\mathrm{e}^{-k}\le\lambda\le\mathrm{e}^{-(k-1)}$, it follows that

$$\frac{M_\rho(K_0(\varepsilon);\varepsilon)}{M_\rho(K_0(\varepsilon);\lambda^{-1}\varepsilon)} \le \prod_{j=0}^{k-1}\frac{M_\rho(K_0(\varepsilon);\mathrm{e}^j\varepsilon)}{M_\rho(K_0(\varepsilon);\mathrm{e}^{j+1}\varepsilon)} \le R^k \le R\,\lambda^{-\ln(R)} \quad\text{for}\quad \lambda^{-1}\varepsilon\le\varepsilon_6. \tag{5.6}$$

But now (5.6) combines with (5.2) to show that there is a $C_7>0$ such that

$$\begin{aligned}
\sup_{t\in\mathcal{G}_q}\#\big(\mathcal{G}_q\cap B_\rho(t,\ell q)\big) &\le \sup_{t\in K_0(q)}\#\big(G_q\cap B_\rho(t,\ell q)\big)\\
&\le\; C_7\,M_\rho(K_0(q);q)\,/\,M_\rho(K_0(q);\ell q)\\
&\le\; C_7\,R\,\ell^{\ln(R)} \quad\text{for}\quad 0<q\le\ell q\le\varepsilon_6.
\end{aligned} \tag{5.7}$$

Define $q=q(u)=Q\,u^{-\alpha/(\alpha-1)}$ where $Q>1$ is a constant that satisfies

$$\sum_{\ell=1}^{\infty} C_7 R\,(\ell+1)^{\ln(R)}\,C_2^2\exp\{-2C_5 Q\ell\}\le \tfrac{1}{2}C_4. \tag{5.8}$$

Since (3.4) yields that

$$\mathbf{P}\{X(s)>u,\,X(t)>u\}\le C_2^2\,\exp\{-2C_5\,u^{\alpha/(\alpha-1)}\rho(s,t)\}\,\mathbf{P}\{X(\tilde{t})>u\}$$

for u large, an application of Bonferroni's inequality [cf. (3.5)] then gives

$$\begin{aligned}
&\mathbf{P}\{\sup\nolimits_{t\in\mathcal{G}_q}X(t)>u\}\\
&\ge \sum_{t\in\mathcal{G}_q}\mathbf{P}\{X(t)>u\}-\sum_{t\in\mathcal{G}_q}\sum_{\ell=1}^{\infty}\sum_{s\,\in\,\mathcal{G}_q\,\cap\,[B_\rho(t,(\ell+1)q)-B_\rho(t,\ell q)]}\mathbf{P}\{X(s)>u,\,X(t)>u\}\\
&\ge n(q)\inf_{t\in\mathcal{G}_q}\mathbf{P}\{X(t)>u\}-n(q)\,\mathbf{P}\{X(\tilde{t})>u\}\sum_{\ell=1}^{\infty}C_2^2\mathrm{e}^{-2C_5Q\ell}\sup_{t\in\mathcal{G}_q}\#\big(\mathcal{G}_q\cap B_\rho(t,(\ell+1)q)\big)\\
&\ge \big[\text{by (3.3) and (5.7)}\big]\\
&\ge n(q)\,C_4\,\mathbf{P}\{X(\tilde{t})>u\}-n(q)\,\mathbf{P}\{X(\tilde{t})>u\}\sum_{\ell\ge1,\,(\ell+1)q\le\varepsilon_6}C_2^2\mathrm{e}^{-2C_5Q\ell}\,C_7R\,(\ell+1)^{\ln(R)}\\
&\qquad -n(q)^2\,\mathbf{P}\{X(\tilde{t})>u\}\sum_{\ell\ge[\varepsilon_6/q]}C_2^2\mathrm{e}^{-2C_5Q\ell}\\
&\ge \big[\text{by (5.5) and (5.8)}\big]\\
&\ge n(q)\,\mathbf{P}\{X(\tilde{t})>u\}\Big(C_4-\tfrac{1}{2}C_4-C_2^2C_6\,n(1)\,q^{-\gamma_3}\exp\{-2C_5Q[\varepsilon_6/q]\}/\big(1-\mathrm{e}^{-2C_5Q}\big)\Big)
\end{aligned}$$

for u large. The theorem thus follows from noting that (5.4) and (5.6) imply

$$n(q)\;=\;n(Qu^{-\alpha/(\alpha-1)})\;\ge\;M_\rho\big(K(u^{-\alpha/(\alpha-1)});u^{-\alpha/(\alpha-1)}\big)\,/\,\big(R\,Q^{\ln(R)}M_\rho(K;\varepsilon_5)\big).\;\Box$$

6. Canonical distance and entropy II. Our upper bound rely on the assumption that $\mathbf{P}\{X(t)>u{+}\varDelta,\,X(s)\le u\}\to0$ sufficiently fast when distance$(s,t)\to0$. To prove this fact we use the estimate

$$\mathbf{P}\{X(t)>u{+}\varDelta,\,X(s)\le u\}\le\mathbf{P}\{X(t)+x[X(t)-X(s)]>u{+}x\varDelta\}. \tag{6.1}$$

As is (2.1), (6.1) is crucial when $\alpha<2$, but is not needed in the Gaussian case.

Guided by (6.1) we are led to consider the 'distance'

$\underline{\varrho_x(s,t)}\equiv\max\{\|F_t{+}x(F_t{-}F_s)\|_\alpha{-}\|F_t\|_\alpha,\,\|F_s{+}x(F_s{-}F_t)\|_\alpha{-}\|F_s\|_\alpha\}$ between $s,t\in T$:

Given $\hat{T}\subseteq T$ and $\varepsilon>0$ we write $\underline{\mathcal{E}(\hat{T},x;\varepsilon)}$ for the minimal cardinality of an ε-net N for $\hat{T}$ wrt. ϱ_x such that there to each $t\in\hat{T}$ is an $s=s(N,t)\in N$ satisfying

$$\varrho_x(s,t)<\varepsilon\quad\text{and}\quad X(t)+x[X(t)-X(s)]\in S_\alpha\big(\|F_t{+}x(F_t{-}F_s)\|_\alpha,-1\big). \tag{6.2}$$

When $\alpha=2$ (6.2) is void and $\mathcal{E}(\hat{T},x;\cdot)=E_{\varrho_x}(\hat{T};\cdot)$. But for $\alpha<2$ (6.2) requires

$$F_t(r)+x\big(F_t(r)-F_s(r)\big)\geq 0 \quad \text{a.s. } (m) \text{ for } r\in S. \tag{6.3}$$

Since $\varrho_x(s,t)<\varepsilon$ implies $\|F_t-F_s\|_\alpha\leq x^{-1}\big(\varepsilon+2\|F_t\|_\alpha\big)$, and more generally that s and t are 'close', it does not seem unreasonable to expect that (6.3) holds when $\varrho_x(s,t)<\varepsilon$ with x large. Indeed, (6.3) holds trivially [and $\mathcal{E}(\hat{T},x;\cdot)=E_{\varrho_x}(\hat{T};\cdot)$] in the common situation when for each pair $s,t\in T$ we either have $F_t(r)\geq F_s(r)$ for all $r\in S$ or else $F_s(r)\geq F_t(r)$ for all $r\in S$.

7. The upper bound. Our upper bound requires that there are constants ε_6, C_8, $\Lambda>0$, $\gamma_4,\gamma_5\in[0,1)$ and $\lambda\in(0,1]$ such that

(7.1)

$$\mathcal{E}\big(K_\ell(\varepsilon),x;\varepsilon\big)\leq C_8\exp\{x^{\gamma_4}+\ell^{\gamma_5}\}\,\mathcal{E}\big(K_0(\varepsilon),\Lambda;\lambda\varepsilon\big) \quad \text{for } \ell\in\mathbb{N},\ x\geq\Lambda \text{ and } \varepsilon\leq\varepsilon_0.$$

Theorem 3. *Let* $\{X(t)\}_{t\in T}$ *satisfy* (1.1)-(1.3). *If* (7.1) *hold we then have*

$$\varlimsup_{u\to\infty}\mathbf{P}\{\sup_{t\in K}X(t)>u\}\Big/\Big[\mathcal{E}\big(K_0(u^{-\alpha/(\alpha-1)}),\Lambda;\lambda\,u^{-\alpha/(\alpha-1)}\big)\,\mathbf{P}\{X(\tilde{t})>u\}\Big]<\infty.$$

Proof. Choose $\gamma_6\in(\gamma_4,1)$ and put $q=q(u)\equiv u^{-\alpha/(\alpha-1)}$ and $w=w(u)\equiv u^{-1/(\alpha-1)}$. Further define $x_n\equiv 2^n\Lambda$,

$$\nu_n\equiv\big(2^{1-\gamma_6}-1\big)\textstyle\sum_{i=1}^n 2^{-(1-\gamma_6)i}, \quad\text{and}\quad \eta_n\equiv\nu_{n+1}-\nu_n=\big(2^{1-\gamma_6}-1\big)2^{-(1-\gamma_6)(n+1)},$$

and take q-nets $\mathcal{N}_{n,\ell}(q)$ for $K_\ell(q)$ wrt. ϱ_{x_n} such that $\#\mathcal{N}_{n,\ell}(q)=\mathcal{E}\big(K_\ell(q),x_n;q\big)$ and (6.2) holds. Using (3.2) and that $(1-x)^{-1}\geq 1+x$ for $x<1$, (3.1) then yields

$$\mathbf{P}\Big\{X(t)>u-\big(1-\nu_{n+1}\big)w,\ X(s)\leq u-\big(1-\nu_n\big)w\Big\} \tag{7.2}$$

$$=\sum_{k=0}^\infty\mathbf{P}\Big\{X(t)>u-\big(1-\nu_{n+1}\big)w,\ u-\big(k+2-\nu_n\big)w<X(s)\leq u-\big(k+1-\nu_n\big)w\Big\}$$

$$\leq\sum_{k=0}^\infty\mathbf{P}\Big\{X(t)-X(s)>\big(\eta_n+k\big)w,\ X(t)>u-\big(1-\nu_{n+1}\big)w\Big\}$$

$$\leq\sum_{k=0}^\infty\mathbf{P}\Big\{X(t)+x_n\big(X(t)-X(s)\big)>u+\big[x_n\big(\eta_n+k\big)-1\big]w\Big\}$$

$$=\sum_{k=0}^\infty\mathbf{P}\Big\{S_\alpha\Big(\big\|F_t+x_n\big(F_t-F_s\big)\big\|_\alpha,-1\Big)>u+\big[x_n\big(\eta_n+k\big)-1\big]w\Big\}$$

$$\leq\sum_{k=0}^\infty\mathbf{P}\Big\{S_\alpha\Big(\|F_{\tilde{t}}\|_\alpha-\ell q+\varrho_{x_n}(s,t),-1\Big)>u+\Big[\Lambda\big(2^{1-\gamma_6}-1\big)2^{n\gamma_6-1}+\Lambda 2^n k-1\Big]w\Big\}$$

$$\leq \sum_{k=0}^{\infty} \mathbf{P}\Big\{ S_\alpha\big(\|F_{\tilde t}\|_\alpha,\, -1\big) > u\Big((1-q) + \varLambda\big(2^{n\gamma_6-1}+2^n k\big)q\Big)\Big(1-\frac{\varrho_{x_n}(s,t)}{\|F_{\tilde t}\|_\alpha} + \frac{\ell q}{\|F_{\tilde t}\|_\alpha}\Big)\Big\}$$

