On the complexity of the description of representations of $*$-algebras by unbounded operators

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Abstract

The notion of $*$-wildness is extended to classes of unbounded representations of finitely presented $*$-algebras using the notion of unbounded elements which generate a C^*-algebra. Examples of $*$-wild unbounded representations are presented. In particular, the problem of unitary classification of non-integrable representations of $\mathbb{C}[x_1, x_2]$ is shown to be equivalent to the unitary classification of $*$-representations of $C^*(\mathcal{F}_2)$, $*$-wild classes of representations of the $*$-algebra $\mathbb{C}\langle x_1, x_2 \mid [x_1, [x_1, x_2]] = 0, x_i^* = x_i, i = 1, 2\rangle$ and a $*$-algebra generated by idempotents are discussed.

In the theory of representations of algebras it was suggested to consider the representation problem to be wild if it contains the problem of describing up to a similarity a pair of matrices without relations. The complexity of the structure of representations of a $*$-algebra by bounded operators in a complex Hilbert space H was discussed in [OS, KS, PS]. It was proposed to choose, for a standard *$*$-wild problem*, the problem of describing pairs of self-adjoint (or unitary) operators up to a unitary equivalence. The representations of such an algebra are treated as complicated ones for the reasoning that it contains as a subproblem the problem of describing $*$-representations of any affine $*$-algebra. Often $*$-algebras have no bounded representations at all, or the set of bounded representations has uninteresting structure. In general, we have to deal with representations by unbounded operators. The aim of this paper is to extend the notion of $*$-wildness to unbounded representations or classes of unbounded representations of $*$-algebras. The complexity of unbounded representations of some $*$-algebras have been already discussed in [ST1, NT]. In particular, it was shown that the problem of describing non-integrable representations of the commutative algebra with two self-adjoint generators contains, as a sub-problem, the problem of classification up to a unitary equivalence representations of a free $*$-algebra with two self-adjoint generators, while any irreducible integrable representation is one-dimensional.

Many $*$-algebras of interest are introduced in terms of generators and relations, i.e. algebraic equalities imposed on the generators. Here we shall mostly concentrate our attention to this type of $*$-algebras.

The paper is organised in the following way. For the reader's convenience and to fix the notations we remind first (Sect. 1, 2) some necessary definitions and results on $*$-wild algebras and relations, and C^*-algebras generated by a finite set of unbounded elements.

Section 3 contains basic concepts and statements of the paper. Developing [OS, KS, PS] we give a definition of $*$-wildness of classes of unbounded representations for finitely presented $*$-algebras. Recall that proving that a unital $*$-algebra $\mathfrak{A}$ is $*$-wild we construct a $*$-homomorphism $\psi\colon \mathfrak{A} \to M_n(C^*(\mathcal{F}_2))$, $n \in \mathbb{N}$, which generates a full functor from the category of $*$-representations $\mathrm{Rep}(C^*(\mathcal{F}_2))$ of the enveloping C^*-algebra $C^*(\mathcal{F}_2)$ to the category of bounded $*$-representations of $\mathfrak{A}$. Here $\mathcal{F}_2$ is a free group with two generators. In order to include unbounded representations we consider "$*$-homomorphisms" ψ from $\mathfrak{A}$ into the elements which are affiliated with the C^*-algebra $CB(H)\otimes C^*(\mathcal{F}_2)$, where $CB(H)$ is the set of compact operators in a separable Hilbert space H. We study two properties of such mappings: (**P.1**) the corresponding functor F_ψ from $\mathrm{Rep}(C^*(\mathcal{F}_2))$ to a category of unbounded representations of $\mathfrak{A}$ is full and (**P.2**) $\psi(t_1), \dots, \psi(t_n)$ generate $CB(H)\otimes C^*(\mathcal{F}_2)$, where $t_1, \dots, t_n$ are generators of the $*$-algebra $\mathfrak{A}$. It is shown that the second condition is stronger then the first one (see Theorem 3 and Remark 4). Moreover, $*$-wildness of a unital $*$-algebra $\mathfrak{A}$ is equivalent to existence of a "$*$-homomorphism" having the property **P.2**. The condition **P.2** is the cornerstone in the definition of $*$-wildness of classes of unbounded representations. Using the general theory on C^*-algebras generated by unbounded elements (developed by S.L.Woronowicz) we derive different properties of $*$-wild classes of representations.

Section 4 provides us examples of $*$-wild representations. Example 1 is devoted to non-integrable representations of the commutative $*$-algebra $\mathbb{C}[x_1, x_2]$ with two generators. Using the constructions from [Schm, ST1] we prove that there exists a $*$-homomorphism $\phi\colon \mathbb{C}[x_1, x_2] \to (CB(H)\otimes C^*(\mathcal{F}_2))^\eta$ such that the corresponding functor F_ϕ is full. Example 2 deals with the $*$-algebra $\mathfrak{A} = \mathbb{C}\langle x_1, x_2 \mid [x_1, [x_1, x_2]] = 0, x_i^* = x_i, i = 1, 2\rangle$. We prove that already representations π defined on a domain formed by analytic vectors for $\pi(x_1)$ and $\pi(x_2)$ is $*$-wild. In Example 3 we discuss a class of unbounded representations generated by four idempotents with zero sum. For the basic definitions and notions of the theory of representations of $*$-algebras and C^*-algebras we refer the reader to [D, Ped, Schm].

Throughout the paper H is a separable Hilbert space, $CB(H)$ and $B(H)$ denote the algebra of compact and bounded operators respectively acting on H. For $*$-algebras A and B the algebraic tensor product of A and B is denoted by $A \odot B$. We write also $\mathrm{Rep}(\mathfrak{A})$ for the category of all nondegenerate $*$-representations of $\mathfrak{A}$, with bounded nondegenerate representations as objects and intertwining operators as morphisms.

1. $*$-Wild algebras and relations. We follow [OS] in introducing the notion of $*$-wild algebras (see also [KS, PS]). Within this section we assume the following convention: all $*$-algebras are unital and representations of $*$-algebras are unital $*$-homomorphisms into $B(H)$.

