The random cluster model on a general graph and a phase transition characterization of nonamenability

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Abstract

The random cluster model on a general infinite graph with bounded degree wired at infinity is considered and a "ghost vertex" method is introduced in order to explicitly construct random cluster measures satisfying the Dobrushin-Lanford-Ruelle condition for $q \geq 1$. It is proved that on a regular nonamenable graph there is a q_0 such that for $q \geq q_0$ there is a phase transition for an entire interval of values of p, whereas on a quasi-transitive amenable graph there is a phase transition for at most a countable number of values of p. In particular, a transitive graph is nonamenable if and only if there is a phase transition for an entire interval of p-values for large enough q. It is also observed that these results have a Potts model interpretation. In particular a transitive graph is nonamenable if and only if the q-state Potts model on that graph has the property that for q large enough there is an entire interval of temperatures for which the free Gibbs state is not a convex combination of the q Gibbs states obtained from one-spin boundary conditions.

It is also proved that on the regular tree, $\mathbf{T}_n$, with $q \geq 1$ and p close enough to 1, there is unique random cluster measure despite the presence of more than one infinite cluster. This partly proves Conjecture 1.9 of [11].

1 Introduction

The purpose of this paper is two-fold; to introduce a technique that overcomes the difficulties involved in explicitly constructiong random cluster measures on a general infinite graph with bounded degree and to give a characterization of nonamenability for transitive graphs in terms of a phase transition for the random cluster model.

Let us begin by introducing the necessary graph theoretical concepts. First of all however, let us state that *all infinite graphs named G in this paper are assumed to be connected and to have bounded degree.* (However the graph H introduced in Section 2 by adding a ghost vertex will not satisfy these assumptions.) An infinite graph, $G = (V, E)$, is said to be *quasi-transitive* if there is a finite set, $A = \{v_1, \ldots, v_k\}$, of vertices such that for each $u \in V$ there is a graph automorphism taking u to v_i for some $v_i \in A$. (In other words, G is quasi-transitive if the automorphism group of G acting on V has finitely many orbits.) If the set A can be taken to be a singleton set, then G is said to be *transitive*. The graph G is called *regular* if all vertices have the same degree.

DEFINITION 1.1 *Let $G = (V, E)$ be an infinite graph. The* Cheeger constant *for G, $\kappa(G)$, is given by*

$$\kappa(G) = \inf_{W \in V, |W| < \infty} \frac{|\partial_E W|}{|W|}$$

where $\partial_E W$ is the edge boundary *of W, i.e. the set of edges connecting W to $V \setminus W$. If $\kappa(G) = 0$, then G is said to be* amenable *and in case $\kappa(G) > 0$, G is said to be* nonamenable.

Remark. The definition of the Cheeger constant is usually given in terms of the vertex boundary ∂W rather than the edge boundary, but the present definition will turn out to be more convenient for our purposes.

The first connection between probability theory and amenability of groups was obtained by Kesten (see [15] and [16]) where he proved that if one takes a finite symmetric generating set for a finitely generated group, then the group is nonamenable if and only if the return probabilities for simple random walk on the resulting Cayley graph decay exponentially (or equivalently the spectral radius for the resulting Markov operator on L_2 has spectral radius strictly less than one). This result was extended in [7] to any graph of bounded degree where it was shown that the return probabilities for simple random walk on the graph decay exponentially if and only if the graph is nonamenable.

Recently, another connection between amenability of groups and probability theory has been obtained. In [2], it is shown that a group is amenable if and only if for all $\alpha < 1$, there is a G–invariant site percolation on one (all) of its Cayley graphs such that the probability of a site being on is larger than α but for which there are no infinite components. (This result was motivated by an earlier result for regular trees in [10]). See [2] for details and where the above is stated in a more general setting. A conjecture concerning percolation on groups is that a group is nonamenable if and only if for one (all) of its Cayley graphs, there is a nontrivial interval of parameters p such that i.i.d. percolation with parameter p yields infinitely many infinite clusters. See [3] for details and a more general conjecture (Conjecture 6) as well as [12] for a related result.

The paper [23] proves a multiple phase transition in the Ising model on some hyperbolic graphs in that for high temperatures there is a unique Gibbs state and for low temperatures the free Gibbs state is a convex combination of the plus and minus states, whereas for an interval of intermediate temperatures the free measure is not a convex combination of the plus and minus states. It will be a consequence of our results that the latter phenomenon occurs for the q-state Potts model on a transitive graph for large enough q if and only if the graph is nonamenable.

In [14] a characterization of nonamenability in terms of a phase transition for the Ising model with a strictly positive external field is given, namely that a quasi-transitive graph is nonamenable if and only if such a phase transition occurs at low enough (but nonzero) temperatures. In particular this result is valid for all Cayley graphs of groups.

The main result of the present paper is, together with the construction of *wired infinity random cluster measures* on a general graph G, the following relation between amenability and phase transition in the *wired infinity random cluster model.* A wired infinity random cluster measure is defined in the usual Dobrushin-Lanford-Ruelle spirit in such a way that *all infinite clusters are considered as one*, i.e. connected to each other at infinity. (The idea of regarding all infinite clusters as one was introduced by Häggström in [11] where the random cluster model on a homogeneous tree is considered.) The precise definition will be given in Section 2, Definition 2.2. In Section 2 we also introduce the promised method for finding such measures on a general infinite graph.

THEOREM 1.2 *Let $G = (V, E)$ be an infinite graph.*

(a) If G is nonamenable and regular with degree d, then there exists a $q_0 < \infty$ such that for $q > q_0$ and $p/(1-p) \in [q^{2/(d+\kappa/2)}, q^{2/(d-\kappa/2)}]$ the wired infinity random cluster model on E exhibits a phase transition, *i.e. there exists more than one wired infinity random cluster measure on $\{0,1\}^E$.*

(b) If G is amenable and quasi-transitive, then for all $q \in [1, \infty)$ there is a unique wired infinity random cluster measure on $\{0,1\}^E$ for all but at most countably many values of p.

In particular, if G is transitive then G is nonamenable if and only if the conclusion in (a) holds.

The proof of Theorem 1.2 is given in Sections 3.1. and 3.2. For part (a) we will use a Peierls type of argument and for (b) we translate the convexity and differentability of pressure argument of [14, Theorem 1.5(b)]. Part (b) is well known for $G = \mathbf{Z}^d$ and this case is proved in [8] where the technique of which our proof is an extension is used. A special case of (a) is proved in [11], namely when G is the homogeneous tree, $\mathbf{T}_n$.

In Section 3.3 we prove the following theorem which partially proves [11, Conjecture 1.9]. The result is relevant here since it negatively answers a question that arises naturally in the light of Theorem 1.2(a), namely if the presence of more than one infinite cluster necessarily entails a phase transition for the wired infinity random cluster model. (However one has to be careful with what to mean with the term "phase transition" here. See the remark after Lemma 4.3.)

THEOREM 1.3 *Let $G = (V, E) = \mathbf{T}_n$, the regular tree with degree $n + 1$. Let $q \geq 1$ and set $p' = p(p + (1-p)q)^{-1}$. Then for all p such that*

$$p' \geq \frac{1 - n^{-1/(n-1)}}{1 - n^{-n/(n-1)}}$$

there is a unique random cluster measure with parameters p and q.

Grimmett [8] proves an analogous result for $G = \mathbf{Z}^d$.

In Section 4 we translate Theorem 1.2 into the above mentioned Potts model result.

Before moving on into Section 2, let us introduce the concept of stochastic monotonicity. If μ and ν are two measures defined on the same partially ordered measurable space A, such that $\int_A f d\mu \leq \int_A f d\nu$ for all increasing measurable functions f, then we say that μ is stochastically dominated by ν, and we write $\mu \leq_d \nu$.