$$\leq \sum_{k=0}^{\infty} \mathbf{P}\Big\{ S_\alpha\big(\|F_{\tilde t}\|_\alpha,\, -1\big) > u\Big[1+\frac{1}{2}\Big(\varLambda\Big[2^{n\gamma_6-1}+2^n k\Big]-2+\frac{\ell-2}{\|F_{\tilde t}\|_\alpha}\Big)q\Big]\Big\}$$

$$\leq \sum_{k=0}^{\infty} C_2^2\, 2^{\alpha/2(\alpha-1)} \exp\Big\{-\frac{C_3\alpha\big(\varLambda[2^{n\gamma_6-1}+2^n k]-2+[\ell-2]\|F_{\tilde t}\|_\alpha^{-1}\big)}{2\,(\alpha-1)\,\|F_{\tilde t}\|_\alpha^{\alpha/(\alpha-1)}}\Big\}\mathbf{P}\big\{X(\tilde t)>u\big\}$$

for $t \in K_\ell(q)$ and $s = s\big(\mathcal{N}_{n,\ell}(q);t\big)$, provided that $1-q-q\|F_{\tilde t}\|_\alpha^{-1} \geq \frac{1}{2}$. When $t\in K_\ell(q)$ and $q\leq\frac{1}{2}$ another (easier) application of (3.1) and (3.2) further yield

$$\text{(7.3)}\quad \mathbf{P}\{X(t)>u-w\} \leq \mathbf{P}\Big\{S_\alpha\big(\|F_{\tilde t}\|_\alpha-\ell q,-1\big) > u-w\Big\}$$
$$\leq C_2^2\, 2^{\alpha/2(\alpha-1)} \exp\Big\{-\frac{C_3\alpha\big(\ell\,\|F_{\tilde t}\|_\alpha^{-1}-1\big)}{(\alpha-1)\,\|F_{\tilde t}\|_\alpha^{\alpha/(\alpha-1)}}\Big\}\,\mathbf{P}\big\{X(\tilde t)>u\big\}.$$

Given a $t\in K_\ell(q)$ the fact that $\varrho_x(s,t) \geq x\|F_t-F_s\|_\alpha - 2\|F_t\|_\alpha$ implies

$$\|F_t-F_s\|_\alpha \;\leq\; \tfrac{1}{2}x_n^{-1}q + x_n^{-1}\|F_t\|_\alpha \;\leq\; 2^{-n}\varLambda^{-1}\big(\tfrac{1}{2}q+\|F_{\tilde t}\|_\alpha\big) \quad \text{for } s=s\big(\mathcal{N}_{n,\ell}(q);t\big).$$

Hence the p-closure $\mathrm{clos}_p(\mathcal{N})$ of $\mathcal{N}=\mathcal{N}(q)\equiv\bigcup_{\ell=0}^{\infty}\bigcup_{n=0}^{\infty}\mathcal{N}_{n,\ell}(q)$ contains K.

Since $X(t)-X(s) \in S_\alpha\big(\|F_t-F_s\|_\alpha,\, \beta_{X(t)-X(s)}\big)$, $X(t)$ is $\mathbf{P}$-continuous wrt. the p-topology. Further T-separability of $X(t)$ implies p-separability of $X(t)$. Since $\mathrm{clos}_p(\mathcal{N})\supseteq K$, a well-known argument now shows that $\mathcal{N}$ separates $\{X(t)\}_{t\in K}$.

Combining (7.1)-(7.3) in an argument inspired by Samorodnitsky (1991) we get

$$\mathbf{P}\big\{\sup_{t\in K} X(t)>u\big\}$$
$$\leq \mathbf{P}\Big\{\bigcup_{\ell=0}^{\infty}\bigcup_{n=0}^{\infty}\Big\{\sup_{t\in\mathcal{N}_{n,\ell}} X(t)>u-(1-\nu_n)w\Big\}\Big\}$$
$$\leq \mathbf{P}\Big\{\bigcup_{\ell=0}^{\infty}\Big[\bigcup_{n=0}^{\infty}\Big\{\sup_{t\in\mathcal{N}_{n+1,\ell}} X(t)>u-(1-\nu_{n+1})w\Big\}\cap\Big\{\sup_{s\in\mathcal{N}_{n,\ell}} X(s)\leq u-(1-\nu_n)w\Big\}\Big]\Big\}$$
$$+\,\mathbf{P}\Big\{\bigcup_{\ell=0}^{\infty}\bigcup_{t\in\mathcal{N}_{0,\ell}}\{X(t)>u-w\}\Big\}$$
$$\leq \sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty}\sum_{t\in\mathcal{N}_{n+1,\ell}} \mathbf{P}\Big\{X(t)>u-(1-\nu_{n+1})w,\; X\big(s(\mathcal{N}_{n,\ell};t)\big)\leq u-(1-\nu_n)w\Big\}$$
$$+\sum_{\ell=0}^{\infty}\sum_{t\in\mathcal{N}_{0,\ell}} \mathbf{P}\big\{X(t)>u-w\big\}$$
$$\leq \sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} C_2^2\, 2^{\alpha/2(\alpha-1)} \exp\Big\{-\frac{C_3\alpha\big(\varLambda[2^{n\gamma_6-1}+2^n k]-2+[\ell-2]\|F_{\tilde t}\|_\alpha^{-1}\big)}{2\,(\alpha-1)\,\|F_{\tilde t}\|_\alpha^{\alpha/(\alpha-1)}}\Big\}$$

$$\times C_8 \exp\{\ell^{\gamma_5} + \varLambda^{\gamma_4} 2^{\gamma_4 n}\} \mathcal{E}(K_0(q), \varLambda; \lambda^2 q)\, \mathbf{P}\{X(\tilde t) > u\}$$

$$+ \sum_{\ell=0}^{\infty} C_2^2\, 2^{\alpha/2(\alpha-1)} \exp\left\{ -\frac{C_3 \alpha\big(\ell\, \|F_{\tilde t}\|_\alpha^{-1} - 1\big)}{(\alpha-1)\, \|F_{\tilde t}\|_\alpha^{\alpha/(\alpha-1)}} \right\}$$

$$\times C_8 \exp\{\ell^{\gamma_5}\} \mathcal{E}(K_0(q), \varLambda; \lambda^2 q)\, \mathbf{P}\{X(\tilde t) > u\} \quad \text{for } u \text{ large.} \quad \square$$

8. Canonical distances III. Here we derive two-term expansions of ρ and ϱ_x. The first terms coincide except for multiplicative constants, and define a distance

$$\delta(s,t) \equiv \max\big\{ \langle (F_t - F_s) F_t^{\alpha-1}\rangle / \|F_t\|_\alpha^{\alpha-1},\ \langle (F_s - F_t) F_s^{\alpha-1}\rangle / \|F_s\|_\alpha^{\alpha-1} \big\} \quad \text{for } s,t \in T.$$

Now let $k_{\hat\alpha} \equiv \sup\big\{ x \geq -\frac12 : |x|^{-\hat\alpha}\big((1+x)^\alpha - 1 - \alpha x\big)\big\}$ for $\hat\alpha \in [\alpha, 2]$, and define

$$\circ, \bullet : T^2 \to T \ \text{ by } \ \begin{array}{l} s\circ t = s \ \text{ and } \ s\bullet t = t \ \text{ when } \ \delta(s,t) = \big\langle (F_t - F_s) F_t^{\alpha-1}\big\rangle / \|F_t\|_\alpha^{\alpha-1} \\ s\circ t = t \ \text{ and } \ s\bullet t = s \ \text{ when } \ \delta(s,t) > \big\langle (F_t - F_s) F_t^{\alpha-1}\big\rangle / \|F_t\|_\alpha^{\alpha-1} \end{array}.$$

Proposition 3. *We have*

$$\delta(s,t) - k_{\hat\alpha} 2^{1-\hat\alpha} \alpha^{-1} \big\langle |F_t - F_s|^{\hat\alpha} F_{s\bullet t}^{\alpha-\hat\alpha}\big\rangle / \|F_{s\bullet t}\|_\alpha^{\alpha-1} \leq \rho(s,t) \leq \alpha\, \delta(s,t) \tag{8.1}$$

for $s,t \in T$. *Under the additional requirement that*

$$\kappa \big\langle (F_{s\bullet t} - F_{s\circ t}) F_{s\bullet t}^{\alpha-1}\big\rangle \geq k_{\hat\alpha} 2^{1-\hat\alpha} \tfrac{1}{\alpha} \big\langle |F_t - F_s|^{\hat\alpha} F_{s\bullet t}^{\alpha-\hat\alpha}\big\rangle \quad \textit{for } (s,t) \in \mathcal{T}, \tag{8.2}$$

for some $\kappa \in (0,1)$ *and* $\mathcal{T} \subseteq T^2$, *we also have*

$$(1-\kappa)\, \delta(s,t) \leq \rho(s,t) \leq \alpha\, \delta(s,t) \quad \textit{for } (s,t) \in \mathcal{T}. \tag{8.3}$$

If in particular $F_{s\bullet t}(r) \geq F_{s\circ t}(r)$ *for* $(r,s,t) \in S \times \mathcal{T}$, *then* (8.3) *holds with* $\kappa = \frac{1}{\alpha}$.

Proof. Since $k_\alpha = 1$, (8.2) holds with $\kappa = \frac{1}{\alpha}$ and $\hat\alpha = \alpha$ when $F_{s\bullet t} \geq F_{s\circ t}$. Further (8.1) and (8.2) imply (8.3). Hence it is enough to prove (8.1). But noting that

$$\begin{array}{llll} & 1 - \alpha x \leq (1-x)^\alpha \leq 1 - \alpha x + k_{\hat\alpha} |x|^{\hat\alpha} & \text{for } x \in (-\infty, \tfrac12], & \text{and} \\ (8.4) & 1 - x^+ \leq (1-x)^{1/\alpha} \leq 1 - \tfrac{1}{\alpha} x & \text{for } x \in (-\infty, 1], & \end{array}$$

we readily obtain

$$\begin{aligned} &(8.5) \qquad \|2F_t\|_\alpha - \|F_s + F_t\|_\alpha \\ &= \|2F_t\|_\alpha - \big[\big\langle (2F_t)^\alpha [1 - (F_t - F_s)/(2F_t)]^\alpha \big\rangle\big]^{1/\alpha} \\ &\begin{cases} \leq \|2F_t\|_\alpha \big[1 - \big(1 - \frac{\alpha}{2} \langle (F_t - F_s) F_t^{\alpha-1}\rangle / \|F_t\|_\alpha^\alpha\big)^{1/\alpha}\big] \\ \geq \|2F_t\|_\alpha \Big[1 - \Big(1 - \frac{\alpha}{2}\big[\big\langle (F_t - F_s) F_t^{\alpha-1}\big\rangle - k_{\hat\alpha} 2^{1-\hat\alpha} \big\langle |F_t - F_s|^{\hat\alpha} F_t^{\alpha-\hat\alpha}\big\rangle\big] / \|F_t\|_\alpha^\alpha\Big)^{1/\alpha}\Big] \end{cases} \\ &\begin{cases} \leq \alpha \big\langle (F_t - F_s) F_t^{\alpha-1}\big\rangle^+ / \|F_t\|_\alpha^{\alpha-1} \\ \geq \big\langle (F_t - F_s) F_t^{\alpha-1}\big\rangle / \|F_t\|_\alpha^{\alpha-1} - k_{\hat\alpha} 2^{1-\hat\alpha} \frac{1}{\alpha} \big\langle |F_t - F_s|^{\hat\alpha} F_t^{\alpha-\hat\alpha}\big\rangle / \|F_t\|_\alpha^{\alpha-1} \end{cases} . \quad \square \end{aligned}$$