Let $\mathfrak{A}$ be a $*$-algebra. A pair $(\tilde{\mathfrak{A}}; \phi\colon \mathfrak{A} \to \tilde{\mathfrak{A}})$, where $\tilde{\mathfrak{A}}$ is a $*$-algebra and ϕ is a unital $*$-homomorphism, is called an *enveloping $*$-algebra* of the algebra $\mathfrak{A}$ if, for any $*$-representation $\pi\colon \mathfrak{A} \to B(H)$ of $\mathfrak{A}$, there exists a unique $*$-representation $\tilde{\pi}\colon \tilde{\mathfrak{A}} \to B(H)$ such that the diagram

$$\begin{array}{ccc} \tilde{\mathfrak{A}} & & \\ {\scriptstyle\phi}\uparrow & \searrow^{\tilde{\pi}} & \\ \mathfrak{A} & \xrightarrow{\ \pi\ } & B(H) \end{array}$$

is commutative.

Enveloping algebra of a C^*-algebra is unique and coincides with the algebra itself.

Let $\pi : \mathfrak{A} \to B(H)$ be a representation of $\mathfrak{A}$. It induces the representation $id \otimes \pi\colon CB(H_0) \odot \mathfrak{A} \to B(H_0 \otimes H)$ of the algebra $CB(H_0) \odot \mathfrak{A}$ on the Hilbert space $H_0 \otimes H$, $\dim H_0 < \infty$. The representation $id \otimes \pi$ determines the representation $\tilde{\pi}$ of the enveloping algebra $(\widetilde{CB(H_0) \odot \mathfrak{A}}, \phi)$ on the same Hilbert space. Now let ψ be a unital $*$-homomorphism of a $*$-algebra $\mathfrak{B}$ into the algebra $\widetilde{CB(H_0) \odot \mathfrak{A}}$. Then $\tilde{\pi} \circ \psi : \mathfrak{B} \to B(H_0 \otimes H)$ defines a representation of $\mathfrak{B}$. We can construct a functor $F_\psi\colon \mathrm{Rep}(\mathfrak{A}) \to \mathrm{Rep}(\mathfrak{B})$ in the following way:

- $F_\psi(\pi) = \tilde{\pi} \circ \psi$, for any $\pi \in \mathrm{Rep}(\mathfrak{A})$,
- $F_\psi(A) = I \otimes A$ for any operator A intertwining π_1, π_2.

(I is the identity operator in $B(H_0)$).

A $*$-algebra $\mathfrak{B}$ *majorizes* a $*$-algebra $\mathfrak{A}$ ($\mathfrak{B} \succ \mathfrak{A}$) if there exist a finite-dimensional Hilbert space H_0, an enveloping algebra $\widetilde{CB(H_0) \odot \mathfrak{A}}$ of the algebra $CB(H_0) \odot \mathfrak{A}$, and a $*$-homomorphism $\psi\colon \mathfrak{B} \to \widetilde{CB(H_0) \odot \mathfrak{A}}$ such that the functor $F_\psi\colon \mathrm{Rep}(\mathfrak{A}) \to \mathrm{Rep}(\mathfrak{B})$ is full.

It follows from the definition that two representations π_1, π_2 of $\mathfrak{A}$ are unitarily equivalent iff the representations $F_\psi(\pi_1)$, $F_\psi(\pi_2)$ of $\mathfrak{B}$ are unitarily equivalent, a representation π of $\mathfrak{A}$ is irreducible iff the representation $F_\psi(\pi)$ is irreducible. Thus the problem of unitary classification of the representations of the $*$-algebra $\mathfrak{B}$ contains, as a subproblem, the problem of unitary classification of the representations of the $*$-algebra $\mathfrak{A}$.

Example 1. 1. $\mathfrak{S}_2 = \mathbb{C}\langle a, b \mid a = a^*, b = b^*\rangle \succ \mathfrak{S}_m = \mathbb{C}\langle a_1, \dots a_m \mid a_i = a_i^*, i = 1, \dots, m\rangle$ *for any* $m = 1,\ 2,\ \dots$.

2. $\mathfrak{S}_2 \succ \mathfrak{U}_m = \mathbb{C}\langle u_1, \dots, u_m, u_1^*, \dots, u_m^* \mid u_i u_i^* = e, u_i^* u_i = e, i = 1, \dots, m\rangle$ *for any* $m = 1,\ 2,\ 3,\ \dots$.

3. $\mathfrak{U}_2 \succ \mathfrak{S}_2$ *(see [OS][Theorem 38,39,Corollary 7]).*

This result allowed, as a model of complexity for problems of unitary classification of representations of $*$-algebras, to choose the problem of unitary classification of representations of the algebra $\mathfrak{U}_2$ or, which is the same thing, its enveloping C^*-algebra $C^*(\mathcal{F}_2)$, where $\mathcal{F}_2$ is a free group with two generators.

Definition 1. *A $*$-algebra $\mathfrak{A}$ is called $*$-wild if $\mathfrak{A} \succ C^*(\mathcal{F}_2)$.*

Note that if $\mathfrak{A}$ is $*$-wild we can always find a $*$-homomorphism ψ of $\mathfrak{A}$ into the C^*-algebra $CB(H_0) \otimes C^*(\mathcal{F}_2)$ ($= M_n(C^*(\mathcal{F}_2))$ if $\dim H_0 = n$) (we take the completion of $CB(H_0) \odot C^*(\mathcal{F}_2)$ in a C^*-norm, it does not depend on the choice of such norm). We also would like to mention the following theorem ([OS]).

Theorem 1. *A C^*-algebra $\mathfrak{A}$ is $*$-wild if and only if there exist a C^*-ideal $J \subset \mathfrak{A}$ and $n \in \mathbb{N}$ such that $\mathfrak{A}/J \simeq M_n(C^*(\mathcal{F}_2))$.*

For other results and examples on $*$-wild algebras we send the reader to [OS].

2. C^*-algebras generated by a finite number of affiliated elements. The notion of a C^*-algebra generated by a finite set of unbounded affiliated elements was introduced and investigated by S. L. Woronowicz ([Wor4], see also [Wor2]). We remind here some definitions and facts from [Wor4].

Let H be a Hilbert space, $C^*(H)$ the set of separable nondegenerate C^*-subalgebra of $B(H)$, and $A \in C^*(H)$. The set of all *multipliers* $M(A)$ of A is defined by

$$M(A) = \{a \in B(H) \mid ab, ba \in A, \text{ for any } b \in A\}.$$

In particular, $M(CB(H)) = B(H)$.

