Mixing properties of the generalized T, T^{-1}–process

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Abstract

Consider a general random walk on $\mathbb{Z}^d$ together with an i.i.d. random coloring of $\mathbb{Z}^d$. The T, T^{-1}–process is the one where time is indexed by $\mathbb{Z}$, and at each unit of time we see the step taken by the walk together with the color of the newly arrived at location. S. Kalikow proved that if $d = 1$ and the random walk is simple, then this process is not Bernoulli (although it was previously known to be K). We generalize his result by proving that it is not Bernoulli in $d = 2$, Bernoulli but not Weak Bernoulli in $d = 3$ and 4, and Weak Bernoulli in $d \geq 5$. These properties are related to the intersection behavior of the past and the future of simple random walk. We obtain similar results for general random walks on $\mathbb{Z}^d$, leading to an almost complete classification. For example, in $d = 1$, if a step of size x has probability proportional to $1/|x|^\alpha$ $(x \neq 0)$, then the T, T^{-1}–process is not Bernoulli when $\alpha \geq 2$, Bernoulli but not Weak Bernoulli when $3/2 \leq \alpha < 2$ and Weak Bernoulli when $1 < \alpha < 3/2$.

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1 Introduction and main definitions

1. Definition of the T, T^{-1}–process. We begin by describing the process that will be the object of our study.

For a fixed integer $d \geq 1$, let $\{X_i\}_{i \in \mathbb{Z}}$ be i.i.d. random variables taking values in $\mathbb{Z}^d$ and having marginal law m. Let $\{S_n\}_{n \in \mathbb{Z}}$ be the corresponding **random walk** on $\mathbb{Z}^d$ defined by

$$S_0 = 0, \;\; S_n = \sum_{i=1}^{n} X_i \;\; (n \geq 1), \;\; S_n = -\sum_{i=n+1}^{0} X_i \;\; (n \leq -1),$$

i.e., X_i is the step at time i, S_n is the position at time n. Next, let $\{C_z\}_{z \in \mathbb{Z}^d}$ be i.i.d. random variables taking values $+1$ and -1 with probability $1/2$ each. We think of this as a **random coloring** of $\mathbb{Z}^d$, i.e., C_z is the color of location z. The walk and the coloring are assumed to be independent. Throughout the sequel, the symbols P, E will denote probability and expectation w.r.t. walk and/or coloring.

Now consider the process

$$\{Z_i\}_{i \in \mathbb{Z}} \text{ with } Z_i = (X_i, C_{S_i}).$$

We shall call this the **T, T^{-1}–process associated with** m. (The name is explained in Section 1.5.) The goal of our paper will be to study the mixing properties of this process.

Since both the step taken by the walk and the color of the newly arrived at location are recorded, knowing the past $\{Z_i\}_{i \leq 0}$ of the T, T^{-1}–process is the same as knowing the past $\{S_i\}_{i \leq 0}$ of the walk and the colors of all the locations in $\{S_i\}_{i \leq 0}$. Asking about the mixing properties of $\{Z_i\}_{i \in \mathbb{Z}}$ therefore means asking what effect this knowledge has on the future $\{Z_i\}_{i > 0}$. Since the steps are i.i.d., the latter boils down to the question of what can be said about the colors encountered in the future given that the coloring is known on a certain (random) subset of $\mathbb{Z}^d$. If the random walk is recurrent and irreducible, then this subset is all of $\mathbb{Z}^d$.

We end this subsection with a result proved in [26].

Theorem 1.1 *([26]) For any m, the T, T^{-1}–process associated with m has a trivial right tail.*

Trivial right tail means that $\cap_{n\geq 1}\sigma(Z_n, Z_{n+1}, \dots)$ only contains sets of probability 0 or 1, where $\sigma(Z_n, Z_{n+1}, \dots)$ is the σ–algebra generated by the random variables $\{Z_n, Z_{n+1}, \dots\}$. So Theorem 1.1 already tells us that the T, T^{-1}–process has reasonably strong mixing properties.

In ergodic theory, within the class of stationary processes whose one–dimensional marginal has finite entropy a process with a trivial right tail is called a K–**automorphism** (K) (see [38] p.207; the entropy of a random variable X is defined to be $-\sum_i p_i \log p_i$, when $p_1, p_2, \dots$ are the atoms of the distribution of X, and is taken to be ∞ when the distribution is not purely discrete.) We mention that K–automorphisms can also be defined outside the class of stationary processes whose one–dimensional marginal has finite entropy, but doing this would take us to far afield.

2. Definition of Very Weak Bernoulli and Weak Bernoulli. We identify a process $\{Y_n\}_{n\in\mathbb{Z}}$ taking values in a complete metric space F with a complete probability measure μ on $F^{\mathbb{Z}}$ in the obvious way. (The σ–algebra is the completed Borel σ–algebra w.r.t. μ. The Borel structure refers of course to the product topology.) Stationarity of the process corresponds to this measure being translation invariant.

To define the concept of Very Weak Bernoulli we need the following notion of distance between probability measures, called the $\overline{d}$-distance in ergodic theory.

Definition 1.2 *If $\mu_1, \mu_2 \in \mathcal{P}(F^N)$, with F a countable set and N a positive integer, then*

$$\overline{d}(\mu_1, \mu_2) = \inf_{\nu\in\mathcal{P}(F^N\times F^N):\ \nu_1=\mu_1,\nu_2=\mu_2} \left\{ \int \left(\frac{1}{N}\sum_{i=1}^{N} 1_{\{\eta_i\neq\xi_i\}} \right) \nu(d\eta, d\xi) \right\},$$

where ν_1 and ν_2 are the 1-st resp. 2-nd marginal of ν, a typical element of $F^N \times F^N$ is denoted by $(\eta, \xi) = (\{\eta_i\}_{i=1}^N, \{\xi_i\}_{i=1}^N)$, and $\mathcal{P}(E)$ denotes the set of probability measures on E.

The infimum runs over all couplings (or joinings) ν of μ_1 and μ_2. The r.h.s. without the infimum measures the expected percentage of errors under the coupling ν.

If $S \subseteq \mathbb{Z}$ is finite, then we let $\sigma(S)$ denote the sub σ–field of $\Omega = F^{\mathbb{Z}}$ generated by the atoms $\{\omega \in \Omega : \omega = \eta \text{ on } S\}$ where η ranges over F^S, and we let $\text{Atom}(S)$ denote the collection of these $|F|^{|S|}$ atoms.

Definition 1.3 *A translation invariant measure $\mu \in \mathcal{P}(F^{\mathbb{Z}})$ with F a countable set is called* **Very Weak Bernoulli** *(*VWB*) if for all $\epsilon > 0$ there exists a positive integer $N = N(\epsilon)$ such that: If $n \geq N$ and $S \subseteq (-\infty, 0] \cap \mathbb{Z}$ with S finite, then*

$$\overline{d}(\mu|_{(0,n]}, \mu|_{(0,n]}/A) < \epsilon$$

for all $A \in \text{Atom}(S)$ except for an ϵ–portion as measured by μ.

Here $\mu|_{(0,n]}$ denotes the measure on $F^{(0,n]\cap\mathbb{Z}}$ obtained by projecting μ onto the coordinates $(0, n] \cap \mathbb{Z}$, $\mu|_{(0,n]}/A$ means $\mu|_{(0,n]}$ conditioned on A, and the proviso in the last line means that the union of the atoms A where the above inequality fails has μ–measure $\leq \epsilon$. In words, VWB means that for large n and for any $m \leq 0$, the future up to time n conditioned on the past down to time m can (for most pasts) be coupled with the unconditioned future with an arbitrarily small percentage of errors (which pasts may depend on n and m).

The concept of Weak Bernoulli, to be defined next, arose in the ergodic theory community when the Ornstein isomorphism theorem for i.i.d. processes ([29] p.6) was extended to more general processes (see [9]). However, it was in fact formulated earlier by Kolmogorov under the name "absolute regularity" (see [2] Section 4.4). It is sometimes also referred to as "β–mixing" (see [4]).

Definition 1.4 *A translation invariant measure $\mu \in \mathcal{P}(F^{\mathbb{Z}})$ with F a complete metric space is called* **Weak Bernoulli** *(*WB*) if*

$$\lim_{n\to\infty} \|\mu_{(-\infty,0]\cup[n,\infty)} - \mu_{(-\infty,0]} \times \mu_{[n,\infty)}\| = 0,$$

where $\|\cdot\|$ denotes the total variation norm of a finite signed measure.

In words, WB means that the past and the future beyond time n are asymptotically independent as $n \to \infty$.

It will be useful for us later to have a characterization of WB in terms of couplings.

Proposition 1.5 *([2] Theorem 4.4.7) A stationary process $\{Y_n\}_{n\in\mathbb{Z}}$ is* WB *if and only if there is a process $\{Y'_n, Y''_n\}_{n\in\mathbb{Z}}$ such that:*
(i) $\{Y_n\}_{n\in\mathbb{Z}}$, $\{Y'_n\}_{n\in\mathbb{Z}}$ and $\{Y''_n\}_{n\in\mathbb{Z}}$ are equal in distribution,
(ii) $\{Y'_n\}_{n\in\mathbb{Z}}$ and $\{Y''_n\}_{n\leq 0}$ are independent,
(iii) a.s. there exists a positive integer N such that $Y'_n = Y''_n$ for all $n \geq N$.

From Proposition 1.5 it is obvious that WB implies VWB. Even for finite state stationary processes the reverse is not true in general. The first counterexample was given in [36]. Another example comes from [32]. An example in the context of skew products (see Section 1.5 for the definition of a skew product) was given in [35]. We shall see still more examples in this paper.

3. Bernoulli vs. Very Weak Bernoulli. Two stationary processes $(F^{\mathbb{Z}}, \mu)$ and $(G^{\mathbb{Z}}, \nu)$ are **isomorphic** if there exists an invertible measure-preserving map from one to the other that is defined a.e. and that commutes with shifts.

Definition 1.6 *A stationary process is called* **Bernoulli** *(*B*) if it is isomorphic to an i.i.d. process.*

It is generally difficult to see directly if a process is Bernoulli or not. However, the following fact is important and helpful.

Theorem 1.7 *([29] p.44, [31]) Let the state space F be finite. Then* B *is equivalent to* VWB.

(An alternative and simpler proof of this fact is given in [20].) For infinite state processes the situation is slightly different.

Theorem 1.8 *Let the state space F be countable. Then* VWB *implies* B. *Conversely, if the one–dimensional marginal of the process has finite entropy, then* B *implies* VWB. *However, the latter may fail without the entropy restriction.*

PROOF: Theorems 4 and 5 on p.53 in [29] state the following continuity principle. If $Y^{(m)} = \{Y_n^{(m)}\}_{n\in\mathbb{Z}}$ is a sequence of stationary processes that are all functions of a stationary process $X = \{X_n\}_{n\in\mathbb{Z}}$, such that each $Y^{(m)}$ is B and such that the sigma–fields $\sigma(Y^{(m)})$ are increasing in m with X being measurable (up to sets of measure 0) w.r.t. the sigma–field generated by $\cup_m \sigma(Y^{(m)})$, then X is also B. Next, we note that it is easily proved that if a countably infinite state process is VWB, then the process obtained by collapsing any infinite set of states into a single state is also VWB. Combining these two facts with Theorem 1.7 above, the first implication immediately follows. The second implication (under the entropy constraint) can be obtained by a straightforward modification of the proof in [20]. Similarly, it is trivial to find examples (necessarily failing the entropy condition) that are B but not VWB. For instance, take any 'block construction' of a finite state Bernoulli process and add at the first position in a block a number that describes the block exactly. This is a countably infinite state Bernoulli process for which a.s. the past determines the future and so we lose the VWB property. □

Within the class of stationary processes whose one–dimensional marginal has finite entropy, B implies K ([5] p.280). Therefore the import of Sections 1.2-1.3 can for this class be be summarized as follows:

$$WB \Longrightarrow VWB \Longleftrightarrow B \Longrightarrow K.$$

For general countable state processes all of the above holds except the $\Longleftarrow$. (The fact that B implies K requires the definition of a K–automorphism in this more general context, which we have not given.)

Despite the fact that B and VWB are not equivalent in general, it turns out that they are equivalent for the T, T^{-1}–process.

Theorem 1.9 *The T, T^{-1}–process is* B *if and only if it* VWB.

We shall sketch the proof in Section 1.6. Most of this sketch is due to Jean-Paul Thouvenot (personal communication). The proof is a digression, since the m's for which we prove that the T, T^{-1}–process associated with m is not VWB all have finite entropy (see Theorem 2.8 below and the comments afterwards) and hence the equivalence with not B follows from Theorem 1.8 anyway.