Now define $K_{\hat\alpha} \equiv \sup\{x\in\mathbb{R} : |x|^{-\hat\alpha}(|1+x|^\alpha - 1 - \alpha x)\}$ for $\hat\alpha \in [\alpha, 2]$, and

$$s\diamond t = t \quad \text{when} \quad \langle |F_t - F_s|^{\hat\alpha} F_t^{\alpha-\hat\alpha}\rangle / \|F_t\|_\alpha^{\alpha-1} \geq \langle |F_t - F_s|^{\hat\alpha} F_s^{\alpha-\hat\alpha}\rangle / \|F_s\|_\alpha^{\alpha-1},$$

$$s\diamond t = s \quad \text{when} \quad \langle |F_t - F_s|^{\hat\alpha} F_t^{\alpha-\hat\alpha}\rangle / \|F_t\|_\alpha^{\alpha-1} < \langle |F_t - F_s|^{\hat\alpha} F_s^{\alpha-\hat\alpha}\rangle / \|F_s\|_\alpha^{\alpha-1}.$$

Proposition 4. *Writing* $\mathbb{T} \equiv \{(s,t)\in T^2 : \varrho_x(s,t) < \frac{1}{2\alpha}\|F_{s\bullet t}\|_\alpha\}$ *we have*

$$(8.6) \quad \varrho_x(s,t) \leq 2\max\{x\,\delta(s,t),\, K_{\hat\alpha} x^{\hat\alpha} \langle |F_t - F_s|^{\hat\alpha} F_{s\diamond t}^{\alpha-\hat\alpha}\rangle / \|F_{s\diamond t}\|_\alpha^{\alpha-1}\} \quad \textit{for } s,t\in T,$$

$$(8.7) \quad \varrho_x(s,t) \geq \tfrac{1}{2} x\,\delta(s,t) \qquad \textit{for } (s,t)\in\mathbb{T}.$$

Under the additional requirement that

$$(8.8) \qquad \kappa\,\langle (F_{s\bullet t} - F_{s\circ t}) F_{s\bullet t}^{\alpha-1}\rangle \geq \langle (F_{s\bullet t} - F_{s\circ t})^- F_{s\bullet t}^{\alpha-1}\rangle \quad \textit{for } (s,t)\in\mathcal{T},$$

for some choice of $\kappa\in[1,\infty)$ *and* $\mathcal{T}\subseteq T^2$, *we also have*

$$(8.9) \qquad \varrho_x(s,t) \geq \tfrac{1}{8\kappa}\max\{x\,\delta(s,t),\, x^\alpha \|F_t - F_s\|_\alpha^\alpha / \|F_{s\bullet t}\|_\alpha^{\alpha-1}\} \quad \textit{for } (s,t)\in\mathcal{T}\cap\mathbb{T}.$$

If in particular $F_{s\bullet t}(r) \geq F_{s\circ t}(r)$ *for* $(r,s,t)\in S\times\mathcal{T}$, *then* (8.9) *holds with* $\kappa = 1$.

Proof. In view of the elementary facts that

$$1 + \alpha x \leq |1+x|^\alpha \leq 1 + \alpha x + K_{\hat\alpha}|x|^{\hat\alpha} \quad \text{for } x\in\mathbb{R}, \quad \text{and}$$

$$(8.10) \qquad 1 + \tfrac{1}{2\alpha}(x\wedge 1) \leq (1+x)^{1/\alpha} \leq 1 + \tfrac{1}{\alpha}x \quad \text{for } x \geq 0,$$

we readily deduce

$$(8.11) \qquad \big\|F_t + x(F_t - F_s)\big\|_\alpha - \|F_t\|_\alpha$$

$$= \big[\langle F_t^\alpha |1 + x(F_t - F_s)/F_t|^\alpha\rangle\big]^{1/\alpha} - \|F_t\|_\alpha$$

$$\begin{cases} \leq \|F_t\|_\alpha \Big[\Big(1 + \alpha x \langle (F_t - F_s) F_t^{\alpha-1}\rangle / \|F_t\|_\alpha^\alpha + K_{\hat\alpha} x^{\hat\alpha} \langle |F_t - F_s|^{\hat\alpha} F_t^{\alpha-\hat\alpha}\rangle / \|F_t\|_\alpha^\alpha\Big)^{1/\alpha} - 1\Big] \\ \geq \|F_t\|_\alpha \big[\big(1 + \alpha x \langle (F_t - F_s) F_t^{\alpha-1}\rangle / \|F_t\|_\alpha^\alpha\big)^{1/\alpha} - 1\big] \end{cases}$$

$$\begin{cases} \leq x \langle (F_t - F_s) F_t^{\alpha-1}\rangle^+ / \|F_t\|_\alpha^{\alpha-1} + K_{\hat\alpha} \tfrac{1}{\alpha} x^{\hat\alpha} \langle |F_t - F_s|^{\hat\alpha} F_t^{\alpha-\hat\alpha}\rangle / \|F_t\|_\alpha^{\alpha-1} \\ \geq \tfrac{1}{2\alpha}\|F_t\|_\alpha \big[\big(\alpha x \langle (F_t - F_s) F_t^{\alpha-1}\rangle / \|F_t\|_\alpha^\alpha\big) \wedge 1\big] \end{cases}.$$

But it is a straightforward matter to conclude (8.6) and (8.7) from these inequalities.

To proceed we assume that (8.8) holds and observe the easy fact that

$$|1+x|^\alpha \geq 1 + \alpha(1 - \tfrac{1}{4\kappa})x^+ - \alpha(1 + \tfrac{1}{4\kappa})x^- + (4\kappa)^{\alpha-2}|x|^\alpha \quad \text{for } x\in\mathbb{R}.$$

Via an inspection of the equality in (8.11) we therefore obtain

$$\big\|F_{s\bullet t} + x(F_{s\bullet t} - F_{s\circ t})\big\|_\alpha^\alpha$$

$$\geq \|F_{s\bullet t}\|_\alpha^\alpha + \alpha(1 - \tfrac{1}{4\kappa})\,x\,\langle (F_{s\bullet t} - F_{s\circ t})^+ F_{s\bullet t}^{\alpha-1}\rangle - \alpha(1 + \tfrac{1}{4\kappa})\,x\,\langle (F_{s\bullet t} - F_{s\circ t})^- F_{s\bullet t}^{\alpha-1}\rangle$$

$$+ (4\kappa)^{\alpha-2} x^\alpha \|F_{s\bullet t} - F_{sot}\|_\alpha^\alpha$$
$$= \|F_{s\bullet t}\|_\alpha^\alpha + \tfrac{\alpha}{4\kappa}\, x \left\langle (F_{s\bullet t} - F_{sot}) F_{s\bullet t}^{\alpha-1} \right\rangle + (4\kappa)^{\alpha-2} x^\alpha \|F_t - F_s\|_\alpha^\alpha$$
$$+ \tfrac{\alpha}{4\kappa}\, x \Big((4\kappa - 2) \big\langle (F_{s\bullet t} - F_{sot}) F_{s\bullet t}^{\alpha-1} \big\rangle - 2 \big\langle (F_{s\bullet t} - F_{sot})^- F_{s\bullet t}^{\alpha-1} \big\rangle \Big).$$

Here the last term is non-negative for $(s,t)\in\mathcal{T}$ by (8.8), and so (8.10) yields

$$\varrho_x(s,t) \geq \|F_{s\bullet t}\|_\alpha \Big[\Big(1 + \tfrac{\alpha}{4\kappa} x \delta(s,t)/\|F_{s\bullet t}\|_\alpha + (4\kappa)^{\alpha-2} x^\alpha \|F_t - F_s\|_\alpha^\alpha / \|F_{s\bullet t}\|_\alpha^\alpha \Big)^{1/\alpha} - 1 \Big]$$
$$\geq \tfrac{1}{2\alpha} \|F_{s\bullet t}\|_\alpha \Big[\Big(\tfrac{\alpha}{4\kappa} x \delta(s,t)/\|F_{s\bullet t}\|_\alpha + (4\kappa)^{\alpha-2} x^\alpha \|F_t - F_s\|_\alpha^\alpha / \|F_{s\bullet t}\|_\alpha^\alpha \Big) \wedge 1 \Big].$$

Taking $(s,t)\in\mathcal{T}\cap\mathbb{T}$, (8.9) thus follows from the trivial fact that $(4\kappa)^{\alpha-2} \geq \frac{\alpha}{4\kappa}$. □

Proposition 5. *If $F_{s\bullet t}(\cdot) \geq F_{sot}(\cdot)$ for $(s,t)\in\mathcal{T}$, for some $\mathcal{T}\subseteq T^2$, then we have*

$$(1 - \tfrac{1}{\alpha})\, \delta(s,t) \leq \|F_{s\bullet t}\|_\alpha - \|F_{sot}\|_\alpha \leq \alpha\, \delta(s,t) \quad \textit{for} \quad (s,t)\in\mathcal{T}.$$

Proof. Using the first set of inequalities in (8.5), with F_t and F_s replaced by $\frac{1}{2}F_{s\bullet t}$ and $F_{sot} - F_t$ respectively, and with $\hat{\alpha} = \alpha$, we readily deduce that

$$(\alpha - k_\alpha) \left\langle (F_{s\bullet t} - F_{sot}) F_{s\bullet t}^{\alpha-1} \right\rangle \leq \|F_{s\bullet t}\|_\alpha^\alpha - \|F_{sot}\|_\alpha^\alpha \leq \alpha \left\langle (F_{s\bullet t} - F_{sot}) F_{s\bullet t}^{\alpha-1} \right\rangle.$$

Here $k_\alpha = 1$ as before. The proposition thus follows from noting that (8.4) implies

$$\text{(8.12)} \qquad \|F_{s\bullet t}\|_\alpha - \|F_{sot}\|_\alpha = \|F_{s\bullet t}\|_\alpha \Big[1 - \big(1 - [\|F_{s\bullet t}\|_\alpha^\alpha - \|F_{sot}\|_\alpha^\alpha] / \|F_{s\bullet t}\|_\alpha^\alpha \big)^{1/\alpha} \Big]$$
$$\begin{cases} \leq [\|F_{s\bullet t}\|_\alpha^\alpha - \|F_{sot}\|_\alpha^\alpha] / \|F_{s\bullet t}\|_\alpha^{\alpha-1} \\ \geq \big(\|F_{s\bullet t}\|_\alpha^\alpha - \|F_{sot}\|_\alpha^\alpha \big) / \big(\alpha \|F_{s\bullet t}\|_\alpha^{\alpha-1} \big) \end{cases}. \qquad \square$$

9. Non-anticipative moving averages of stable motion. Let

$$\text{(9.1)} \qquad X(t) \equiv \textstyle\int_{\mathbb{R}} F_t(r)\, dM(r) \quad \text{for} \quad t\in\mathbb{R},$$

where M has skewness $\beta(r) = -1$ and Lebesgue control measure, and where

$$\text{(9.2)} \qquad F_t(r) = f(t-r)\, I_{\mathbb{R}^+}(t-r) \quad \text{for some} \quad f:\mathbb{R}\to\mathbb{R}^+ \quad \text{satisfying} \quad \|f\|_\alpha = 1.$$

Further assume that f satisfies the left Lipschitz condition

$$\text{(9.3)} \qquad M_1 \equiv \sup\big\{ [f(s) - f(t)] / [(t-s)\, f(t)] : 0 \leq s < t < \infty \big\} < \infty,$$

and that f is absolutely continuous with derivative f' such that

$$\text{(9.4)} \qquad h^{-1}\big(f(\cdot + h) - f(\cdot) \big) \to_{\mathbb{L}^\alpha(\mathbb{R})} f'(\cdot) \quad \text{as} \quad h \to 0.$$

