NUMERICAL SHADOWING NEAR THE GLOBAL ATTRACTOR FOR A SEMILINEAR PARABOLIC EQUATION

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ABSTRACT. We use the shadowing approach to study the long-time behavior of numerical approximations of semilinear parabolic equations. We show that the corresponding nonlinear semigroup has a Lipschitz shadowing property in a neighborhood of its global attractor. The proof is based on reduction to an inertial manifold and application of shadowing techniques developed for finite-dimensional systems. When applied to a semilinear parabolic problem in one space variable, approximated by a standard finite element method in space and by backward Euler time-stepping, our result yields, for any computed trajectory near the attractor, an exact shadow trajectory with an optimal error bound uniformly in time.

1. INTRODUCTION

In this paper we use the shadowing approach to study the long-time behavior of numerical approximations (discretized both in space and time) of semilinear parabolic equations.

Let $S(t)$ be the nonlinear semigroup defined by the partial differential equation

$$(1)\qquad \begin{aligned} &u_t - u_{xx} = f(u), && 0 < x < 1,\ t > 0,\\ &u(0,t) = u(1,t) = 0, && t > 0,\\ &u(x,0) = u_0(x), && 0 < x < 1, \end{aligned}$$

so that $u(x,t) = \big(S(t)u_0\big)(x)$, and let $U_n(x) = \big(S_{hk}(t_n)u_0\big)(x)$ denote the numerical approximation obtained by a standard finite element method in the spatial variable and backward Euler time stepping. Our goal is to estimate the differences between exact solutions and numerical approximations uniformly over infinitely long time intervals. For this purpose we prove that the discrete semigroup $S(1)$ has a Lipschitz shadowing property in a neighborhood of its global attractor $\mathcal{A}$. The proof of this result is accomplished by first reducing the shadowing problem to its analogue on an inertial manifold containing $\mathcal{A}$, and then applying shadowing techniques developed for finite-dimensional structurally stable systems [19].

Roughly speaking, when applied to the semilinear parabolic problem (1), our shadowing result implies that, given a computed trajectory $U_n = S_{hk}(t_n)u_0$, we can find a time t_N and an initial value u_N such that $u(t_n) = S(t_n - t_N)u_N$ is close

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to $U_n = S_{hk}(t_n)u_0$ for $t_n \geq t_N$, with an error bound of optimal order uniformly in time.

In earlier work, see, e.g., Beyn [4], [5], Alouges and Debussche [1], [2], Larsson and Sanz-Serna [14], [15], similar results were obtained, but only in a neighborhood of a part of the attractor, such as a hyperbolic fixed point, periodic orbit, or connecting orbit.

In order to prove the Lipschitz shadowing property in a whole neighborhood of the attractor $\mathcal{A}$ we rely heavily on the assumption that S has Morse-Smale structure on $\mathcal{A}$. This is the main reason why we restrict our attention to the semilinear parabolic problem (1) in one space variable, for which this is known to be generically true; see below. Hale and Raugel [10] proved perturbation results for the whole attractor (also based on the Morse-Smale structure), but their error bounds are of lower order of convergence. However, although our present result gives optimal order error bounds for shadowing trajectories, it does not immediately lead to an optimal order bound for the distance between the attractors of S and S_{hk}. This problem will be addressed in future work.

2. The Shadowing Problem

We consider the initial-boundary value problem (1). We use the standard Hilbert space $L_2 = L_2(0,1)$ with inner product and norm

$$(v,w) = \int_0^1 v(x)w(x)\,dx, \quad \|v\| = \Big(\int_0^1 |v(x)|^2\,dx\Big)^{1/2}, \tag{2}$$

and the standard Sobolev spaces $H^l = H^l(0,1)$, $l = 1,2$, with norms

$$\|v\|_l = \Big(\sum_{j=0}^{l} \|v^{(j)}\|^2\Big)^{1/2}, \tag{3}$$

and $H_0^1 = \{v \in H^1 : v(0) = v(1) = 0\}$. Below we denote H_0^1 by $\mathcal{H}$, we write $|v|$ instead of $\|v\|_1$, and $\operatorname{dist}(Y,X)$ is the distance between $Y, X \subset \mathcal{H}$.

Defining the operator $A = -D_x^2$ with domain $\mathcal{D}(A) = \mathcal{H} \cap H^2$ we write (1) as an initial value problem in $\mathcal{H}$,

$$\dot{u} + Au = F(u),\ t > 0; \quad u(0) = u_0, \tag{4}$$

and denote its solution by $u(t) = S(t)u_0$. We assume that f belongs to C^2. Then the induced mapping $F : \mathcal{H} \to L_2$ satisfies a local Lipschitz condition: for any bounded set $B \subset \mathcal{H}$ there is a constant $C(B)$ such that

$$\|F(v) - F(w)\| \leq C(B)|v - w|, \quad \text{for } v, w \in B.$$

A standard argument based on writing (4) as fixed point equation,

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}F(u(s))\,ds,$$

now gives local solvability: for any bounded set $B_0 \subset \mathcal{H}$ there is $t^* = t^*(B_0) > 0$ and a bounded set $B \subset \mathcal{H}$ such that for all $u_0 \in B_0$ (4) has a unique solution $u(t) = S(t)u_0 \in B$ for $t \in [0, t^*]$.

We further assume that F is such that solutions of (4) satisfy a global a priori bound: for any bounded set $B_0 \subset \mathcal{H}$ there is a bounded set $B \subset \mathcal{H}$ such that

$$S(t)B_0 \subset B, \quad t \in [0, \infty). \tag{5}$$

For example, it is sufficient to assume that there is a constant C such that

$$sf(s) \leq C, \quad \text{for } s \in \mathbf{R}. \tag{6}$$

Then local solutions are extended globally in t and, hence, $S(t)$ is a semigroup of nonlinear operators on $\mathcal{H}$. It is also known that under our conditions $S(t)u_0$ depends smoothly on u_0 and that for a fixed $T > 0$ and for any bounded set $B \subset \mathcal{H}$ there exists $C(T, B)$ such that

$$|S(t)v - S(t)w| \leq C(T, B)|v - w|, \quad \text{for } v, w \in B,\ t \in [0, T]. \tag{7}$$

We next describe two Properties a and b that are crucial to our analysis.

Property a. $S(t)$ has a global attractor $\mathcal{A}$ in $\mathcal{H}$, i.e., there is a compact subset $\mathcal{A}$ of $\mathcal{H}$ with the following properties:

a1. $\mathcal{A}$ is *invariant*, i.e., $S(t)\mathcal{A} = \mathcal{A}$ for $t \in \mathbf{R}$;

a2. $\mathcal{A}$ is *uniformly globally attractive*, i.e., for any $\epsilon > 0$ and for any bounded set $B \subset \mathcal{H}$ there exists $T > 0$ such that

$$\operatorname{dist}(S(t)B, \mathcal{A}) < \epsilon \quad \text{for } t \geq T.$$

a3. $\mathcal{A}$ is *Lyapunov stable*, i.e., for any neighborhood W of $\mathcal{A}$ there is a neighborhood U of $\mathcal{A}$ such that $S(t)U \subset W$ for $t \in [0, \infty)$.

Note that **a3** is a consequence of **a2**.

The existence of a global attractor $\mathcal{A}$ implies the existence of a bounded absorbing set $\mathcal{H}_0$, such that for each bounded set $B_0 \subset \mathcal{H}$ there is T such that

$$S(t)B_0 \subset \mathcal{H}_0 \quad \text{for } t \in [T, \infty).$$

Condition (6) implies [12] that $S(t)$ has a global attractor.

The second Property b is that $S(t)$ has *Morse-Smale structure* on $\mathcal{A}$, as described below. First we give some definitions.