4. Brief overview of results. The following theorem due to S. Kalikow was the main motivation for the present paper. As usual, simple random walk refers to the case where at each step the walk chooses uniformly from its $2d$ nearest neighbors.

Theorem 1.10 *([17]) The T, T^{-1}–process associated with simple random walk in $d = 1$ is not* B.

Remark:
Theorem 1.10 above solved a problem that had been open for over 10 years.

It showed that even for a 'natural' process like the T, T^{-1}–process it is possible to be K (recall Theorem 1.1) and yet fail to be B. Examples of processes that are K but not B were known earlier (see [28] and [30]), but they were clearly constructed for the purpose. The result proved in [17] is actually a little stronger in that it is shown that the process does not even satisfy a weaker property called **loosely Bernoulli**, but we shall not deal with the latter concept here.

The main results of our paper are formulated in Section 2. Here we give a brief outline.

1. We show that a necessary and sufficient condition for the T, T^{-1}–process to be WB can be given in terms of the intersection properties of the underlying random walk. More specifically, let I be the intersection set of the forward part (future) and the backward part (past) of the walk. Then the process is WB if and only if $|I| < \infty$ a.s. The expected value $E|I|$ can be computed as a simple sum involving the Green's function of the random walk. While $E|I| < \infty$ of course implies that $|I| < \infty$ a.s., the converse is not true in general and a counterexample will be presented. However, we show that the converse is true for a large class of random walks, including all the symmetric ones as well as the ones with finite variance. In Section 5 we characterize to some extent the random walks for which the Green's function sum is finite resp. infinite.

2. We show that transience of the random walk is a sufficient condition for the T, T^{-1}–process to be VWB. We also show that recurrence together with some 'mild assumptions', namely, the existence of a positive moment of m and a certain property involving the intersections of the random walk, implies that the T, T^{-1}–process is not VWB. In Section 7 we show that this latter assumption holds when at least one component of the random walk is in the domain of attraction of a stable law. This will extend the result in [17]. Transience is equivalent to the Green's function being finite.

The problems that we study are intimately connected with questions concerning intersection properties of random walks. Such questions can be quite difficult, as evidenced by the existence of the book [21].

5. The T, T^{-1}–transformation (for ergodic theorists only). This subsection is a brief digression into ergodic theory. It is not needed in order to read any other part of the paper and so the reader may want to skip it and move on to Section 2.

An ergodic theorist will be more interested in the T, T^{-1}–transformation (from which the T, T^{-1}–process arises). We briefly explain what this is and what our results tell us about it.

Definition 1.11 *A* **dynamical system** *is a quadruple* $(\Omega, \mathcal{B}, \mu, T)$*, where* Ω *is a set,* $\mathcal{B}$ *is a* σ*–field of subsets of* Ω*,* μ *is a probability measure on* $\mathcal{B}$*, and* T *is a bijective bimeasurable measure-preserving transformation on* Ω*.*

An important example of a dynamical system that often arises in ergodic theory is a skew product.

Definition 1.12 *Let* $(\Omega, \mathcal{B}, \mu, T)$ *be a dynamical system. Let* $\{T_\omega\}_{\omega\in\Omega}$ *be a family of bijective bimeasurable measure-preserving transformations on the measure space* $(\Omega', \mathcal{B}', \mu')$*, with* $(\omega, \omega') \to T_\omega(\omega')$ *jointly measurable. Then the resulting* **skew product** *is the dynamical system* $(\Omega \times \Omega', \mathcal{B} \times \mathcal{B}', \mu \times \mu', \tilde{T})$ *where*

$$\tilde{T}(\omega, \omega') = (T(\omega), T_\omega(\omega')).$$

(One easily checks that $\tilde{T}$ is measure–preserving.)

The T, T^{-1}–transformation is a particular family of skew products. Let $\Theta = ((\mathbb{Z}^d)^{\mathbb{Z}}, \mathcal{B}, \mu)$, where

$$\mu = m^{\mathbb{Z}} \text{ for some probability measure } m \text{ on } \mathbb{Z}^d$$

and $\mathcal{B}$ is the completed Borel σ–algebra w.r.t. μ. Let $\Theta' = (\{+1, -1\}^{\mathbb{Z}^d}, \mathcal{B}', \nu)$, where

$$\nu = [(1/2)\delta_{+1} + (1/2)\delta_{-1}]^{\mathbb{Z}^d}$$

and $\mathcal{B}'$ is the completed Borel σ–algebra w.r.t. ν. Abbreviate $\Omega = (\mathbb{Z}^d)^{\mathbb{Z}}$ and $\Omega' = \{+1, -1\}^{\mathbb{Z}^d}$. Let T be the bijective bimeasurable measure-preserving transformation on Θ given by the left shift

$$(T\omega)(n) = \omega(n + 1) \text{ for all } \omega \in \Omega \text{ and } n \in \mathbb{Z}.$$

For $\omega \in \Omega$, let T_ω be the bijective bimeasurable measure-preserving transformation on Θ' given by

$$\Big(T_\omega(\omega')\Big)(z) = \omega'(z + \omega(0)) \text{ for all } \omega' \in \Omega' \text{ and } z \in \mathbb{Z}^d.$$

Here a typical element $\omega \in \Omega$ is written

$$\{\ldots, \omega(-1), \omega(0), \omega(1), \ldots\}$$

and similarly for $\omega' \in \Omega'$. The resulting skew product is called the T, T^{-1}–**transformation with measure** μ.

Note that the only freedom we have with the above system is the marginal m of μ. This marginal represents the step distribution of the random walk, while ν represents the law of the i.i.d. random coloring. The term T, T^{-1} now comes from the original case studied by Kalikow, which was $d = 1$ and $m = (1/2)\delta_{+1} + (1/2)\delta_{-1}$. Then we are essentially shifting the second coordinate in the skew product either to the left or to the right, i.e., we apply T or T^{-1} with T denoting the shift on the color sequence.

Given a dynamical system $(\Omega, \mathcal{B}, \mu, T)$ and a countable partition Q of Ω, we obtain in a natural way a stationary process $\{Y_n\}_{n\in\mathbb{Z}}$, defined on the probability space $(\Omega, \mathcal{B}, \mu)$ and taking values in Q, by letting $Y_n(\omega)$ be the partition element of Q containing $T^n(\omega)$. We say that Q **generates** if $\mathcal{B}$ is contained (up to sets of measure 0) in the σ–algebra generated by $\{Y_n\}_{n\in\mathbb{Z}}$. When Q and Q' are two finite partitions that generate, then Theorem 1.7 implies that the stationary process associated with Q is VWB if and only if the one associated with Q' is. This says that the property of VWB is an isomorphism invariant for finite state stationary processes and hence is an intrinsic property of the dynamical system. However, the existence of VWB processes that are not WB implies (using Theorem 1.7) that a similar equivalence for WB fails to be true. Because of this fact, an ergodic theorist might not consider the concept of WB natural, since it is not an intrinsic property of the dynamical system. However, from a probabilist point of view it is obviously important.

The next observation we ask the reader to make is that the T, T^{-1}–*process* associated with m can be obtained by taking the T, T^{-1}–*transformation* with $\mu = m^{\mathbb{Z}}$ and letting Q be the countable partition of $\Omega \times \Omega'$ generated by the pair $(\omega(0), \omega'(0))$. (Note that Q is a finite partition if m has finite support, which means that the corresponding random walk has bounded step size.)

The advantage of having a partition Q generate is that the dynamical system is then isomorphic to the stationary process associated with Q. However, in our T, T^{-1}–transformation the partition based on $(\omega(0), \omega'(0))$ does not necessarily generate. In fact, it generates if and only if $\{S_n\}_{n\in\mathbb{Z}} = \mathbb{Z}^d$ a.s., a condition that is slightly weaker than recurrence and irreducibility together. Therefore, if we prove that the T, T^{-1}-process is B, then this does not necessarily imply that the T, T^{-1}–transformation is B. (If the T, T^{-1}-process is not B, then it follows from [27] p.350 that the T, T^{-1}–transformation is not B.) However, to prove that the T, T^{-1}–transformation is B, we can proceed as follows. Fix k and define a process $\{Z_n^k\}_{n\in\mathbb{Z}}$ containing somewhat more information than the T, T^{-1}–process, namely, at time n we record the step of the walk together with the color of the location the walker newly arrives at *and* the colors of all the locations within k units. In other words, $Z_n^k = (X_n, \{C_z\}_{z\in B_k+S_n}))$ where $B_k = [-k, k]^d \cap \mathbb{Z}^d$.

The proof that for transient random walk $\{Z_n\}_{n\in\mathbb{Z}}$ is VWB (see Theorem 2.6 below) carries over immediately to show that $\{Z_n^k\}_{n\in\mathbb{Z}}$ is VWB for each k. It is also clear that $\mathcal{B} \times \mathcal{B}'$ is the σ–algebra generated by $\cup_k \mathcal{F}_k$, where $\mathcal{F}_k$ is the σ–algebra generated by $\{Z_n^k\}_{n\in\mathbb{Z}}$. It therefore follows from Theorem 1.8 above and from Theorems 4 and 5 on p.53 in [29] that the T, T^{-1}–transformation is B. (For the content of the latter two theorems, see the proof of Theorem 1.8 above.)

Also the proof that $\{Z_n\}_{n\in\mathbb{Z}}$ is WB under the appropriate assumptions on the random walk (see Theorem 2.2 below) carries over immediately to show that $\{Z_n^k\}_{n\in\mathbb{Z}}$ is WB for each k.

6. Sketch of the proof of Theorem 1.9 (for ergodic theorists only). Finite codings of i.i.d. processes are trivially VWB. Moreover, it is easy to show that any coding can be approximated in the $\overline{d}$–metric by finite codings for which the state space of the image process is finite. (The $\overline{d}$–metric was only defined for a finite number of variables but can easily be extended to stationary processes by requiring the couplings to be jointly stationary.) Therefore we need only show that the set of VWB processes is closed in the $\overline{d}$–metric. This is proved in [20] for finite state processes and extends immediately to processes whose one–dimensional marginal has finite entropy. The argument can, however, also be carried out for general countable state processes, provided we can extend an entropy theorem due to Rohklin stating the following: if $\{A_n, B_n\}$ is a jointly stationary finite state process, then

$$\lim_{n\to\infty} H(A_1|\{A_m\}_{m\le 0}, \{B_m\}_{m\le -n}) = H(A_1|\{A_m\}_{m\le 0}),$$

where $H(\cdot|\cdot)$ denotes conditional entropy. In particular, the argument in [20] could be carried over if we could verify the above when $\{A_n\}$ is any finite state process and $\{B_n\}$ is the T, T^{-1}–process $\{Z_n\}$.

In fact, Rohklin's proof could be carried over after we prove the following: if $\{C_n, D_n\}$ is a jointly stationary ergodic process with $\{C_n\}$ a finite state process and $\{D_n\}$ an i.i.d. process, then

$$\lim_{n\to\infty} H(C_1|\{C_m\}_{m\le 0}, \{D_m\}_{m\le -n}) = H(C_1|\{C_m\}_{m\le 0}). \tag{1}$$

To prove (1), we first use the relativized Sinai theorem to find a process $\{E_n\}$ that is a stationary coding of $\{C_n, D_n\}$, such that $\{E_n\}$ is i.i.d. and independent of $\{D_n\}$ and such that its entropy is

$$H(C_1|\{C_m\}_{m\le 0}, \{D_m\}_{m\in\mathbb{Z}}).$$

It suffices to prove that the two quantities in (1) are within a fixed $\epsilon > 0$. Fix such an ϵ and use the fact that

$$H(C_1|\{C_m\}_{m\le 0}, \{D_m, E_m\}_{m\in\mathbb{Z}}) = 0$$

to construct a process $\{F_n\}$ (defined on the same probability space) such that $\{C_n, D_n, E_n, F_n\}$ is jointly stationary, the entropy of F_1 is less than ϵ, and $\{C_n\}$ is measurable w.r.t. $\{D_n, E_n, F_n\}$. It is easy to show that if $\{D_n, E_n, F_n, G_n\}$ is jointly stationary, where the partition corresponding to G_1 can be expressed as the partition corresponding to $\{D'_n, E_n, F_n\}_{|n|\le N}$ for some N with $\{D'_n\}$ a process simply obtained by collapsing the state space of $\{D_n\}$ down to a finite number of states, then

$$|\lim_{n\to\infty} H(G_1|\{G_m\}_{m\le 0}, \{D_m\}_{m\le -n}) - H(G_1|\{G_m\}_{m\le 0})| \le \epsilon. \tag{2}$$

Next, a simple computation shows that if (2) holds, then it also holds when $\{G_n\}$ is replaced by any process that is a finite state factor (coding) of $\{G_n\}$. It is also easy to show that the above property is preserved under taking $\overline{d}$–limits in the $\{G_n\}$–variable. Finally, since $\{C_n\}$ is measurable w.r.t.

to $\{D_n, E_n, F_n\}$, it is a $\bar{d}$–limit of the above type processes and hence (1) follows. □

7. A comment about more general groups. We finally mention that given an arbitrary random walk on an arbitrary discrete group G, one can define an associated T, T^{-1}–process completely analogously to the case $G = \mathbb{Z}^d$. All of the results that we obtain for $G = \mathbb{Z}^d$ also hold for general groups (provided they make sense in this more general context). The proofs are in fact identical. However, we decided for concreteness to restrict our discussion to $\mathbb{Z}^d$. See [6] for a classification of recurrence vs. transience of random walks on countable Abelian groups.