Application 1. *Consider the α-stable moving average $\{X(t)\}_{t\in\mathbb{R}}$ given by* (9.1) *where $\{F_t(\cdot)\}_{t\in\mathbb{R}}$ satisfies* (9.2)-(9.4) *with $f(0) > 0$. For each $T > 0$ we then have*

$$0<\varliminf_{u\to\infty}\frac{\mathbf{P}\{\sup_{t\in[0,T]}X(t)>u\}}{u^{\alpha/(\alpha-1)}\,\mathbf{P}\{X(0)>u\}}\leq\varlimsup_{u\to\infty}\frac{\mathbf{P}\{\sup_{t\in[0,T]}X(t)>u\}}{u^{\alpha/(\alpha-1)}\,\mathbf{P}\{X(0)>u\}}<\infty.$$

Proof. Using (9.2) and (9.4), elementary considerations reveal that

$$\text{(9.5)}\quad \big\langle(F_s-F_t)F_s^{\alpha-1}\big\rangle\sim-(t-s)\int_{-\infty}^{s}f'(s-r)f(s-r)^{\alpha-1}\,dr=\alpha^{-1}f(0)^{\alpha}(t-s)$$

$$\text{(9.6)}\quad \big\langle(F_t-F_s)F_t^{\alpha-1}\big\rangle\sim(t-s)\int_{-\infty}^{t}f'(t-r)f(t-r)^{\alpha-1}\,dr+\int_{s}^{t}f(s-r)f(t-r)^{\alpha-1}\,dr$$
$$\sim(1-\alpha^{-1})\,f(0)^{\alpha}(t-s)$$

uniformly as $t-s\downarrow 0$. It follows that $s\bullet t=s\wedge t$ and $s\circ t=s\vee t$, and that

$$\text{(9.7)}\qquad \delta(s,t)\sim\alpha^{-1}f(0)^{\alpha}|t-s|\quad\text{uniformly for } |t-s| \text{ small.}$$

Also observe the fact that [by (9.4)]

$$\text{(9.8)}\quad \|F_t-F_s\|_{\alpha}^{\alpha}=\int_{-\infty}^{s}\big|f(t-r)-f(s-r)\big|^{\alpha}\,dr+\int_{s}^{t}f(t-r)^{\alpha}\,dr$$
$$\sim(t-s)^{\alpha}\|f'I_{\mathbb{R}+}\|_{\alpha}^{\alpha}+f(0)^{\alpha}(t-s)\quad\text{uniformly as } t-s\downarrow 0.$$

Take $\hat{\alpha}=\alpha$ and $\kappa\in(2^{1-\alpha},1)$. Since $k_{\alpha}=1$, (9.5)-(9.8) then imply that (8.2) holds uniformly for $|t-s|$ small. By Proposition 3, (8.3) thus holds uniformly for $|t-s|$ small. But (8.3) and (9.7) combine to prove that

$$\text{(9.9)}\qquad C_9^{-2}\,|t-s|\;\leq\;C_9^{-1}\,\delta(s,t)\;\leq\;\rho(s,t)\;\leq\;C_9\,\delta(s,t)\;\leq\;C_9^{2}\,|t-s|$$

uniformly for $|t-s|$ small, for some $C_9>1$. Consequently we have

$$\text{(9.10)}\qquad C_9^{-2}(b-a)/\varepsilon\;\leq\;M_{\rho}([a,b];\varepsilon)\;\leq\;1+C_9^{2}(b-a)/\varepsilon\quad\text{for } [a,b]\subseteq[0,T].$$

In view of (8.6) and (8.7), (9.7) and (9.8) show that

$$\text{(9.11)}\qquad C_{10}^{-1}x\,|t-s|\leq\varrho_x(s,t)\leq C_{10}\big(x\vee x^{\alpha}\big)|t-s|$$

uniformly for $|t-s|$ small when $\varrho_x(s,t)\leq\frac{1}{2\alpha}$. Further (9.3) and (9.11) give

$$F_t(r)+x\big(F_t(r)-F_s(r)\big)$$
$$=f(t-r)+x\big(f(t-r)-f(s-r)\big)\;\geq\;f(t-r)\big[1-M_1x(t-s)\big]\;\geq\;f(t-r)\big[1-M_1C_{10}\varepsilon\big]\;\geq\;0$$

for $0\leq s-r\leq t-r$ and $\varrho_x(s,t)<\varepsilon$ with ε small. Hence (6.3) holds and

$$\text{(9.12)}\qquad \tfrac{1}{2}C_{10}^{-1}(b-a)y/\varepsilon\;\leq\;\mathcal{E}([a,b],y;\varepsilon)\;=\;E_{\varrho_y}([a,b];\varepsilon)\;\leq\;1+\tfrac{1}{2}C_{10}(b-a)y^{\alpha}/\varepsilon\quad\text{for } y\geq 1.$$

Clearly, (9.10) shows that $M_{\rho}([0,T];\cdot)$ satisfies (4.2) and (5.1). Taking $\hat{\rho}(s,t)=|t-s|$, (9.9) further combines with Proposition 2 to prove (4.5) and (5.2). Hence

Theorem 2 gives the lower bound. Using (9.12) we further deduce that (7.1) holds with $\Lambda=\lambda=1$, and so Theorem 3 yields the upper bound. $\square$

10. $\mathbb{C}^2$-moving averages of stable motion. Suppose that

$$F_t(r) = f(t-r) \quad \text{for some } f:\mathbb{R}\to\mathbb{R}^+ \text{ satisfying } \|f\|_\alpha = 1. \tag{10.1}$$

Here f is assumed to be absolutely continuous with derivative f'. We also assume that f' is absolutely continuous with derivative f'', and that

$$(f')^2 f^{\alpha-2} \in \mathbb{L}^1(\mathbb{R}) \quad \text{and} \quad f'' f^{\alpha-1} \in \mathbb{L}^1(\mathbb{R}). \tag{10.2}$$

Further we require that f satisfies the Lipschitz condition

$$M_2 \equiv \sup\big\{|f(s)-f(t)|\big/[|t-s|\,f(t)] : s,t\in\mathbb{R},\ s\neq t\big\} < \infty. \tag{10.3}$$

Finally we have to require that f 'obeys it's Taylor expansion' in the sense that

$$h^{-2}\big\langle[f(\cdot)+h\,f'(\cdot) - f(h+\cdot)]^2 f(\cdot)^{\alpha-2}\big\rangle \to 0 \tag{10.4}$$

$$h^{-2}\big\langle[f(\cdot)+h\,f'(\cdot)+\tfrac{1}{2}h^2 f''(\cdot) - f(h+\cdot)]\, f(\cdot)^{\alpha-1}\big\rangle \to 0 \tag{10.5}$$

as $h\to 0$.

Application 2. *Consider the α-stable moving average $\{X(t)\}_{t\in\mathbb{R}}$ given by* (9.1) *where M has skewness -1 and Lebesgue control measure, and where F_t satisfies* (10.1)-(10.5). *For each $T>0$ we then have*

$$0 < \varliminf_{u\to\infty} \frac{\mathbf{P}\{\sup_{t\in[0,T]} X(t)>u\}}{u^{\alpha/[2(\alpha-1)]}\,\mathbf{P}\{X(0)>u\}} \leq \varlimsup_{u\to\infty} \frac{\mathbf{P}\{\sup_{t\in[0,T]} X(t)>u\}}{u^{\alpha/[2(\alpha-1)]}\,\mathbf{P}\{X(0)>u\}} < \infty.$$

Proof. Putting $F_t'(\cdot) \equiv f'(t-\cdot)$ and $F_t''(\cdot) \equiv f''(t-\cdot)$, Hölder's inequality and (10.2) show that $F_t' F_t^{\alpha-1} \in \mathbb{L}^1(\mathbb{R})$. Thus we have $\langle F_t' F_t^{\alpha-1}\rangle = 0$, and (10.5) yields

$$\begin{aligned} &\langle (F_t-F_s)F_t^{\alpha-1}\rangle \qquad (10.6)\\ &= \big\langle \big[F_t+(s-t)F_t'+\tfrac{1}{2}(s-t)^2F_t'' - F_s\big]F_t^{\alpha-1}\big\rangle - (s-t)\langle F_t'F_t^{\alpha-1}\rangle - \tfrac{1}{2}(s-t)^2\langle F_t''F_t^{\alpha-1}\rangle\\ &\sim -\tfrac{1}{2}\langle F_t''F_t^{\alpha-1}\rangle\,(s-t)^2\\ &= \tfrac{1}{2}(\alpha-1)\,\langle (F_t')^2F_t^{\alpha-2}\rangle\,(s-t)^2 \quad \text{uniformly as } |t-s|\downarrow 0. \end{aligned}$$

Using (10.4) and the fact that $(x+y)^2 \leq \frac{\alpha+1}{\alpha-1}x^2 + \frac{\alpha+1}{2}y^2$, we further obtain

$$\begin{aligned} &\langle (F_t-F_s)^2F_t^{\alpha-2}\rangle \qquad (10.7)\\ &\leq \tfrac{\alpha+1}{\alpha-1}\big\langle[F_t+(s-t)F_t' - F_s]^2F_t^{\alpha-2}\big\rangle + \tfrac{\alpha+1}{2}\,(s-t)^2\langle(F_t')^2F_t^{\alpha-2}\rangle\\ &\sim \tfrac{\alpha+1}{2}\,\big\langle(F_t')^2F_t^{\alpha-2}\big\rangle\,(s-t)^2 \qquad \text{uniformly as } |t-s|\downarrow 0. \end{aligned}$$

Observing that $k_{\hat\alpha} \le \alpha–1$ when $\hat\alpha = 2$, (10.6) and (10.7) imply that (8.2) holds for $\kappa \in \big((\alpha+1)/(2\alpha), 1\big)$ uniformly for $|t-s|$ small. Hence also (8.3) holds uniformly for $|t-s|$ small. But (8.3) combines with (10.6) to show that

$$(10.8) \qquad C_{11}^{-2}(t-s)^2 \le C_{11}^{-1}\delta(s,t) \le \rho(s,t) \le C_{11}\delta(s,t) \le C_{11}^2(t-s)^2$$

uniformly for $|t-s|$ small, for some $C_{11} > 1$. Consequently

$$(10.9) \quad C_{11}^{-1}(b-a)/\sqrt{\varepsilon} \le M_\rho([a,b];\varepsilon) \le 1 + C_{11}(b-a)/\sqrt{\varepsilon} \quad \text{for } [a,b] \subseteq [0,T].$$

Clearly (10.9) shows that $M_\rho([0,T];\cdot)$ satisfies (4.2) [and (5.1)]. Further, by (10.8), $\hat\rho(s,t) \equiv (t–s)^2$ satisfies (4.3). Since also (4.4) holds with $\hat\Delta = 2$, Proposition 2 proves (4.5) [and (5.2)]. Thus the lower bound follows from Theorem 2.