Note that a fixed point $p = p(x)$ of $S(t)$ is a solution of the boundary value problem:

$$p_{xx} + f(p) = 0, \quad p(0) = p(1) = 0. \tag{8}$$

As usual, we say that a fixed point p is a *hyperbolic fixed point* of $S(t)$ if the spectrum of the derivative $DS(t)(p)$ does not intersect the unit circle [11]. In the present situation one can reformulate this condition as follows. A fixed point $p = p(x)$ of $S(t)$ is hyperbolic if and only if 0 is not an eigenvalue of the linear variational operator

$$v \mapsto v_{xx} + f'(p(x))v$$

with the Dirichlet boundary conditions. In this case we say that $p(x)$ is a hyperbolic solution of (8).

If p is hyperbolic, then the stable manifold of p with respect to $S(t)$, defined by

$$W_S^s(p) = \{u \in \mathcal{H} : S(t)u \to p \text{ as } t \to \infty\},$$

and the unstable manifold of p, defined by

$$W_S^u(p) = \{u \in \mathcal{H} : S(t)u \text{ exists for } t \leq 0 \text{ and } S(t)u \to p \text{ as } t \to -\infty\},$$

are smooth immersed submanifolds of $\mathcal{H}$ [11]. If p, q are hyperbolic fixed points of $S(t)$, then we say that $W_S^u(p)$ and $W_S^s(q)$ are transversal if for any $v \in W_S^u(p) \cap W_S^s(q)$ the direct sum of the tangent spaces $T_v W_S^u(p)$ and $T_v W_S^s(q)$ equals $\mathcal{H}$ [12].

Property b. $S(t)$ has *Morse-Smale structure* on $\mathcal{A}$, i.e.,

b1. $S(t)$ has a finite number of fixed points $\pi_1, \ldots, \pi_N$, and these points are hyperbolic;

b2. every trajectory of $S(t)$ tends to a fixed point, i.e.,

$$\mathcal{H} = \bigcup_{1 \le i \le N} W_S^s(\pi_i)$$

b3. the attractor $\mathcal{A}$ is the union of the unstable manifolds of the fixed points of $S(t)$, i.e.,

$$\mathcal{A} = \bigcup_{1 \le i \le N} W_S^u(\pi_i)$$

b4. $W_S^s(\pi_i)$ and $W_S^u(\pi_j)$ are transversal for $i, j \in \{1, \ldots, N\}$.

It is known that $S(t)$ has Property b *generically* with respect to the nonlinearity f. Let us explain the meaning of this statement.

Fix an integer $q \ge 0$. We introduce the C^q *strong Whitney topology* on the set of functions $f : \mathbf{R} \to \mathbf{R}$ of class C^q as follows. For any functions $f, g \in C^q(\mathbf{R})$ and for any compact set $K \subset \mathbf{R}$ define the number

$$\rho_K^q(f, g) = \sum_{r=0}^{q} \sup_{v \in K} |f^{(r)}(v) - g^{(r)}(v)|.$$

The base of neighborhoods of a function f in the C^q strong Whitney topology consists of the sets

$$\{g : \rho_{K_n}^q(f, g) < \epsilon_n\},$$

where $\{K_n\}$ is a countable family of compact subsets of $\mathbf{R}$ such that any point of $\mathbf{R}$ belongs to a finite number of the sets K_n, the equality $\bigcup_n K_n = \mathbf{R}$ holds, and $\{\epsilon_n\}$ is a sequence of positive numbers.

A subset Y of a topological space X is called *residual* if Y contains a countable intersection of open and dense subsets of X. If P is a property of elements of X, then we say that this property is *generic* if the set $\{x \in X : x \text{ satisfies } P\}$ is residual. It is known [13] that every residual subset of the space of functions $f : \mathbf{R} \to \mathbf{R}$ of class C^q with the C^q strong Whitney topology is dense in this space.

Brunovsky and Chow [6] showed that there exists a residual subset $\mathcal{G}$ of the set of functions of class C^2 on $\mathbf{R}$ with the C^2 strong Whitney topology having the following two properties.

- If $f \in \mathcal{G}$, then any solution $p(x)$ of (8) is hyperbolic;
- if $f \in \mathcal{G}$, then the set of $\mu > 0$ for which the problem

$$p_{xx} + \mu f(p) = 0, \quad p(0) = p(1) = 0, \tag{9}$$

has nonhyperbolic solutions is countable.

For a special class of functions f, it is possible to give an explicit description of the set $\{\mu\}$ that corresponds to nonhyperbolic solutions of problem (9). Chaffee and Infante [7] considered (1) with a nonlinearity $\mu f \in C^2$ and f satisfying the conditions

- $f(0) = 0, \ f'(0) = 1;$
- $uf''(u) < 0$ for $u \ne 0$;
- $\overline{\lim}_{u \to \pm\infty} f(u)/u \le 0.$

It is shown in [12] that problem (9) has only hyperbolic solutions if $\mu \neq n^2\pi^2$, $n = 1, 2, \ldots$.

Thus, for a C^2-generic nonlinearity all fixed points of $S(t)$ are hyperbolic. It follows from results of Henry [12] and Angenent [3] that for hyperbolic fixed points their stable and unstable manifolds are always transversal. Hence, for a generic nonlinearity f in (1) satisfying condition (6), the global attractor $\mathcal{A}$ has properties **b1** and **b4**. It is easy to establish also properties **b2** and **b3**.

Let $\sigma(u) = S(1)u$ for $u \in \mathcal{H}$. Our goal is to study conditions under which the semi-dynamical system σ has a variant of the Lipschitz shadowing property on a neighborhood of the global attractor of $S(t)$. Fix $d > 0$. As usual, we say that a sequence $\{u_j \in \mathcal{H} : j \geq 0\}$ is a d-pseudotrajectory of σ if

$$|\sigma(u_j) - u_{j+1}| \leq d \quad \text{for } j \geq 0. \tag{10}$$

We say that the system σ has a Lipschitz shadowing property on a set W if there exist positive constants d_0, L_0 such that for any d-pseudotrajectory $\{u_j\} \subset W$ with $d \leq d_0$ there is a point u such that

$$|\sigma^j(u) - u_j| \leq L_0 d \quad \text{for } j \geq 0. \tag{11}$$

Our proof of the shadowing property is based on the existence of a smooth inertial manifold constructed as follows. There exists an orthogonal projection P with finite-dimensional range $P\mathcal{H}$, and a C^1-mapping $\Phi : P\mathcal{H} \to Q\mathcal{H}$ (where $Q = I - P$) such that $\mathcal{M}$, the graph of Φ, has the following two properties.

- $S(t)\mathcal{M} \subset \mathcal{M}, t \geq 0$;
- $\mathcal{M}$ is exponentially attractive, i.e., for any bounded set $B \subset \mathcal{H}$ there exist positive constants C, a (depending on B) such that

$$\text{dist}(S(t)u, \mathcal{M}) \leq Ce^{-at}\text{dist}(u, \mathcal{M}), \quad \text{for } u \in B,\ t \in [0, \infty). \tag{12}$$

Obviously, it follows from our definitions that $\mathcal{A} \subset \mathcal{M}$. It is known [9], [8] that if the nonlinearity F in (4) is modified so that it vanishes outside a ball in $\mathcal{H}$, then the corresponding modified semigroup has an inertial manifold.