2 The random cluster model on a general graph

The random cluster model was first introduced in the 70's by Fortuin and Kasteleyn (see [6]) as a tool to handle Ising and Potts models on $\mathbf{Z}^d$. The definition of a random cluster measure on a finite graph is the following. (As usual an edge with $\eta(e) = 1$ is said to be open and an edge with $\eta(e) = 0$ is said to be closed.)

DEFINITION 2.1 *Let $G = (V, E)$ be a finite graph. For $p \in [0,1]$ and $q > 0$, the random cluster measure $\mu_G^{p,q}$ on $\{0,1\}^E$ is given by*

$$\mu_G^{p,q}(\eta) = \frac{1}{Z_G^{p,q}} (\prod_{e \in E} p^{\eta(e)} (1-p)^{1-\eta(e)}) q^{k(\eta)} \tag{1}$$

for all $\eta \in \{0,1\}^E$. Here $Z_G^{p,q}$ is the normalizing constant $\sum_{\eta \in \{0,1\}^E} (\prod_{e \in E} p^{\eta(e)} (1-p)^{1-\eta(e)}) q^{k(\eta)}$ and $k(\eta)$ is the number of connected components in the open subgraph of G given by η

In case G is infinite this definition does not work but it can be generalized by taking so called thermodynamic limits. Fix a finite set $S \subseteq E$, $p \in [0,1]$, $q > 0$ and a configuration $\xi \in \{0,1\}^{E \setminus S}$ on the edges off S and define the *free infinity* random cluster measure on S with *boundary condition* ξ by

$$\phi_{S,\xi}^{p,q}(\eta) = \frac{1}{U_{S,\xi}^{p,q}} (\prod_{e \in S} p^{\eta(e)} (1-p)^{1-\eta(e)}) q^{l(\eta,\xi)} \tag{2}$$

where $U_{S,\xi}^{p,q}$ is the proper normalizing constant and $l(\eta, \xi)$ is the number of connected components in the configuaration given by η on S and ξ on $E \setminus S$ that intersect $V(S) = \{v \in V : \exists e \in S \text{ such that } e \text{ is incident to } v\}$. Define also the *wired infinity* random cluster measure on S with boundary condition ξ by

$$\mu_{S,\xi}^{p,q}(\eta) = \frac{1}{Z_{S,\xi}^{p,q}} (\prod_{e \in S} p^{\eta(e)} (1-p)^{1-\eta(e)}) q^{k(\eta,\xi)} \tag{3}$$

where $Z^{p;q}_{S,\xi}$ is the normalizing constant and $k(\eta, \xi)$ is the number of *finite* connected components in the configuaration given by η on S and ξ on $E \setminus S$ that intersect $V(S)$. The definition of infinite volume (free infinity/wired infinity) random cluster measures is now that the conditional probabilities are to satisfy (2)/(3). Here X and Y are $\{0,1\}^E$-valued random variables with distribution ϕ and μ respectively and P is the underlying probability measure.

DEFINITION 2.2 *Let $G = (V, E)$ be an infinite graph. A probability measure, ϕ, on $\{0,1\}^E$ is said to be a free infinity random cluster measure with parameters p and q if*

$$P(X(S) = \eta | X(E \setminus S) = \xi) = \phi^{p;q}_{S,\xi}(\eta) \tag{4}$$

for all finite $S \subseteq E$, all $\eta \in \{0,1\}^S$ and ϕ-a.e. $\xi \in \{0,1,\}^{E \setminus S}$. Similarly a probability measure, μ, on $\{0,1\}^E$ is said to be a wired infinity random cluster measure with parameters p and q if

$$P(Y(S) = \eta | Y(E \setminus S) = \xi) = \mu^{p;q}_{S,\xi}(\eta) \tag{5}$$

for all finite $S \subseteq E$, all $\eta \in \{0,1\}^S$ and μ-a.e. $\xi \in \{0,1,\}^{E \setminus S}$.

This definition is analogous to the usual Dobrushin-Lanford-Ruelle definition of an infinite volume Gibbs measure. Free infinity random cluster measures are random cluster measures in the sense of Grimmett [8] (where the case $G = \mathbf{Z}^d$ is considered). They are obtained by regarding all infinite clusters as separate. Wired infinity random cluster measures on the other hand, are obtained by regarding all infinite clusters as one, i.e. as wired at infinity. These measures were introduced by Häggström [11] for $G = \mathbf{T}_n$, the homogeneous tree.

Let us now for a while consider the case when G is quasi-transitive and amenable. Free infinity random cluster measures are then known to exist for $q \geq 1$. We refer to [8] for details on $\mathbf{Z}^d$. In [8] the following explicit construction of free infinity random cluster measures (on $\mathbf{Z}^d$, but it works for any quasi-transitive amenable graph) is given. Let $S_1, S_2, \ldots$ be finite subsets of E such that $S_n \uparrow E$ and define the probability measures $\phi_{0,n}$ and $\phi_{1,n}$ on $\{0,1\}^E$ by first assigning all edges off S_n the value 0 and 1 respectively and then assigning values to the edges of S_n according to (2) with $\xi \equiv 0$ and $\xi \equiv 1$ respectively. (We suppress the superscripts p and q here in order not to burden the notation and regard them as understood.) By standard monotonicity arguments based on Holley's Theorem (see e.g. [19]) $\phi_{0,1} \leq_d \phi_{0,2} \leq_d \dots$ and $\phi_{1,1} \geq_d \phi_{1,2} \geq_d \dots$ so that the weak limits (w.r.t. the weak topology on $\{0,1\}^E$) ϕ_0 and ϕ_1 exist. By monotonicity these limits are independent of the particular sequence $\{S_n\}$ and $\phi_0 \leq_d \phi \leq_d \phi_1$ for any free infinity random cluster measure ϕ with the same parameters. A consequence of the latter fact is that the question of phase transition in the free infinity model boils down to the question of whether $\phi_0 = \phi_1$ or not. A phase transition is known to take place on $\mathbf{Z}^d$ for large q when $p = p_c(q)$, the critical value for percolation, (see [8]) but not for more than at most a countable number of values of p.

To prove that ϕ_0 and ϕ_1 are indeed free infinity random cluster measures in the sense of Definition 2.2 is an exercise in using the definitions of conditional probability and weak convergence. A crucial fact, proved in [8], for that argument is that $\phi_{S,\xi}(\eta)$ regarded as a function of ξ is continuous at ϕ_i-a.e. ξ, $i = 0, 1$. For $q \geq 1$ monotonicity arguments imply that ϕ_i is automorphism invariant, so the Burton-Keane Theorem (see [4]) applies to show that ϕ_i-a.e. ξ contains at most one infinite cluster and from this continuity follows. (The Burton-Keane Theorem was originally stated for $G = \mathbf{Z}^d$ but extends easily to all quasi-transitive amenable graphs.)

A completely analogous construction yields wired infinity random cluster measures μ_{00} and μ_1 having the corresponding properties. (The notation μ_{00} is used in order to save μ_0 for a third measure appearing below.) Moreover, the Burton-Keane Theorem implies that an automorphism invariant measure is a free infinity random cluster measure if and only if it is also a wired infinity random cluster measure. In particular $\phi_1 = \mu_1$ and $\phi_0 = \mu_{00}$ so that there is a phase transition in the free infinity model if and only if there is a phase transition in the wired infinity model for the same parameters.