2 Main results

In this section we state our main results. Proofs are deferred to Sections 3–4 and 6–7.

The random walk is **recurrent** if $P(S_n = 0 \text{ for some } n > 0) = 1$ and **transient** otherwise. We write $S[k, \ell]$ to denote the set $\{S_k, S_{k+1}, \ldots, S_\ell\}$, i.e., the collection of all the locations that are hit during the time interval $[k, \ell]$. The random walk starting from x is denoted by $\{S_n^x\}_{n \in \mathbb{Z}}$ and defined by $S_n^x = x + S_n$. Throughout this paper we assume that the random walk is **irreducible**, which means that given any $x, y \in \mathbb{Z}^d$ there is an $n \geq 0$ such that $P(S_n^x = y) > 0$.

1. Weak Bernoulli. Theorem 2.2 below gives a necessary and sufficient condition for the T, T^{-1}–process $\{Z_n\}$ to be WB in terms of the intersection properties of the underlying random walk $\{S_n\}$.

Definition 2.1 *We say that the random walk $\{S_n\}$ has* **property** $\heartsuit$ *if*

$$|S[0, \infty) \cap S(-\infty, 0]| < \infty \text{ a.s.}$$

Remarks:

(a) The two sets in the above intersection are independent.

(b) It follows from the Hewitt–Savage zero-one law (see [7] p.174) that the event in the above definition has probability 0 or 1.

(c) A $d = 1$ nearest neighbor random walk with nonzero drift satisfies property $\heartsuit$.
(d) A recurrent random walk cannot satisfy property $\heartsuit$.
(e) Simple random walk satisfies property $\heartsuit$ if and only if $d \geq 5$ (see [21] Section 3).

Theorem 2.2 *The* T, T^{-1}*-process* $\{Z_n\}$ *associated with the random walk* $\{S_n\}$ *is* WB *if and only if* $\{S_n\}$ *has property* $\heartsuit$.

2. Relationship between property $\heartsuit$ and the Green's function. We begin with some notation. Let

$$\begin{aligned} p_n(x,y) &= P(S_n^x = y) \\ f_n(x,y) &= P(S_n^x = y, S_m^x \neq y \text{ for } 0 \leq m < n). \end{aligned}$$

Let $p_n(x) = p_n(0,x)$ and $f_n(x) = f_n(0,x)$, and

$$\begin{aligned} G(x) &= \textstyle\sum_{n \geq 0} p_n(x) \\ F(x) &= \textstyle\sum_{n \geq 0} f_n(x). \end{aligned}$$

G is the **Green's function** of the random walk. $G(x)$ is equal to the expected number of visits to x starting from 0, $F(x)$ is the probability of ever reaching x starting from 0 (note that $F(0) = 1$ because $f_0(0) = 1$ and $f_n(0) = 0$ for $n > 0$). The step distribution is $m(x) = p_1(x)$.

Remarks:
(f) By irreducibility, $G(x) = \infty$ for all x if the random walk is recurrent and $G(x) < \infty$ for all x if the random walk is transient.
(g) The renewal relation $p_n(x) = \sum_{m=0}^n f_m(x) p_{n-m}(0)$ implies that for transient random walk $F(x) = G(x)/G(0)$.
(h) For transient random walk G determines m (see [37] Section 2) and so distinct transient random walks have distinct Green's functions.

We are now ready to relate property $\heartsuit$ with properties of the Green's function. The first observation is that (see Remark (g))

$$E\Big(|S[0,\infty) \cap S(-\infty,0]|\Big) = \sum_{x \in \mathbb{Z}^d} F(x)F(-x) = \frac{1}{G^2(0)} \sum_{x \in \mathbb{Z}^d} G(x)G(-x). \quad (3)$$

We therefore obtain the following corollary.

Corollary 2.3 *If* $\sum_{x\in\mathbb{Z}^d} G(x)G(-x) < \infty$, *then the random walk* $\{S_n\}$ *has property* $\heartsuit$.

The interesting question is whether the converse is true. It turns out that the answer is no.

Proposition 2.4 *There exists a random walk* $\{S_n\}$ *in* $d = 1$ *with*

$$\begin{aligned} E\Big(|S[0,\infty) \cap S(-\infty,0]|\Big) &= \infty \\ |S[0,\infty) \cap S(-\infty,0]| &< \infty \ a.s. \end{aligned}$$

On the other hand, the converse is true for a large class of random walks.

Theorem 2.5 *Assume that* $\sum_{x\in\mathbb{Z}^d} G(x)G(-x) = \infty$. *Assume that the random walk* $\{S_n\}$ *satisfies at least one of the following two properties:*
(i) symmetric (i.e., $m(x) = m(-x)$ *for all* x*),*
(ii) finite variance (i.e., $\sum_{x\in\mathbb{Z}^d} |x|^2 m(x) < \infty$*).*
Then $\{S_n\}$ *does not satisfy property* $\heartsuit$.

Remarks:
(i) It turns out that the only reason why property $\heartsuit$ need not correspond to $\sum_{x\in\mathbb{Z}^d} G(x)G(-x) < \infty$ is that the random walk run backwards need not be the same as the random walk run forwards (see Corollary 4.1 below).
(j) It is possible to improve on conditions (i-ii) in Theorem 2.5. For instance, it can be shown that property $\heartsuit$ also fails when $\sum_{x\in\mathbb{Z}^d} G(x)|G(x)-G(-x)| < \infty$ and $\sup_{n\geq 0} n^{1+\delta}p_n(0) < \infty$ for some $\delta > 0$. We shall, however, not pursue this line.

In Section 5 we shall characterize to some extent which random walks satisfy the Green's function criterion $\sum_{x\in\mathbb{Z}^d} G(x)G(-x) < \infty$.

3. Very Weak Bernoulli. Our next result gives a sufficient condition for $\{Z_n\}$ to be VWB (which we also believe to be necessary).

Theorem 2.6 *If the random walk* $\{S_n\}$ *is transient, then the* T, T^{-1}*–process* $\{Z_n\}$ *is* VWB.

Our final result, an almost converse to Theorem 2.6, tells us that if $\{S_n\}$ is recurrent and satisfies some mild assumptions, then the T,T^{-1}–process $\{Z_n\}$ is not VWB. To describe this, we introduce a property that concerns the self–intersection behavior of the random walk and that plays a role analogous to property $\heartsuit$.

Definition 2.7 *A random walk* $\{S_n\}$ *has* **property** $\spadesuit$ *if there exist constants* $C,\gamma>0$ *such that for all integers* $M,N\geq 1$ *and all* $r\in(0,1)$

$$P(E^M_{N,r})\leq C\frac{1}{N^\gamma r^2},$$

where $E^M_{N,r}$ *is the event*

$$\Big\{\exists\mathcal{I}\subseteq\{1,\ldots,N\},\,|\mathcal{I}|\geq rN:$$

$$S[(i-1)M,iM]\cap S[(j-1)M,jM]\neq\emptyset\quad\forall i,j\in\mathcal{I}\Big\}.$$

Remarks:
(k) The upper bound in the definition of $\spadesuit$ is uniform in M.
(l) In [17] it is shown that simple random walk in $d=1$ satisfies property $\spadesuit$ with $C=20$ and $\gamma=1/3$.
(m) If some coordinate of the random walk satisfies property $\spadesuit$, then the random walk itself also does.

The main result of this section is the following:

Theorem 2.8 *Let* $\{S_n\}$ *be a recurrent random walk satisfying property* $\spadesuit$ *and* $\sum_{x\in\mathbb{Z}^d}|x|^\delta m(x)<\infty$ *for some* $\delta>0$. *Then the* T,T^{-1}*–process* $\{Z_n\}$ *is not* VWB.

We already know from Theorem 2.6 that recurrence is a necessary condition for $\{Z_n\}$ not to be VWB. Despite the fact that recurrent random walks can have arbitrarily fat tails (see [11] or [34]), the assumption of a finite δ–moment is rather weak (although it can be shown to imply that m has finite entropy). As far as property $\spadesuit$ is concerned, we believe that this is no restriction at all.

Conjecture 2.9 *Every random walk* $\{S_n\}$ *satisfies property* $\spadesuit$.

4. Sufficient condition for ♠. In this section we formulate a sufficient condition for ♠, thereby giving content to Theorem 2.8.

Definition 2.10 *A one-dimensional random walk* $\{S_n\}$ *is* **in the domain of attraction of a random variable** Y *if there exist two sequences of constants* $\{b_n\} \subseteq \mathbb{R}$ *and* $\{a_n\} \subseteq \mathbb{R}$ *with* $a_n > 0$ *such that*

$$\frac{S_n - b_n}{a_n} \Rightarrow Y \quad (n \to \infty)$$

($\Rightarrow$ *means convergence in distribution). The possible limiting random variables* Y *are called* **stable laws**.

Remarks:

(n) Before we proceed, let us recall the following facts about stable laws (see [7] Section 2.7). The normal distribution as well as the constants are stable laws. Modulo translations and scalings, all other stable laws can be parametrized by two parameters: $\alpha \in (0, 2)$, called the index, and $\theta \in [0, 1]$, measuring the skewness. The density functions of these stable laws are known only in very few cases. The stable law Y corresponding to $\alpha \in (0, 2)$ and $\theta \in [0, 1]$ has the properties

$$\begin{array}{lcl} \lim_{x\to\infty} x^\alpha P(|Y| > x) & \in & (0, \infty) \\ \lim_{x\to\infty} \frac{P(Y>x)}{P(|Y|>x)} & = & \theta. \end{array}$$

(o) The case $\alpha = 2$ corresponds to the normal distribution. If $E(X_1^2) < \infty$, then it follows from the central limit theorem that $\{S_n\}$ is in the domain of attraction of a normal distribution where the sequences can be taken to be $a_n = \sqrt{n}$, $b_n = nE(X_1)$.

(p) It can deduced from [8] pp.574–581 that if $\{S_n\}$ is in the domain of attraction of some stable law with index $\alpha \in (0, 2]$, then the sequence $\{a_n\}$ is of the form $a_n = n^{\frac{1}{\alpha}} L(n)$ with L a **slowly varying** function, i.e., $\lim_{t\to\infty} \frac{L(tx)}{L(t)} = 1$ for all $x > 0$.

Theorem 2.11 *Let* $\{S_n\}$ *be a one–dimensional random walk that is in the domain of attraction of a nondegenerate stable law where the centering sequence* $\{b_n\}$ *can be taken to be* $\equiv 0$. *If, in addition,* $\alpha \geq 1$ *and/or* $\theta \in (0, 1)$, *then* $\{S_n\}$ *satisfies property* ♠.

Remarks:
(q) Random walks in the domain of attraction of a stable law with index $\alpha > 1$ must have a step distribution m with finite mean ([7] p.153). In that case the centering constants can be taken to be $\equiv 0$ if and only if m has zero mean ([3] p.347 and [7] p.159). The centering constants can also be taken to be $\equiv 0$ if m is symmetric ([7] p.153).
(r) We shall see in the proof of Theorem 2.11 that γ in property ♠ can be taken arbitrarily close to $1/(1+\alpha)$.
(s) Random walks in the domain of attraction of a stable law with index $\alpha < 1$ are not recurrent and centering is unnecessary (see [3] p.347).

Theorems 2.8 and 2.11 together with Remarks (m) and (q) imply the following corollary.