Let T be a closed operator acting on a Hilbert space H. We say that T is *affiliated* with $A \in C^*(H)$ if the *z-transform* $z_T = T(I + T^*T)^{1/2}$ of T belongs to $M(A)$, $z_T^* z_T \leq I$

and $(I - z_T^* z_T)A$ is dense in A. We write $T\eta A$. The set of all elements affiliated with A will be denoted by A^η.

Let A be a C^*-algebra, $B \in C^*(H)$. The *set of morphisms* $\mathrm{Mor}(A, B)$ consists of all $\pi \in \mathrm{Rep}(A, H)$ such that $\pi(A)B$ is dense in B, where $\mathrm{Rep}(A, H)$ is the set of all nondegenerate representations of A on H. In particular, $\mathrm{Mor}(A, CB(H)) = \mathrm{Rep}(A, H)$. If $\varphi \in \mathrm{Mor}(A, B)$, then ϕ can be uniquely extended to a mapping from A^η to B^η.

The notions of multiplier algebra, affiliated elements are independent of the choice of embedding of C^*-algebras into $B(H)$ (see [Wor4]).

Let A be a C^*-algebra and $T_1, \dots, T_n$ be elements affiliated with A. We say that A *is generated* by $T_1, \dots, T_n$ if for any Hilbert space H, $B \in C^*(H)$ and any $\pi \in \mathrm{Rep}(A, H)$ the condition $\pi(T_i)\eta B$, $i = 1, \dots, n$ implies $\pi \in \mathrm{Mor}(A, B)$. In what follows we will use the following sufficient condition for a C^*-algebra A to be generated by elements affiliated with A.

Theorem 2. *Let A be a C^*-algebra and $T_1, \dots, T_n$ be elements affiliated with A. The subset of $M(A)$ composed of all elements of the form $(I + T_i^* T_i)^{-1}$ and $(I + T_i T_i^*)^{-1}$, $i = 1, \dots, n$ will be denoted by Γ. Assume that*

1. *$T_1, \dots, T_n$ separate representations of A: if φ_1, φ_2 are different elements of Rep(A, H) then $\varphi_1(T_i) \neq \varphi_2(T_i)$ for some $i = 1, \dots, n$.*

2. *There exist elements $r_1, \dots, r_k \in \Gamma$ such that the product $r_1 \dots r_k \in A$.*

Then A is generated by $T_1, \dots, T_n$.

Remark 1. A C^*-algebra A is generated by $T_1, \dots, T_n \eta A$ with $||T_i|| < \infty$ for all $i = 1, \dots, n$ if and only if A is unital and A coincides with the norm closure of all algebraic combinations of I, $T_1, \dots, T_n$.

Unfortunately, there is no canonical way for given unbounded operators $T_1, \dots, T_n$ on H to construct a C^*-algebra A generated by $T_1, \dots, T_n \eta A$. Moreover, even the existence of such C^*-algebra is not guaranteed. See [Wor4] for other discussion on C^*-algebras generated by affiliated elements.

3. The complexity of unbounded representations of $*$-algebras. In this section we shall extend the notion of $*$-wild problem to unbounded representations. We restrict our attention to finitely generated unital $*$-algebras.

Let $\mathfrak{A}$ be a unital $*$-algebra generated by elements $t_1 \dots, t_n$, $t_1^*, \dots, t_n^*$ and relations

$$w_j(t_1, \dots, t_n, t_1^*, \dots, t_n^*) = 0, \qquad j = 1, \dots, m. \tag{1}$$

Here w_j are polynomials over $\mathbb{C}$ in the noncommuting variables $t_1, \dots, t_n$, $t_1^*, \dots, t_n^*$ and e, where e is the unity of $\mathfrak{A}$.

Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space H. A family of closed operators $T_1, \dots, T_n, T_1^*, \dots, T_n^*$ defined on $\mathcal{D}$ is called a *representation* π *of* $\mathfrak{A}$ *on* $\mathcal{D}$ if $\mathcal{D}$ is invariant with respect to all operators of the family, $\mathcal{D}$ is a core for T_i, T_i^*, $i = 1, \dots, n$ and the relations

$$w_j(T_1, \dots, T_n, T_1^*, \dots, T_n^*)\varphi = 0, \qquad j = 1, \dots, m.$$

hold for any $\varphi \in \mathcal{D}$ with the identity operator on H instead of the unity e. We denote by $\mathrm{Rep}_{unb}(\mathfrak{A})$ the category of unbounded representations of $\mathfrak{A}$. The objects of $\mathrm{Rep}_{unb}(\mathfrak{A})$ are representations defined above and the morphisms of $\mathrm{Rep}_{unb}(\mathfrak{A})$ are bounded operators $C \in B(H, \tilde{H})$ intertwining representations π and $\tilde{\pi}$ which act on Hilbert spaces H and $\tilde{H}$ respectively, i.e.

$$CT_i \subseteq \tilde{T}_i C \qquad CT_i^* \subseteq \tilde{T}_i^* C, \quad i = 1, \dots, n$$

(we write $C \in I(\pi, \tilde{\pi})$). We say that two representations π_1 and π_2 are unitarily equivalent if there exists a unitary operator of $H(\pi_1)$ onto $H(\pi_2)$ such that $U \in I(\pi_1, \pi_2)$ and $U^{-1} \in I(\pi_2, \pi_1)$. In this case we write $\pi_1 \simeq \pi_2$.

Proving that the problem of unitary classification of bounded representations of a $*$-algebra A is $*$-wild we construct a unital $*$-homomorphism $\psi\colon A \to CB(H) \otimes C^*(\mathcal{F}_2)$ which generates a full functor $F_\psi\colon \mathrm{Rep}(C^*(\mathcal{F}_2)) \to \mathrm{Rep}(\mathfrak{A})$. In order to include unbounded representations we shall replace the above $*$-homomorphism by "$*$-homomorphisms" into the set of affiliated elements $(CB(H) \otimes C^*(\mathcal{F}_2))^\eta$ (here H is not necessarily finite-dimensional).

Namely, let $\psi\colon \mathfrak{A} \to (CB(H) \otimes C^*(\mathcal{F}_2))^\eta$ be a unital mapping from $\mathfrak{A}$ to $(CB(H) \otimes C^*(\mathcal{F}_2))^\eta$ such that there exists a dense linear subset D of $CB(H) \otimes C^*(\mathcal{F}_2)$ satisfying the following conditions:

- D is invariant with respect to $\psi(T_i)$, $i = 1, \dots, n$,
- $w_j(\psi(t_1), \dots, \psi(t_n), \psi(t_1)^*, \dots, \psi(t_n)^*)a = 0, \qquad j = 1, \dots, m, \quad a \in D,$
- D is a core for $\psi(t_1), \dots, \psi(t_n)$.