Let us now turn back to the general situation. In this case there might be a positive probability for having more than one infinite cluster. This is known to be the case for e.g. i.i.d. percolation on $\mathbf{T}_n$, a fact which follows from a simple branching process argument, and on $\mathbf{T}_n \times \mathbf{Z}$. For the latter statement see [9] where it is shown that for an interval of edge densities there are infinitely many infinite clusters whereas for high edge densities there is a unique infinite cluster. It has been conjectured that for any nonamenable graph there is an entire interval of edge densities for which more than one infinite cluster appears. Possible nonuniqueness of infinite clusters causes some new problems. First of all it is clear that in this case the free infinity model and the wired infinity model disagree and we have to decide on what model to use. There is no way of telling which model is the "correct" one, but in [11] it is observed that on $\mathbf{T}_n$ the free infinity approach necessarily yields product measure with density $p(p+(1-p)q)^{-1}$ for all values of p and q (and thereby uniqueness of random cluster measures) whereas it is proved that the wired infinity approach gives a much richer behavior. In particular it is shown that for $q > 2$ there is a phase transition for an entire interval of p-values. Since we intend to characterize nonamenability in terms of a phase transition for the random cluster model we are therefore forced to *stick to the wired infinity model in this paper.* Therefore *we will henceforth use the convention that a random cluster measure is understood to be a wired infinity random cluster measure.*

A second problem with nonuniqueness of infinite clusters is that the continuity of $\mu_{S,\xi}(\eta)$ in ξ fails. (This happens also if we use the free infinity model.) In our wired infinity world $\mu_{S,\xi}(\eta)$ is upper semicontinuous and it can be shown that when we repeat the constructions of μ_{00} and μ_1 above for a general graph, then μ_1 is a random cluster measure. This might be false, however, for μ_{00}. (For the free infinity model $\phi_{S,\xi}(\eta)$ is lower semicontinuous and ϕ_0 is a free infinity random cluster measure but ϕ_1 might fail to be. Thus, in order to be consistent we should perhaps have used the notation ϕ_{11} instead of ϕ_1.)

We will now introduce a method of explicitly constructing wired infinity random cluster measures in the sense of Definition 2.2 that correspond to μ_0 and μ_1 in the amenable case, i.e. two random cluster measures obtained as weak limits of certain measures with free and wired boundary conditions respectively such that their definitions do not depend on the particular sequence $\{S_n\}$. The idea is to introduce an imaginary extra vertex, v_0 incident to all vertices of V. We will call v_0 a "ghost vertex". This term was introduced by Aizenman and Barsky in their proof of exponential decay of the radius distribution for the cluster containing the origin in subcritical i.i.d. bond percolation on $\mathbf{Z}^d$. (See [1].) We thus consider the new graph $H = (\bar{V}, \bar{E})$ where $\bar{V} = V \cup \{v_0\}$ and $\bar{E} = E \cup E_0$ where $E_0 = \{(v, v_0); v \in V\}$. (For each $\bar{S} \subseteq \bar{E}$ we write $\bar{S} = S \cup S_0$ in the same way.) We define a new three-parameter class of measures on $\{0,1\}^{\bar{E}}$. The $\{0,1\}^{\bar{E}}$-valued random variable, X, below is understood to have distribution ν.

DEFINITION 2.3 *We say that a probability measure, ν, on $\{0,1\}^{\bar{E}}$ is a (random cluster) ghost-measure with parameters $r \in [0,1]$, $p \in [0,1]$ and $q > 0$ if, for all finite $\bar{S} \subseteq \bar{E}$, all $\eta \in \{0,1\}^{\bar{S}}$ and ν-a.e. $\xi \in \{0,1\}^{\bar{E} \setminus \bar{S}}$*

$$P(X(\bar{S}) = \eta | X(\bar{E} \setminus \bar{S}) = \xi) = \nu^r_{\bar{S},\xi}(\eta) \tag{6}$$

where

$$\nu^r_{\bar{S},\xi}(\eta) = \frac{1}{Z^{r,p,q}_{\bar{S},\xi}}(\prod_{e \in S_0} r^{\eta(e)}(1-r)^{1-\eta(e)})(\prod_{e \in S} p^{\eta(e)}(1-p)^{1-\eta_e})q^{k(\eta,\xi)} \tag{7}$$

where $k(\eta, \xi)$ is the number of finite connected components that intersect $V(\bar{S})$.

In words, a ghost-measure is nothing but a random cluster measure on the edges of H but with different "p-values" for different edges depending on whether they are in E or in E_0.

Let us now mimik the standard construction of random cluster measures on quasi-transitive amenable graphs above. Let $S_n \uparrow E$ and set $\bar{S_n} = S_n \cup S_{n,0}$, where $S_{n,0}$ is the set of edges going from v_0 to one of the vertices of $V(S_n)$. Fix $r > 0$ (this is essential) and define the measures $\nu^r_{0,n}$, $n = 1, 2, \ldots$, according to (6) with $\bar{S} = \bar{S_n}$ and $\xi \equiv 0$ and define $\nu^r_{1,n}$ analogously with $\xi \equiv 1$. It is clear that $\nu^r_{i,n}$, $i = 0, 1$, have conditional probabilities according to (6) for $\bar{S} \subseteq \bar{S_n}$. By standard monotonicity arguments we have for $q \geq 1$ that $\nu^r_{0,1} \leq_d \nu^r_{0,2} \leq_d \ldots$ and $\nu^r_{1,1} \geq_d \nu^r_{1,2} \geq_d \ldots$ so that the weak limits ν^r_i exist and are independent of the sequence $\{\bar{S_n}\}$.

LEMMA 2.4 *Let $q \geq 1$ and $r > 0$. The projections of ν^r_i onto $\{0,1\}^{E_0}$, $i = 0, 1$, stochastically dominates product measure with density $r(r + (1-r)q)^{-1}$ and are stochastically dominated by product measure with density r. If $q < 1$ the situation the reverse result holds.*

Proof. It follows from standard arguments that the result holds for the projection of $\nu^r_{i,n}$ onto $\{0,1\}^{S_0}$ for any finite S_0 such that $S_0 \subseteq S_n$, so the result follows from weak convergence. □

LEMMA 2.5 *Let $r > 0$. If X^r is a random variable with distribution ν^r_i, $i = 0, 1$, then X^r a.s. has a unique infinite cluster.*

Proof. By Lemma 2.4 any infinite cluster is a.s. connected to v_0 and thus a.s. connected to any other infinite cluster. □

Now the next lemma follows on copying the proof of Lemma 3.4 of [8].

LEMMA 2.6 *Let $r > 0$. For any finite $\bar{S} \subseteq \bar{E}$ and any $\eta \in \{0,1\}^{\bar{S}}$, $\nu^r_{\bar{S},\xi}(\eta)$ is continuous at ν^r_i-a.e. $\xi \in \{0,1\}^{\bar{E}\setminus\bar{S}}$.*

PROPOSITION 2.7 *Let $q \geq 1$ and $r > 0$. The measures ν^r_0 and ν^r_1 are random cluster ghost-measures with parameters r, p and q.*

Proof. Let X be distributed according to ν^r_0 and let $X_1, X_2, \ldots$ be distributed according to $\nu^r_{0,1}, \nu^r_{0,2}, \ldots$ respectively, all defined on the same probability space with the underlying probability measure P. For a cylinder set $B \in \mathcal{B}(\{0,1\}^{\bar{E}\setminus\bar{S}})$, we have

$$P(\{X(\bar{S}) = \eta\} \cap \{X(\bar{E} \setminus \bar{S}) \in B\}) = \int_B P(X(\bar{S}) = \eta | X(\bar{E} \setminus \bar{S}) = \xi)\nu^r_0(d\xi)$$

$$= \lim_{n\to\infty} P(\{X_n(\bar{S}) = \eta\} \cap \{X_n(\bar{E} \setminus \bar{S}) \in B\})$$

$$= \lim_{n\to\infty} \int_B P(X_n(\bar{S}) = \eta | X_n(\bar{E} \setminus \bar{S}) = \xi)\nu^r_{0,n}(d\xi)$$

$$= \lim_{n\to\infty} \int_B \nu^r_{\bar{S},\xi}(\eta)\nu^r_{0,n}(d\xi) = \int_B \nu^r_{\bar{S},\xi}(\eta)\nu^r_0(d\xi)$$

where the last inequality follows from Lemma 2.6 and the definition of weak convergence. Since the class of cylinder sets is closed under finite intersections and generate $\mathcal{B}(\{0,1\}^{\bar{E}\setminus\bar{S}})$, this proves that $P(X(\bar{S}) = \eta | X(\bar{E} \setminus \bar{S}) = \xi) = \nu^r_{\bar{S},\xi}(\eta)$ for ν^r_0-a.e. ξ as desired. The proof for ν^r_1 is analogous. □

Remark. For $q < 1$ the monotonicity of the sequences $\{\nu^r_{i,n}\}$ fails so it is not clear that the limits exist. However, the compactness of the family of probability measures on $\{0,1\}^{\bar{E}}$ implies the existence of subsequential weak limits. For these limits the proof of Proposition 2.7 goes through unchanged.