Corollary 2.12 *Let $\{S_n\}$ be a recurrent random walk with $\sum_{x\in\mathbb{Z}^d}|x|^\delta m(x) < \infty$ for some $\delta > 0$ and having some coordinate in the domain of attraction of a nondegenerate stable law with index α. If either of the following holds:*
(i) $\alpha > 1$ (in which case the coordinate must have zero mean)
(ii) $\alpha = 1$ and the coordinate is symmetric,
then $\{Z_n\}$ is not VWB. *In particular, if $\{S_n\}$ is any zero mean finite variance random walk in $d = 1$ or 2, then $\{Z_n\}$ is not* VWB.

Example 2.13 Let $d = 1$ and $m(x) = C/x^2$ for $x \neq 0$. The corresponding random walk is recurrent ([37] Example 8.2) and is in the domain of attraction of the Cauchy distribution ([7] Theorem 2.7.7). By the above corollary, this gives us a random walk with infinite mean for which the associated T, T^{-1}–process is not VWB.

3 Proofs of Theorems 2.2 and 2.6

1. Large blocks in random walk intersections. To prove Theorem 2.2, we need the following lemma. This says that if the intersection set of the past and the future of a random walk is infinite, then it contains infinitely many blocks of arbitrary size.

Definition 3.1 *Given a set $A \subseteq \mathbb{Z}^d$, the k-interior of A is defined as*

$$int_k(A) = \{x \in A : B_k + x \subseteq A\},$$

where $B_k = [-k,k]^d \cap \mathbb{Z}^d$.

Lemma 3.2 *Assume that* $\{S_n\}$ *does not satisfy property* $\heartsuit$. *Then for all* $k \geq 0$

$$\left| int_k\Big(S[0,\infty) \cap S(-\infty,0]\Big)\right| = \infty \text{ a.s.}$$

PROOF: As the claim is trivial in the recurrent case, we may assume that the random walk is transient. Fix $k \geq 0$. By irreducibility, there exists a positive integer N and a $\delta > 0$ such that $P(B_k \subseteq S[0,N]) > \delta$. Moreover, since B_k is symmetric around the origin, it follows that $P(B_k \subseteq S[-N,0])= P(B_k \subseteq S[0,N])$.

We first show that

$$\left| int_k\Big(S[0,\infty)\Big) \cap S(-\infty,0]\right| = \infty \text{ a.s.} \tag{4}$$

Let $T_0 = 0$ and $E_0 = \{B_k \subseteq S[0,N]\}$. Inductively, for $r \geq 0$ let

$$\begin{aligned} T_{r+1} &= \inf\{m \geq T_r + N : \ S_m \in S(-\infty,0]\} \\ E_{r+1} &= \{B_k + S_{T_{r+1}} \subseteq S[T_{r+1}, T_{r+1}+N]\}. \end{aligned}$$

Since the random walk does not satisfy property $\heartsuit$, all the T_r's are finite a.s. Clearly, the events E_r are independent and each has probability at least δ. Hence, by Borel-Cantelli, a.s. infinitely many of them occur. So, the future of the random walk infinitely often fills a box of size k around some site in the past of the random walk. By transience this happens around infinitely many *distinct* sites. Hence we have proved (4).

We next repeat the above argument, but backwards in time. In fact, let $U_0 = \sup\{m \leq 0 : S_m \in int_k(S[0,\infty))\}$ and $F_0 = \{B_k + S_{U_0} \subseteq S[U_0 - N, U_0]\}$. Inductively, for $r \geq 0$ let

$$\begin{aligned} U_{r+1} &= \sup\{m \leq U_r - N : \ S_m \in int_k(S[0,\infty))\} \\ F_{r+1} &= \{B_k + S_{U_{r+1}} \subseteq S[U_{r+1} - N, U_{r+1}]\}. \end{aligned}$$

By (4), all the U_r's are finite a.s. Clearly, the events F_r are independent and each has probability at least δ. Hence, by Borel-Cantelli, a.s. infinitely many of them occur. This, together with transience and the fact that $int_k(A \cap B) = int_k(A) \cap int_k(B)$ for all $k \geq 0$ and all sets A, B, completes the proof. □

2. Proof of Theorem 2.2. The proof comes in two steps.
1. We first assume that property ♡ holds and verify the conditions in Proposition 1.5 to prove that $\{Z_n\}$ is WB.

Let
$$\{X'_i\}_{i\in\mathbb{Z}},\ \{C'_z\}_{z\in\mathbb{Z}^d},\ \{X''_i\}_{i\in\mathbb{Z}},\ \{C''_z\}_{z\in\mathbb{Z}^d}$$
be independent copies of the steps of our random walk and of the colors of our random coloring. The process $\{Z'_n, Z''_n\}_{n\in\mathbb{Z}}$ that we construct below will be a function of the above four families of random variables. Let $\{S'_n\}_{n\in\mathbb{Z}}$ and $\{S''_n\}_{n\in\mathbb{Z}}$ be the random walks associated with the above increments (see Section 1.1). Define
$$\begin{aligned} Z'_n &= (X'_n, C'_{S'_n}) \quad (n\in\mathbb{Z}) \\ Z''_n &= (X''_n, C''_{S''_n}) \quad (n\le 0). \end{aligned}$$
For $n>0$, let the first component of Z''_n be X'_n. Then, clearly, condition (ii) and most of condition (i) in Proposition 1.5 are satisfied. We now need to define the second component of Z''_n for $n>0$ in such a way that $\{Z''_n\}_{n\in\mathbb{Z}}$ is equal in distribution to $\{Z_n\}_{n\in\mathbb{Z}}$ and $Z'_n = Z''_n$ for large n a.s. It is easily checked (using property ♡) that this can be accomplished by letting the second component of Z''_n be
$$\begin{array}{ll} C''_{S'_n} & \text{if } S'_n = S''_k \text{ for some } k\le 0 \\ C'_{S'_n} & \text{otherwise.} \end{array}$$
The idea here is that if at time n we are back at a location at which we have been before then we must use the C''-coloring, while if we are at a new location then we are free to use the C'-coloring (which of course we want to do).
2. We next assume that $\{Z_n\}$ is WB and that property ♡ fails. We show that this leads to a contradiction.

If $\{Z_n\}$ is WB, then by Proposition 1.5 there is a process $\{Z'_n, Z''_n\}_{n\in\mathbb{Z}}$ such that:
(i) $\{Z_n\}_{n\in\mathbb{Z}}$, $\{Z'_n\}_{n\in\mathbb{Z}}$ and $\{Z''_n\}_{n\in\mathbb{Z}}$ are equal in distribution,
(ii) $\{Z'_n\}_{n\in\mathbb{Z}}$ and $\{Z''_n\}_{n\le 0}$ are independent,
(iii) a.s. there exists a positive integer N such that $Z'_n = Z''_n$ for all $n\ge N$.
Write
$$\begin{aligned} Z'_n &= (X'_n, Y'_n) \\ Z''_n &= (X''_n, Y''_n). \end{aligned}$$

Let $\{S'_n\}_{n\in\mathbb{Z}}$ and $\{S''_n\}_{n\in\mathbb{Z}}$ be the random walks associated with the two first components. Let Ω' be the event that for all $k > 0$ and all $n \in \mathbb{Z}$,

$$S'_{n+k} = S'_n \Longrightarrow Y'_{n+k} = Y'_n \text{ and } S''_{n+k} = S''_n \Longrightarrow Y''_{n+k} = Y''_n. \qquad (5)$$

Trivially, $P(\Omega') = 1$ by (i) (which says nothing other than that if the random walk visits the same site twice then it must see the same color). We may now assume that Ω' is our entire probability space. Next, for $z \in \mathbb{Z}^d$, let

$$R'(z) = \begin{cases} Y'_n & \text{if } S'_n = z \text{ for some } n \geq 0 \\ \emptyset & \text{if } S'_n \neq z \text{ for all } n \geq 0 \end{cases}$$

and

$$R''(z) = \begin{cases} Y''_n & \text{if } S''_n = z \text{ for some } n \leq 0 \\ \emptyset & \text{if } S''_n \neq z \text{ for all } n \leq 0. \end{cases}$$

Note that $R'(z)$ and $R''(z)$ are well defined because of (5). In words, the function R' tells us the colors of the first process but only for those locations the forward random walk of the first process reaches. Similarly for the function R'' and the second process. Now, for $k \geq 0$, let Ω_k be the event that

$$\Big|\{z \in int_k\Big(S'[0,\infty) \cap S''(-\infty,0]\Big) :$$

$$R'(w) = +1 \text{ and } R''(w) = -1 \text{ for all } w \in B_k + z\}\Big| = \infty.$$

(Note that Ω_k is measurable with respect to $\{Z'_n\}_{n\geq 0}$ and $\{Z''_n\}_{n\leq 0}$.) In words, Ω_k is the event that there are infinitely many translates of B_k with the property that: (a) they are contained in the forward walk of the first process and the backward walk of the second process, (b) they are colored $+1$ in the first process and -1 in the second process.

We claim that $P(\Omega_k) = 1$ for all k. Indeed, since property $\heartsuit$ fails, Lemma 3.2 together with (i) and (ii) implies that $|int_k(S'[0,\infty) \cap S''(-\infty,0])| = \infty$. Since the walk and the coloring are independent for $\{Z_n\}$, it follows (again via (i) and (ii)) that $P(\Omega_k) = 1$. This in turn implies that $P(\cap_{k\geq 0}\Omega_k) = 1$ and hence that

$$P\left(\{\cap_{k\geq 0}\Omega_k\} \cap \{Z'_n = Z''_n \text{ for all large } n\}\right) = 1.$$

The proof is now completed by observing that

$$\{\cap_{k\geq 0}\Omega_k\} \cap \{Z'_n = Z''_n \text{ for all large } n\} = \emptyset,$$

giving us the desired contradiction. □

3. Transient random walks have zero density intersections. To prove Theorem 2.6, we need the following lemma.

Lemma 3.3 *If $\{S_n\}$ is transient, then*

$$\lim_{N\to\infty} E\Big(\frac{1}{N}\sum_{n=0}^{N-1} 1_{\{S_n\in S(-\infty,0]\}}\Big) = 0.$$

Proof: It is clearly enough to show that $\lim_{n\to\infty} P(S_n \in S(-\infty,0]) = 0$. However, by time reversal, we have

$$P(S_n \in S(-\infty,0]) = P(-S_k = 0 \text{ for some } k \geq n)$$

which goes to zero as $n \to \infty$ by transience. □

4. Proof of Theorem 2.6. We verify the VWB property. In fact, we prove that there is a process $\{Z'_n, Z''_n\}_{n\in\mathbb{Z}}$ such that:
(i) $\{Z_n\}_{n\in\mathbb{Z}}$, $\{Z'_n\}_{n\in\mathbb{Z}}$ and $\{Z''_n\}_{n\in\mathbb{Z}}$ are equal in distribution,
(ii) $\{Z'_n\}_{n\in\mathbb{Z}}$ and $\{Z''_n\}_{n\leq 0}$ are independent,
(iii) $\lim_{N\to\infty} E\Big(\frac{1}{N}\sum_{i=1}^{N} 1_{\{Z'_i\neq Z''_i\}}\Big) = 0$.
It is easily verified that this condition is sufficient for $\{Z_n\}$ to be VWB (see Definitions 1.2 and 1.3 in Section 1.2).

The first half of the proof of Theorem 2.2 provides us with a process $\{Z'_n, Z''_n\}_{n\in\mathbb{Z}}$ satisfying (i) and (ii). Next the explicit construction of this process and Lemma 3.3 guarantee that (iii) holds as well. □

4 Proofs of Proposition 2.4 and Theorem 2.5

Proof of Proposition 2.4. Consider the random walk whose step distribution is given by

$$m = b\delta_{-1} + \sum_{n=0}^{\infty} a_n\delta_n,$$

i.e., a left–continuous random walk. We assume that:
(i) $b + \sum_{n=0}^{\infty} a_n = 1$,

(ii) $b > \sum_{n=0}^{\infty} n\, a_n$ (so that the mean is finite and negative),
(iii) $\sum_{n=1}^{\infty} n(n-1)\, a_n = \infty$ (so that the variance is infinite).
Property (ii) implies that the random walk is transient with $S_n \to -\infty$ a.s. as $n \to \infty$ and $S_n \to \infty$ a.s. as $n \to -\infty$. Therefore $|S[0,\infty) \cap S(-\infty,0]| < \infty$ a.s. Next, the left–continuity implies that

$$|S[0,\infty) \cap S(-\infty,0]| = 1 + \sup_{n\geq 0} S_n - \inf_{n\leq 0} S_n \geq \sup_{n\geq 0} S_n$$

(the latter is a.s. finite by (ii)). However, it is proved in [19] that under (i) and (iii),

$$E\Big(\sup_{n\geq 0} S_n\Big) = \infty.$$

□

Proof of Theorem 2.5. We distinguish the two cases stated in the theorem.