Combining (10.6) and (10.7) with (8.6) and (8.7) we obtain

$$(10.10) \quad C_{12}^{-1}(t-s)^2 x \le \varrho_x(s,t) \le C_{12}(t-s)^2\big(x \vee x^2\big) \quad \text{uniformly for } |t-s| \text{ small.}$$

Taking $x \ge 1$ and $|t-s| < x^{-1}\sqrt{\varepsilon/C_{12}}$, (10.3) further shows that

$$F_t(r) + x\big(F_t(r)–F_s(r)\big) = f(t–r) + x\big(f(t–r)–f(s–r)\big) \ge f(t–r)\big[1–M_2 x|t–s|\big] \ge 0$$

for $\varepsilon \le M_2^{-2}C_{12}$, so that (6.3) holds. Using (10.10) we therefore readily conclude

$$\tfrac{1}{2}(b–a)\sqrt{y/C_{12}\varepsilon} \le \mathcal{E}([a,b],y;\varepsilon) = E_{\varrho_y}([a,b];\varepsilon) \le 1 + \tfrac{1}{2}(b–a)y\sqrt{C_{12}/\varepsilon} \quad \text{for } y \ge 1.$$

Thus (7.1) hold with $\Lambda = \lambda = 1$ and Theorem 3 yields the upper bound. □

11. Fractional stable motions with index of self-similarity $> 1/\alpha$**.** The most commonly used α-stable analogue of fractional Brownian motion is

$$(11.1) \qquad X_{a,b}^H(t) \equiv \int_{\mathbb{R}} \Big(a\big([(t+r)^+]^H - [r^+]^H\big) - b\big([(t+r)^-]^H - [r^-]^H\big)\Big)\, dM(r),$$

where M has Lebesgue control measure and constant skewness, and where $H \in (-1/\alpha, 0) \cup (0, 1-1/\alpha)$ and $a, b \ge 0$. This process is self-similar with index $H + 1/\alpha$ and has stationary increments [e.g., Samorodnitsky and Taqqu (1994, Proposition 7.4.2)]. After stable Lévy motion, $X_{a,b}^H(t)$ is the most important stable process.

Application 3. *For the totally skewed fractional α-stable motion*

$$X(t) \equiv X_{1,0}^H = \int_{\mathbb{R}} \big([(t+r)^+]^H - [r^+]^H\big)\, dM(r) \quad \text{for } t \in \mathbb{R}^+,$$

where $H \in (0, 1-1/\alpha)$ *and* M *has skewness* -1, *we have*

$$0 < \varliminf_{u\to\infty} \frac{\mathbf{P}\{\sup_{t\in[0,1]} X(t) > u\}}{\mathbf{P}\{X(1) > u\}} \le \varlimsup_{u\to\infty} \frac{\mathbf{P}\{\sup_{t\in[0,1]} X(t) > u\}}{\mathbf{P}\{X(1) > u\}} < \infty.$$

Proof. Only the upper bound requires a proof. That bound will follow if we prove

$$L_6 \equiv \varlimsup_{u\to\infty} \mathbf{P}\{\sup_{t\in[\frac{1}{2},1]} X(t) > u\} / \mathbf{P}\{X(1) > u\} < \infty. \tag{11.2}$$

This is a consequence of the easy observation that by self-similarity (for u large)

$$\begin{aligned} \mathbf{P}\{\sup_{t\in[0,1]} X(t) > u\} &\le \sum_{k=0}^{\infty} \mathbf{P}\{\sup_{t\in[2^{-(k+1)},2^{-k}]} X(t) > u\} \\ &= \sum_{k=0}^{\infty} \mathbf{P}\{\sup_{t\in[\frac{1}{2},1]} X(t) > 2^{k(H+1/\alpha)} u\} \\ &\le 2L_6 \sum_{k=0}^{\infty} \mathbf{P}\{X(1) > 2^{k(H+1/\alpha)} u\} \qquad \text{[by (11.2)]} \\ &\sim 2L_6\, \mathbf{P}\{X(1) > u\} \quad \text{as } u\to\infty \qquad \text{[by (3.1)]}. \end{aligned} \tag{11.3}$$

To prove (11.2) we first note that a two-fold application of the inequality

$$x^H - y^H \ge H\,(x-y)\,x^{H-1} \quad \text{for } x,y>0, \tag{11.4}$$

yields

$$\begin{aligned} \langle (F_t - F_s) F_t^{\alpha-1} \rangle &\ge \int_0^{\infty} \big[(t+x)^H - (s+x)^H\big]\,\big[(t+x)^H - x^H\big]^{\alpha-1}\,dx \\ &\ge H\,(t-s)\,H^{\alpha-1} t^{\alpha-1} \int_0^{\infty} (t+x)^{H-1} (t+x)^{(H-1)(\alpha-1)}\,dx \\ &= \tfrac{H^\alpha}{(1-H)\alpha-1}\, t^{H\alpha}\,(t-s) \quad \text{for } 0 \le s < t. \end{aligned} \tag{11.5}$$

Invoking the inequality (8.4), with $1/\alpha$ replaced by H, we further obtain

$$\begin{aligned} \|F_t - F_s\|_\alpha^\alpha &= \int_{-t}^{-s} (t+x)^{H\alpha}\,dx + \int_{-s}^{\infty} \big[(t+x)^H - (s+x)^H\big]^\alpha\,dx \\ &= \tfrac{1}{H\alpha+1}\,(t-s)^{H\alpha+1} + \int_{-s}^{\infty} (t+x)^{H\alpha} \left[1 - \left(1 - \frac{t-s}{t+x}\right)^H\right]^\alpha dx \\ &\le \tfrac{1}{H\alpha+1}\,(t-s)^{H\alpha+1} + (t-s)^\alpha \int_{-s}^{\infty} (t+x)^{H\alpha} (t+x)^{-\alpha}\,dx \\ &= \tfrac{1}{H\alpha+1}\,(t-s)^{H\alpha+1} + \tfrac{1}{(1-H)\alpha-1}\,(t-s)^{H\alpha+1} \end{aligned} \tag{11.6}$$

for $0 \le s < t$. On the other hand an inspection of (11.6) shows that

$$\|F_t - F_s\|_\alpha^\alpha \ge \tfrac{1}{H\alpha+1}\,(t-s)^{H\alpha+1} \quad \text{for } 0 \le s < t. \tag{11.7}$$

Finally we observe that, by three applications of (8.4), with $1/\alpha$ replaced by H,

$$\langle (F_t - F_s) F_t^{\alpha-1} \rangle \tag{11.8}$$

$$= \tfrac{1}{H\alpha+1}(t-s)^{H\alpha+1} + \int_{-s}^{0}(s+x)^H\bigg[\Big(1+\frac{t-s}{s+x}\Big)^H - 1\bigg](t+x)^{H(\alpha-1)}\,dx$$

$$+ \int_0^\infty (s+x)^{H\alpha}\bigg[\Big(1+\frac{t-s}{s+x}\Big)^H - 1\bigg]\bigg[\Big(1+\frac{t-s}{s+x}\Big)^H - \Big(1-\frac{s}{s+x}\Big)^H\bigg]^{\alpha-1} dx$$

$$\le \tfrac{1}{H\alpha+1}(t-s)^{H\alpha+1} + H(t-s)t^{H(\alpha-1)}\int_{-s}^{0}(s+x)^{H-1}\,dx$$

$$+ H(t-s)\int_0^\infty (s+x)^{H\alpha-1}\Big(\frac{H(t-s)}{s+x} + \frac{s}{s+x}\Big)^{\alpha-1} dx$$

$$\le \tfrac{1}{H\alpha+1}(t-s)^{H\alpha+1} + (t-s)t^{H(\alpha-1)}s^H + H(t-s)\int_0^\infty (s+x)^{H\alpha-1}\Big(\frac{t}{s+x}\Big)^{\alpha-1} dx$$

$$= \tfrac{1}{H\alpha+1}(t-s) + (t-s) + \tfrac{H}{(1-H)\alpha-1}s^{1-(1-H)\alpha}(t-s) \quad \text{for } 0<s<t\le 1.$$

Obviously we have $s\bullet t = s\vee t$ and $s\circ t = s\wedge t$. Combining (11.5)-(11.8), it therefore follows that there is a $C_{13}>1$ such that

$$C_{13}^{-1}(t-s) \le \delta(s,t) \le C_{13}(t-s) \quad \text{for } \tfrac12\le s<t\le 1. \tag{11.9}$$

Since $F_{s\bullet t}(\cdot)\ge F_{s\circ t}(\cdot)$, and $s\diamond t = s\wedge t$ for $\hat\alpha=\alpha$, (8.6) and (8.9) further combine with (11.9) and (11.6)-(11.7) to show that there is a $\varrho_0\in(0,1)$ such that

$$\varrho_x(s,t) \le C_{14}\max\Big\{x(s\vee t)^{H-1+1/\alpha}|t-s|,\ x^\alpha(s\wedge t)^{-H(\alpha-1)-1+1/\alpha}|t-s|^{H\alpha+1}\Big\}$$
$$\le C_{14}^2\max\{x\,|t-s|,\ x^\alpha|t-s|^{H\alpha+1}\}$$
$$\varrho_x(s,t) \ge C_{14}^{-1}\max\Big\{x(s\vee t)^{H-1+1/\alpha}|t-s|,\ x^\alpha(s\vee t)^{-H(\alpha-1)-1+1/\alpha}|t-s|^{H\alpha+1}\Big\}$$
$$\ge C_{14}^{-2}\max\{x\,|t-s|,\ x^\alpha|t-s|^{H\alpha+1}\}$$

for $s,t\in K\equiv[\frac12,1]$ with $\varrho_x(s,t)\le\varrho_0$. Since (6.3) holds for $s\le t$ it follows that

$$\frac{\frac12(b-a)}{C_{14}^2\min\{\varepsilon/y,(\varepsilon/y^\alpha)^{1/(H\alpha+1)}\}} \le \mathcal{E}\big([a,b],y;\varepsilon\big) \le 1+\frac{C_{14}^2(b-a)}{\min\{\varepsilon/y,(\varepsilon/y^\alpha)^{1/(H\alpha+1)}\}} \tag{11.10}$$

for $[a,b]\subseteq[\frac12,1]$, $y\ge1$ and $\varepsilon\in(0,\varrho_0]$.

In view of (11.9) an application of Proposition 5 reveals that

$$C_{13}^{-1}(1-\tfrac1\alpha)(1-s) \le \|F_1\|_\alpha - \|F_s\|_\alpha \le C_{13}(1-s) \quad \text{for } 0\le s<1.$$

Hence the set $K_0(\sigma)=\big\{s\in[\frac12,1]: \|F_s\|_\alpha\ge\|F_1\|_\alpha-\sigma\big\}$ satisfies

$$\big[\tfrac12\vee(1-C_{15}^{-1}\sigma),\,1\big] \subseteq K_0(\sigma) \subseteq \big[\tfrac12\vee(1-C_{15}\sigma),\,1\big]. \tag{11.11}$$

Invoking (11.10) we therefore deduce that

$$\text{(11.12)}\qquad \begin{aligned}\mathcal{E}\big(K_0(\ell\varepsilon),y;\varepsilon\big) &\leq 1+C_{16}\min\{\tfrac{1}{2},\ell\varepsilon\}\,/\min\{\varepsilon/y,(\varepsilon/y^{\alpha})^{1/(H\alpha+1)}\}\\ \mathcal{E}\big(K_0(\varepsilon),y;\varepsilon\big) &\geq C_{16}^{-1}\min\{\tfrac{1}{2},\varepsilon\}\,/\min\{\varepsilon/y,(\varepsilon/y^{\alpha})^{1/(H\alpha+1)}\}\end{aligned},$$

and thus (7.1) holds with $\Lambda=\lambda=1$. Since (11.12) gives $C_{16}^{-1}\leq\mathcal{E}\big(K_0(\varepsilon),1;\varepsilon\big)\leq 1+C_{16}$, (11.2) now follows from Theorem 3. □

12. Log-fractional stable motion. Here we study

$$\text{(12.1)}\qquad X(t)\equiv\int_0^\infty\big(\ln(t+r)-\ln(r)\big)\,dM(r)\quad\text{for } t\in\mathbb{R}^+,$$

where M has Lebesgue control measure and skewness -1. Log-fractional motions arise as normalized limits of fractional motions when $H\to 0$.