In the next step, where we assume throughout that $q \geq 1$, we let $r \downarrow 0$ and obtain the weak limits $\nu_i = \lim_{r\downarrow 0} \nu^r_i$, $i = 0, 1$. The existence of these limits follows from the fact that ν^r_i is stochastically decreasing in r, a fact which in turn follows from a standard application of Holley's Theorem. We claim that the projections onto $\{0,1\}^E$ of these two measures are random cluster measures. For the proof of that, the following lemma is convenient. (This is just Lemma 2.4 of [11] where this is stated for $G = \mathbf{T}_n$ but the proof is valid on any graph.)

LEMMA 2.8 *Let μ be a probability measure on $\{0,1\}^E$ and let X be a $\{0,1\}^E$-valued random variable with distribution μ. If, for all $e \in E$ and μ-a.e. $\xi \in \{0,1\}^{E\setminus e}$,*

$$P(X(e) = 1 | X(E \setminus e) = \xi) = pI_{C_e}(\xi) + p(p + (1-p)q)^{-1} I_{C_e^c}(\xi)$$

where C_e is the set of configurations, ξ', in $\{0,1\}^E$ where the end vertices of e are either connected in ξ' or in two different infinite connected components of ξ', then μ is a random cluster measure with parameters p and q.

THEOREM 2.9 *Let, for $i = 0,1$, μ_i be the projection onto $\{0,1\}^E$ of ν_i. Then μ_0 and μ_1 are random cluster measures with parameters p and q.*

Proof. As in Proposition 2.7 we do the proof for μ_0. The proof for μ_1 is analogous.

Let $r_n \downarrow 0$ and let $X_1, X_2, \ldots$ be distributed according to $\nu_0^{r_1}, \nu_0^{r_2}, \ldots$ respectively and let X be distributed according to ν_0. Fix an edge $e = (u,v) \in E$. Let C_e be as in Lemma 2.8, let $\bar{C}_e$ be the set of configurations, $\xi' \in \{0,1\}^{\bar{E}\setminus e}$, such that u and v are either connected in ξ' or in two different infinite connected components of ξ' and set, for $n = 1,2,\ldots$, $\bar{C}_e^{(n)}$ to be the set of configurations in $\{0,1\}^{\bar{E}\setminus e}$ such that u and v are either connected by a path of open edges in $\bar{S}_n$ or both connected to $\bar{E} \setminus \bar{S}_n$. Note that $\bar{C}_e^{(n)} \downarrow \bar{C}_e$ and that since $X(E_0) \equiv 0$ a.s. we have for any set $B \in \{0,1\}^{E\setminus e}$ that $P(X(\bar{E} \setminus e) \in \bar{C}_e \cap B \times \{0,1\}^{E_0}) = P(X(E \setminus e) \in C_e \cap B)$.

Now fix any cylinder set $B \in \mathcal{B}(\{0,1\}^{E\setminus e})$ and set $p' = p(p + (1-p)q)^{-1}$. Then

$$\int_B P(X(e) = 1 | X(E \setminus e) = \xi)\nu_0(d\xi) = P(\{X(e) = 1\} \cap \{X(E \setminus e) \in B\})$$

$$= \lim_{n\to\infty} P(\{X_n(e) = 1\} \cap \{X_n(E \setminus e) \in B\})$$

$$= \lim_{n\to\infty} \int_{B\times\{0,1\}^{E_0}} P(X_n(e) = 1 | X_n(E \setminus e) = \xi, X_n(E_0) = \xi')\nu_0^{r_n}(d(\xi,\xi'))$$

$$= \lim_{n\to\infty} (pP(X_n(\bar{E} \setminus e) \in \bar{C}_e \cap B \times \{0,1\}^{E_0})$$

$$+ p'P(X_n(\bar{E} \setminus e) \in \bar{C}_e^c \cap B \times \{0,1\}^{E_0}))$$

by weak convergence and Proposition 2.7. Since standard monotonicity arguments imply that $P(X_n(\bar{E} \setminus e) \in \bar{C}_e \cap B \times \{0,1\}^{E_0})$ is decreasing in n and we also have that $\bar{C}_e^{(m)}$ decreases in m we have

$$\lim_{n\to\infty} P(X_n(\bar{E} \setminus e) \in \bar{C}_e \cap B \times \{0,1\}^{E_0})$$

$$= \lim_{n\to\infty}\lim_{m\to\infty} P(X_n(\bar{E}\setminus e) \in \bar{C}_e^{(m)} \cap B \times \{0,1\}^{E_0})$$

$$= \lim_{m\to\infty}\lim_{n\to\infty} P(X_n(\bar{E}\setminus e) \in \bar{C}_e^{(m)} \cap B \times \{0,1\}^{E_0})$$

$$= \lim_{m\to\infty} P(X(\bar{E}\setminus e) \in \bar{C}_e^{(m)} \cap B \times \{0,1\}^{E_0}) = P(X(E\setminus e) \in C_e \cap B)$$

by weak convergence, as $\bar{C}_e^{(m)}$ is a cylinder set, and the above. Thus

$$\int_B P(X(e) = 1 | X(E\setminus e) = \xi)\nu_0(d\xi)$$

$$= pP(X(E\setminus e) \in C_e \cap B) + p'P(X(E\setminus e) \in C_e^c \cap B)$$

$$= \int_B (pI_{C_e}(\xi) + p'I_{C_e^c}(\xi))\nu_0(d\xi)$$

i.e. $P(X(e) = 1|X(E\setminus e) = \xi) = pI_{C_e}(\xi) + p'I_{C_e^c}(\xi)$ ν_0-a.e. and the result follows from Lemma 2.8. □

Remark. We do not need the ghost vertex for the construction of μ_1. Set $\nu_{1,n}^0$ to be $\nu_{1,n}^r$ with $r = 0$. Then $\nu_1 = \inf_r \inf_n \nu_{1,n}^r = \inf_n \inf_r \nu_{1,n}^r = \inf_n \nu_{1,n}^0$. Now let $\mu_{1,n}$ be the projection onto $\{0,1\}^E$ of $\nu_{1,n}^0$ and it follows that $\mu_1 = \inf_n \mu_{1,n}$. On the other hand, if we let $\mu_{0,n}$ be the projection of $\nu_{0,n}^0$, then $\mu_{00} := \lim_{n\to\infty}\mu_{0,n}$ does not always equal μ_0; we are not allowed to reverse the order of a supremum and an infimum.

3 Proofs of Theorem 1.2 and Theorem 1.3

3.1 Phase transition in the nonamenable case

The time has come to prove Theorem 1.2(a) which said that if G is nonamenable and regular, then for q large enough there is an entire interval of values of p for which there is more than one random cluster measure. We start with two lemmas. The first one is due to Kesten [17] and the second one is an immediate consequence of the definition of the Cheeger constant.