Fix an absorbing set $\mathcal{H}_0$ for $\mathcal{A}$. It is known that there is a bounded set $\mathcal{H}'$ such that

$$S(t)\mathcal{H}_0 \subset \mathcal{H}' \quad \text{for } t \geq 0;$$

cf. (5). We denote by $\mathcal{H}^*$ the 1-neighborhood of $\mathcal{H}'$ and modify F so that the new nonlinearity coincides with F in $\mathcal{H}^*$ and vanishes on a neighborhood of infinity. We keep the notation F and $S(t)$ for the modified system. It follows that the new system has an inertial manifold $\mathcal{M}$ and its trajectories beginning at points of $\mathcal{H}_0$ coincide with the corresponding trajectories of the original system. Let inequality (12) be satisfied for $u \in \mathcal{H}^*$.

Theorem 1. *Assume that $S(t)$ has Properties a and b. Then there exist constants $d_0, L_0 > 0$ and a neighborhood W of $\mathcal{A}$ in $\mathcal{H}$ such that if $\{u_j : j \geq 0\} \subset W$ is a d-pseudotrajectory of σ with $d \leq d_0$ (see* (10)*), and*

$$\text{dist}(u_0, \mathcal{M}) \leq 2d, \tag{13}$$

then there is a point $u \in \mathcal{M}$ such that (11) *holds.*

To prove this theorem, we first reduce the shadowing problem to an analogous problem on the finite-dimensional manifold $\mathcal{M}$, and then establish the desired shadowing property on $\mathcal{M}$.

The assumption (13), which is not needed in the finite-dimensional case (see [19] and Theorem 2 below), occurs in the reduction; see (19). We do not know if it is really necessary in the infinite-dimensional case.

3. Reduction to the Inertial Manifold

Take an integer $T > 0$ such that, with C, a as in (12),

$$\nu := Ce^{-aT} < \tfrac{1}{2}. \tag{14}$$

Below we impose another restriction on T; this restriction is "absolute", i.e., it depends only on the attractor $\mathcal{A}$; see (33).

For $u_0 \in \mathcal{H}$ let $p_0 = Pu_0$. Denote by $p(t, p_0)$ the solution of the finite-dimensional system in $P\mathcal{H}$,

$$\dot{p} + Ap = PF(p + \Phi(p)),\ t > 0; \quad p(0, p_0) = p_0. \tag{15}$$

Then (see [9]) for $u_0 \in \mathcal{M}$ we have

$$u(t) = S(t)u_0 = p(t, p_0) + \Phi(p(t, p_0)). \tag{16}$$

We introduce the following notation. For $p_0 = Pu_0, u_0 \in \mathcal{H}$ we set

$$\sigma_1^*(p_0) = p(1, p_0),\ \sigma_T^*(p_0) = p(T, p_0)$$

(the integer T was fixed in (14)), and for $m_0 \in \mathcal{M}$ we set $p_0 = Pm_0$ and

$$\phi_1(m_0) = \sigma_1^*(p_0) + \Phi(\sigma_1^*(p_0)),\ \phi(m_0) = \sigma_T^*(p_0) + \Phi(\sigma_T^*(p_0)), \tag{17}$$

It follows from the inclusions $f, \Phi \in C^1$ that σ_1^*, σ_T^* (and ϕ_1, ϕ) are C^1-mappings of $P\mathcal{H}$ (of $\mathcal{M}$, respectively) to itself. In addition, by (16) for $u_0 \in \mathcal{M}$ we have

$$\phi_1(u_0) = S(1)u_0 = \sigma(u_0),$$

i.e., ϕ_1 is the restriction of σ to $\mathcal{M}$.

Obviously, $\mathcal{A}$ is the global attractor of σ in $\mathcal{H}$ and of ϕ_1 in $\mathcal{M}$ (the definitions are parallel to the one for S).

Set $C_1 = C(1, \mathcal{H}_0), C_T = C(T, \mathcal{H}_0)$; see (7). Since the system (15) is linear in a neighborhood of infinity, all its solutions are defined for $t \in \mathbf{R}$. It follows that ϕ, ϕ^{-1} are diffeomorphisms of $\mathcal{M}$ of class C^1.

Fix a neighborhood $\mathcal{M}_0 \subset \mathcal{H}_0$ of $\mathcal{A}$ in $\mathcal{M}$. Since $\mathcal{A}$ is Lyapunov stable, there exists a neighborhood $\mathcal{M}_1 \subset \mathcal{M}_0$ of $\mathcal{A}$ such that $S(t)u \in \mathcal{M}_0$ for $u \in \mathcal{M}_1$ and $t \geq 0$. It follows that for $p_0 \in \Pi_1 = P\mathcal{M}_1$ we have $p(t, p_0) \in \Pi_0 = P\mathcal{M}_0$ for $t \geq 0$. Let K_1 be a Lipschitz constant of $\sigma_T^* = p(T, \cdot)$ on Π_1.

Due to the same reason there exists $d^0 > 0$ such that the inequality $\operatorname{dist}(u, \mathcal{A}) \leq d^0$ implies the inclusions $\sigma^n(u) \in \mathcal{H}_0$ for $n \geq 0$. Let W_1 be the d^0-neighborhood of $\mathcal{A}$. Assume, in addition, that the d^0-neighborhood of W_1 is a subset of $\mathcal{H}_0$ and that P projects this neighborhood of W_1 into Π_1.

The following Lemmas 1 and 2 show that our problem of Lipschitz shadowing for d-pseudo-trajectories of σ is reduced to the same problem for Cd-pseudotrajectories of σ^T.

Take a d-pseudotrajectory $\{u_j\} \subset W_1$ of σ with $d \leq d^0$. Let $z_j = u_{jT}$.

Lemma 1. *Let $C_2 = 1 + C_1 + \cdots + C_1^{T-1}$, $C_3 = C_2/(1-\nu)$. Then*

(i) *the sequence $\{z_j\}$ is a $C_2 d$-pseudotrajectory of σ^T;*

(ii) *if $\operatorname{dist}(z_0, \mathcal{M}) \leq 2d$, then $\operatorname{dist}(z_j, \mathcal{M}) \leq 2C_3 d$ for $j \geq 0$.*

Proof. First note that the choice of d^0 and W_1 implies that $\sigma^n(u_j) \in \mathcal{H}_0$ for $n \geq 0$. Let us prove statement (i). By definition,

$$|\sigma(z_j) - u_{jT+1}| = |\sigma(u_{jT}) - u_{jT+1}| \leq d. \tag{18}$$

Now we estimate

$$\begin{aligned}|\sigma^2(z_j) - u_{jT+2}| &\leq |\sigma(u_{jT+1}) - u_{jT+2}| + |\sigma(\sigma(u_{jT})) - \sigma(u_{jT+1})| \\ &\leq d + C_1 d,\end{aligned}$$

since both u_{jT+1} and $\sigma(u_{jT})$ are in $\mathcal{H}_0$, and (18) holds. Similarly,

$$\begin{aligned}|\sigma^3(z_j) - u_{jT+3}| &\leq |\sigma(u_{jT+2}) - u_{jT+3}| + |\sigma(\sigma^2(u_{jT})) - \sigma(u_{jT+2})| \\ &\leq d + C_1(d + C_1 d) = d + C_1 d + C_1 d^2,\end{aligned}$$

and so on. Thus, we arrive at the inequality

$$|\sigma^T(z_j) - z_{j+1}| \leq d(1 + C_1 + \cdots + C_1^{T-1}) = C_2 d.$$

This shows that $\{z_j\}$ is a $C_2 d$-trajectory of σ^T.