(i): The symmetric case $m(x) = m(-x)$.
We shall use the second moment method, whereby one obtains a bound on the second moment and then applies the Cauchy-Schwarz inequality.

Let $V_N = |S[0,N] \cap S[-N,0]|$ and $V = |S[0,\infty) \cap S(-\infty,0]|$. Clearly, $\lim_{N\to\infty} V_N = V$ a.s. and $\lim_{N\to\infty} EV_N = EV$. The latter is infinite by our assumption that $\sum_{x\in\mathbb{Z}^d} G(x)G(-x) = \infty$. Now, Lemma 3.1 in [25] tells us that (because of the symmetry)

$$E(V_N^2) \leq 4(EV_N)^2 \text{ for all } N. \tag{6}$$

We show that this implies $V = \infty$ a.s., as desired.

By Cauchy–Schwarz, we have

$$\begin{aligned} EV_N &= E\Big(V_N 1\{V_N \geq \tfrac{1}{2}EV_N\}\Big) + E\Big(V_N 1\{V_N < \tfrac{1}{2}EV_N\}\Big) \\ &\leq (EV_N^2)^{\frac{1}{2}} P\Big(V_N \geq \tfrac{1}{2}EV_N\Big)^{\frac{1}{2}} + \tfrac{1}{2}EV_N. \end{aligned}$$

This together with (6) implies that

$$P\Big(V \geq \frac{1}{2}EV_N\Big) \geq P\Big(V_N \geq \frac{1}{2}EV_N\Big) \geq \frac{(EV_N)^2}{4E(V_N^2)} \geq \frac{1}{16}.$$

Letting $N \to \infty$ and using the fact that $\lim_{N\to\infty} EV_N = EV = \infty$, we conclude that

$$P(V = \infty) \geq \frac{1}{16}.$$

Finally, noting that the event $\{V = \infty\}$ is an exchangeable event w.r.t. the steps of the backward resp. the forward walk, the Hewitt–Savage zero-one law tells us that $P(V = \infty)$ is 0 or 1, and hence we conclude it is 1.

Remark: Lemma 3.1 in [25] and the above argument in fact give us the following corollary.

Corollary 4.1 *If $S_n^{(1)}, S_n^{(2)}, \ldots, S_n^{(k)}$ are k independent copies of the same arbitrary random walk, then*

$$|\cap_{i=1}^k S^{(i)}[0,\infty)| < \infty \textit{ a.s. if and only if } E\left(|\cap_{i=1}^k S^{(i)}[0,\infty)|\right) < \infty.$$

The latter is in turn equivalent to $\sum_{x\in\mathbb{Z}^d}[G(x)]^k < \infty$.

<u>(ii): The finite variance case $\sum_x |x|^2 m(x) < \infty$.</u>
Under the finite variance assumption, it follows from Theorems 5.2–5.4 below that $\sum_{x\in\mathbb{Z}^d} G(x)G(-x) = \infty$ if and only if $d \leq 4$ and m has zero mean. Let us therefore look closer at this class.

For $d = 1$ or 2, the random walk is recurrent and hence does not satisfy property ♡. For $d = 4$, equation (6) follows from [24] pp.511-519 (see also [25] p.666) and hence the exact same argument as above shows that ♡ fails. For $d = 3$, we can extend the random walk 'by adding some steps in the fourth direction' and thereby obtain an irreducible zero mean finite variance random walk in $d = 4$. By the above, this extended random walk will have $\sum_{x\in\mathbb{Z}^d} G(x)G(-x) = \infty$ and so from what we just saw will not satisfy property ♡. Therefore the original walk clearly does not either. (This conclusion can also be seen directly from [23] pp.496-497, where the limits of the moments of $V_N/E(V_N)$ as $N \to \infty$ are given in terms of Brownian intersections.) □

5 Characterization of the Green's function criterion

In this section we list some cases for which

$$\sum_{x\in\mathbb{Z}^d} G(x)G(-x) < \infty, \qquad (\diamondsuit)$$

by appealing to various results scattered over the literature. We shall no longer be concerned with when $\diamondsuit$ and $\heartsuit$ are equivalent. This was already done in Section 4. We recall that all random walks are assumed to be irreducible.

Lemma 5.1 *If the random walk $\{S_n\}$ satisfies*

$$p_n(x) \le A(n+1)^{-\alpha} \textit{ for all } x \in \mathbb{Z}^d, n \ge 0$$

for some constant A and with $\alpha > 2$, then $\diamondsuit$ holds.

PROOF: Write

$$\begin{aligned}\textstyle\sum_x G(x)G(-x) &= \textstyle\sum_{m,n\ge 0}\sum_x p_m(x)p_n(-x)\\ &= \textstyle\sum_{m,n\ge 0} p_{m+n}(0)\\ &\le \textstyle\sum_{m,n\ge 0} A(m+n+1)^{-\alpha}\\ &= \textstyle A\sum_{j\ge 1} j^{-\alpha+1}.\end{aligned}$$

□

Theorem 5.2 *$\diamondsuit$ holds for an arbitrary random walk $\{S_n\}$ in $d \ge 5$.*

PROOF: Recall that m is the step distribution of $\{S_n\}$. Let $\{\tilde{S}_n\}$ be the modified random walk whose step distribution is

$$(1/2)m + (1/2)\delta_0.$$

Proposition 7.6 in [37] tells us that the assumption of Lemma 5.1 holds for $\{\tilde{S}_n\}$ with $\alpha = d/2$ (and some constant A). Since $d \ge 5$, Lemma 5.1 implies that $\diamondsuit$ holds for $\{\tilde{S}_n\}$. Next, it is easy to check that the Green's function for $\{\tilde{S}_n\}$ is twice the Green's function for $\{S_n\}$ and hence $\diamondsuit$ also holds for $\{S_n\}$.
□

Theorem 5.3 *Let $\{S_n\}$ be an arbitrary random walk in $d \leq 4$ whose step distribution m has zero mean and finite variance. Then $\diamondsuit$ fails.*

PROOF: For $d = 1$ or 2 such a random walk is recurrent by [37] Theorem 8.1. For $d = 3$, [37] Proposition 26.1 states that

$$G(x) \asymp (1 + |x|)^{-1}$$

(the $\asymp$ means that the ratio of the two sides is bounded between two positive constants independent of x), easily implying that $\diamondsuit$ fails. For $d = 4$, it was pointed out in [22] that zero mean and finite variance is not quite enough to conclude that

$$G(x) \asymp (1 + |x|)^{-2}$$

(although this is true if one assumes a little more). However, it is shown in [22] that zero mean and finite variance does imply that

$$G(x) \geq C(1 + |x|)^{-2} \text{ for some } C > 0,$$

again easily implying that $\diamondsuit$ fails. □

Remarks:

(a) Combining Theorems 2.2, 2.5, 5.2 and 5.3 (or Theorem 2.2 and Remark (e) in Section 2.1), we obtain one of the claims made in the abstract of our paper, namely that the T, T^{-1}–process associated with simple random walk is WB if and only if $d \geq 5$.

(b) Similarly, combining Theorems 2.2, 2.5 and 5.3, we obtain the fact that the T, T^{-1}–process associated with any zero mean finite variance random walk in $d \leq 4$ is not WB.

(c) It is obvious that any random walk having at least one coordinate whose step distribution has a finite nonzero mean satisfies property $\heartsuit$ (even though $\diamondsuit$ may fail).

Theorem 5.4 *Let $\{S_n\}$ be an arbitrary random walk having at least one coordinate whose step distribution has a negative mean and finite variance. Then $\diamondsuit$ holds.*

PROOF: Let $\{S'_n\}$ be the random walk obtained by projecting $\{S_n\}$ onto any coordinate and let G' be the respective Green's function. It is easy to see that

$$\sum_{x \in \mathbb{Z}^d} G(x)G(-x) \leq \sum_{x \in \mathbb{Z}} G'(x)G'(-x)$$

and hence it suffices to prove the theorem when the random walk is one–dimensional. In that case clearly,

$$|S[0,\infty) \cap S(-\infty,0]| \leq 1 + \sup_{n\geq 0} S_n - \inf_{n\leq 0} S_n$$

and so, in order to prove $\diamondsuit$, it suffices to show that $E(\sup_{n\geq 0} S_n) < \infty$ (note that $E(\sup_{n\geq 0} S_n) = -E(\inf_{n\leq 0} S_n)$ and recall (3)). However, this follows from [19] by the negative mean finite variance assumption. □

Note that the example in the proof of Proposition 2.4 shows that the conditions in Theorem 5.4 are sharp.

One cannot really hope to characterize in much fuller detail than in Theorems 5.2-5.4 which random walks satisfy property $\diamondsuit$. We conclude our description by looking at a certain 1–parameter class of random walks of a special form, namely

$$m(x) \asymp (1+|x|)^{-(d+\beta)},$$

where $d \geq 1$ is the dimension and $\beta > 0$ is arbitrary. For this class the steps have a finite mean (finite variance) if and only if $\beta > 1$ ($\beta > 2$).

Theorem 5.5 *([39]) If $\beta \in (0, \min\{d,2\})$, then*

$$G(x) \asymp (1+|x|)^{\beta-d}.$$

Easy algebra and Theorem 5.5 give us the following corollary.

Corollary 5.6 *Let $d \leq 4$. Then $\diamondsuit$ holds when $\beta \in (0, d/2)$ and fails when $\beta \in [d/2, \min\{d,2\})$.*

For our special family, the theorems in this section now cover all the cases except for $d = 1$ with $\beta \in [1,2]$ and $d = 2,3,4$ with $\beta = 2$. For $d = 1$ and $\beta \in (1,2]$, the random walk has a finite mean, zero or nonzero (this obviously cannot be ascertained from the assumed form of m and either is possible). If the mean is zero, then it is recurrent by [37] Theorem 8.1, and so property $\diamondsuit$ of course fails. If the mean is nonzero, then we cannot say anything. For the case $d = 1$ and $\beta = 1$, we cannot say anything except when $m(x) \sim C(1+|x|^2)^{-1}$. In this case [37] Example 8.2 shows that the random walk is recurrent and so property $\diamondsuit$ again fails. Lastly, we are not sure about the cases $d = 2,3,4$ with $\beta = 2$, although it seems that $\diamondsuit$ should certainly fail at least when $d = 2$ or 3, since in these cases Corollary 5.6 indicates that the respective 'critical β's' are 1 and 3/2 (while in $d = 4$ it is 2).

Remark:
(d) Combining Theorems 1.9, 2.2 and 2.5 and Corollaries 2.12 and 5.6 together with a standard theorem about domains of attraction ([7] Theorem 2.7.7), we obtain the last claim made in the abstract of our paper.

6 Proof of Theorem 2.8

In this section we extend Kalikow's proof of Theorem 1.10 in [17] to much more general random walks. The proof is somewhat involved and is therefore organized in six subsections. The basic idea is a renormalization type argument leading up to a contradiction.

We give a slightly different presentation than Kalikow does. In addition, we need to take care of the fact that the steps of our random walk may be unbounded, which introduces a new ingredient into the argument. Other than that, the ideas all come from [17].

1. Notation and main proposition. In order to carry out the proof, it will be easiest to have our random walk defined on a canonical probability space. We therefore let $\Omega = (\mathbb{Z}^d)^{\mathbb{Z}}$ be given the product probability measure all of whose marginals are m (the step distribution). We further let X_i be the random variable on Ω given by $X_i(w) = w_i$ and $S_n(w)$ the random variable on Ω given by $S_n(w) = \sum_{i=1}^{n} w_i$. Of course, these have the distribution of the steps resp. the positions of the random walk (see Section 1.1). We shall abbreviate $S[a,b](w) = \{S_n(w)\}_{a \le n \le b}$ and $w(a,b] = \{w_i\}_{a < i \le b}$.