LEMMA 3.1 *Let $G = (V,E)$ be an infinite graph with maximum degree d and fix an arbitrary edge $e_0 \in E$. For $m = 1,2,\ldots$, let $\mathcal{C}_m(e_0)$ be the family of connected subsets of E of size m containing e_0. Then*

$$|\mathcal{C}_m(e_0)| \le (e(2d+1))^m.$$

LEMMA 3.2 *Let $G = (V, E)$ be a regular graph with degree d. For a finite set, $W \subseteq V$, let $E_-(W)$ be the set of edges with both end vertices in W and let $E_+(W) = E_-(W) \cup \partial_E W$ be the set of edges with at least one end vertex in W. Then*

$$\frac{|E_-(W)|}{|W|} \leq \frac{d-\kappa}{2}$$

and

$$\frac{|E_+(W)|}{|W|} \geq \frac{d+\kappa}{2}.$$

Proof of Theorem 1.2(a). We will eventually specify how large a q to choose but from now on we assume that $p/(1-p) \in [q^{2/(d+\kappa/2)}, q^{2/(d-\kappa/2)}]$.

Let X be a random variable with distribution μ_0, let X^r have distribution ν_0^r and X_n^r be distributed according to $\nu_{0,n}^r$ and let P be the underlying probability measure. We start by proving that for any $\rho > 0$ we may pick q so large that for any edge, e_0, we have $P(X(e_0) = 1) \leq \rho$. Pick the sequence $\{S_n\}$ such that $S_n \uparrow E$ and $e_0 \in S_1$ and pick $\gamma << \kappa \wedge 1$. Set $r = \gamma/2$ and fix n so large that

$$|P(X_n^r(e_0) = 1) - P(X^r(e_0) = 1)| \leq \rho/2.$$

Observe that

$$\{X_n^r(e_0) = 1\} = \bigcup_{m=1}^{\infty} \bigcup_{C \in \mathcal{C}_m(e_0)} \{X_n^r \in A_C\}$$

where A_C is the set of configurations where the connected component of open edges that e_0 is found in (disregarding possible edges to v_0) is exactly C. Let B_C be the set of configurations where no more than $\gamma|V(C)|$ of the edges connecting $V(C)$ to v_0 are open and let $D_C = A_C \cap B_C$. By Lemma 2.4 and Markov's inequality we have that $P(X_n^r \in A_C) \leq 2P(X_n^r \in D_C)$. Now, for $C \subseteq S_n$, let us compare the probabilities for $\{X_n^r \in D_C\}$ and $\{X_n^r(C \cup \partial_E V(C)) \equiv 0\} \cap \{X_n^r \in B_C\}$. For any outcome in the latter event compared with the corresponding outcome in the former (i.e. the outcome which agrees on the edges connecting $V(C)$ to v_0) we lose $(p/(1-p))^{|E_-(V(C))|} = (p/(1-p))^{|C|}$ from loosing open edges but we win at least $q^{(1-\gamma)|V(C)|}$ from winning more clusters. Therefore

$$P(X_n^r(e_0) = 1) \leq 2\sum_{m=1}^{\infty} \sum_{C \in \mathcal{C}_m(e_0)} \frac{P(X_n^r \in D_C)}{P(\{X_n^r(C \cup \partial_E V(C)\} \cap \{X_n^r \in B_C\}) \equiv 0)}$$

$$\leq 2\sum_{m=1}^{\infty} \sum_{C \in \mathcal{C}_m(e_0)} \frac{(p/(1-p))^{|C|}}{q^{(1-\gamma)|V(C)|}}$$

$$\leq 2\sum_{m=1}^{\infty}\sum_{C\in\mathcal{C}_m(e_0)}\frac{(p/(1-p))^m}{q^{2m(1-\gamma)/(d-\kappa)}}$$

$$\leq 2\sum_{m=1}^{\infty}\sum_{C\in\mathcal{C}_m(e_0)}(q^{2/(d-\kappa/2)-2(1-\gamma)/(d-\kappa)})^m$$

$$\leq 2\sum_{m=1}^{\infty}(e(2d+1)q^{2/(d-\kappa/2)-2(1-\gamma)/(d-\kappa)})^m$$

by Lemma 3.2 and Lemma 3.1 and since the exponent of q is less than 1 we can pick q large enough to make sure that this expression is less than $\rho/2$. Thus $P(X(e_0)=1)\leq P(X^r(e_0)=1)\leq\rho$ as desired.

Note that as an immediate consequence of the above result, $\mu_0(e_0\leftrightarrow\infty)\leq 2d\rho$. Here $\{e_0\leftrightarrow\infty\}$ is the event that at least one of the end vertices of e_0 is connected to infinity by a path of open edges. Now in order to prove phase transition we prove that $\mu_1(e_0\leftrightarrow\infty)>2d\rho$ for small enough ρ. We do so by proving that $\mu_{1,n}(e_0\leftrightarrow\partial V(S_n)\geq 1/2)$, say, for all n. (Recall the remark after Theorem 2.9.)

Let Y_n be distributed according to $\mu_{1,n}$ and let $\mathcal{W}$ denote the random subset of vertices that are not connected to $\partial V(S_n)$ in the Y_n-configuration. We want to prove that $P(e_0\in E_-(\mathcal{W}))<1/2$. Given $\mathcal{W}=W$, $Y_n(E_-(W))$ is distributed according to $\mu_{0,E_-(W)}$, i.e. the measure on $\{0,1\}^{E_-(W)}$ with free boundary condition, see the remark after Theorem 2.9. (Note that the event $\{\mathcal{W}=W\}$ is measurable with respect to the σ-algebra generated by $\{Y_n(e):e\in E\setminus E_-(W)\}$.) This measure is dominated by the projection of $\nu^r_{0,E_-(W)}$ for any $r>0$. Therefore, if $E_-(W)$ contains e_0, the conditional probability that $Y_n(e_0)=1$ is at most ρ by the above result. Thus, by Markov's inequality, $P(\mathcal{W}=W)\leq 2P(\{\mathcal{W}=W\}\cap B)$ where B is the event that at most $2\rho|E_-(W)|$ of the edges of $E_-(W)$ are open.

Let us now compare the outcomes, η, of $\{\mathcal{W}=W\}\cap B$ with the outcomes $\bar{\eta}$ where we flip the values of all the edges of $E_+(W)$, i.e. $\bar{\eta}(e)=\eta(e)$ for $e\in S_n\setminus E_+(W)$ and $\bar{\eta}(e)=1-\eta(e)$ for $e\in E_+(W)$. Changing η to $\bar{\eta}$ gives us at least $(1-4\rho)|E_+(W)|$ more open edges and we lose no more than $|W|$ clusters. (Remember that all vertices outside S_n are regarded as connected to each other.) Thus

$$\frac{P(Y_n=\eta)}{P(Y_n=\bar{\eta})}\leq\frac{q^{|W|}}{(p/(1-p))^{(1-4\rho)(d+\kappa)|W|/2}}$$

$$\leq\frac{q^{|W|}}{q^{(1-4\rho)(d+\kappa)|W|/(d+\kappa/2)}}\leq(q^{1-(1-4\rho)(d+\kappa)/(d+\kappa/2)})^{|W|}:=a^{|W|}$$

by Lemma 3.2. Due to the algebraic fact that $\alpha_i/\beta_i \leq c, i = 1, \ldots, k$ implies that $(\sum_{i=1}^k \alpha_i)/(\sum_{i=1}^k \beta_i) \leq c$ we get that $P(\mathcal{W} = W) \leq 2a^{|W|}$ for all W such that $e_0 \in E_-(W)$. Since the exponent of q above is less than 1 for ρ small enough we can by picking q large make a arbitrarily small. By using Lemma 3.1 and rewriting the event $\{e_0 \in E_-(\mathcal{W})\}$ in the same spirit as above we get

$$P(e_0 \in E_-(\mathcal{W})) \leq \sum_{m=1}^{\infty} (e(2d+1)a)^m \leq 1/2$$

for sufficiently small a. The proof is complete. □

Remark. The above proof can be generalized slightly. If all but at most finitely many of the vertices have degree d, then Lemma 3.2 essentially holds for all but finitely many sets W. Therefore the first part can be carried out for all but at most finitely many edges e so it is clear that this will not cause anything more than technical problems. Thus Theorem 1.2(a) holds for e.g. an n-ary tree with a proper root or for the kind of hyperbolic graphs considered in [23].