In the sequel, whenever we write a $*$-homomorphism ψ from $\mathfrak{A}$ to $(CB(H) \otimes C^*(\mathcal{F}_2))^\eta$ we mean a unital mapping $\psi\colon \mathfrak{A} \to (CB(H) \otimes C^*(\mathcal{F}_2))^\eta$ satisfying the above conditions.

As before, the mapping ψ generates a functor $F_\psi\colon \mathrm{Rep}(C^*(\mathcal{F}_2)) \to \mathrm{Rep}_{unb}(\mathfrak{A})$:

- $F_\psi(\pi)(t_i) = (id \otimes \pi)(\psi(t_i))$ for any representation $\pi \in \mathrm{Rep}(C^*(\mathcal{F}_2))$, where $id \otimes \pi$ is the unique extension to the affiliated elements,
- $F_\psi(A) = I \otimes A$ for any A intertwining π_1, π_2.

If π is a representation of $C^*(\mathcal{F}_2)$ on a Hilbert space $H(\pi)$ then $F_\psi(\pi)(t_i)$, $i = 1, \dots, n$ define a representation of $\mathfrak{A}$ on $\mathcal{D} = \{(id \otimes \pi)(D)\varphi, \varphi \in H \otimes H(\pi)\}$, which is dense in $H \otimes H(\pi)$.

Consider the following two properties:

P.1. The functor F_ψ is full.

P.2. $\psi(t_1), \dots, \psi(t_n)$ generate $CB(H) \otimes C^*(\mathcal{F}_2)$.

Theorem 3. P.2 $\Rightarrow$ P.1.

Proof. To prove the statement it is enough to show that given an operator $C \in B(H)$, a representation $\pi \in \mathrm{Rep}(C^*(\mathcal{F}_2))$ such that $CF_\psi(\pi)(t_i) \subseteq F_\psi(\pi)(t_i)C$, $CF_\psi(\pi)(t_i)^* \subseteq F_\psi(\pi)(t_i)^*C$, we have $C = I \otimes A$, where $A \in I(\pi, \pi)$.

We first show that C commutes with $(id \otimes \pi)(a)$ for any $a \in CB(H) \otimes C^*(\mathcal{F}_2)$. Define the following set:

$$\mathfrak{B}_C = \left\{ a \in M(\mathfrak{B}) \colon \begin{array}{l} C(id \otimes \pi)(a) = (id \otimes \pi)(a)C, \\ C(id \otimes \pi)(a^*) = (id \otimes \pi)(a^*)C \end{array} \right\}$$

Here $\mathfrak{B} = CB(H) \otimes C^*(\mathcal{F}_2)$. $\mathfrak{B}_C$ is a C^*-subalgebra of $M(\mathfrak{B})$. Let us show now that $\mathfrak{B}_C = M(\mathfrak{B})$.

(1) $\mathfrak{B}_C$ is nonempty, since the z-transforms $z_{\psi(t_i)}$, $z^*_{\psi(t_i)}$, $i = 1, \dots, n$ belong to $\mathfrak{B}_C$.

(2) $\mathfrak{B}_C$ separates representations of $\mathfrak{B}$. Indeed, let π_1, π_2 are two different representations of $\mathfrak{B}$. Assuming that $\pi_1(q) = \pi_2(q)$ for any $q \in \mathfrak{B}_C$ we get $\pi_1(z_{\psi(t_i)}) = \pi_2(z_{\psi(t_i)})$, $i = 1, \dots, n$ which implies $\pi_1(\psi(t_i)) = \pi_2(\psi(t_i))$ for any $i = 1, \dots, n$. Since $\psi(t_1), \dots \psi(t_n)$ generate $\mathfrak{B}$, we get $\pi_1 = \pi_2$. A contradiction.

(3) $\mathfrak{B}_C$ is strictly closed. In fact, let $\{x_\lambda\} \in \mathfrak{B}$ strictly converge to x, i.e. $||a(x_\lambda - x)|| \to 0$ and $||(x_\lambda - x)a|| \to 0$ for any $a \in \mathfrak{B}$. If $\pi \in \mathrm{Rep}(\mathfrak{B}, H)$, we get $||\pi(x_\lambda a) - \pi(xa)|| \to 0$ and

$$(\pi(x_\lambda) - \pi(x))\pi(a)\varphi \to 0, \quad a \in \mathfrak{B},\ \varphi \in H.$$

Since π is a nondegenerate representation of $\mathfrak{B}$, the set $\{\pi(a)\varphi \colon a \in \mathfrak{B}, \varphi \in H\}$ is dense in H and we deduce that

$$(\pi(x_\lambda) - \pi(x))\varphi \to 0, \quad \varphi \in H.$$

Therefore, if $C\pi(x_\lambda) = \pi(x_\lambda)C$ we get $C\pi(x) = \pi(x)C$ and $x \in \mathfrak{B}_C$.

According to [Wor4][Proposition 2.2] (Stone-Weierstrass Theorem), (1), (2), (3) imply

$$\mathfrak{B}_C = M(\mathfrak{B})$$

and hence $[C, (id \otimes \pi)(a)] = 0$ for any $a \in CB(H) \otimes C^*(\mathcal{F}_2)$. This gives $C = I \otimes A$, where $[A, \pi(a)] = 0$ for any $a \in C^*(\mathcal{F}_2)$ and hence the functor F_ψ is full. $\square$

We shall see below (Example 2) that the inverse implication is not true in general.

In the sequence we shall consider classes R of representations from $\mathrm{Rep}_{unb}(\mathfrak{A})$ which are closed with respect to the direct sum and taking subrepresentation, i.e. R satisfies the following conditions:

a) if $\pi_1 \in R$ and $\pi_1 \simeq \pi_2$ then $\pi_2 \in R$;

b) if $(\pi_\lambda)_{\lambda\in\Lambda}$ is a family of representations from R, where Λ is a countable set of indexes then $\oplus_{\lambda\in\Lambda}\pi_\lambda \in R$;

c) if $\pi_1 \oplus \pi_2 \in R$ then $\pi_i \in R$, $i = 1, 2$.