In the proof several parameters will occur:

$$
\begin{aligned}
&n_0, L \in \mathbb{N},\ p_0 \in (0,1),\\
&(\alpha_k)_{k \ge 0} \subseteq \mathbb{N} \text{ with } \lim_{k\to\infty} \alpha_k = \infty,\\
&(\beta_k)_{k \ge 0} \subseteq (0, 1/4] \text{ with } \lim_{k\to\infty} \beta_k = 0,\\
&(m_k)_{k \ge 1} \subseteq \mathbb{N} \text{ with } \lim_{k\to\infty} m_k = \infty.
\end{aligned}
$$

These will be chosen appropriately later (see Section 6.6). Given the above, let $(n_k)_{k \ge 1}$ be the sequence of integers defined by

$$n_{k+1} = \alpha_k n_k \quad (k \ge 0) \tag{7}$$

and $(\theta_k)_{k\geq 0}$ be the sequence of events defined by

$$\theta_0 = \{w \in \Omega : \ |S[1, n_0](w)| \geq L\}$$

and

$$\theta_{k+1} = \theta'_{k+1} \cap \theta''_{k+1} \cap \theta'''_{k+1} \quad (k \geq 0)$$

with

$$\begin{aligned}
\theta'_{k+1} &= \Big\{w \in \Omega : \ \forall \mathcal{I} \subseteq \{1, 2, \ldots, \alpha_k\}, |\mathcal{I}| \geq \alpha_k \beta_k : \\
&\quad \exists i, j \in \mathcal{I} \text{ with } S[(i-1)n_k, in_k](w) \cap S[(j-1)n_k, jn_k](w) = \emptyset\Big\} \\
\theta''_{k+1} &= \Big\{w \in \Omega : \ \Big|i \in \{1, 2, \ldots, \alpha_k\} : w((i-1)n_k, in_k] \notin \theta_k\Big| \leq \alpha_k \beta_k\Big\} \\
\theta'''_{k+1} &= \Big\{w \in \Omega : \ S[0, n_{k+1}](w) \subseteq B_{m_{k+1}}\Big\}.
\end{aligned}$$

(Recall that $B_\ell = [-\ell, \ell]^d \cap \mathbb{Z}^d$.) The event θ_k depends on the random walk up until time n_k (i.e., on $w(0, n_k]$) and hence in the above we are identifying θ_k with a subset of $(\mathbb{Z}^d)^{n_k}$. The definition is recursive since θ''_{k+1} is defined in terms of θ_k.

The symbol w suggests the word *walk*. We shall use the symbol c to suggest the word *color*. For $c \in \{+1, -1\}^{\mathbb{Z}^d}$, we let $(w, c(w))$ denote the element of $(\mathbb{Z}^d \times \{+1, -1\})^{n_k}$ whose components are

$$(w, c(w))_m = \Big(w_m, c_{S_m(w)}\Big) \quad (1 \leq m \leq n_k),$$

where c_z denotes the color of location z. Let c_1 be a *random* element from the probability space $(\{+1, -1\}^{\mathbb{Z}^d}, \nu)$ where ν is product measure with marginal $(1/2)\delta_{+1} + (1/2)\delta_{-1}$. For $k \geq 0$, $w_1 \in \theta_k$, $c_2 \in \{+1, -1\}^{\mathbb{Z}^d}$ and $p \in (0, 1)$, define the event

$$A^{k, w_1, c_2, p} = \Big\{c_1 \in \{+1, -1\}^{\mathbb{Z}^d} : \exists w_2 \in \theta_k \text{ such that}$$

$$\tfrac{1}{n_k} \textstyle\sum_{m=1}^{n_k} 1\{(w_1, c_1(w_1))_m \neq (w_2, c_2(w_2))_m\} \leq p\Big\}.$$

In words, this is the event that there is some walk w_2 in θ_k such that the n_k–futures of $(w_1, c_1(w_1))$ and $(w_2, c_2(w_2))$ have $\overline{d}$–distance at most p. Note that $A^{k, w_1, c_2, p}$ is measurable w.r.t. c_1 restricted to $S[1, n_k](w_1)$ (i.e, the sites visited by the random walk associated with w_1 over the time interval $[1, n_k]$).

We next introduce the function

$$f_k(p) = \sup\Big\{P(A^{k,w_1,c_2,p}) : w_1 \in \theta_k, c_2 \in \{+1,-1\}^{\mathbb{Z}^d}\Big\}.$$

The following proposition shows why this is a key object.

Proposition 6.1 *If the random walk is recurrent and*

$$\begin{aligned} \lim_{k\to\infty} f_k(p_\infty) &= 0 \textit{ for some } p_\infty > 0 \\ \lim_{k\to\infty} P(\theta_k^c) &= 0, \end{aligned}$$

then the process $\{Z_n\}$ is not VWB.

PROOF: To prove this result it will be expedient to use an equivalent definition of VWB given in [35]. Namely, a stationary process $(Y_i)_{i\in\mathbb{Z}}$ is VWB if for all $\epsilon > 0$ there exists a positive integer $N = N(\epsilon)$ such that for all $n \geq N$,

$$\overline{d}\Big(\{Y_i\}_{i\in(0,n]}, \{Y_i\}_{i\in(0,n]}/\{Y_i\}_{i\in(-\infty,0]}\Big) < \epsilon$$

except on a set of pasts of probability at most ϵ.

In our case, conditioning on the past of the T, T^{-1}–process is exactly the same as conditioning on the entire coloring, since the random walk is recurrent. Let $\{Z_n^c\}$ denote the T, T^{-1}–process conditioned on a given coloring c. We shall show that for all $c \in \{+1,-1\}^{\mathbb{Z}^d}$,

$$\liminf_{k\to\infty} \overline{d}\Big(\{Z_n\}_{n\in(0,n_k]}, \{Z_n^c\}_{n\in(0,n_k]}\Big) \geq p_\infty > 0.$$

This certainly violates the above condition.

Let $\Omega_1^k = (\mathbb{Z}^d)^{n_k}$, $\Omega_2^k = \{+1,-1\}^{n_k}$ and $\Omega^k = \Omega_1^k \times \Omega_2^k$. Fix c and consider any sequence of couplings of $\{Z_n\}_{n\in(0,n_k]}$ and $\{Z_n^c\}_{n\in(0,n_k]}$. These are just measures P_k on $\Omega^k \times \Omega^k$ with the appropriate marginals. If $E_k \subseteq \Omega^k \times \Omega^k$ is the event that the two processes are within p_∞ in $\overline{d}$–distance, then it suffices to show that

$$\lim_{k\to\infty} P_k(E_k) = 0.$$

To prove this, let

$$U_1 = (\theta_k^c \times \Omega_2^k) \times \Omega^k \text{ and } U_2 = \Omega^k \times (\theta_k^c \times \Omega_2^k).$$

Trivially,

$$P_k(U_1) = P(\theta_k^c) \text{ and } P_k(U_2) = P(\theta_k^c).$$

Next, it follows from the definitions of f_k and E_k that

$$P_k(E_k \backslash (U_1 \cup U_2)) \leq f_k(p_\infty).$$

This gives us

$$P_k(E_k) \leq f_k(p_\infty) + 2P(\theta_k^c),$$

which approaches 0 as $k \to \infty$ by assumption. □

Thus, we are left with checking the assumptions in Proposition 6.1.

2. Estimate for $P(\theta_k^c)$. The following lemma provides us with an estimate for each of the three events of which θ_k is the intersection.

Lemma 6.2 *Let $\{S_n\}$ be a recurrent random walk satisfying property ♠ with constants C and γ, and assume that $C_\delta = E|X_1|^\delta < \infty$ for some $\delta > 0$. Then for all $k \geq 0$,*

$$(i)\ P(\theta'^c_{k+1}) \leq C \frac{1}{\alpha_k^\gamma \beta_k^2}$$

$$(ii)\ P(\theta''^c_{k+1}) \leq \left(\tfrac{1}{2}\right)^{\alpha_k \beta_k} \text{ provided } P(\theta_k^c) \leq \tfrac{1}{8}\beta_k$$

$$(iii)\ P(\theta'''^c_{k+1}) \leq C_\delta n_{k+1} \left(\tfrac{m_{k+1}}{n_{k+1}}\right)^{-\delta}.$$

PROOF: (i) This is immediate from the definition of θ'_{k+1} and property ♠.
(ii) Abbreviate $p_k = P(\theta_k^c)$ and define (recall (7))

$$Y_i = 1\{w((i-1)n_k, in_k] \notin \theta_k\} \ (1 \leq i \leq \alpha_k).$$

Clearly, the Y_i's are i.i.d. {0,1}–valued with $P(Y_i = 1) = p_k$. A standard large deviation argument now gives

$$\begin{aligned} P(\theta''^c_{k+1}) &= P\Big(\sum_{i=1}^{\alpha_k} Y_i > \alpha_k \beta_k\Big) \\ &\leq \inf_{\lambda > 0} e^{-\lambda \alpha_k \beta_k} [(1-p_k) + p_k e^\lambda]^{\alpha_k} \end{aligned}$$

$$\begin{aligned} &\leq \inf_{\lambda>0} \exp[-\lambda\alpha_k\beta_k + (e^\lambda - 1)\alpha_k p_k] \\ &= \exp\left[-\alpha_k\beta_k\Big\{\log\Big(\frac{\beta_k}{p_k}\Big) - \Big(1 - \frac{p_k}{\beta_k}\Big)\Big\}\right]. \end{aligned}$$

Since $p_k \leq \frac{1}{8}\beta_k$ by assumption, the term between braces is $> \log 8 - 1 > \log 2$, proving the claim.
(iii) Estimate

$$\begin{aligned} P(\theta'''^c_{k+1}) &= P\Big(S[0, n_{k+1}] \not\subseteq B_{m_{k+1}}\Big) \\ &\leq n_{k+1} P\Big(|X_1| \geq \frac{m_{k+1}}{n_{k+1}}\Big) \\ &\leq n_{k+1}\Big(\frac{m_{k+1}}{n_{k+1}}\Big)^{-\delta} E|X_1|^\delta. \end{aligned}$$

□

3. Estimate for $f_k(p)$. The next lemma is a key step in the proof. It provides us with an important recursive inequality that we shall need in order to carry through the argument.

Lemma 6.3 *Let $k \geq 0$ and suppose that $\frac{p_{k+1}}{p_k} \leq 1 - 3\beta_k$. Then*

$$f_{k+1}(p_{k+1}) \leq \alpha_k^2 |B_{m_{k+1}}|^2 [f_k(p_k)]^2.$$

PROOF: For $w \in \theta_{k+1}$, let $\mathcal{D}_w$ and $\mathcal{E}_w$ be the index sets defined by

$$\mathcal{D}_w = \Big\{i \in \{1, 2, \ldots \alpha_k\} : w((i-1)n_k, in_k] \in \theta_k\Big\},$$

$$\mathcal{E}_w = \Big\{(i,j) \in \mathcal{D}_w \times \mathcal{D}_w : S[(i-1)n_k, in_k](w) \cap S[(j-1)n_k, jn_k](w) = \emptyset\Big\}.$$

We shall show that if $w_1 \in \theta_{k+1}$ and $c_2 \in \{+1, -1\}^{\mathbb{Z}^d}$, then

$$A^{k+1, w_1, c_2, p_{k+1}} \subseteq \bigcup_{(i,j) \in \mathcal{E}_{w_1}} \bigcup_{I,J \in B_{m_{k+1}}}$$

$$\Big\{ \Big(\tau^{-S_{(i-1)n_k}(w_1)} A^{k, \sigma^{(i-1)n_k}(w_1), \tau^I(c_2), p_k}\Big) \tag{8}$$

$$\cap \Big(\tau^{-S_{(j-1)n_k}(w_1)} A^{k, \sigma^{(j-1)n_k}(w_1), \tau^J(c_2), p_k}\Big) \Big\},$$

where σ denotes the left-shift on $(\mathbb{Z}^d)^{\mathbb{Z}}$ and τ denotes the natural action of $\mathbb{Z}^d$ on $\{+1,-1\}^{\mathbb{Z}^d}$. For $(i,j) \in \mathcal{E}_{w_1}$ the two events in the last line are independent. Since the coloring is stationary and since $|\mathcal{E}_w| \leq |\mathcal{D}_w|^2 \leq \alpha_k^2$, it now is immediate that (8) implies the statement of the lemma.