3.2 No phase transition in the amenable case

We will generalize the the proofs of [14, Section 3.2] and [8, Section 4] which are in turn extensions of methods originally introduced in [18] and [21]. Since the present proof does not contain anything new, the presentation will be kept compact.

Let G be any amenable graph with maximum degree d. Let $\{S_n\}$ be a sequence of subsets of E such that $S_n \uparrow E$ and $|\partial_E V(S_n)|/|S_n| \to 0$. Consider the measures $\mu_1 = \lim_{n\to\infty} \mu_{1,n}$ and $\mu_{00} = \lim_{n\to\infty} \mu_{0,n}$ where $\mu_{1,n}$ and $\mu_{0,n}$ are defined as in the remark after Theorem 2.9. As noted there $\mu_{1,n}$ is a random cluster measure and if uniqueness of the (possible) infinite cluster is in force, then so is μ_{00}. In any case $\mu_{00} \leq_d \mu \leq_d \mu_1$ for all random cluster measures μ.

Recall the normalizing constant $Z^{p,q}_{S_n,\xi}$ of (3), i.e.

$$Z^{p,q}_{S_n,\xi} = \sum_{\eta\in\{0,1\}^{S_n}} \Big(\prod_{e\in S_n} p^{\eta(e)}(1-p)^{1-\eta(e)}\Big) q^{k(\eta,\xi)}$$

where $k(\eta, \xi)$ is the number of finite connected components that intersect $V(S_n)$. Let $a(\eta)$ denote the set of open edges of η and set

$$Y^{p,q}_{S_n,\xi} = (1-p)^{-|S_n|} Z^{p,q}_{S_n,\xi} = \sum_{\eta\in\{0,1\}^{S_n}} q^{k(\eta,\xi)} e^{\pi|a(\eta)|} \tag{8}$$

where $\pi = \log(p/(1-p))$. Let

$$f^{\xi}(n,\pi,q) = |S_n|^{-1} \log Y^{p,q}_{S_n,\xi}. \tag{9}$$

Now fix q and fix a π_0 and consider the sequence $\{f^0(n,\pi,q)\}$. By inspection of (9) there is a $K = K(\pi_0,q)$ such that $f^0(n,\pi,q) \in [0,K]$ for every n and $\pi \in [0,\pi_0]$. By compactness there exists a sequence $\{n_i\}$ such that $\lim_{i\to\infty} f^0(n_i,\pi,q)$ exists for all rational $\pi \in [0,\pi_0]$.

Now for fixed n and π we have

$$\frac{\partial}{\partial\pi} f^0(n,\pi,q) = |S_n|^{-1}\mu_{0,n}(|a(\eta)\cap S_n|) \tag{10}$$

and

$$\frac{\partial^2}{\partial\pi^2} f^0(n,\pi,q) = |S_n|^{-1}(\mu_{0,n}(|a(\eta)\cap S_n|^2) - (\mu_{0,n}(|a(\eta)\cap S_n|))^2) \geq 0.$$

Thus $f^0(n,\pi,q)$ is convex in π for each n and it follows from e.g. [5,Theorem V1.3.3(a)] that $\lim_{i\to\infty} f^0(n_i,\pi,q)$ exists for all $\pi \in [0,\pi_0]$ and is convex in π. Denote this limit $f(\pi,q)$. This limit function may depend on $\{S_n\}$ and $\{n_i\}$ and also on the fact that we have been working with free boundary conditions. However if $\{\xi_n\}$ is an arbitrary sequence of boundary conditions, then $|k(\eta,\xi_n) - k(\eta,0)| \leq |\partial V(S_n)|$ and so $|f^{\xi_n}(n,\pi,q) - f^0(n,\pi,q)| \leq (\log q)|\partial V(S_n)|/|S_n| \to 0$. Hence $\lim_{i\to\infty} f^{\xi_{n_i}}(n_i,\pi,q)$ equals $f(\pi,q)$ for any sequence of boundary conditions.

Being convex implies that $f(\pi,q)$ is differentiable for all but at most countably many values of π. Now fix such a π. By Lemma IV.6.3 in [5] and the above we have that

$$\frac{\partial}{\partial\pi} f^{\xi_{n_i}}(n_i,\pi,q) \to \frac{\partial}{\partial\pi} f(\pi,q)$$

for any boundary conditions. Applying this to (10) and the analogous equation for wired boundary condition and taking the difference yields

$$\lim_{i\to\infty} |S_{n_i}|^{-1}(\mu_{1,n_i}(|a(\eta)\cap S_{n_i}|) - \mu_{0,n_i}(|a(\eta)\cap S_{n_i}|)) = 0$$

for all but countably many values of p such that $\pi \in [0,\pi_0]$. However since π_0 was arbitrary we have established the following result.

Proposition 3.3 *Let $G = (V, E)$ be an amenable graph with uniformly bounded degree and fix $q \geq 1$. Then for any sequence $\{S_n\}$ of finite subsets of E such that $S_n \uparrow E$ and $|\partial_E V(S_n)|/|S_n| \to 0$, there is a subsequence, $\{n_i\}$, such that*

$$\lim_{i\to\infty} |S_{n_i}|^{-1}(\mu_{1,n_i}(|a(\eta) \cap S_{n_i}|) - \mu_{0,n_i}(|a(\eta) \cap S_{n_i}|)) = 0$$

for all but at most countably many values of p.

Proof of Theorem 1.2(b). Let $\{S_n\}$ be as in Proposition 3.3 and fix p and $q \geq 1$. Write $V = V_1 \cup \ldots \cup V_k$ where the V_j's are the orbits of the automorphism group of G acting on V. By [2, Proposition 3.6] there are strictly positive constants, $\alpha_1, \ldots, \alpha_k$ such that $|V(S_n) \cap V_j|/|S_n| \to \alpha_j$ for $j = 1, \ldots, k$. Now write $E = E_1 \cup \ldots \cup E_l$ where the E_m's are the orbits of the automorphism group of G acting on E. (We have $l \leq dk$.) Since $|\partial_E V(S_n)|/|S_n| \to 0$ we have that $|S_n \cap E_m|/|S_n| \to \beta_m$ for $m = 1, \ldots, l$ where $\beta_m \geq \beta := \min_j \alpha_j/d$ for all m.

Now if $\mu_{00} \neq \mu_1$, then there is an m such that $\mu_1(\eta(e) = 1) - \mu_{00}(\eta(e) = 1) := \epsilon > 0$ for all $e \in E_m$. (The measures μ_{00} and μ_1 are automorphism invariant.) However if this is the case then by stochastic monotonicity

$$|S_n|^{-1}\mu_{1,n}(|a(\eta) \cap S_n|) \geq |S_n|^{-1}\mu_1(|a(\eta) \cap S_n|)$$

$$\geq |S_n|^{-1}\mu_{00}(|a(\eta) \cap S_n|) + \beta\epsilon \geq |S_n|^{-1}\mu_{0,n}(|a(\eta) \cap S_n|) + \beta\epsilon$$

which contradicts Proposition 3.3 for all but at most countably many values of p. Theorem 1.2(b) follows (and since G is amenable and quasi-transitive, the Burton-Keane uniqueness Theorem is valid and what we have proved is in fact equivalent to Theorem 1.2(b).) □

3.3 Proof of Theorem 1.3

We do the proof for $G = \mathbf{T}_2$; it extends in a straightforward way to $\mathbf{T}_n$, $n \geq 3$.

For arbitrary p and $q \geq 1$, fix any random cluster measure, μ with those parameters. We claim that if $\mu(\eta(e) = 1) = \mu_1(\eta(e) = 1)$ for every $e \in E$, then $\mu = \mu_1$. This is the case since by Strassen's Theorem (see e.g. [20]) we can define random variables X and X_1 on a common probability space with underlying probability measure P in such a way that X has distribution μ, X_1 has distribution μ_1 and $X \leq X_1$ a.s.