To prove statement (ii), let $b_j = \text{dist}(z_j, \mathcal{M})$. By assumption, we have $b_0 \leq 2d$. Now we estimate

$$\begin{aligned}b_{j+1} &\leq \text{dist}(\sigma^T(z_j), z_{j+1}) + \text{dist}(\sigma^T(z_j), \mathcal{M}) \\ &\leq C_2 d + \nu \text{dist}(z_j, \mathcal{M}) = C_2 d + \nu b_j;\end{aligned}$$

see (12) and (14). It follows that

$$\begin{aligned}&b_0 \leq 2d \leq 2C_2 d, \ b_1 \leq C_2 d + \nu b_0 \leq 2C_2 d(1+\nu), \ \ldots, \\ &b_n \leq 2C_2 d(1 + \nu + \cdots + \nu^n) \leq \frac{2C_2 d}{1-\nu}.\end{aligned}$$

This completes the proof. □

Lemma 2. *Assume that for some $u \in \mathcal{H}$ we have*

$$\sigma^n(u) \in \mathcal{H}_0, \ n \geq 0, \ \textit{and} \ |\sigma^{jT}(u) - z_j| \leq L'd, \ j \geq 0.$$

Then

$$|\sigma^j(u) - u_j| \leq C_5 d, \quad j \geq 0,$$

where $C_5 = 1 + C_4 + \cdots + C_4^T$, $C_4 = C_1 L'$.

Proof. Take $n \in (jT, (j+1)T]$. One easily obtains the inequalities

$$\begin{aligned}|\sigma^{jT+1}(u) - u_{jT+1}| &\leq |\sigma^{jT+1}(u) - \sigma(u_{jT})| + |\sigma(u_{jT}) - u_{jT+1}| \\ &\leq C_1|\sigma^{jT}(u) - u_{jT}| + d \leq C_4 d + d = d(1 + C_4).\end{aligned}$$

Similarly,

$$\begin{aligned}|\sigma^{jT+2}(u) - u_{jT+2}| &\leq |\sigma(\sigma^{jT+1}(u)) - \sigma(u_{jT+1})| + |\sigma(u_{jT+1}) - u_{jT+2}| \\ &\leq C_1(C_4 d + d) + d \leq d(1 + C_4 + C_4^2).\end{aligned}$$

Finally, we arrive at the desired estimate

$$|\sigma^n(u) - u_n| \leq d(1 + C_4 + \cdots + C_4^T).$$

□

The following lemma will be used in Section 5 to show that assumption (13) can be satisfied.

Lemma 3. *Let the ball $B = \{u \in \mathcal{H} : |u| \le R\}$ be a subset of $\mathcal{H}_0$. For any $d > 0$ there exists a number $N = N(R, d)$ such that if $\{u_j\} \subset \mathcal{H}_0$ is a d-pseudotrajectory of σ^T with $u_0 \in B$, then* $\operatorname{dist}(u_n, \mathcal{M}) \le 2d$ *for some* $n \in [0, N]$.

Proof. Let $b_j = \operatorname{dist}(u_j, \mathcal{M})$ and set $\nu_1 = \nu + \frac{1}{2}$. It follows from (14) that $\nu_1 < 1$. Select $N > 0$ such that $\nu_1^N R < 2d$.

To obtain a contradiction, assume that $b_j > 2d$ for $0 \le j \le N$. Then for $0 \le j \le N - 1$ we have, by (12) and (14),

$$b_{j+1} \le \operatorname{dist}(\sigma^T(u_j), \mathcal{M}) + |\sigma^T(u_j) - u_{j+1}| \le \nu b_j + d.$$

By our assumption, $b_j > 2d$, we have $d < b_j/2$, and hence

$$b_{j+1} \le (\nu + \tfrac{1}{2})b_j = \nu_1 b_j \quad \text{for } 0 \le j \le N - 1.$$

It follows that

$$b_N \le \nu_1^N b_0 \le \nu_1^N R < 2d.$$

The obtained contradiction proves the lemma. □

We continue to discuss the sequence $\{z_j\}$ introduced before Lemma 1. Take $d^1 = d^0/(2C_3)$. We assume in the sequel that $d \le d^1$.

Let $z'_j = Pz_j$ and set $v_j = z'_j + \Phi(z'_j) \in \mathcal{M}$. We shall prove that $\{v_j\}$ is a $C'd$-pseudotrajectory of ϕ, defined in (17), with C' independent of d.

Since $\operatorname{dist}(z_j, \mathcal{M}) \le 2C_3 d$ (by Lemma 1), we can find $w_j \in \mathcal{M}$ such that

$$|w_j - z_j| \le 2C_3 d. \tag{19}$$

We obtain from the choice of d^1 that $w_j \in \mathcal{H}_0$. Now it follows from (7) and (19) that

$$|\sigma^T(w_j) - \sigma^T(z_j)| \le C_6 d, \tag{20}$$

where $C_6 = 2C_T C_3$, $C_T = C(T, \mathcal{H}_0)$. From Lemma 1 and from (20) we conclude that

$$|\sigma^T(w_j) - z_{j+1}| \le |\sigma^T(z_j) - z_{j+1}| + |\sigma^T(w_j) - \sigma^T(z_j)| \le C_7 d, \tag{21}$$

where $C_7 = C_2 + C_6$. Let $w'_j = Pw_j$. By our choice, $z'_j, w'_j \in \Pi_1$. It follows from the definition of ϕ in (17) that

$$P\sigma^T(w_j) = P\phi(w_j) = \sigma_T^*(Pw_j) = \sigma_T^*(w'_j). \tag{22}$$

We want to estimate the value

$$|\sigma_T^*(z'_j) - z'_{j+1}| \le |\sigma_T^*(z'_j) - \sigma_T^*(w'_j)| + |\sigma_T^*(w'_j) - Pz_{j+1}|. \tag{23}$$

The first term in (23) does not exceed $K_1 C_3 d$, because K_1 is a Lipschitz constant of σ_T^* on Π_1 and

$$|z'_j - w'_j| = |Pz_j - Pw_j| \le C_3 d;$$

see (19)). By (22) we have

$$|\sigma_T^*(w'_j) - Pz_{j+1}| = |P\sigma^T(w_j) - Pz_{j+1}|.$$

Hence, the second term in (23) does not exceed

$$|\sigma^T(w_j) - z_{j+1}| \le C_7 d;$$

see (21). Thus, we obtain the estimate

$$|\sigma_T^*(z'_j) - z'_{j+1}| \le C_8 d, \tag{24}$$

where $C_8 = K_1 C_3 + C_7$.

Let K_2 be a Lipschitz constant of Φ on Π_1. We obtain from (24) the estimate

$$|v_{j+1} - \phi(v_j)| \le |z'_{j+1} - \sigma^*_T(z'_j)| + |\Phi(z'_{j+1}) - \Phi(\sigma^*_T(z'_j))| \le C_9 d, \tag{25}$$

where $C_9 = (1 + K_2)C_8$.

Denote by r the metric on $\mathcal{M}$ generated by the distance in $\mathcal{H}$. Obviously, there is a constant $K_3 > 0$ such that

$$K_3^{-1} r(v, v') \le |v - v'| \le r(v, v') \quad \text{for } v, v' \in \mathcal{M}_1.$$

It follows from (25) that $r(v_{j+1}, \phi(v_j)) \le K_3 C_9 d$. Thus, $\{v_j\}$ is a K_3C_9d-pseudotrajectory of ϕ on $\mathcal{M}$. Let us show that Theorem 1 follows from the following statement.