Definition 2. *Let $\mathfrak{A}$ be a unital $*$-algebra generated by $t_1, \dots, t_n$ and relations (1). We say that a class $R \subset \mathrm{Rep}_{unb}(\mathfrak{A})$ is $*$-wild if there exists a $*$-homomorphism $\psi\colon \mathfrak{A} \to (CB(H) \otimes C^*(\mathcal{F}_2))^\eta$ such that $(id \otimes \pi)(\psi)$ belongs to R for any $\pi \in Rep(C^*(\mathcal{F}_2))$ and $\psi(t_1), \dots, \psi(t_n)$ generate the C^*-algebra $CB(H) \otimes C^*(\mathcal{F}_2)$.*

Proposition 1. *The class of bounded representations of $\mathfrak{A}$ is $*$-wild if and only if $\mathfrak{A}$ is $*$-wild.*

Proof. Assume that there exists a $*$-homomorphism $\psi\colon \mathfrak{A} \to (CB(H) \otimes C^*(\mathcal{F}_2))^\eta$ such that the operators $(id \otimes \pi)(\psi(t_i))$, $i = 1, \dots, n$ are bounded for any $\pi \in \mathrm{Rep}(C^*(\mathcal{F}_2))$ and $\psi(t_1), \dots, \psi(t_n)$ generate $\mathfrak{B} = CB(H) \otimes C^*(\mathcal{F}_2)$. Clearly, $\psi(t_i)$, $i = 1, \dots, n$ are bounded. Then it follows from [Wor4, Example 1] that $\mathfrak{B}$ is unital and coincides with the norm closure of all algebraic combinations of I, $\psi(t_1), \dots, \psi(t_n)$. This implies that H is finite-dimensional and ψ is a $*$-homomorphism from $\mathfrak{A}$ to $CB(H) \otimes C^*(\mathcal{F}_2)$. Moreover, by Theorem 3, the functor generated by ψ is full which means that the $*$-algebra $\mathfrak{A}$ is $*$-wild.

Conversely, suppose that $\mathfrak{A}$ is $*$-wild. Then there exists a $*$-homomorphism $\psi\colon \mathfrak{A} \to CB(H) \otimes C^*(\mathcal{F}_2)$ with $\dim H < \infty$. Let $\tilde{\mathfrak{B}}$ be the norm closure of the set of all algebraic combinations of I, $\psi(t_1), \dots, \psi(t_n)$. Then $\tilde{\mathfrak{B}}$ is a C^*-subalgebra of $CB(H)\otimes C^*(\mathcal{F}_2)$. Since $\mathfrak{A}$ is $*$-wild, $\pi(\psi(\mathfrak{A}))' = \pi(CB(H)\otimes C^*(\mathcal{F}_2))'$ for any representation π of $CB(H)\otimes C^*(\mathcal{F}_2)$ (see the proof of [OS, Theorem 50]). This implies $\pi(\tilde{\mathfrak{B}})' = \pi(CB(H) \otimes C^*(\mathcal{F}_2))'$. By [OS, Lemma 14], the inclusion $i\colon \tilde{\mathfrak{B}} \to CB(H) \otimes C^*(\mathcal{F}_2)$ is a surjection, and hence $\tilde{\mathfrak{B}} = CB(H) \otimes C^*(\mathcal{F}_2)$. Since the C^*-algebra $CB(H) \otimes C^*(\mathcal{F}_2)$ is unital, by Remark 1, we see that $\psi(t_1), \dots, \psi(t_n)$ generate $CB(H) \otimes C^*(\mathcal{F}_2)$ in the sense of affiliated elements. The proof is complete. □

Let $\mathfrak{A}$ be a $*$-algebra generated by $t_1, \dots, t_n$ and relations (1). We say that a class $R \subset \mathrm{Rep}_{unb}(\mathfrak{A})$ is *manageable* if there exists a separable C^*-algebra $\mathfrak{B}$ (unital or non-unital) and $T_1, \dots, T_n \,\eta\, \mathfrak{B}$ such that $\mathfrak{B}$ is generated by $T_1, \dots, T_n$ and if any representation $\pi \in R$ is generated by a representation of $\mathfrak{B}$, i.e. if there exists $\tilde{\pi} \in \mathrm{Rep}(\mathfrak{B}, H)$ such that $\pi(t_i) = \tilde{\pi}(T_i)$, $i = 1, \dots, n$ (see [Wor4]).

Proposition 2. *Let R be a manageable class of $*$-representations of $\mathfrak{A}$. Let $\mathfrak{B}$ be the C^*-algebra defined above. Then R is $*$-wild if and only if there exist a C^*-ideal J and a Hilbert space H such that*

$$\mathfrak{B}/J \simeq CB(H) \otimes C^*(\mathcal{F}_2). \tag{2}$$

Proof. Let R be $*$-wild. Then there exists $\psi \colon \mathfrak{A} \to (CB(H) \otimes C^*(\mathcal{F}_2))^\eta$ such that $\psi(t_1), \ldots, \psi(t_n)$ generate the C^*-algebra $CB(H) \otimes C^*(\mathcal{F}_2)$ and $(id \otimes \pi)(\psi) \in R$ for any representation $\pi \in \mathrm{Rep}(C^*(\mathcal{F}_2))$. Let us take π_0 an embedding of $C^*(\mathcal{F}_2)$ into $B(H_0)$. Let $\tilde{\pi}_0$ be the representation of $\mathfrak{B}$ such that $\tilde{\pi}_0(T_i) = (id \otimes \pi_0)(\psi(t_i))$, $i = 1, \ldots, n$. Since $T_1, \ldots, T_n$ generate $\mathfrak{B}$ and $id \otimes \pi_0$ is injection, by [Wor4][Proposition 4.5] there exists $\pi' \in Mor(\mathfrak{B}, CB(H) \otimes C^*(\mathcal{F}_2))$ such that

$$\tilde{\pi} = (id \otimes \pi)\pi'$$

and $\psi(t_i) = \pi'(T_i)$. Then since $\psi(t_1), \ldots, \psi(t_n)$ generate $CB(H) \otimes C^*(\mathcal{F}_2)$, applying [Wor4][Proposition 3.2] we conclude that $\pi'(\mathfrak{B}) = CB(H) \otimes C^*(\mathcal{F}_2)$ and hence there exists a $*$-ideal $J = \ker \pi'$ such that (2) holds. The converse is obvious. $\square$

Remark 2. If we have a $*$-homomorphism $\psi \colon \mathfrak{A} \to (CB(H) \otimes C^*(\mathcal{F}_2))^\eta$ which generates a full functor $F_\psi : \mathrm{Rep}(C^*(\mathcal{F}_2)) \to R \subset \mathrm{Rep}_{unb}(\mathfrak{A})$, then the problem of unitary classification of $*$-representations R of $\mathfrak{A}$ is difficult and contains as a subproblem the problem of unitary classification of all representations of $C^*(\mathcal{F}_2)$. Namely, the representation $((id \otimes \pi)(\psi)(t_1), \ldots, (id \otimes \pi)(\psi)(t_n))$, $\pi \in C^*(\mathcal{F}_2)$, is irreducible iff π is irreducible, two such representations are unitarily equivalent iff the corresponding representations of $C^*(\mathcal{F}_2)$ are unitarily equivalent.