However by assumption $P(X(e) \neq X_1(e)) = P(X_1(e) = 1) - P(X(e) = 1) = 0$ for every e so by countable additivity $X = X_1$ a.s.

Now fix $q \geq 1$ and an edge $e = (u, v)$ and let T_u and T_v be the left and right subtrees, i.e. the trees descending from u and v respectively. In order to prove that $\mu(\eta(e) = 1) \geq \mu_1(\eta(e) = 1)$ for large enough p, we shall prove that for large p, e will with probability 1 be completely surrounded by some finite set W of vertices of which all are connected to infinity via open edges. (Formally we define the statement "e is completely surrounded by W" as meaning that every path from e to infinity intersects W. Note that our claim is stronger than just saying that e is completely surrounded by open edges which is the case as soon as the closed edges do not percolate.) Let us call such a set a *wiring set* if in addition no vertex of W is a descendant of any other vertex in W. Here we say that w' is a descendant of w and that w is an ascendant of w' if every path from u to w' goes through w. If in addition w is adjacent to w' then we say that w is the mother of w' and that w' is a daughter of w'. Then, knowing that a wiring set exists, the conditional distribution of X inside it will stochastically dominate the projection of μ_1.

Let us first extend X to a random variable on $\{0,1\}^{V \cup E}$ by declaring a vertex w to be open if there is a path of open edges from w to infinity through the set of descendants of w. If w is not declared open then we call it closed. Second, let Y be another $\{0,1\}^{V \cup E}$-valued random variable (defined on the same probability space) by first letting $Y(E)$ be an i.i.d. percolation with edge density $p' = p(p + (1-p)q)^{-1}$ and then declaring vertices to be open or closed in the same way as we did for $X(V)$. Since $X(E) \geq_d Y(E)$ it follows that $X \geq_d Y$. Next, fix p such that $p' = (1 - 2^{-1/(2-1)})/(1 - 2^{2/(2-1)}) = 2/3$. Then basic branching process theory tells us that $P(Y(w) = 0)$ is given by the smallest solution of the equation

$$s = (2s/3 + 1/3)^2$$

i.e. $P(Y(w) = 0) = 1/4$. We now ask ourselves: Can there be a path from e to infinity using only vertices, w, with $Y(w) = 0$? If we can answer this question with a no, that would also imply that the same question for the $X(V)$-configuration is answered negatively and thereby establish the existence of a wiring set for the $X(E)$-configuration. The answer is not obvious, however, because of the strong dependence between the $Y(w)$'s. We shall make use of the fact that the distribution of the statuses of the descendants of a vertex w given the statuses of w and all other vertices just depends on $Y(w)$. For a fixed w, let w_a and w_b denote the two daughters of w. Then, for $x_a, x_b \in \{0,1\}$,

$$\begin{aligned} &P(Y(w_a) = x_a, Y(W_b) = x_b | Y(w) = 0) \\ &= \frac{P(Y(w_a) = x_a, Y(w_b) = x_b, Y(w) = 0)}{P(Y(w) = 0)} \\ &= \frac{((3/4)(1/3))^{x_a+x_b}(1/4)^{2-x_a-x_b}}{1/4} = (1/2)^{x_a+x_b}(1/2)^{2-x_a-x_b} \end{aligned}$$

i.e. the conditional distribution of $(Y(w_a), Y(w_b))$ given $Y(w) = 0$ is product measure with density $1/2$.

Now let us try to find a path from u to infinity through the left tree, T_u, through only Y-closed vertices using the follwing search algorithm. First order the vertices, $\{w_1, w_2, \ldots\}$ in such a way that $w_1 = u$, w_2 and w_3 are the daughters of u, $w_4, \ldots, w_7$ are the granddaughters of u, etc. Start the search by checking the value of $Y(w_1)$. Then, at each step, we check the next vertex in the ordering for which we have not already found an ascendant, w, with $Y(w) = 1$. If at some step there is no such vertex, then the search terminates and in this case we know that there is no path from u to infinity through T_u using only Y-closed vertices. If in addition an analogous search for the same kind of path through T_v also terminates, then we have established the existence of a wiring set. However from the above it follows that for each new vertex w we check, the conditional probability that $Y(w) = 0$ given what we have seen so far is at most $1/2$. This means that given the order $\{w_{k_1}, w_{k_2}, \ldots\}$ in wich we happen to check the vertices, we always have

$$(Y(w_{k_1}), Y(w_{k_2}), \ldots) \geq_d (Z(w_{k_1}), Z(w_{k_2}), \ldots)$$

where the Z_{w_k}'s are i.i.d. with $Pr(Z(w_k = 0) = 1 - Pr(Z(w_k) = 1) = 1/2$, $k = 1, 2, \ldots$. Since the $Z(w_k)$'s correspond to applying the same search algorithm to search for an infinite cluster of open vertices in an i.i.d. site percolation with density $1/2$, it follows from an application of Strassen's Theorem that the search algorithm will terminate a.s. for T_u as well as T_v. (The critical value for percolation on $\mathbf{T}_n$ is well known to be $1/n$ with no percolation at the critical value.) Since $X(E) \geq_d Y(E)$ another application of Strassen's Theorem entails that e will a.s. be surrounded by a wiring set in the $X(E)$-configuration as desired.

Now fix $\epsilon > 0$ and let $S_k \uparrow E$ and fix k so large that the probability that we can find a wiring set, W, that surrounds e, in the $X(E)$-configuration such that $W \subseteq S_k$ is at least $1 - \epsilon$. Let $\mathcal{W}$ be the random set of vertices defined as the outmost wiring set inside S_k, i.e. the unique wiring set in which all vertices, w, are either in $\partial V(S_k)$ or connected to $\partial V(S_k)$ by a path of closed vertices. The event $\{\mathcal{W} = W\}$ is measurable with respect to the σ-algebra generated by

$$\{X(w') : w' \text{ descendant of } w \text{ for some } w \in W\}.$$

Therefore, with $\bar{W} = W \cup \{w_0 : w_0 \text{ ascendant to } w \text{ for some } w \in W\}$, the distribution of $X(E_-(\bar{W}))$ given $\mathcal{W} = W$ is $\mu_{1,E_-(\bar{W})}$ which stochastically dominates $\mu_{1,k}$. Since $\mathcal{W}$ is degenerate with probability at most ϵ, we have

$$\mu(\eta(e) = 1) = P(X(e) = 1) \geq \sum_{W \subseteq S_k} P(X(e) = 1 | \mathcal{W} = W) P(\mathcal{W} = W)$$

$$\geq (1 - \epsilon)\mu_{1,k}(\eta(e) = 1) \geq (1 - \epsilon)\mu_1(\eta(e) = 1)$$

proving that $\mu(\eta(e) = 1) = \mu_1(\eta(e) = 1)$ and since e was arbitrary it follows that $\mu = \mu_1$ for $p' = 2/3$. That the result also holds for all p such that $p' \geq 2/3$ now easily follows from monotonicity arguments. $\Box$

4 A consequence for the Potts model

We assume that the reader is familiar with the Potts model, but in order to introduce our notation, let us give a formal definition. The parameter β, the inverse temperature, is a positive real number and q is a positive integer.

Definition 4.1 *Let λ be a probability measure on $\{1, \ldots, q\}^V$, let Y be a random variable distributed according to λ and let P be the underlying probability measure. We say that λ is a Gibbs measure for the q-state Potts model with inverse temperature β if, for every finite $W \subseteq V$, every $\omega \in \{1, \ldots, q\}^W$ and λ-a.e. $\omega' \in \{1, \ldots, q\}^{V \setminus W}$,*

$$P(Y(W) = \omega | Y(V \setminus W) = \omega') = \frac{1}{Z^\beta_{W,\omega}} e^{-2\beta D(\omega)} \tag{11}$$

where $Z^\beta_{W,\omega}$ is a normalizing constant and $D(\omega) = \sum_{u,v \in W \cup \partial W : u \sim v} I_{\{\omega(u) \neq \omega(v)\}}$, the number of pairs of adjacent vertices with different spins.