The process $X(t)$ is self-similar with index $1/\alpha$, but [by e.g., Samorodnitsky and Taqqu (1994, Proposition 7.3.6)] does not have stationary increments.

Application 4. *For the log-fractional α-stable motion given by* (12.1) *we have*

$$0<\varliminf_{u\to\infty}\frac{\mathbf{P}\{\sup_{t\in[0,1]}X(t)>u\}}{\mathbf{P}\{X(1)>u\}}\leq\varlimsup_{u\to\infty}\frac{\mathbf{P}\{\sup_{t\in[0,1]}X(t)>u\}}{\mathbf{P}\{X(1)>u\}}<\infty.$$

Proof. It is enough to prove (11.2). To that end we first observe that

$$\begin{aligned}\text{(12.2)}\qquad &\|F_t-F_s\|_\alpha^\alpha\\ &=\int_0^\infty\frac{\alpha(t-s)x}{(s+x)(t+x)}\left[\ln\left(1+\frac{t-s}{s+x}\right)\right]^{\alpha-1}dx &&\text{[by partial integration]}\\ &\leq\int_0^\infty\frac{\alpha(t-s)^\alpha}{x^{\alpha-1}(t+x)}\,dx &&\text{[since }\ln(1+x)\leq x\text{ for }x\geq 0]\\ &=\tfrac{\pi\alpha}{\sin[\pi(2-\alpha)]}\,t^{1-\alpha}\,(t-s)^\alpha\quad\text{for }0\leq s<t &&\text{[by Erdélyi et al. (1954, p. 308, Eq. 3)].}\end{aligned}$$

Moreover the inequality

$$\text{(12.3)}\qquad \ln(1+x)\geq x\left[1-\big(1\wedge(\tfrac{1}{2}x)\big)\right]\quad\text{for } x\geq 0$$

ensures that [compare with (12.2)]

$$\begin{aligned}\text{(12.4)}\qquad &\|F_t-F_s\|_\alpha^\alpha\\ &=\int_0^\infty\frac{\alpha(t-s)x}{(s+x)(t+x)}\left[\ln\left(1+\frac{t-s}{s+x}\right)\right]^{\alpha-1}dx\\ &\geq\int_0^\infty\frac{\alpha(t-s)x}{(s+x)(t+x)}\left(\frac{t-s}{s+x}\right)^{\alpha-1}\left[1-\left(1\wedge\left(\frac{t-s}{2(s+x)}\right)\right)\right]^{\alpha-1}dx\\ &\geq\int_0^\infty\frac{\alpha(t-s)^\alpha x}{(t+x)^{\alpha+1}}\left[1-\left(1\wedge\left(\frac{t-s}{2(s+x)}\right)\right)\right]dx &&\text{[by (8.4) with }1/\alpha\text{ replaced by }\alpha-1]\end{aligned}$$

$$\geq \int_0^\infty \frac{\alpha(t-s)^\alpha x}{(t+x)^{\alpha+1}}\,dx - \int_0^\infty \frac{\alpha(t-s)^{\alpha+1}}{2(t+x)^{\alpha+1}}\,dx$$

$$= \tfrac{1}{\alpha-1}\, t^{1-\alpha}(t-s)^\alpha - \tfrac{1}{2} t^{-\alpha}(t-s)^{\alpha+1} \qquad \text{[by Erdélyi et al. (1954, p. 310, Eq. 19)]}$$

$$\geq \tfrac{3-\alpha}{2(\alpha-1)}\, t^{1-\alpha}(t-s)^\alpha \quad \text{for } 0\leq s<t.$$

Combining (12.2) with Hölder's inequality we easily get the upper estimate

$$\text{(12.5)} \quad \big\langle (F_t-F_s)F_t^{\alpha-1}\big\rangle \leq \|F_t-F_s\|_\alpha\, \|F_t\|_\alpha^{\alpha-1} \leq \tfrac{\pi\alpha}{\sin[\pi(2-\alpha)]}\,(t-s) \quad \text{for } 0\leq s<t.$$

On the other hand

$$\text{(12.6)} \qquad \big\langle (F_t-F_s)F_t^{\alpha-1}\big\rangle$$

$$= \int_0^\infty \ln\Big(\frac{t+x}{s+x}\Big)\Big[\ln\Big(\frac{t+x}{x}\Big)\Big]^{\alpha-1} dx$$

$$\geq \int_0^\infty \ln\Big(1+\frac{t-s}{t+x}\Big)\Big[\ln\Big(1+\frac{t}{t+x}\Big)\Big]^{\alpha-1} dx$$

$$\geq \text{[by (12.3), and by (8.4) with } 1/\alpha \text{ replaced by } \alpha-1]$$

$$\geq \int_0^\infty \frac{t-s}{t+x}\Big[1-\Big(1\wedge\Big(\frac{t-s}{2(t+x)}\Big)\Big)\Big]\Big(\frac{t}{t+x}\Big)^{\alpha-1}\Big[1-\Big(1\wedge\Big(\frac{t}{2(t+x)}\Big)\Big)\Big]\,dx$$

$$\geq \int_0^\infty \frac{t^{\alpha-1}(t-s)}{(t+x)^\alpha}\,dx - \int_0^\infty \frac{t^\alpha(t-s)}{2(t+x)^{\alpha+1}}\,dx - \int_0^\infty \frac{t^{\alpha-1}(t-s)^2}{2(t+x)^{\alpha+1}}\,dx$$

$$\geq \tfrac{1}{\alpha(\alpha-1)}\,(t-s) \quad \text{for } 0\leq s<t.$$

Clearly $s\bullet t=s\vee t$, $s\circ t=s\wedge t$ and $F_{s\bullet t}(\cdot)\geq F_{s\circ t}(\cdot)$, while $s\diamond t=s\wedge t$ when $\hat\alpha=\alpha$. Using (12.2) and (12.4)-(12.6) we get upper and lower bounds for $\delta(s,t)$ and $\|F_t-F_s\|_\alpha^\alpha/\|F_t\|_\alpha^{\alpha-1}$, $s,t\in K\equiv[\frac{1}{2},1]$, which only differ by multiplicative constants. Invoking Proposition 4, these bounds show that there is a $\varrho_1\in(0,1)$ such that

$$C_{17}^{-1}\max\big\{x\,|t-s|,\,(x\,|t-s|)^\alpha\big\} \leq \varrho_x(s,t) \leq C_{17}\max\big\{x\,|t-s|,\,(x\,|t-s|)^\alpha\big\}$$

when $\varrho_x(s,t)\leq\varrho_1$. Since (6.3) holds when $s\leq t$, it follows that

$$\tfrac{1}{2}C_{17}^{-1}(b-a)y/\varepsilon \leq \mathcal{E}([a,b],y;\varepsilon) \leq 1+C_{17}(b-a)y/\varepsilon \quad \text{for } [a,b]\subseteq K \text{ and } y\geq 1.$$

Noting that Proposition 5 and the bounds on $\delta(s,t)$ imply (11.11), we conclude

$$\text{(12.7)} \qquad C_{18}^{-1}\big(\tfrac{1}{2}\wedge\sigma\big)\,y\,\varepsilon^{-1} \leq \mathcal{E}\big(K_0(\sigma),y;\,\varepsilon\big) \leq 1+C_{18}\big(\tfrac{1}{2}\wedge\sigma\big)\,y\,\varepsilon^{-1}.$$

Since $K_\ell(\varepsilon)\subseteq K_0(\ell\varepsilon)$ it follows that (7.1) holds, and so Theorem 3 gives (11.2). $\square$

13. Fractional stable motions with index of self-similarity $< 1/\alpha$**.** When $H<0$ the process $X_{a,b}^H(t)$ in (11.1) is not totally skewed for any $a,b\geq 0$, but there are several totally skewed modifications of $X_{a,b}^H(t)$. To not unnecessarily extend an already long journey of estimates we consider the simplest possible modification

$$(13.1)\qquad X(t)\equiv\int_{\mathbb{R}}\big([r^+]^H-[(r-t)^+]^H\big)^+\,dM(r)=\int_0^t r^H\,dM(r)\quad\text{ for }\ t\in\mathbb{R}^+.$$

Again M has Lebesgue control measure and skewness -1, but now $H\in(-1/\alpha,0)$. Further $X(t)$ is self-similar with index $H+1/\alpha$ and non-stationary increments.

Application 5. *For the fractional* α*-stable motion given by* (13.1) *we have*

$$0<\varliminf_{u\to\infty}\frac{\mathbf{P}\{\sup_{t\in[0,1]}X(t)>u\}}{\mathbf{P}\{X(1)>u\}}\leq\varlimsup_{u\to\infty}\frac{\mathbf{P}\{\sup_{t\in[0,1]}X(t)>u\}}{\mathbf{P}\{X(1)>u\}}<\infty.$$

Proof. Again it is enough to prove (11.2). To that end we observe the trivial facts that $s\bullet t=s\vee t$ and $s\circ t=s\diamond t=s\wedge t$, with $F_{s\bullet t}(r)\geq F_{s\circ t}(r)$ for $r\in S$ and

$$\|F_t\|_\alpha^{\alpha-1}\delta(s,t)\;=\;\|F_t-F_s\|_\alpha^\alpha\;=\;\int_s^t x^{H\alpha}\,dx\;=\;\tfrac{1}{1+H\alpha}\big(t^{1+H\alpha}-s^{1+H\alpha}\big)$$

for $\frac12\leq s<t\leq 1$. By application of Propositions 4 and 5 we thus conclude that

$$C_{19}^{-1}(x\vee x^\alpha)\,|t-s|\;\leq\;\varrho_x(s,t)\;\leq\;C_{19}(x\vee x^\alpha)\,|t-s|,$$

and that (11.11) holds. Since (6.3) holds for $s\leq t$, it follows that

$$C_{20}^{-1}\big(\tfrac12\wedge\sigma\big)\,y^\alpha\,\varepsilon^{-1}\;\leq\;\mathcal{E}\big(K_0(\sigma),y;\,\varepsilon\big)\;\leq\;1+C_{20}\big(\tfrac12\wedge\sigma\big)\,y^\alpha\,\varepsilon^{-1}\quad\text{ for }\ y\geq 1$$

[cf. (12.7)]. Hence (7.1) holds, so that Theorem 3 implies (11.2). $\square$

14. Gaussian processes. Here we derive simplified versions of Theorem 1-3 when $\alpha=2$. We also discuss to what extent these results are new.

Let $\{X(t)\}_{t\in T}$ be zero-mean Gaussian, so that $\alpha=2$ and $\|F_t\|_\alpha=\sqrt{2\,\mathbf{E}\{X(t)^2\}}$. Take a compact $K\subseteq T$ and let $\tilde t\in K$ satisfy $\mathbf{E}\{X(\tilde t)^2\}=\sup_{t\in K}\mathbf{E}\{X(t)^2\}>0$. Further define the distance $\mathbf{c}(s,t)\equiv\mathbf{E}\big\{[X(s)-X(t)]^2\big\}$ between $s,t\in T$.