Theorem 2. *There exists a neighborhood M of $\mathcal{A}$ in $\mathcal{M}$ and numbers $d', L > 0$ such that if $\{v_j\} \subset M$ is a d-pseudotrajectory of ϕ with $d \le d'$, then there is a point $v \in \mathcal{M}$ with the property*

$$r(\phi^j(v), v_j) \le Ld, \quad j \ge 0.$$

Indeed, let $\{u_j\}$ be as in Theorem 1, define $\{z_j\}$, ϕ, and $\{v_j\}$ as above. By using Lemma 1 we showed that $\{v_j\}$ is a $C'd$-pseudotrajectory of ϕ on $\mathcal{M}$. Theorem 2 then gives $v \in \mathcal{M}$ such that

$$r(\phi^j(v), v_j) \le L'd, \quad j \ge 0 \quad (L' = LC'd),$$

and hence

$$|\phi^j(v) - v_j| \le L'd, \quad j \ge 0. \tag{26}$$

Let us estimate

$$|z_j - v_j| \le |z_j - w_j| + |w_j - v_j|.$$

By (19), the first term (and also $|z'_j - w'_j|$) does not exceed $2C_3d$. Since $w_j = w'_j + \Phi(w'_j)$, $v_j = z'_j + \Phi(z'_j)$, we estimate the second term by $2C_3d(1 + K_2)$. Now we obtain from (26) the inequality

$$|z_j - \sigma^{jT}(v)| = |z_j - \phi^j(v)| \le \big(L' + 2C_3(2 + K_2)\big)d = L''d.$$

Lemma 2 now implies that $|\sigma^j(v) - u_j| \le C_5 L''d = L'''d$. This proves our reduction statement.

4. Shadowing on the Inertial Manifold

In this section we prove Theorem 2.

Properties **b1**–**b4** of the attractor $\mathcal{A}$ of σ imply the following properties of the mapping ϕ_1 (defined in (17)) on $\mathcal{M}$.

b1'. $\mathcal{A}$ contains fixed points $\pi_1, \dots, \pi_N$ of ϕ_1, and they are hyperbolic (we denote by $W^s(\pi_i), W^u(\pi_i)$ their stable and unstable manifolds);

b2'.

$$\mathcal{M} = \bigcup_{1 \le i \le N} W^s(\pi_i); \tag{27}$$

b3'.

$$\mathcal{A} = \bigcup_{1 \le i \le N} W^u(\pi_i); \tag{28}$$

b4'. $W^s(\pi_i)$ and $W^u(\pi_j)$ are transversal for $i, j \in \{1, \dots, N\}$.

Let us prove only **b4'**. Take a point

$$x \in W^s(\pi_i) \cap W^u(\pi_j).$$

As before, we denote by T_pN the tangent space of N at p. Let

$$T_1^s = T_x W^s(\pi_i), \quad T_2^s = T_x W_S^s(\pi_i),$$
$$T_1^u = T_x W^u(\pi_j), \quad T_2^u = T_x W_S^u(\pi_j).$$

By **b4**, $W_S^s(\pi_i)$ and $W_S^u(\pi_j)$ are transversal at x, i.e.,

$$T_2^s \oplus T_2^u = \mathcal{H}. \tag{29}$$

We claim that

$$T_1^z = T_x\mathcal{M} \cap T_2^z, \quad z = s, u. \tag{30}$$

Note that the norm in $T_p\mathcal{M}$, $p \in \mathcal{M}$, generated by the metric r on $\mathcal{M}$ coincides with the norm in $T_p\mathcal{M}$ induced from $\mathcal{H}$.

The equality

$$T_1^s = \{w \in T_p\mathcal{M} : |D\phi_1^k(x)w| \to 0, \ k \to \infty\}$$

is proved in [17, Chap. 13]. Similarly one proves that

$$T_2^s = \{w \in \mathcal{H} : |D\sigma^k(x)w| \to 0, \ k \to \infty\}.$$

Obviously these equalities imply (30) for $z = s$. For $z = u$ the proof is similar. Now it follows from (29) that

$$T_1^s \oplus T_1^u = T_x\mathcal{M},$$

i.e., $W^s(\pi_i)$ and $W^u(\pi_j)$ are transversal at x.

Since any fixed point π_i of ϕ_1 is hyperbolic, there exist the "stable" and "unstable" subspaces E_i^s and E_i^u of $T_{\pi_i}\mathcal{M}$ with the standard properties,

- E_i^z, $z = s, u$, are $D\phi_1$-invariant;
- $E_i^s \oplus E_i^u = T_{\pi_i}\mathcal{M}$;
- there are $K_i > 0, \lambda_i \in (0,1)$ such that

$$|D\phi_1^m(\pi_i)v| \le K_i\lambda_i^m|v|, \quad v \in E_i^s, \ m \ge 0, \tag{31}$$

$$|D\phi_1^{-m}(\pi_i)v| \le K_i\lambda_i^m|v|, \quad v \in E_i^u, \ m \ge 0. \tag{32}$$

Take $\lambda_0 \in (0,1)$ and an integer T_0 such that

$$K_i\lambda_i^{T_0} < \lambda_0, \quad 1 \le i \le N. \tag{33}$$

We fix T satisfying (14) so that $T \ge T_0$ (note that T_0 is an "absolute" constant).

Since $\phi = \phi_1^T$ (see (17)), it follows from (31) and (32) that

$$\|D\phi|_{E_i^s}\| \le \lambda_0, \quad \|D\phi^{-1}|_{E_i^u}\| \le \lambda_0. \tag{34}$$

For a set $X \subset \mathcal{M}$ we denote

$$O^+(X) = \bigcup_{k\ge 0} \phi^k(X), \ O^-(X) = \bigcup_{k \le 0} \phi^k(X), \ O(X) = O^+(X) \cup O^-(X).$$

The proof of the following statement repeats the proof of parts I, III, and IV of [20, Theorem 10.1]. We denote by $T\mathcal{M}$ the tangent bundle of $\mathcal{M}$.

Lemma 4. *There exists $\lambda_1 \in (\lambda_0, 1)$, neighborhoods Z_i of π_i in $\mathcal{M}$, and continuous subbundles S_i, U_i of $T\mathcal{M}$ on $\overline{Z}_i \cup O(Z_i)$ such that*

(i) *$S_i(x) \oplus U_i(x) = T_x\mathcal{M}$ for $x \in Z_i$;*

(ii) S_i, U_i *are* $D\phi$*-invariant on* $O(Z_i)$ *(consequently, the equalities in* (i) *hold for* $x \in O(Z_i)$*);*

(iii)
$$\|D\phi|_{S_i(x)}\| \le \lambda_1, \quad \|D\phi^{-1}|_{U_i(x)}\| \le \lambda_1 \quad \textit{for } x \in Z_i;$$

(iv)
$$S_i(x) \subset S_j(x), \quad U_j(x) \subset U_i(x) \quad \textit{for } x \in O^+(Z_i) \cap O^-(Z_j).$$

It is shown in [18] that there exists a neighborhood $\mathcal{M}_2 \subset \mathcal{M}_1$ of $\mathcal{A}$ such that $\phi(\overline{\mathcal{M}_2}) \subset \mathcal{M}_2$. We define
$$Z = \bigcup_{1 \le i \le N} Z_i.$$

Let τ be a Birkhoff constant for ϕ on $\overline{\mathcal{M}_2}$, i.e., a positive number with the property
$$\operatorname{card}\{k \in \mathbf{Z} : \phi^k(x) \in \overline{\mathcal{M}_2} \setminus Z\} \le \tau$$
for $x \in \overline{\mathcal{M}_2}$ (here card is the cardinality). Let $\mathcal{M}_3 = \phi(\mathcal{M}_2)$.

Let
$$N_0 = \max_{x \in \overline{\mathcal{M}_2}} (\|D\phi(x)\|, \|D\phi^{-1}(x)\|).$$

Take $p \in \overline{\mathcal{M}_2}$, there exists $m_0 \in [0, \tau]$ such that $q = \phi^{m_0}(p) \in Z_i$ and $\phi^m(p) \notin Z_l$ with $i \ne l$ for $0 \le m < m_0$. Define linear subspaces $S(p), U(p)$ of $T_p\mathcal{M}$ by
$$S(p) = D\phi^{-m_0}(q) S_i(q), \quad U(p) = D\phi^{-m_0}(q) U_i(q).$$
Now we define a mapping $e_p : T_p\mathcal{M} \to \mathcal{M}$ for $p \in \mathcal{M}$ by
$$e_p(v) = P(p+v) + \Phi(P(p+v)), \quad v \in T_p\mathcal{M}.$$
Since Φ is of class C^1, e_p is also of class C^1.