Gibbs measures for the Potts model are constructed in the same way as random cluster measures. Let $W_n \uparrow V$ and let, for $k = 1, \dots, q$, $\lambda_{k,n}$ be the measure given by first setting $\omega(v) = k$ for all $v \in V \setminus W_n$ and then assigning spins to $v \in W_n$ according to (11) with $W = W_n$ and $\omega' \equiv k$. By monotonicity properties the limits $\lambda_k = \lim_{n\to\infty} \lambda_{k,n}$ exist and since the interactions of the Potts model are only local (which is not the case for the random cluster model) it is straightforward to verify that the λ_k's indeed satisfy Definition 4.1. This also goes for the *free* measure, λ_f, which is obtained as the limit $\lim_{n\to\infty} \lambda_{f,n}$ where $\lambda_{f,n}$ is just the Potts measure on the finite graph $(W_n, E_-(W_n))$, i.e. the vertices off W_n do not have any influence. (It is not obvious that the limit exists through the whole sequence, $\{W_n\}$, but this existence is a consequence of Lemma 4.2(b) below and the monotonicity of $\{\mu_{0,n}\}$.)

There is a close correspondence between the Potts model and the random cluster model. It is captured by the following well known lemma, which was first proved by Swendsen and Wang [22]. Here $p = 1 - e^{-2\beta}$ and $\mu_{0,n}$ and $\mu_{1,n}$ are the projections onto $\{0,1\}^E$ of $\nu^0_{0,n}$ and $\nu^0_{1,n}$ as in Section 2 with $S_n = E_-(W_n)$. Recall that $\lim_{n\to\infty} \mu_{1,n} = \mu_1$ but that about $\lim_{n\to\infty} \mu_{0,n}$, which we denote μ_{00}, we only know that it is dominated by μ_0. (It might, as for $G = \mathbf{T}_n$ with large p, be strictly stochastically smaller.)

LEMMA 4.2 *(a) Construct a $\{0,1\}^{S_n}$-valued random variable, X, by*

- *(i) picking Y according to $\lambda_{j,n}$,*
- *(ii) letting $X(e) = 1$ with probability p if the end vertices of e have the same spin in Y, and with probability 0 otherwise, independently for different edges.*

If $j \in \{1, \dots, q\}$, then X is distributed according to (the projection of) $\mu_{1,n}$ and if $j = f$, then X is distributed according to (the projection of) $\mu_{0,n}$.

(b) Construct a $\{1, \dots, q\}^{W_n}$-valued random variable, Y, by

- *(i) picking X according to $\mu_{i,n}$,*
- *(ii) letting all vertices that are part of the same connected component of X get the same spin, uniformly chosen from $\{1, \dots, q\}$, independently for different connected components. (Remember here that infinite clusters are regarded as connected at infinity.)*

If $i = 0$, then Y is distributed according to (the projection of) $\lambda_{f,n}$ and if $i = 1$, then Y is distributed according to (the projection of) $\sum_{k=1}^{q} \lambda_k/q$.

One well known consequence of Lemma 4.2 is that the Potts model exhibits a phase transition if and only if μ_1 percolates. Another consequence which is more interesting from our point of view is the following:

LEMMA 4.3 *The measures λ_f and $\sum_{k=1}^{q} \lambda_k/q$ are equal if and only if $\mu_{00} = \mu_1$.*

Proof. The case $q = 1$ is trivial so assume throughout the proof that $q \geq 2$. Assume first that $\mu_{00} \neq \mu_1$. Then there is an edge, $e = (u, v)$, such that $\mu_1(\eta(e) = 1) - \mu_{00}(\eta(e) = 1) = c > 0$. Since $\mu_{0,n} \leq_d \mu_{00}$ and $\mu_{1,n} \geq_d \mu_1$ for all n we have that $\mu_{1,n}(\eta(e) = 1) - \mu_{0,n}(\eta(e) = 1) \geq c$ for all n. However since $\mu_{i,n}(\eta(e) = 1) = p\mu_{i,n}(C_e) + p(p + (1-p)q)^{-1}\mu_{i,n}(C_e^c)$ for $i = 0, 1$ it follows that $\mu_{1,n}(C_e) - \mu_{0,n}(C_e) \geq c' > 0$ for all n for some c' and so for some $c'' > 0$ $\mu_{1,n}(u \leftrightarrow v) - \mu_{0,n}(u \leftrightarrow v) \geq c''$ for all n. Here the set $\{u \leftrightarrow v\}$ is the set of configurations for which u is connected to v by an open path or where u and v are in different infinite clusters and C_e is as in Lemma 2.8. By Lemma 4.2(b) $\lambda_{k,n}(\omega(u) = \omega(v)) - \lambda_{f,n}(\omega(u) = \omega(v)) \geq c''' > 0$ for all n so by weak convergence $\lambda_k(\omega(u) = \omega(v)) - \lambda_f(\omega(u) = \omega(v)) \geq c'''$. Thus $\lambda_f \neq \sum_{k=1}^{q} \lambda_k/q$.

On the other hand, if $\mu_{00} = \mu_1$, then the probability for having more than one infinite cluster must be 0. The reason for this is that if this is not the case then for any edge e there is a positive probability that if e is open then it connects two otherwise different infinite clusters. However, by conditioning on the configuration off e and using the definitions of μ_{00} and μ_1 this would imply that $\mu_{00}(\eta(e) = 1) < \mu_1(\eta(e) = 1)$, a contradiction. The uniqueness of a possible infinite cluster implies that $\mu_{00}(u \leftrightarrow v) = \lim_{n\to\infty} \mu_{0,n}(u \leftrightarrow v)$ and $\mu_1(u \leftrightarrow v) = \lim_{n\to\infty} \mu_{1,n}(u \leftrightarrow v)$. (The uniqueness of the possible infinite cluster is crucial for the first one of these equalities.) Therefore

$$(\sum_{k=1}^{q} \lambda_k/q)(\omega(u) = \omega(v)) = \lim_{n\to\infty} \lambda_{1,n}(\omega(u) = \omega(v))$$

$$= 1/q + ((q-1)/q) \lim_{n\to\infty} \mu_{1,n}(u \leftrightarrow v) = 1/q + ((q-1)/q) \lim_{n\to\infty} \mu_{0,n}(u \leftrightarrow v)$$

$$= \lambda_f(\omega(u) = \omega(v))$$

for all u and v. Since all one dimensional events trivially have the same $\sum_{k=1}^{q} \lambda_k/q$- and λ_f-probabilities, this implies that the same goes for all two dimensional events. In the same manner it can be shown that all finite dimensional events have the same $\sum_{k=1}^{q} \lambda_k/q$- and λ_f-probabilities, i.e. the two measures are equal. □

Remark. The proof of the above lemma gives a simple argument that if G, p and q are such that there is a positive μ_1-probability for having more than one infinite cluster, then $\mu_{00} \neq \mu_1$. However this does not imply a phase transition for the random cluster model in our sense for such cases as the argument fails to prove that $\mu_0 \neq \mu_1$. On the contrary Theorem 1.3 gives examples of such situations where there is no phase transition.

Since $\mu_{00} \leq_d \mu_0$ an immediate consequence of Lemma 4.3 and Theorem 1.2 is the promised result:

THEOREM 4.4 *Let $G = (V, E)$ be an infinite graph.*

(a) If G is nonamenable and regular with degree d, then there is a q_0 such that for $q \geq q_0$ and $e^{2\beta} - 1 \in [q^{2/(d+\kappa/2)}, q^{2/(d-\kappa/2)}]$, we have $\lambda_f \neq \sum_{k=1}^{q} \lambda_k/q$.

(b) If G is amenable and quasi-transitive, then $\lambda_f = \sum_{k=1}^{q} \lambda_k$ for all but at most contably many values of β.

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