To prove (8), we assume that $c_1 \in A^{k+1,w_1,c_2,p_{k+1}}$. We can then choose a $w_2 \in \theta_{k+1}$ for which

$$\frac{1}{n_{k+1}} \sum_{m=1}^{n_{k+1}} 1\{(w_1, c_1(w_1))_m \neq (w_2, c_2(w_2))_m\} \leq p_{k+1}.$$

This immediately implies that (recall (7))

$$\Big|\Big\{i \in \{1,2,\ldots \alpha_k\} :$$

$$\tfrac{1}{n_k} \textstyle\sum_{m=(i-1)n_k+1}^{in_k} 1\{(w_1, c_1(w_1))_m \neq (w_2, c_2(w_2))_m\} \leq p_k\Big\}\Big|$$

$$\geq \alpha_k(1 - \tfrac{p_{k+1}}{p_k}).$$

Next, let $\mathcal{F}$ denote the latter subset of $\{1,2,\ldots \alpha_k\}$. Since $w_1, w_2 \in \theta_{k+1} \subseteq \theta''_{k+1}$, we have that $|\mathcal{D}_{w_1}| \geq \alpha_k(1-\beta_k)$ and $|\mathcal{D}_{w_2}| \geq \alpha_k(1-\beta_k)$, so

$$|\mathcal{F} \cap \mathcal{D}_{w_1} \cap \mathcal{D}_{w_2}| \geq \alpha_k\Big(1 - \frac{p_{k+1}}{p_k} - 2\beta_k\Big).$$

By assumption, this last expression is at least $\alpha_k\beta_k$. Hence, since $w_1 \in \theta_{k+1} \subseteq \theta'_{k+1}$, it follows that there exist $i,j \in \mathcal{F} \cap \mathcal{D}_{w_1} \cap \mathcal{D}_{w_2}$ with

$$S[(i-1)n_k, in_k](w_1) \cap S[(j-1)n_k, jn_k](w_1) = \emptyset. \tag{9}$$

Consequently, $(i,j) \in \mathcal{E}_{w_1}$.

Next, let

$$I = S_{(i-1)n_k}(w_2) \text{ and } J = S_{(j-1)n_k}(w_2).$$

Since $w_2 \in \theta_{k+1} \subseteq \theta'''_{k+1}$, we have that $I, J \in B_{m_{k+1}}$. Finally, since $i,j \in \mathcal{F} \cap \mathcal{D}_{w_1} \cap \mathcal{D}_{w_2}$, it follows that

$$c_1 \in \Big(\tau^{-S_{(i-1)n_k}(w_1)} A^{k,\sigma^{(i-1)n_k}(w_1),\tau^I(c_2),p_k}\Big)$$

$$c_1 \in \Big(\tau^{-S_{(j-1)n_k}(w_1)} A^{k,\sigma^{(j-1)n_k}(w_1),\tau^J(c_2),p_k}\Big),$$

completing the proof of (8). □

4. Requirements for the parameters. We are now ready to collect our various estimates and to formulate some requirements on our parameters so that the two conditions in Proposition 6.1 are met.

Fix $p_0 > 0$ and let $p_{k+1} = (1 - 3\beta_k)p_k$ (recall that $\beta_k \in (0, 1/4]$ for all $k \geq 0$). Then the condition in Lemma 6.3 is satisfied. Suppose that

$$(I) \quad \sum_{k \geq 0} \beta_k < \infty.$$

Then $p_k \downarrow p_\infty > 0$ as $k \to \infty$. If p_0 could be chosen such that

$$\lim_{k \to \infty} f_k(p_k) = 0, \tag{10}$$

then obviously the first condition in Proposition 6.1 would be met since $p \to f_k(p)$ is clearly increasing.

Iterating the inequality in Lemma 6.3, we get

$$f_k(p_k) \leq [f_0(p_0)]^{2^k} \prod_{\ell=1}^{k} [\alpha_{k-\ell} |B_{m_{k-\ell+1}}|]^{2^\ell}. \tag{11}$$

Suppose now that

$$(II) \quad \text{There exists } C_2 < \infty \text{ such that } \alpha_k |B_{m_{k+1}}| \leq e^{\frac{C_2 2^k}{(k+1)^2}} \ (k \geq 0).$$

Then

$$\begin{aligned} f_k(p_k) &\leq [f_0(p_0)]^{2^k} \exp\Big[\sum_{\ell=1}^{k} \frac{C_2 2^{k-\ell}}{(k-\ell+1)^2} 2^\ell\Big] \\ &= [f_0(p_0)]^{2^k} \exp\Big[2^k C_2 \sum_{\ell=1}^{k} \frac{1}{\ell^2}\Big]. \end{aligned}$$

Hence (10) would follow as soon as $f_0(p_0) < \exp[-C_2 \zeta(2)]$ with $\zeta(2) = \sum_{\ell=1}^{\infty} \frac{1}{\ell^2}$.

Now, for any n_0 and $p_0 < 1/n_0$ and for any $w_1 \in \theta_0$ and $c_2 \in \{+1, -1\}^{\mathbb{Z}^d}$, if the event A^{0,w_1,c_2,p_0} occurs then w_2 (whose existence is guaranteed in the definition of A^{0,w_1,c_2,p_0}) must be identical to w_1 over the interval $[1, n_0]$.

Therefore the *random* coloring c_1 must agree with c_2 on the set $S[1,n_0](w_1)$, an event which has probability $(1/2)^{|S[1,n_0](w_1)|}$. Since $|S[1,n_0](w_1)| \geq L$ for $w_1 \in \theta_0$, it follows from the definition of $f_k(p)$ that the conditions

$$\begin{array}{ll} (III) & p_0 < 1/n_0 \\ (IV) & L > C_2\zeta(2)/\log 2 \end{array}$$

will guarantee that $f_0(p_0) < \exp[-C_2\zeta(2)]$, implying (10) as desired.

Next we turn to the second condition in Proposition 6.1. If $P(\theta_k^c) \leq \frac{1}{8}\beta_k$ $(k \geq 0)$, then by (I) this condition would be met. The case $k = 0$ gives us one requirement involving n_0 and β_0, namely

$$(V) \quad P(|S[1,n_0]| < L) \leq \frac{1}{8}\beta_0.$$

Using the fact that

$$P(\theta_{k+1}^c) \leq P(\theta_{k+1}^{\prime c}) + P(\theta_{k+1}^{\prime\prime c}) + P(\theta_{k+1}^{\prime\prime\prime c}),$$

together with Lemma 6.2, we obtain our last requirement

$$(VI) \quad C\frac{1}{\alpha_k^\gamma \beta_k^2} + \Big(\frac{1}{2}\Big)^{\alpha_k\beta_k} + C_\delta n_{k+1}\Big(\frac{m_{k+1}}{n_{k+1}}\Big)^{-\delta} \leq \frac{1}{8}\beta_{k+1} \quad (k \geq 0).$$

Note here that $(\frac{1}{2})^{\alpha_k\beta_k}$ is only an upper bound for $P(\theta_{k+1}^{\prime\prime c})$ if $P(\theta_k^c) \leq \frac{1}{8}\beta_k$ (see Lemma 6.2 (ii)). However, if this inequality holds for $k \leq k_0$ then (VI) guarantees that it also holds for $k = k_0 + 1$. So we can apply induction.

5. Lemma about the range of the random walk. To verify (I-VI), we need the following property. The proof is due to Harry Kesten (personal communication).

Lemma 6.4 *Abbreviate* $R_n = |S[0,n]|$. *Then*

$$\lim_{N\to\infty} \inf_{n\geq 0} P(R_{Nn} \geq \sqrt{n}) = 1.$$

PROOF: A straightforward calculation gives that $E(R_n^2) \leq 2(ER_n)^2$ for all n (see [25] p.664). Hence, as in the proof of Theorem 2.5 in Section 4,

$$P\Big(R_n \geq \frac{1}{2}ER_n\Big) \geq \frac{(ER_n)^2}{4E(R_n^2)} \geq \frac{1}{8} \text{ for all } n.$$

Pick $N \geq 1$ integer and estimate

$$P\Big(R_{Nn} < \frac{1}{2}ER_n\Big) \leq \Big[P\Big(R_n < \frac{1}{2}ER_n\Big)\Big]^N \leq \Big(\frac{7}{8}\Big)^N.$$

It follows that

$$\lim_{N\to\infty} \inf_{n\geq 0} P\Big(R_{Nn} \geq \frac{1}{2}ER_n\Big) = 1.$$

Next, by reversing the order of the steps in the random walk, we have (recall the notation introduced in Section 2.2)

$$\begin{aligned} ER_n &= \textstyle\sum_{k=0}^n P(S_k \notin \{S_0, S_1, \ldots, S_{k-1}\}) \\ &= \textstyle\sum_{k=0}^n P(0 \notin \{S_1, S_2, \ldots, S_k\}) \\ &= \textstyle\sum_{k=0}^n [1 - \sum_{\ell=1}^k f'_\ell(0)] \end{aligned}$$

where $f'_\ell(0)$ is the probability that the first return to 0 occurs at time ℓ. Using the equation

$$p_m(0) = \delta_{m0} + \sum_{\ell=1}^m f'_\ell(0) p_{m-\ell}(0),$$

we have for any $z \in (0,1)$ that

$$\sum_{k\geq 0} z^k ER_k = \frac{1}{(1-z)^2}\Big[1 - \sum_{\ell\geq 1} z^\ell f'_\ell(0)\Big] = \Big[(1-z)^2 \sum_{m\geq 0} z^m p_m(0)\Big]^{-1}.$$

Putting $z = 1 - \frac{1}{n}$ and using that there exists a $C_1 < \infty$ such that $p_m(0) \leq C_1/\sqrt{m+1}$ for all m ([37] Proposition 7.6), we find

$$\sum_{k\geq 0} \Big(1 - \frac{1}{n}\Big)^k ER_k \geq C_2(n+1)^{\frac{3}{2}}$$

for some $C_2 > 0$ and all n. Finally, because ER_k is increasing in k and $ER_{in} \leq iER_n$ $(i \geq 1)$, the l.h.s. is bounded from above by $(n+1)ER_n \sum_{i\geq 1} ie^{-(i-1)}$. Hence we obtain $ER_n \geq C_3\sqrt{n+1}$ for some $C_3 > 0$ and all n. □

6. Choice of parameters. We now complete the proof of Theorem 2.8 by showing that our parameters can be chosen so that (I-VI) are satisfied. We may of course assume that $C \geq 1$ and $\gamma \leq 1$ in the definition of property ♠.

Put

$$\beta_k = \frac{1}{4(k+1)^2}.$$

This guarantees (I). Next put

$$\alpha_k = (8000C)^{\frac{1}{\gamma}}(k+1)^{\frac{16}{\gamma}},$$

where C and γ come from the definition of property ♠. A trivial computation (left to the reader) shows that each of the first two terms in the l.h.s. of (VI) is at most $\beta_{k+1}/24$.

Now let L be arbitrary. The rest of the parameters will be chosen in terms of L. All the estimates below will hold *uniformly* in L.

Put $n_0 = C_1 L^2$, where C_1 is chosen so that

$$\inf_{L\geq 1} P(|S[1, C_1 L^2]| \geq L) \geq \frac{31}{32},$$

which is possible by Lemma 6.4. Then (V) holds because $\frac{1}{8}\beta_0 = \frac{1}{32}$. Put $p_0 = 1/2n_0$. Then (III) holds. Next, put $C_2 = L/2$. Then (IV) holds. Finally, put (recall (7))

$$m_{k+1} = (100C_\delta)^{\frac{1}{\delta}}(k+2)^{\frac{2}{\delta}}(n_{k+1})^{\frac{1+\delta}{\delta}}.$$

Then another trivial computation (left to the reader) shows that the last term in the l.h.s. of (VI) is at most $\beta_{k+1}/24$. Therefore (VI) holds.

Now, all our parameters except L have been defined (some in terms of L) and the conditions (I) and (III–VI) hold uniformly in L. The last step is to choose L so large that (II) holds. This goes as follows.

It is immediate to check that for all $k \geq 0$,

$$n_k \leq (8000C)^{\frac{k}{\gamma}} k^{\frac{16k}{\gamma}} C_1 L^2.$$

This in turn gives us that

$$m_{k+1} \leq (100C_\delta)^{\frac{1}{\delta}}(k+2)^{\frac{2}{\delta}}\left[(8000C)^{\frac{k+1}{\gamma}}(k+1)^{\frac{16(k+1)}{\gamma}} C_1 L^2\right]^{\frac{1+\delta}{\delta}}.$$

Calling the r.h.s. of the last inequality $a(k,L)$ (as γ, C, C_1, C_δ are fixed), we see that to prove (II) it is enough to show that for some $L \geq 1$,

$$(8000C)^{\frac{1}{\gamma}}(k+1)^{\frac{16}{\gamma}}[2a(k,L)+1]^d \leq e^{\frac{\frac{L}{2}2^k}{(k+1)^2}} \quad (k \geq 0)$$

(d is the dimension). To do this, it suffices to show that for any positive number A there exists an $L \geq 1$ such that

$$A^k k^{Ak} L^A \leq e^{\frac{\frac{L}{2}2^k}{(k+1)^2}} \ (k \geq 0).$$

But the latter is trivial. □

7 Proof of Theorem 2.11

In this section we prove the sufficient condition for property ♠ stated in Theorem 2.11. As stated in Conjecture 2.9, we believe ♠ to hold in general, but unfortunately we are unable to prove this.