Straightforward calculation shows that
$$De_p(0) = I. \tag{35}$$

The following statement is proved similarly to [19, Lemma 2.1].

Lemma 5. *For* $p \in \overline{\mathcal{M}_2}$ *the spaces* $S(p), U(p)$ *have the properties*

(i) $S(p) \oplus U(p) = T_p\mathcal{M}$;

(ii) *There exists* $N_1 > 0$ *such that for* $Q^s(p), Q^u(p)$*, the projectors onto* $S(p), U(p)$ *parallel to* $U(p), S(p)$*, respectively, the inequalities*
$$\|Q^s(p)\| \le N_1, \quad \|Q^u(p)\| \le N_1$$
hold;

(iii)
$$D\phi(p) S(p) \subset S(\phi(p)), \quad D\phi^{-1}(p) U(p) \subset U(\phi^{-1}(p))$$
(the second inclusion holds if, in addition, $p \in \overline{\mathcal{M}_3}$*);*

(iv) *for each* $\mu, a_1 > 0$ *there exists* $a_2 > 0$ *such that if* $p, z \in \phi^\mu(\overline{\mathcal{M}_2})$ *and* $r(z, \phi^\mu(p)) < a_2$*, then there is a linear isomorphism* $\Pi(p,z) : T_z\mathcal{M} \to T_z\mathcal{M}$ *with the properties*
$$\|\Pi(p,z) - I\| < a_1, \quad \Pi(p,z)[De_z^{-1}(q) D\phi^\mu(p) S(p)] \subset S(z),$$
and a linear isomorphism $\Theta(p,z) : T_p\mathcal{M} \to T_p\mathcal{M}$ *with the properties*
$$\|\Theta(p,z) - I\| < a_1, \quad \Theta(p,z)[De_p^{-1}(t) D\phi^{-\mu}(z) U(z)] \subset U(p),$$
where $q = \phi^\mu(p), t = \phi^{-\mu}(z)$.

It is easy to see that for $v \in S_i(x)$ we have

$$|D\phi^m(x)v| \le K\lambda_1^m|v|, \quad m \ge 0,$$

and for $v \in U_i(x)$ we have

$$|D\phi^m(x)v| \le K\lambda_1^{-m}|v|,$$

if $m \le 0$ and $\phi^k(x) \in \overline{\mathcal{M}}_2$ for $k \in [m, 0]$, where $K = (N_0/\lambda_1)^\tau$ (and τ is the Birkhoff constant chosen above). Take $\mu \ge \tau$ such that $K\lambda_1^\mu < \lambda_1$. Let $\xi = \phi^\mu$.

The same reasons as above (see Lemmas 1 and 2) show that ϕ has a Lipschitz shadowing property in a neighborhood of $\mathcal{A}$ if and only if ξ has. Thus, we work below with ξ.

Obviously, Lemma 5 remains true for ξ instead of ϕ, but in statement (iii) $\mathcal{M}_3$ is to be replaced by $\mathcal{M}_4$, where

$$\mathcal{M}_4 = \xi(\mathcal{M}_2).$$

In addition, by the choice of μ, the following inequalities hold:

$$|D\xi(p)v| \le \lambda_1|v|, \quad v \in S(p), p \in \mathcal{M}_2, \tag{36}$$

$$|D\xi^{-1}(p)v| \le \lambda_1|v|, \quad v \in U(p), p \in \mathcal{M}_4. \tag{37}$$

For $a > 0$ we denote

$$\mathcal{E}_a(p) = \{v \in T_p\mathcal{M} : |v| < a\}, \quad \mathcal{D}_a(p) = e_p(\mathcal{E}_a(p)).$$

Obviously, there exists a neighborhood $\mathcal{M}_5 \subset \mathcal{M}_4$ of $\mathcal{A}$ and a number $c > 0$ such that for $p \in \overline{\mathcal{M}}_5$ we have

$$\mathcal{D}_c(p),\ \xi(\mathcal{D}_c(p)),\ \xi^{-1}(\mathcal{D}_c(p)) \subset \mathcal{M}_4,$$

and e_p is a diffeomorphism of $\mathcal{E}_c(p)$ onto $\mathcal{D}_c(p)$ with uniform estimates of $\|De_p\|$ and $\|De_p^{-1}\|$.

Let $M = \mathcal{M}_5$. Take a d-pseudotrajectory $\{v_k : k \ge 0\} \subset M$ of ξ.

Below we denote by d' positive constants that depend on properties of ξ on M and do not depend on $\{v_k\}$. At each step of the proof, we consider d-pseudotrajectories such that d does not exceed the minimal d' previously chosen. Since we choose d' finitely many times, no generality is lost.

In the proof below we apply a "local" shadowing result for a sequence of mappings of Banach spaces ([19, Theorem 1.1]). Let us formulate it. Consider a sequence of Banach spaces $H_k,\ k \ge 0$. We denote by $|\cdot|$ norms in H_k and by $\|\cdot\|$ the corresponding operator norms for linear operators. Consider a sequence of mappings

$$\phi_k : H_k \to H_{k+1}$$

of the form

$$\phi_k(u) = A_k u + w_{k+1}(u),$$

where A_k are linear mappings.

Lemma 6. *Assume that*

(i) *there exist numbers* $\lambda \in (0,1),\ N > 0$, *and projectors* P_k, Q_k *in* H_k *(we denote below* $S_k = P_k H_k,\ U_k = Q_k H_k$*) such that*

$$\|P_k\|, \|Q_k\| \le N, \quad P_k + Q_k = I,$$
$$\|A_k|_{S_k}\| \le \lambda, \quad A_k S_k \subset S_{k+1};$$

(ii) *there exist linear mappings* $B_k : U_{k+1} \to H_k$ *such that*

$$B_k U_{k+1} \subset U_k, \ \|B_k\| \leq \lambda, \ A_k B_k|_{U_{k+1}} = I;$$

(iii) *there exist numbers* $\kappa, \Delta > 0$ *such that*

$$|w_{k+1}(u) - w_{k+1}(u')| \leq \kappa |u - u'| \quad \text{for } |u|, |u'| \leq \Delta,$$

and

$$\kappa N' < 1, \tag{38}$$

where $N' = N\frac{1+\lambda}{1-\lambda}$.

Then there exist constants $d^*, L^* > 0$ *(depending on* λ, N, Δ*) with the following property: if*

$$|\phi_k(0)| \leq d \leq d^* \quad \text{for } k \geq 0,$$

then one can find points $u_k \in H_k$ *such that* $\phi_k(u_k) = u_{k+1}$ *and*

$$|u_k| \leq L^* d.$$

Now we fix $k \geq 0$ and denote $p = v_k, \ z = v_{k+1}, \ H_k = T_{v_k}\mathcal{M}$. Take $d' < c/2$ and such that the inequality

$$r(z, \xi(p)) \leq d' \tag{39}$$

implies the inclusion

$$\xi^{-1}(z) \in \mathcal{D}_c(p). \tag{40}$$

Find $0 < b < c$ such that

$$\xi(\mathcal{D}_b(x)) \subset \mathcal{D}_{d'}(\xi(x)), \ x \in M$$

(obviously, b depends only on ξ). Then it follows from (39) that

$$\xi(\mathcal{D}_b(p)) \subset \mathcal{D}_c(z).$$

Thus, the mapping $\psi_k : \mathcal{E}_b(p) \to H_{k+1}$ given by

$$\psi_k(w) = e_z^{-1} \circ \xi \circ e_p(w)$$

is properly defined. Let us introduce the following notation:

$$q = \xi(p), \ q' = e_z^{-1}(q), \ t = \xi^{-1}(z), \ t' = e_p^{-1}(t),$$

$$D = D\psi_k(0), \ D' = D\xi(t)De_p(t'), \ G = De_p^{-1}(t)D\xi^{-1}(z).$$

Note that t', D', G are well-defined since (40) holds.