Proposition 3. *Let $\mathfrak{B}$ be a C^* -algebra with the property (2), in particular, a $*$-wild unital C^*-algebra, let $\psi \colon \mathfrak{A} \to (CB(H_0) \otimes \mathfrak{B})^\eta$ be a unital $*$-homomorphism such that $\psi(t_1), \ldots, \psi(t_n)$ generate $CB(H_0) \otimes \mathfrak{B}$. Assume that the representation $((id \otimes \pi)(\psi(t_1)), \ldots$ $(id \otimes \pi)(\psi(t_n))$ belongs to a class $R \subset \mathrm{Rep}_{unb}(\mathfrak{A})$ for any $\pi \in \mathrm{Rep}(\mathfrak{B})$. Then R is $*$-wild.*

Proof. By the assumption, there exists a $*$-homomorphism $\varphi \colon \mathfrak{B} \to CB(H_1) \otimes C^*(\mathcal{F}_2)$ such that

$$\varphi(\mathfrak{B}) = CB(H_1) \otimes C^*(\mathcal{F}_2). \tag{3}$$

If we prove that $(id \otimes \varphi)(\psi(t_i))$, $i = 1, \ldots, n$ generate $\mathcal{A} = CB(H_0) \otimes CB(H_1) \otimes C^*(\mathcal{F}_2)$, the assertion follows.

Let $\mathcal{H}$ be a separable Hilbert space and $\pi \in \mathrm{Rep}(\mathcal{A}, \mathcal{H})$. Then there exists a unitary operator $V \in B(\mathcal{H})$ and a representation $\rho \in \mathrm{Rep}(C^*(\mathcal{F}_2))$ such that $\pi = V(id_{H_0} \otimes id_{H_1} \otimes$

$\rho)V^{-1}$. Let $\mathcal{C} \in C^*(\mathcal{H})$ with the property $\pi((id_{H_0} \otimes \varphi)(\psi(t_i)))\eta\ \mathcal{C}$. Then $(id_{H_0} \otimes id_{H_1} \otimes \rho)((id_{H_0} \otimes \varphi)(\psi(t_i)))\ \eta\ V^{-1}\mathcal{C}V =: \tilde{\mathcal{C}}$. Since $\tilde{\rho} := (id_{H_1} \otimes \rho)(\varphi)$ is a representation of $\mathfrak{B}$, $id_{H_0} \otimes \tilde{\rho} \in Mor(CB(H_0) \otimes \mathfrak{B}, \tilde{\mathcal{C}})$. From this and (3) we conclude that $id_{H_0} \otimes id_{H_1} \otimes \rho \in Mor(\mathcal{A}, \tilde{\mathcal{C}})$ and $\pi \in Mor(\mathcal{A}, \mathcal{C})$. The proof is finished. □

Let R be a class of $\mathrm{Rep}_{unb}(\mathfrak{A})$, and let H and H_0 be separable Hilbert spaces, $\dim H = \infty$. We denote by $R(\mathcal{H})$ the set of all representations $\pi \in R$ acting on $\mathcal{H} = H \otimes H_0$. Assume that there exists a $*$-homomorphism $\psi : \mathfrak{A} \to (CB(H) \otimes C^*(\mathcal{F}_2))^\eta$. We shall write $R_\psi(\mathcal{H})$ for the subset of those representations $\pi \in R(\mathcal{H})$ which are generated by ψ i.e. $R_\psi(\mathcal{H}) = \{((\pi)(\psi(t_1)), \dots (\pi)(\psi(t_n)), \pi \in \mathrm{Rep}(CB(H) \otimes C^*(\mathcal{F}_2))\}$. Then $R(\mathcal{H})$ and $R_\psi(\mathcal{H})$ are complete subsets of $(\mathbb{C}^N \otimes CB(\mathcal{H}))^\eta$. We recall that a set of representations $\mathcal{R}$ in $(\mathbb{C}^N \otimes CB(\mathcal{H}))^\eta$ is complete if

a) $\mathcal{R}$ is closed under the adjoint action of isometries: if $\pi \in \mathcal{R}$ and V is an isometry such that $VV^* \in I(\pi, \pi)$ then $V^*\pi V \in \mathcal{R}$;

b) $\mathcal{R}$ is closed under V-direct sums: if $\pi_\lambda \in \mathcal{R}$ for all $\lambda \in \Lambda$ (Λ is denumerable set) then $V(\oplus_{\lambda\in\Lambda}\pi_\lambda)V^* \in \mathcal{R}$ for any unitary operator $V : \oplus_{\lambda\in\Lambda}\mathcal{H} \to \mathcal{H}$.

To formulate next result we recall the definition of the C^*-algebras of continuous operator functions and continuous operator functions vanishing at infinity defined on a complete set $\mathcal{R}$.

For any complete set $\mathcal{R}$ in $(\mathbb{C}^N \otimes CB(\mathcal{H}))^\eta$ we define mappings $F : \mathcal{R} \to (CB(\mathcal{H}))^\eta$ such that $F(V^*\pi V) = V^*F(\pi)V$ for any $\pi \in \mathcal{R}$ and every isometry V with $VV^* \in I(\pi, \pi)$. In this case F is called operator function on $\mathcal{R}$. Following [Wor1] we denote the set of all such mappings by $\mathcal{F}(\mathcal{R})$. A function $F \in \mathcal{F}(\mathcal{R})$ is said to be bounded if $F(\pi) \in \mathbb{C}^N \otimes B(\mathcal{H})$ for any $\pi \in \mathcal{R}$. We endow $\mathcal{R}$ with the topology of almost uniform convergence and define continuous operator functions on $\mathcal{R}$ to be

$$C(\mathcal{R}) = \{F \in \mathcal{F}_{bounded}(\mathcal{R}) \mid \mathcal{R} \ni \pi \to F(\pi) \in (C^N \otimes CB(\mathcal{H}))^\eta \text{ is continuous}\}.$$

So defined set is a C^*-algebra with unit (see [Wor1]). A representation ρ of $C(\mathcal{R})$ is called singular if it is disjoint with any representation of the form $\rho_\pi(F) = F(\pi)$, $\pi \in \mathcal{R}$, $F \in C(\mathcal{R})$.