Proof of Theorem 2.11. Fix $M, N \geq 1$. Let

$$Y_{i,j}^M = 1\Big\{S[(i-1)M, iM] \cap S[(j-1)M, jM] \neq \emptyset\Big\} \ (i, j \geq 1)$$

and

$$Y_N^M = \sum_{1 \leq i < j \leq N} Y_{i,j}^M.$$

Then we can estimate the probability of the event in Definition 2.7 as follows:

$$\begin{aligned} P(E_{N,r}^M) &\leq P\Big(Y_N^M \geq \tfrac{1}{2} rN(rN-1)\Big) \\ &\leq \tfrac{1}{\frac{1}{2}rN(rN-1)} \textstyle\sum_{1 \leq i < j \leq N} P(Y_{i,j}^M = 1). \end{aligned} \tag{12}$$

Next fix $1 \leq i < j$. For arbitrary $h > 0$ we obviously have

$$\begin{aligned} &P(S[(i-1)M, iM] \cap S[(j-1)M, jM] \neq \emptyset) \\ &\leq P(|S_{(i-1)M} - S_{(j-1)M}| \leq 2h) \\ &\quad + P(\max\nolimits_{0 \leq m \leq M} |S_{(i-1)M+m} - S_{(i-1)M}| > h) \\ &\quad + P(\max\nolimits_{0 \leq m \leq M} |S_{(j-1)M+m} - S_{(j-1)M}| > h) \end{aligned}$$

and hence

$$P(Y_{i,j}^M = 1) \leq P(|S_{(j-i)M}| \leq 2h) + 2P\Big(\max_{0 \leq m \leq M} |S_m| > h\Big). \tag{13}$$

We shall estimate each of these terms separately, then choose h appropriately and finally sum over i, j, to get an estimate for the r.h.s. of (12).

Theorem 1 in [12] tells us that there is a $C_1 < \infty$ such that

$$P(S_n = x) \leq \frac{C_1}{a_n} \ (x \in \mathbb{Z}, n \geq 1).$$

This immediately implies

$$P(|S_{(j-i)M}| \leq 2h) \leq \frac{C_1(2h+1)}{a_{(j-i)M}}. \tag{14}$$

Next, it follows easily from the same Theorem 1 in [12] that there are $C_2 < \infty$ and $0 < \lambda \leq \alpha$ such that

$$P(|S_M| > h) \leq C_2\Big(\frac{a_M}{h}\Big)^\lambda.$$

For arbitrary $h > 0$,

$$\begin{aligned} P(S_M > h) &\geq \textstyle\sum_{m=1}^M P(S_m > h, S_k \leq h \text{ for } 0 \leq k < m, S_M \geq S_m) \\ &\geq \zeta P\Big(\max_{0\leq m\leq M} S_m > h\Big), \end{aligned}$$

where $\zeta = \inf_{n\geq 0} P(S_n \geq 0)$. A similar estimate holds for $P(S_M < -h)$ with $\zeta' = \inf_{n\geq 0} P(S_n \leq 0)$. So

$$P\Big(\max_{0\leq m\leq M} |S_m| > h\Big) \leq \Big(\frac{1}{\zeta} + \frac{1}{\zeta'}\Big) P(|S_M| > h).$$

Now, the support of a stable law Y with indices α, θ is all of $\mathbb{R}$ when $\alpha \geq 1$ and/or $\theta \in (0,1)$ (see [3], p.348–350). Consequently, we have $\zeta, \zeta' > 0$ in that case. Hence we obtain that that there is a $C_3 < \infty$ such that

$$2P\Big(\max_{0\leq m\leq M} |S_m| > h\Big) \leq C_3(\frac{a_M}{h})^\lambda \tag{15}$$

with λ as above.

Next, combine (13-15) and pick

$$h = a_{(j-i)M}^{\frac{1}{1+\lambda}} a_M^{\frac{\lambda}{1+\lambda}}.$$

Then a small calculation gives that

$$P(Y_{i,j}^M = 1) \leq C_4 \left(\frac{a_{(j-i)M}}{a_M}\right)^{-\frac{\lambda}{1+\lambda}}$$

for some $C_4 < \infty$. Now, as mentioned in Remark (p) in Section 2.4, we have $a_M = M^{\frac{1}{\alpha}} L(M)$ with L a slowly varying function. The above inequality therefore becomes

$$P(Y^M_{i,j} = 1) \leq C_4 (j-i)^{-\frac{\lambda}{\alpha(1+\lambda)}} \left[\frac{L((j-i)M)}{L(M)} \right]^{-\frac{\lambda}{1+\lambda}}.$$

Finally, Theorem 1.5.6(ii) in [3] states that, given any slowly varying function L, for every $\delta > 0$ there exists a $D = D(\delta)$ such that for all $1 \leq i < j$ and all $M \geq 1$,

$$\frac{1}{D(j-i)^\delta} \leq \frac{L((j-i)M)}{L(M)} \leq D(j-i)^\delta.$$

Therefore, choosing any $\delta < 1/\alpha$, we finally get

$$P(Y^M_{i,j} = 1) \leq C_5 (j-i)^{-\gamma}$$

for some $C_5 < \infty$, with

$$\gamma = \left(\frac{\lambda}{\lambda + 1} \right) \left(\frac{1}{\alpha} - \delta \right).$$

Now recall (12) to find that property ♠ holds for some constant C and with γ as above. □

Remarks:
(a) From [33] p.6 and [12] one can see that the λ arising in the above proof can be taken to be arbitrarily close to α, thereby verifying Remark (r) in Section 2.4. With the help of the methods in [12] it might be possible to prove property ♠ for a class of random walks outside the domain of attraction of a stable law. However, we shall not pursue this line.
(b) Combining Theorems 1.9 and 2.6 and Corollary 2.12, we obtain one of the claims made in the abstract of our paper, namely that the T, T^{-1}–process associated with simple random walk is B if and only if $d \geq 3$.

8 Related Problems

One of the main open questions that remain for the T, T^{-1}–process is the following:

I. For simple random walk in $d = 1$ or 2, is the second coordinate of $\{Z_i\}_{i\in\mathbb{Z}}$, which is $\{C_{S_i}\}_{i\in\mathbb{Z}}$, a Bernoulli process?

One would suspect that, like $\{Z_i\}_{i\in\mathbb{Z}}$, it is not. However, this does not follow from the proof in Section 6. Part of this question is very much related to some interesting recent work in [1], [14], [15], [16] and [18]. In these papers, the following question is studied:

II. For simple random walk in $d = 1$ or 2, can one determine the values $\{C_z\}_{z\in\mathbb{Z}^d}$ if one is given the values $\{C_{S_i}\}_{i\in\mathbb{Z}}$?

If the answer is yes, then we say that C is **retrievable**.

Note that both questions concern the recurrent case, so that from the values $\{Z_i\}_{i\in\mathbb{Z}}$ it would be trivial to find the values $\{C_z\}_{z\in\mathbb{Z}^d}$. However, without the first coordinate of $\{Z_i\}_{i\in\mathbb{Z}}$ we only see the colors that are encountered but we do not see where the walk is.

One way to formalize question II is as follows.

III. Fix $d = 1$ or 2. Let $C, C' : \mathbb{Z}^d \to \{+1, -1\}$ with $C \neq C'$ be two given colorings, and let $\{S_n\}_{n\in\mathbb{Z}}$ be simple random walk. For $\ell \in \mathbb{Z}$, let μ_C^ℓ ($\mu_{C'}^\ell$) be the probability measure on $\{+1, -1\}^{\mathbb{Z}}$ given by the distribution of $\{C_{S_i}\}_{i\geq\ell}$ ($\{C'_{S_i}\}_{i\geq\ell}$). Are μ_C^ℓ and $\mu_{C'}^\ell$ mutually singular measures for all ℓ?

If the answer is yes, then we say that C and C' are **distinguishable**.

It is easy to see that there are $C \neq C'$ for which μ_C and $\mu_{C'}$ are identical. For example, if $C(z) = C'(-z)$ for all $z \in \mathbb{Z}^d$, then obviously μ_C and $\mu_{C'}$ are equal. Or, in $d = 1$, if C' is an even shift of C, then μ_C^ℓ and $\mu_{C'}^\ell$ are not mutually singular for all ℓ. In $d = 2$ there are more interesting examples: if the coloring is obtained by either alternately coloring vertical lines $+1$ and -1 or by tiling with 2×2 squares and alternately coloring these squares $+1$ and -1, then in both cases the process $\{C_{S_i}\}_{i\geq 0}$ is just a sequence of i.i.d. $+1$'s and -1's.

The following conjecture was raised by den Hollander and Keane and independently by Benjamini and Kesten (see [1]).

Conjecture 8.1 *Let $d = 1$. If C and C' are two colorings and if there is no even integer k such that $C(k + n) = C'(n)$ for all n or $C(k + n) = C'(-n)$*

for all n, then C and C' are distinguishable.

The following theorem is a result is this direction. Let ν be product measure on $\{+1,-1\}^{\mathbb{Z}^d}$ with each marginal being $(1/2)\delta_{+1} + (1/2)\delta_{-1}$.

Theorem 8.2 *([1]) In $d = 1$ and 2, every coloring C is distinguishable from ν-a.e. coloring C'.*

(Related results are contained in [14], [15], [16] and [18].)

Theorem 1.10 and Theorem 8.2 have in fact more in common than might first be apparent. The methods of [1] should be able to show that:

$(*)$ In $d = 1$ and 2, ν-a.e. coloring C has the property that μ_C^ℓ and $\int_{C'} \mu_{C'}^\ell d\nu(C')$ are mutually singular for all ℓ.

(The same statement with 'ν-a.e. coloring C' replaced by 'every coloring C' clearly implies Theorem 8.2.)

The proof of Theorem 1.10 in Section 6 essentially shows that for every coloring c of $\mathbb{Z}$ or $\mathbb{Z}^2$, the processes $\{Z_n\}_{n\geq 1}$ and $\{Z_n^c\}_{n\geq 1}$ (which is $\{Z_n\}_{n\geq 1}$ conditioned on c) cannot be coupled well in the $\overline{d}$–metric. Property $(*)$, on the other hand, says that for ν-a.e. coloring c of $\mathbb{Z}$ or $\mathbb{Z}^2$, the second coordinate of $\{Z_n\}_{n\geq 1}$ and the second coordinate of $\{Z_n^c\}_{n\geq 1}$ cannot be coupled so that they eventually agree with positive probability. The difference between Theorem 1.10 and $(*)$ is therefore that:

– the former deals with the 'larger' process $\{Z_n\}$ and shows that it does not satisfy a weaker type of coupling property,

– the latter deals with the 'smaller' process $\{C_{S_i}\}$ and shows that it does not satisfy a stronger type of coupling property.

Therefore the two results, though related, are not directly comparable.

We conclude our paper by reflecting on some possible extensions of the results formulated in Section 2. Throughout our paper we have assumed that the random coloring is i.i.d. This was essential for our methods. It is an interesting problem to find out how much of the i.i.d.-property can be relaxed. For instance, the T, T^{-1}–process associated with a random walk that is transient (resp. satisfies property $\heartsuit$) and with a random coloring that has strong mixing properties other than i.i.d. is it VWB (resp. WB)? If we only want the T, T^{-1}–process to be K, then we can refer to [26]. There it is shown that K holds for arbitrary (irreducible) random walk when the random coloring is totally ergodic.

Another interesting direction is related to 'induced systems'. For example, in [13] the following problem is studied. We again have a random walk and a random coloring, where the latter is assumed to be stationary but not necessarily i.i.d. However, we *only* observe the system at the times when the random walk hits a location colored $+1$, and at those times we report the local coloring. Such a process is stationary and is what is called an induced system in ergodic theory. It is shown in [13], among other things, that under weak conditions the induced system is mixing. This result was recently strengthened in [10] to show that under the same conditions the induced system is even K (and was also extended to general groups). An obvious problem that one could study is whether the induced process is VWB resp. WB. This has so far not been done, not even when the random coloring is i.i.d.

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