Fix $N_2 > 0$ such that

$$\|D\xi(x)\|, \|D\xi^{-1}(x)\| \leq N_2, \quad x \in M.$$

Let $N = \max(N_1, N_2)$ (N_1 is given by Lemma 5).

Find $\nu_0 \in (0, 1)$ such that

$$\lambda = (1 + \nu_0)^2 \lambda_1 < 1$$

and let

$$N' = N\frac{1+\lambda}{1-\lambda}.$$

Take $\kappa > 0$ such that (38) holds and find $\nu < \nu_0$ with the property

$$N(3N + 1)\nu < \tfrac{1}{2}\kappa. \tag{41}$$

It follows from (35) that

$$D = De_z^{-1}(q)D\xi(p). \tag{42}$$

Now we find d' such that inequality (39) implies the inequalities

$$|De_p(t')w| \le (1+\nu)|w|, \quad w \in T_p\mathcal{M}, \tag{43}$$

$$|De_z^{-1}(q)w| \le (1+\nu)|w|, \quad w \in T_q\mathcal{M}, \tag{44}$$

$$|De_p^{-1}(t)w| \le (1+\nu)|w|, \quad w \in T_t\mathcal{M} \tag{45}$$

(we apply (35) and the uniform continuity of De_x, De_x^{-1}),

$$\|D - D'\| < \nu \tag{46}$$

(compare (42)), the definition of D', and take into account the previous argument),

$$\|\Pi(p,z) - I\|, \|\Theta(p,z) - I\|, \|\Theta^{-1}(p,z) - I\| < \nu; \tag{47}$$

see Lemma 5.

Define $A_k : H_k \to H_{k+1}$ by

$$A_k = \Pi(p,z)DQ^s(p) + D'\Theta^{-1}(p,z)Q^u(p).$$

For $w \in S(p)$ we have $w^s = A_k w = \Pi(p,z)Dw \in S(z)$ (see Lemma 5), hence

$$A_k S(p) \subset S(z). \tag{48}$$

Since $|D\xi(p)w| \le \lambda_1|w|$ by (36), it follows from (44) and (47) that

$$|w^s| \le (1+\nu)^2\lambda_1|w| = \lambda|w|.$$

Hence

$$\|A_k|_{S(p)}\| \le \lambda. \tag{49}$$

Now we consider a mapping $B_k : U(z) \to H_k$ defined as follows: $B_k w = \Theta(p,z)Gw$. We easily show that $w^u = B_k w \in U(p)$, hence

$$B_k U(z) \subset U(p), \tag{50}$$

and that $|w^u| \le \lambda|w|$. Hence

$$\|B_k|_{U(z)}\| \le \lambda. \tag{51}$$

It follows from

$$A_k w^u = D'\Theta^{-1}(p,z)\Theta(p,z)Gw = D'Gw = w$$

that

$$A_k B_k|_{U(z)} = I. \tag{52}$$

Represent

$$\psi_k(w) = Dw + \rho(w).$$

Obviously there exists d' such that

$$|\rho(w) - \rho(w')| \le \tfrac{1}{2}\kappa|w - w'| \tag{53}$$

for $|w|, |w'| \le d'$.

Now we represent

$$\psi_k(w) = A_k w + \chi(w), \tag{54}$$

where $\chi(w) = (D - A_k)w + \rho(w)$, $\chi(0) = \psi_k(0)$.

Let us estimate $\|D - A_k\|$. We have

$$\begin{aligned}\|D - A_k\| &= \|D(Q^s(p) + Q^u(p)) - A_k\| \\ &\leq \|DQ^s(p) - \Pi(p,z)DQ^s(p)\| + \|DQ^u(p) - D'Q^u(p)\| \\ &\quad + \|D'Q^u(p) - D'\Theta^{-1}(p,z)Q^u(p)\|.\end{aligned}$$

Since $\|Q^s(p)\|, \|D\| \leq N$, and $\|\Pi(p,z) - I\| < \nu$ (see (47)), we obtain that the first term on the right does not exceed $N^2\nu$. The same reasons (and inequality (46)) show that the second term is estimated by $N\nu$. From the definition of D' and from the inequality $\nu < 1$ we obtain that $\|D'\| < 2N$. Thus, the third term is estimated by $2N^2\nu$.

This gives the inequality

$$\|D - A_k\| \leq N(3N+1)\nu \leq \tfrac{1}{2}\kappa;$$

see (41). Combined with (53), the last inequality shows that

$$|\chi(w) - \chi(w')| \leq \kappa|w - w'| \quad \text{for } |w|, |w'| \leq d'. \tag{55}$$

Since the derivatives $De_x(y), De_x^{-1}(y')$ are uniformly bounded for

$$x \in M,\ y \in \mathcal{E}_c(x),\ y' \in \mathcal{D}_c(x), \tag{56}$$

there exists $N^* > 0$ such that for x, y, y' that satisfy (56) we have

$$r(x, e_x(y)) \leq N^*|x - y|, \quad |x - e_x^{-1}(y')| \leq N^* r(x, y').$$

Since $\psi_k(0) = e_z^{-1}(\xi(p))$, it follows from $r(\xi(p), z) < d$ that

$$|\psi_k(0)| \leq N^* r(z, \xi(p)) \leq N^* d. \tag{57}$$

Now we obtain from Lemma 5 and from (48)–(52), (55), and (57) that ψ_k satisfy all the conditions of Lemma 6 (with the obvious change of notation).

Hence, there exist $d', L' > 0$ (depending only on ξ and M) with the following property: if $\{v_k\}$ is a d-pseudotrajectory of ξ with $d \leq d'$, then there exists a sequence $w_k \in H_k$ such that $\psi_k(w_k) = w_{k+1}$ and $|w_k| \leq L'd$.

Set $w'_k = e_{v_k}(w_k)$. Then it follows from the definition of ψ_k that $w'_{k+1} = \xi(w'_k)$, hence the sequence w'_k is a ξ-trajectory of $v = w'_0$, i.e., $w'_k = \xi^k(v)$.

Now we obtain from the definition of N^* that there exists d_0 such that, if $\{v_k\}$ is a d-pseudotrajectory of ξ with $d \leq d_0$, then the inequalities

$$r(v_k, \xi^k(v)) = r(v_k, w'_k) \leq N^*|w_k| \leq Ld,$$

where $L = N^*L'$, hold for $k \geq 0$. This completes the proof of Theorem 2.

5. The Discretized Problem

In this section we show how to apply the shadowing result when the pseudotrajectory is a numerical approximation defined by means of a standard piecewise linear finite element method and the backward Euler time stepping.