Continuous operator functions vanishing at infinity are defined as follows

$$C_\infty(\mathcal{R}) = \{f \in C(\mathcal{R}) \mid \rho(F) = 0 \text{ for any singular representation } \pi\}.$$

$C_\infty(\mathcal{R})$ is a C^*-algebra ([Wor1]).

Proposition 4. *If ψ is a $*$-homomorphism such that the property* **P.2** *holds then*

$$\mathrm{Rep}(CB(H) \otimes C^*(\mathcal{F}_2), \mathcal{H}) \ni \pi \to (\pi(\psi(t_i)))_{i=1}^n \in \mathcal{R}_\psi(\mathcal{H})$$

is a homeomorphism and

$$CB(H) \otimes C^*(\mathcal{F}_2) \simeq C_\infty(\mathcal{R}_\psi(\mathcal{H})).$$

Proof. It follows from [Wor4, Theorem 6.2] and the fact that

$$CB(H) \otimes C^*(\mathcal{F}_2) \simeq C_\infty(\mathrm{Rep}(CB(H) \otimes C^*(\mathcal{F}_2), \mathcal{H}).$$

□

Corollary 1. *If R is a $*$-wild class of representations then there exists a complete subclass $\mathcal{R}$ of representations acting on a Hilbert space $\mathcal{H}$ such that $C_\infty(\mathcal{R}) \simeq CB(H) \otimes C^*(\mathcal{F}_2)$ for some Hilbert space H.*

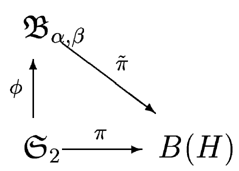
Remark 3. In the contrary with Proposition 4 the structure of the set $\mathcal{R}(\mathcal{H})$ is not necessary isomorphic to $\mathrm{Rep}\, C_\infty(\mathcal{R}(\mathcal{H}))$. The C^*-algebra $C_\infty(\mathcal{R}(\mathcal{H}))$ can be very small and even trivial. But if $\mathcal{R}(\mathcal{H})$ is locally compact (see [Wor1] for the definition) then the mapping $\mathcal{R}(\mathcal{H}) \ni \pi \to \rho_\pi \in \mathrm{Rep}\, C_\infty(\mathcal{R}(\mathcal{H}))$ is a homeomorphism and in this case one can prove that R is $*$-wild iff $C_\infty(\mathcal{R}(\mathcal{H}))$ contains an ideal J such that

$$C_\infty(\mathcal{R}(\mathcal{H}))/J \simeq CB(H) \otimes C^*(\mathcal{F}_2).$$

4. Examples

Example 2. Unbounded representations of the commutative algebra with two generators. Let $\mathfrak{A} = \mathbb{C}[x_1, x_2]$ be the commutative unital algebra of complex polynomials in the real variables x_1, x_2. It is known that any irreducible integrable representation π of the algebra is one-dimensional. Recall that for $\mathbb{C}[x_1, x_2]$ a representation $\pi = (X_1 = X_1^*, X_2 = X_2^*)$ is integrable if spectral projections of X_1 and X_2 commute. An equivalent condition (see [N]) is that $X_1 X_2 \varphi = X_2 X_1 \varphi$, where φ belongs to a dense domain, invariant with respect to the operators of the representation and consisting of analytic vectors for X_1, X_2. The result which was given without proof in [ST1] is that non-integrable representations have more complicated structure and the classification of such representations contain as a subproblem the problem of unitary classification of representations of the $*$-algebra $\mathfrak{S}_2$ (see Example 1). We repeat relevant material from [ST1].

Let α, $\beta > 0$. Consider representations π of $\mathfrak{S}_2$ in a Hilbert space H such that $||\pi(a)|| \leq \alpha$, $||\pi(b)|| \leq \beta$ and $\pi(a), \pi(b) > 0$. Then there exists the unique unital C^*-algebra $\mathfrak{B}_{\alpha,\beta}$ which makes the diagram



commutative for any such $*$-representation π of $\mathfrak{S}_2$. We write

$$\mathfrak{B}_{\alpha,\beta} = C^*(a, b; a > 0, b > 0, ||a|| \leq \alpha, ||\beta|| \leq \beta).$$

Consider the following construction, analogous to the one in [Schm]. Consider $p, q \in M_3(\mathfrak{B}_{\alpha,\beta})$ given by

$$p = \begin{pmatrix} \lambda e & \mu a & 0 \\ \mu a & \lambda e & \mu b \\ 0 & \mu b & \lambda e \end{pmatrix}, \qquad q = \begin{pmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

here λ, $\mu \in \mathbb{R}$ are such that $1/2 < p < 3/4$, e is the unit element in $\mathfrak{B}_{\alpha,\beta}$. Let w_1, $w_2 \in M_2(M_3(\mathfrak{B}_{\alpha,\beta})) \simeq M_6(\mathfrak{B}_{\alpha,\beta})$ be defined by

$$w_1 = \begin{pmatrix} i(e_3 - 2q) & 0 \\ 0 & e_3 \end{pmatrix}, \quad w_2 = \begin{pmatrix} e_3 - 2p & -2(p - p^2)^{1/2} \\ -2(p - p^2)^{1/2} & 2p - e_3 \end{pmatrix},$$

where e_3 is the unit element in $M_3(\mathfrak{B}_{\alpha,\beta})$. Since $1/2 < p < 3/4$, the element w_2 is well-defined.