Corollary 1. *If there is a map* $h:\mathbb{R}^+\to\mathbb{R}^+$ *such that*

$$(14.1)\qquad \lim_{u\to\infty}h(u)^{-1}\ln M_{\mathbf{c}}\big(K_0(u^{-2});u^{-2}h(u)\big)=0,$$

then we have

$$\varliminf_{u\to\infty}\mathbf{P}\{\sup_{t\in K}X(t)>u\}\Big/\Big[M_{\mathbf{c}}\big(K_0(u^{-2});\,u^{-2}h(u)\big)\,\mathbf{P}\{X(\tilde t)>u\}\Big]>0.$$

Proof. Since $s\bullet t = t$ for $\|F_t\|_2 \ge \|F_s\|_2$ and $s\bullet t = s$ for $\|F_t\|_2 < \|F_s\|_2$, we have

$$\begin{aligned}
(14.2) \qquad \rho(s,t) &= \|2F_{s\bullet t}\|_2 \left[1 - \left(1 - \frac{\|2F_{s\bullet t}\|_2^2 - \|F_{s\bullet t} + F_{s\circ t}\|_2^2}{\|2F_{s\bullet t}\|_2^2}\right)^{1/2}\right] \\
&\ge \frac{\|2F_{s\bullet t}\|_2^2 - \|F_{s\bullet t} + F_{s\circ t}\|_2^2}{2\,\|2F_{s\bullet t}\|_2} \\
&= \frac{\frac{1}{2}\mathbf{c}(s,t) + \|F_{s\bullet t}\|_2^2 - \|F_{s\circ t}\|_2^2}{2\,\|F_{s\bullet t}\|_2} \\
&\ge \mathbf{c}(s,t) \big/ \big(4\,\|F_{\tilde{t}}\|_2\big).
\end{aligned}$$

[Here we used the argument (8.12).] It follows that

$$(14.3) \qquad M_\rho(\hat{T};\varepsilon) \ge M_{\mathbf{c}}\big(\hat{T}; 4\|F_{\tilde{t}}\|_2\varepsilon\big) \quad \text{for } \hat{T} \subseteq T \text{ and } \varepsilon > 0.$$

Since $\|F_{s\circ t}\|_2 + \sigma \ge \|F_{s\bullet t}\|_2 \ge \|F_{\tilde{t}}\|_2 - (\ell+1)\sigma$ for $s,t \in K_\ell(\sigma)$, we readily obtain

$$\rho(s,t) \le \frac{\frac{1}{2}\mathbf{c}(s,t) + \big(\|F_{s\bullet t}\|_2 + \|F_{s\circ t}\|_2\big)\big(\|F_{s\bullet t}\|_2 - \|F_{s\circ t}\|_2\big)}{\|F_{s\bullet t}\|_2} \le \frac{\mathbf{c}(s,t)}{2\,[\|F_{\tilde{t}}\|_2 - (\ell+1)\sigma]} + 2\sigma$$

for $s,t \in K_\ell(\sigma)$ [where we used (8.12) again]. Consequently

$$\begin{aligned}
(14.4) \qquad M_\rho(\hat{T};\hat{\varepsilon}) &\le \sum_{\ell=0}^{3} M_\rho\big(\hat{T} \cap K_\ell(\tfrac{1}{4}\varepsilon); \hat{\varepsilon}\big) \\
&\le \sum_{\ell=0}^{3} M_{\mathbf{c}}\big(\hat{T} \cap K_\ell(\tfrac{1}{4}\varepsilon)\,;\, [\|F_{\tilde{t}}\|_2 - (\ell+1)\tfrac{1}{4}\varepsilon]\hat{\varepsilon}\big) \\
&\le 4\, M_{\mathbf{c}}\big(\hat{T}; \tfrac{1}{2}\|F_{\tilde{t}}\|_2\, \hat{\varepsilon}\big) \quad \text{for } \hat{T} \subseteq K_0(\varepsilon) \text{ and } 0 < \varepsilon \le \hat{\varepsilon} \le \tfrac{1}{2}\|F_{\tilde{t}}\|_2.
\end{aligned}$$

By Theorem 1 it is sufficient to find a $g: \mathbb{R}^+ \to \mathbb{R}^+$ such that (2.2) holds and

$$(14.5) \qquad M_{\mathbf{c}}\big(K_0(u^{-2});\, u^{-2}h(u)\big) \le M_\rho\big(K_0(u^{-2});\, u^{-2}g(u)\big).$$

But taking $g(u) = \frac{1}{4}\|F_{\tilde{t}}\|_2^{-1}h(u)$, (14.3) implies that (14.5) holds. Further (2.2) follows from (14.1) and the fact that (14.4) yields

$$M_\rho\big(K_0(u^{-2}); u^{-2}\tfrac{1}{4}\|F_{\tilde{t}}\|_2^{-1}h(u)\big) \le M_{\mathbf{c}}\big(K_0(u^{-2}); \tfrac{1}{8}u^{-2}h(u)\big) \quad \text{for } u \text{ large.} \quad \square$$

Of course, (14.1) can often be verified by application of Proposition 1.

The best general lower bounds in the Gaussian literature are those for polynomially and exponentially increasing entropies by Samorodnitsky (1991, Theorems 4.1.ii and 5.1.iii). For such entropies the bound of Corollary 1 coincide with Samorodnitsky's. But Corollary 1 has the advantage of providing a unified formulation applicable also to other entropies, and Corollary 1 only requires an upper bound on the entropy [through (14.1)], while Samorodnitsky also needs a lower bound.

Corollary 2. *Assume that*

$$\overline{\lim_{\hat\varepsilon\downarrow 0}}\ \sup_{\varepsilon\in(0,\hat\varepsilon]} \frac{M_{\mathbf c}\big(K_0(2\|F_{\tilde t}\|_2^{-1}\varepsilon);\hat\varepsilon\big)}{M_{\mathbf c}\big(K_0(2\|F_{\tilde t}\|_2^{-1}\varepsilon);\lambda^{-1}\hat\varepsilon\big)}<\infty \quad \text{for } \lambda\in(0,1], \tag{14.6}$$

and that there is an $\varepsilon_7>0$ *such that*

$$\sup_{0<\varepsilon\le\hat\varepsilon\le\varepsilon_7,\, t\in K_0(2\|F_{\tilde t}\|_2^{-1}\varepsilon)} \frac{M_{\mathbf c}\big(K_0(2\|F_{\tilde t}\|_2^{-1}\varepsilon);\hat\varepsilon\big)\, M_{\mathbf c}\big(K_0(2\|F_{\tilde t}\|_2^{-1}\varepsilon)\cap B_{\mathbf c}(t,\hat\varepsilon);\varepsilon\big)}{M_{\mathbf c}\big(K_0(2\|F_{\tilde t}\|_2^{-1}\varepsilon);\varepsilon\big)}<\infty. \tag{14.7}$$

Then we have

$$\underline{\lim}_{u\to\infty}\ \mathbf P\big\{\sup_{t\in K}X(t)>u\big\}\Big/\Big[M_{\mathbf c}\big(K_0(u^{-2});\,u^{-2}\big)\,\mathbf P\{X(\tilde t)>u\}\Big]>0.$$

Proof. By Theorem 2 it is sufficient to prove (5.1) and (5.2). But here (5.1) follows from observing that (14.3) and (14.4) imply

$$\begin{aligned}
&\overline{\lim_{\hat\varepsilon\downarrow 0}}\ \sup_{\varepsilon\in(0,\hat\varepsilon]} \frac{M_\rho(K_0(\varepsilon);\hat\varepsilon)}{M_\rho\big(K_0(\varepsilon);\lambda^{-1}\hat\varepsilon\big)}\\
&\le \overline{\lim_{\hat\varepsilon\downarrow 0}}\ \sup_{\varepsilon\in(0,\hat\varepsilon]} \frac{4\,M_{\mathbf c}\big(K_0(\varepsilon);\frac12\|F_{\tilde t}\|_2\hat\varepsilon\big)}{M_{\mathbf c}\big(K_0(\varepsilon);4\|F_{\tilde t}\|_2\lambda^{-1}\hat\varepsilon\big)} = \overline{\lim_{\hat\varepsilon\downarrow 0}}\ \sup_{\varepsilon\in(0,\hat\varepsilon]} \frac{4\,M_{\mathbf c}\big(K_0(2\|F_{\tilde t}\|_2^{-1}\varepsilon);\hat\varepsilon\big)}{M_{\mathbf c}\big(K_0(2\|F_{\tilde t}\|_2^{-1}\varepsilon);8\lambda^{-1}\hat\varepsilon\big)}<\infty.
\end{aligned}$$

Further (14.2) shows that $B_\rho(t,\hat\varepsilon)\subseteq B_{\mathbf c}\big(t,4\|F_{\tilde t}\|_2\hat\varepsilon\big)$, so that (14.3) and (14.4) yield

$$\begin{aligned}
&\sup_{0<\varepsilon\le\hat\varepsilon,\, t\in K_0(\varepsilon)} \frac{M_\rho(K_0(\varepsilon);\hat\varepsilon)\, M_\rho\big(K_0(\varepsilon)\cap B_\rho(t,\hat\varepsilon);\varepsilon\big)}{M_\rho(K_0(\varepsilon);\varepsilon)}\\
&\le \sup_{0<\varepsilon\le\hat\varepsilon,\, t\in K_0(\varepsilon)} \frac{M_{\mathbf c}\big(K_0(\varepsilon);\frac12\|F_{\tilde t}\|_2\hat\varepsilon\big)\, M_{\mathbf c}\big(K_0(\varepsilon)\cap B_{\mathbf c}(t,4\|F_{\tilde t}\|_2\hat\varepsilon);\frac12\|F_{\tilde t}\|_2\varepsilon\big)}{M_{\mathbf c}\big(K_0(\varepsilon);4\|F_{\tilde t}\|_2\varepsilon\big)}\\
&\le \sup_{0<\varepsilon\le\hat\varepsilon,\, t\in K_0(2\|F_{\tilde t}\|_2^{-1}\varepsilon)} \frac{M_{\mathbf c}\big(K_0(2\|F_{\tilde t}\|_2^{-1}\varepsilon);\hat\varepsilon\big)\, M_{\mathbf c}\big(K_0(2\|F_{\tilde t}\|_2^{-1}\varepsilon)\cap B_{\mathbf c}(t,8\hat\varepsilon);\varepsilon\big)}{M_{\mathbf c}\big(K_0(2\|F_{\tilde t}\|_2^{-1}\varepsilon);8\varepsilon\big)}.
\end{aligned}$$

Here the right hande side is finite by (14.6) and (14.7), and so (5.2) holds. □

Lower bounds for polynomial entropies in the literature either assume stationarity (e.g., Albin, 1994, Corollary 2) or involve a nuisance function making the bound unsharp (e.g., Theorem 1 and Samorodnitsky, 1991, Theorem 4.1.ii). Corollary 2 contributes by applying to non-homogeneous processes without a nuisance function.