The finite element method is based on the weak formulation of the initial boundary value problem (1),

$$\begin{aligned}&u(t) \in \mathcal{H},\ u(0) = u_0, \\ &(u_t, v) + (u_x, v_x) = (f(u), v), \quad \forall v \in \mathcal{H},\ t > 0,\end{aligned}$$

obtained by integration by parts, taking the boundary condition into account; the inner product was defined in (2). Let $\{\mathcal{H}_h\}_{h>0}$ denote a family of finite dimensional subspaces of $\mathcal{H}$, where each $\mathcal{H}_h$ consists of continuous piecewise polynomials of degree ≤ 1 with respect to a mesh $0 = x_0 < x_1 < \cdots < x_M = 1$, and where $h = \max(x_i - x_{i-1})$. The backward Euler method with constant step k and times $t_j = jk$, $j = 0, 1, \ldots$, is

$$\begin{aligned} &U_j \in \mathcal{H}_h;\ U_0 = v_h, \\ &k^{-1}(U_j - U_{j-1}, \chi) + (U_{j,x}, \chi_x) = (f(U_j), \chi), \quad \forall \chi \in \mathcal{H}_h,\ t_j > 0. \end{aligned} \tag{58}$$

Introducing the operator $A_h : \mathcal{H}_h \to \mathcal{H}_h$ and the orthogonal projector $P_h : L_2 \to \mathcal{H}_h$ defined by

$$(A_h\psi, \chi) = (\psi_x, \chi_x),\ (P_h\phi, \chi) = (\phi, \chi), \quad \forall \psi, \chi \in \mathcal{H}_h,\ \phi \in L_2,$$

we may write (58) as

$$\begin{aligned} &U_j \in \mathcal{H}_h;\ U_0 = v_h; \\ &k^{-1}(U_j - U_{j-1}) + A_h U_j = P_h F(U_j), \quad t_j > 0. \end{aligned} \tag{59}$$

In a similar way as for the differential equation (4) one obtains a unique local solution of (59) for $k \leq k^*$. We denote its solution (for as long as it exists) by $U_j = S_{hk}(t_j)v_h$. Again in a similar way as for the differential equation (4) one shows that for any bounded set $B_0 \subset \mathcal{H}$ there is a bounded set $B \subset \mathcal{H}$ such that, for $k \leq k^*(B_0)$,

$$S_{hk}(t_n)(B_0 \cap \mathcal{H}_h) \subset B,\ t_n \geq 0. \tag{60}$$

This means that the local solutions can be extended in time so that $S_{hk}(t_n)$ exists for all $t_n \geq 0$. Our main result is the following.

Theorem 3. *Let $B \subset \mathcal{H}$ be a bounded set. There are positive numbers h^*, k^* and C such that for any $h \leq h^*$, $k \leq k^*$ and $v_0 \in B \cap \mathcal{H}_h$ there exist $T = T(h, k, B)$ and $u \in \mathcal{H}$ such that*

$$|S_{hk}(t_n)v_0 - S(t_n - T)u| \leq C(h + k), \quad t_n \geq T.$$

We need some preparations for the proof. From [16, Theorem 5.3] we deduce the following error bound. Recall that $|\cdot|$ denotes the norm of H_0^1; see (3).

Lemma 7. *Let $0 < T_1 < T_2$ and $B_0 \subset \mathcal{H}$ be a bounded set. Assume that $v \in B_0$, $v_h \in B_0 \cap \mathcal{H}_h$. Then, for $k \leq k^*(B_0)$,*

$$|S(t_n)v - S_{hk}(t_n)v_h| \leq C(B_0, T_1, T_2)\big(|v - v_h| + h + k\big), \quad t_n \in [T_1, T_2]. \tag{61}$$

Proof. Theorem 5.3 of [16] provides the following error bound. Let $B \subset \mathcal{H}$ be a bounded set. If $S(t_n)v \in B$, $S_{hk}(t_n)v_h \in B$ for $t_n \in [0, T_2]$, then for $k \leq k^*(B)$,

$$|S(t_n)v - S_{hk}(t_n)v_h| \leq C(B, T_2)\big(|v - v_h| + ht_n^{-1/2} + kt_n^{-1}\big), \quad t_n \in (0, T_2]. \tag{62}$$

Restricting the time interval and using the a priori bounds (5) and (60) we obtain an error estimate of the desired form (61). The singularities $t_n^{-1/2}$, t_n^{-1} in (62) arise because the initial value $v \in \mathcal{H}$ is not sufficiently smooth for the error bound to hold uniformly as $t_n \to 0$. □

In the next lemma we show that the approximate solution moves into a neighborhood of the attractor $\mathcal{A}$ after a finite time and remains there.

Lemma 8. *Let $B \subset \mathcal{H}$ be a bounded set and let $\epsilon > 0$. There are positive numbers T_0, k^*, h^*, depending on ϵ and B, such that, for $h \le h^*$, $k \le k^*$, we have*

$$\operatorname{dist}(S_{hk}(t_n)(B \cap \mathcal{H}_h), \mathcal{A}) < \epsilon, \quad t_n \in [T_0, \infty).$$

Proof. See, e.g., the proof of [16, Theorem 4.5]. □

Proof of Theorem 3. Without loss of generality we assume that $t = 1$ is a mesh point and write $\sigma(v) = S(1)v$ as before. Let d_0, L_0, W be given by Theorem 1. We shall extract from $S_{hk}(t_n)v_0$ a pseudotrajectory of σ for which we can apply Theorem 1.

Using Lemma 8 we find T_0, h^*, k^* depending on B and W such that

$$S_{hk}(t_n)v_0 \subset W, \quad \text{for } t_n \ge T_0, \ h \le h^*, \ k \le k^*.$$

We restrict h^*, k^* further so that

$$C(W,1,1)(h^* + k^*) \le d_0;$$

cf. Lemma 7. For $h \le h^*$, $k \le k^*$, $v_0 \in B \cap \mathcal{H}_h$ we have

$$v'_j := S_{hk}(T_0 + j)v_0 \in W \quad \text{for } j \ge 0,$$

and, by Lemma 7,

$$|\sigma(v'_j) - v'_{j+1}| = |S(1)v'_j - S_{hk}(1)v'_j| \le C(W,1,1)(h+k),$$

so that v'_j is a d-pseudotrajectory of σ with $d = C(W,1,1)(h+k) \le d_0$. Lemma 3 gives N depending on d and W such that there exists $m \in [0, N]$ with

$$\operatorname{dist}(v'_m, \mathcal{M}) \le 2d.$$

Take $u'_0 = S_{hk}(m)v'_0 = S_{hk}(T_0 + m)v_0$ and define $u'_j = S_{hk}(j)u'_0$. We have constructed u'_j so that it satisfies the assumptions of Theorem 1 and hence there is $u' \in \mathcal{H}$ such that

$$|\sigma^j(u') - u'_j| \le L_0 d \quad \text{for } j \ge 0. \tag{63}$$

We define $T = T_0 + N + 1$ and

$$\begin{aligned} \tilde{u} &= S(N-m)u', & \tilde{u}_0 &= S_{hk}(N-m)u'_0, \\ u &= S(1)\tilde{u}, & u_0 &= S_{hk}(1)\tilde{u}_0. \end{aligned}$$

Let $l \ge 0$ and take $t_n \in [T + l, T + l + 1]$. Then

$$\begin{aligned} S_{hk}(t_n)v_0 &= S_{hk}(t_n - T - l + 1)S_{hk}(l)\tilde{u}_0 = S_{hk}(t_n - T - l + 1)u'_j, \\ S(t_n - T)u &= S(t_n - T - l + 1)S(l)\tilde{u} = S(t_n - T - l + 1)\sigma^j(u'), \end{aligned}$$

where $t_n - T - l + 1 \in [1, 2]$ and $j = l + N - m$. Lemma 7 and (63) then give

$$\begin{aligned} |S(t_n - T)u - S_{hk}(t_n)v_0| &\le C(W,1,2)\big(|\sigma^j(u') - u'_j| + h + k\big) \\ &\le C(W,1,2)\big(L_0 d + h + k\big) \le C(h+k). \end{aligned}$$

□

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