Corollary 3. *If there are constants* $\gamma_7,\gamma_8\in[0,1)$ *and* $\lambda\in(0,1]$ *such that*

$$E_{\mathbf c}\big(K_\ell(\varepsilon);\tfrac16\|F_{\tilde t}\|_2x^{-2}\varepsilon\big)\le C_{20}\exp\{x^{\gamma_7}+\ell^{\gamma_8}\}\,E_{\mathbf c}\big(K_0(\varepsilon);3\|F_{\tilde t}\|_2\lambda\,\varepsilon\big) \tag{14.8}$$

for $\ell\in\mathbb N$ *and* $x\ge1$, *and if* $\{X(t)\}_{t\in K}$ *is a.s. bounded, then we have*

$$\overline{\lim_{u\to\infty}}\ \mathbf P\big\{\sup_{t\in K}X(t)>u\big\}\Big/\Big[E_{\mathbf c}\big(K_0(u^{-2});\tfrac16\|F_{\tilde t}\|_2\lambda\,u^{-2}\big)\,\mathbf P\{X(\tilde t)>u\}\Big]<\infty.$$

Proof. Writing $\tilde{K} \equiv \{t \in K : \mathbf{E}\{X(t)^2\} \geq \frac{1}{2}\mathbf{E}\{X(\tilde{t})^2\}\}$ it is well-known that

$$\mathbf{P}\{\sup_{t\in\tilde{K}} X(t) > u\} \sim \mathbf{P}\{\sup_{t\in K} X(t) > u\} \quad \text{as } u \to \infty$$

when $\{X(t)\}_{t\in K}$ is a.s. bounded. Hence it is sufficient to prove

$$\text{(14.9)} \quad \varlimsup_{u\to\infty} \mathbf{P}\{\sup_{t\in\tilde{K}} X(t) > u\} \Big/ \Big[E_{\mathbf{c}}\big(K_0(u^{-2}); \tfrac{1}{6}\|F_{\tilde{t}}\|_2 \lambda\, u^{-2}\big)\, \mathbf{P}\{X(\tilde{t}) > u\}\Big] < \infty.$$

Since $K_{\hat{\alpha}} = 1$ when $\hat{\alpha} = \alpha = 2$, (8.11) shows that

$$\begin{aligned} \varrho_x(s,t) &\leq x\,\delta(s,t) + x^2\mathbf{c}(s,t) \,/\, \min\{\|F_s\|_2, \|F_t\|_2\} \\ &= \tfrac{1}{2}x\,\big[\mathbf{c}(s,t) + \mathbf{d}(s,t)\big] \,/\, \|F_{s\bullet t}\|_2 + x^2\mathbf{c}(s,t) \,/\, \|F_{s\circ t}\|_2 \\ &\leq 3\,\big[x\,\mathbf{d}(s,t) + x^2\mathbf{c}(s,t)\big] \,/\, \|F_{\tilde{t}}\|_2 \quad \text{for } x \geq 1 \text{ and } s,t \in \tilde{K}, \end{aligned}$$

where $\mathbf{d}(s,t) \equiv \big|\|F_t\|_2^2 - \|F_s\|_2^2\big|$. Consequently

$$\text{(14.10)} \qquad \mathcal{E}(\hat{T}, x; \varepsilon) \leq E_{x\mathbf{d}+x^2\mathbf{c}}\big(\hat{T}; \tfrac{1}{3}\|F_{\tilde{t}}\|_2 \varepsilon\big) \quad \text{for } x \geq 1 \text{ and } \hat{T} \subseteq \tilde{K}.$$

Now let $m_0 \in \mathbb{N}$ satisfy $m_0 - 1 \leq 12\lambda^{-1}x < m_0$, so that $\bigcup_{m=0}^{m_0-1} K_{\ell m_0+m}(\varepsilon/m_0) = K_\ell(\varepsilon)$ and $x\,\mathbf{d}(s,t) \leq \frac{1}{6}\|F_{\tilde{t}}\|_2 \lambda\varepsilon$ for $s,t \in K_{\ell m_0+m}(\varepsilon/m_0)$. Then we have

$$\begin{aligned} \text{(14.11)} \quad E_{x\mathbf{d}+x^2\mathbf{c}}\big(K_\ell(\varepsilon); \tfrac{1}{3}\|F_{\tilde{t}}\|_2\lambda\varepsilon\big) &\leq \sum_{m=0}^{m_0-1} E_{x\mathbf{d}+x^2\mathbf{c}}\big(K_{\ell m_0+m}(\varepsilon/m_0); \tfrac{1}{3}\|F_{\tilde{t}}\|_2\lambda\varepsilon\big) \\ &\leq \sum_{m=0}^{m_0-1} E_{\mathbf{c}}\big(K_{\ell m_0+m}(\varepsilon/m_0); \tfrac{1}{6}\|F_{\tilde{t}}\|_2\lambda x^{-2}\varepsilon\big) \\ &\leq (12\lambda^{-1}x+1)\, E_{\mathbf{c}}\big(K_\ell(\varepsilon); \tfrac{1}{6}\|F_{\tilde{t}}\|_2\lambda x^{-2}\varepsilon\big). \end{aligned}$$

If on the other hand $\varrho_x(s,t) = \varepsilon\|F_{s\bullet t}\|_2$ where $\varepsilon \leq 1$ and $x \geq 1$, then we get

$$\begin{aligned} &\big\|F_{s\bullet t} + x(F_{s\bullet t} - F_{s\circ t})\big\|_2^2 \leq (1+\varepsilon)^2\|F_{s\bullet t}\|_2^2 \\ \Leftrightarrow\ & x\,\big[\mathbf{c}(s,t) + \mathbf{d}(s,t)\big] + x^2\mathbf{c}(s,t) \leq (2\varepsilon+\varepsilon^2)\|F_{s\bullet t}\|_2^2 \Rightarrow x^2\mathbf{c}(s,t) \,/\, \|F_{\tilde{t}}\|_2 \leq 3\varrho_x(s,t). \end{aligned}$$

It follows that

$$\text{(14.12)} \qquad \mathcal{E}(\hat{T}, x; \lambda\varepsilon) \geq E_{\mathbf{c}}\big(\hat{T}; 3\|F_{\tilde{t}}\|_2 x^{-2}\lambda\varepsilon\big) \quad \text{for } x \geq 1 \text{ and } \varepsilon \leq 1.$$

Combining (14.10)-(14.12) we obtain

$$\frac{\mathcal{E}\big(\tilde{K}_\ell(\varepsilon), x; \varepsilon\big)}{\mathcal{E}\big(\tilde{K}_0(\varepsilon), 1; \lambda\varepsilon\big)} \leq \frac{(12\lambda^{-1}x+1)\, E_{\mathbf{c}}\big(K_\ell(\varepsilon); \tfrac{1}{6}\|F_{\tilde{t}}\|_2 x^{-2}\varepsilon\big)}{E_{\mathbf{c}}\big(K_0(\varepsilon); 3\|F_{\tilde{t}}\|_2\lambda\varepsilon\big)} \quad \text{for } \varepsilon \text{ small,}$$

and thus (14.8) shows that $\{X(t)\}_{t\in\tilde{K}}$ satisfies (7.1) with $\Lambda = 1$. In view of Theorem 3, (14.9) therefore follows from another application of (14.10)-(14.11). □

Using that $\mathbf{c}(r,t) \le 2[\mathbf{c}(r,s)+\mathbf{c}(s,t)]$ it is easy to prove $M_{\mathbf{c}}(\hat{T};8\varepsilon) \le E_{\mathbf{c}}(\hat{T};\varepsilon) \le M_{\mathbf{c}}(\hat{T};\varepsilon)$. Hence the difference between the bounds of Corollary 2 and 3 is small.

Compared with the upper bounds by Samorodnitsky (1991, Theorems 4.1.i and 5.1.i-ii), Corollary 3 has the advantage of providing a unified treatment applicable not only to polynomially or exponentially 'behaved' entropies.

Samorodnitsky's results only assume upper bounds on the entropy while Corollary 3 also require lower bounds [through (14.8)]. The reason is that Samorodnitsky assumes an entropy bounded from above by a polynomially or exponentially behaved function $E(\cdot)$ which satisfies versions of (14.8). He then derives an upper bound for the tail-behaviour expressed in terms of E. But this bound is only sharp if the entropy also is bounded from below by E.

Application 6. (Fractional Browninan motion) *For a zero-mean Gaussian process* $\{X(t)\}_{t\ge 0}$ *with* $\mathbf{E}\{X(s)X(t)\} = \frac{1}{2}\big(s^\gamma + t^\gamma - |t-s|^\gamma\big)$ *for some* $\gamma \in (0,2]$, *we have*

$$0 < \varliminf_{u\to\infty} \frac{\mathbf{P}\{\sup_{t\in[0,1]} X(t) > u\}}{u^{2\,(1/\gamma-1)^+}\mathbf{P}\{X(1) > u\}} \le \varlimsup_{u\to\infty} \frac{\mathbf{P}\{\sup_{t\in[0,1]} X(t) > u\}}{u^{2\,(1/\gamma-1)^+}\mathbf{P}\{X(1) > u\}} < \infty.$$

This result can be derived from e.g., Konstant and Pitebarg (1993, Section 2). But their reults only apply in Euclidian settings under very special conditions on the covariance structure. Previously general bounds like Corollary 2 and 3 have not been sharp enough to yield the true tail-behaviour, and so it is interesting to see how our bounds work here.

Proof. Take an $[a,b] \subseteq K \equiv [0,1]$. Since $\mathbf{c}(s,t) = |t-s|^\gamma$ and $\|F_t\|_2 = t^{\gamma/2}$, we have

$$\frac{b-a}{\varepsilon^{1/\gamma}} \le M_{\mathbf{c}}([a,b];\varepsilon) \le 1 + \frac{b-a}{\varepsilon^{1/\gamma}} \quad \text{and} \quad \frac{b-a}{2\,\varepsilon^{1/\gamma}} \le E_{\mathbf{c}}([a,b];\varepsilon) \le 1 + \frac{b-a}{2\,\varepsilon^{1/\gamma}}.$$

Moreover $K_\ell(\varepsilon) \subseteq K_0(\ell\varepsilon) = [(1-\ell\varepsilon)^{2/\gamma}, 1]$. Using the elementary inequalities $1-(1-\varepsilon)^{2/\gamma} \ge \gamma^{-1}\varepsilon$ for ε small, and $1-(1-\ell\varepsilon)^{2/\gamma} \le 2\gamma^{-1}\ell\varepsilon$, it follows that

$$\max\{1, \tfrac{1}{2}\gamma^{-1}\varepsilon\,\hat{\varepsilon}^{-1/\gamma}\} \le E_{\mathbf{c}}(K_0(\varepsilon);\hat{\varepsilon}) \le E_{\mathbf{c}}(K_0(\ell\varepsilon);\hat{\varepsilon}) \le 1 + \gamma^{-1}\ell\varepsilon\,\hat{\varepsilon}^{-1/\gamma}$$

for ε small, so that (14.8) holds. Similarly we obtain

$$\text{(14.13)} \quad \max\{1, \gamma^{-1}\varepsilon\,\hat{\varepsilon}^{-1/\gamma}\} \le M_{\mathbf{c}}(K_0(\varepsilon);\hat{\varepsilon}) \le 1 + 2\,\gamma^{-1}\varepsilon\,\hat{\varepsilon}^{-1/\gamma} \quad \text{for } \varepsilon \text{ small},$$

while the fact that $K_0(\varepsilon) \cap B_{\mathbf{c}}(t,\hat{\varepsilon}) = [(1-\varepsilon)^{2/\gamma}, 1] \cap [t-\hat{\varepsilon}^{1/\gamma}, t+\hat{\varepsilon}^{1/\gamma}]$ yields

$$\text{(14.14)} \quad M_{\mathbf{c}}\big(K_0(2\|F_{\tilde{t}}\|_2^{-1}\varepsilon) \cap B_{\mathbf{c}}(t,\hat{\varepsilon});\, \varepsilon\big) \le 1 + \min\{2\gamma^{-1}2\|F_{\tilde{t}}\|_2^{-1}\varepsilon,\, 2\hat{\varepsilon}^{1/\gamma}\}\,\varepsilon^{-1/\gamma}.$$

But (14.13)-(14.14) readily show that (14.6)-(14.7) hold, and that

$$M_{\mathbf{c}}(K_0(u^{-2});u^{-2}) \;\geq\; \max\bigl\{1,\, \gamma^{-1}u^{2(1/\gamma-1)}\bigr\} \;\geq\; \min\{1,\gamma^{-1}\}\, u^{2(1/\gamma-1)^+}.$$

Hence the desired result follows from application of Corollaries 2 and 3. □

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