Let H be an infinite-dimensional separable Hilbert space and let $\{f_k, k \in \mathbb{Z}\}$ be an orthonormal basis in H. Let P_i be the projection onto $\mathbb{C}\langle f_k\rangle$ and v be the shift operator, i.e., $vf_k = f_{k+1}$, $k \in \mathbb{Z}$. We define now v_1, v_2, $v_3 \in L(H) \odot M_6(\mathfrak{B}_{\alpha,\beta})$ by

$$v_1 = v \otimes e_6, \quad v_2 = v(I - P_1) \otimes e_6 + vP_1 \otimes w_1,$$
$$v_3 = v(I - P_1 - P_2) \otimes e_6 + vP_1 \otimes w_1 + vP_2 \otimes w_2,$$

here e_6 is the unity in $M_6(\mathfrak{B}_{\alpha,\beta})$.

Finally, we define operators U_1 and $U_2 \in L(H) \odot L(H) \odot M_6(\mathfrak{B}_{\alpha,\beta}) \subset L(\mathcal{H}) \odot \mathfrak{B}_{\alpha,\beta}$, $\mathcal{H} = \oplus_{i=1}^{6} H \otimes H$ by

$$U_1 = v \otimes E, \quad U_2 = (\sum_{i=-\infty}^{1} P_i) \otimes v_1 + P_2 \otimes v_2 + (\sum_{i=3}^{+\infty} P_i) \otimes v_3$$

Here E is the unity in $L(H) \odot M_6(\mathfrak{B}_{\alpha,\beta})$. U_1, U_2 are easily seen to be unitary elements of $M(CB(\mathcal{H}) \otimes \mathfrak{B}_{\alpha,\beta})$.

Proposition 5. *There exist selfadjoint elements X_1, X_2 η $CB(\mathcal{H}) \otimes \mathfrak{B}_{\alpha,\beta}$ such that U_1, U_2 are the Cayley transforms of X_1 and X_2 respectively.*

Proof. We denote by A the C^*-algebra $CB(\mathcal{H}) \otimes \mathfrak{B}_{\alpha,\beta}$. According to [WN][Proposition 5.1, Theorem 5.2], it suffices to show that $(I - U_i^*)A$ is dense in A, $i = 1, 2$. Suppose for the moment that the statement is false. Then $(I - U_i^*)A$ is a proper right ideal in A and there exists a pure state on A such that $f((I - U_i^*)a) = 0$, for any $a \in A$ ([D, Theorem 2.9.5]). Using the GNS procedure we can construct a representation $\pi \in \mathrm{Rep}(A, H)$ and a cyclic vector $\varphi \in H$ such that $f(a) = (\varphi \mid \pi(a)\varphi)$ for all $a \in A$. Thus

$$0 = f((I - U_i^*)a) = (\varphi \mid \pi(I - U_i^*)\pi(a)\varphi) = ((I - \pi(U_i))\varphi \mid \pi(a)\varphi)$$

Since φ is a cyclic vector, $\pi(A)\varphi$ is dense in H. This implies $(I - \pi(U_i))\varphi = 0$ and hence $\varphi \in \ker(I - \pi(U_i))$. Any nondegenerate representation π of $CB(\mathcal{H}) \otimes \mathfrak{B}_{\alpha,\beta}$ is of the form $Vid \otimes \pi_0 V^*$ where π_0 is a nondegenerate representation of $\mathfrak{B}_{\alpha,\beta}$. It is easy to check that $\ker((id \otimes \pi_0)(U_i) - I) = \{0\}$, for any π_0. A contradiction. By [WN][Proposition 5.1], U_1, U_2 are the Cayley transform of selfadjoint operators $X_1 \eta A$ and $X_2 \eta A$ respectively. Moreover, $D(X_i) = (I - U_i^*)CB(\mathcal{H}) \otimes \mathfrak{B}_{\alpha,\beta}$, $i = 1, 2$. □

Proposition 6. *ψ: $\mathbb{C}[x_1, x_2] \ni x_i \to X_i \in (CB(\mathcal{H}) \otimes \mathfrak{B}_{\alpha,\beta})^\eta$ is a $*$-homomorphism.*

Proof. Let us consider the following set $A_{1,1} = [U_1, U_2]A$, where $A = CB(\mathcal{H}) \otimes \mathfrak{B}_{\alpha,\beta}$. Using a simple computation one can show that

$$[U_1, U_2] = P_3 v \otimes P_3 v \otimes (e_6 - w_2) + P_2 v \otimes P_2 v \otimes (e_6 - w_1)$$

and $Q_{1,1} = P_3 \otimes P_3 \otimes ((e_6 - w_2)/2) + P_2 \otimes P_2 \otimes \begin{pmatrix} e_3 & 0 \\ 0 & 0 \end{pmatrix} \in A$ is the projection of A onto $A_{1,1}$. Define now

$$D = (U_1^* - I)(U_2^* - I)(I - Q_{1,1})A.$$

The proposition follows from the following lemma.

Lemma 1. *D is dense in A, D is a core for X_1, X_2 and $X_1X_2a = X_2X_1a$, for any $a \in D$.*

Proof. The proof is similar to that of [Schm][Lemma 9.3.2, 9.3.3, 9.3.4]. We begin by showing that $X_1X_2a = X_2X_1a$ for any $a \in D$. By definition of the projection $Q_{1,1}$, we have $(I - Q_{1,1})[U_1, U_2] = (I - Q_{1,1})[U_1 - I, U_2 - I] = 0$ which implies

$$[U_1^* - I, U_2^* - I](I - Q_{1,1}) = 0$$

and

$$(U_1^* - I)(U_2^* - I)b = (U_2^* - I)(U_1^* - I)b, \quad b \in (I - Q_{1,1})A.$$

Let $a \in D$. Then $a = (U_1^* - I)(U_2^* - I)b = (U_2^* - I)(U_1^* - I)b$ with $b \in (I - Q_{1,1})A$. Remembering that each element of $D(X_i)$ is of the form $(I - U_i^*)a$, $a \in A$, $i = 1, 2$, we see that $a \in D(X_1X_2) \cap D(X_2X_1)$ and

$$b = ((X_2 - i)/2i)((X_1 - i)/2i)a = ((X_1 - i)/2i)((X_2 - i)/2i)a$$

which implies $X_2X_1a = X_1X_2a$ for any $a \in D$.

To prove that D is a core for X_1 and X_2 we have to show that $D = (I - z_i^* z_i)^{1/2} D_i$, where D_i is a dense subset in A, here z_i is the z-transform of X_i, $i = 1,2$. Let $U_i - I = V_i|U_i - I|$ be the polar decomposition of $U_i - I$ (see [WN][Proposition 0.2]). Then $V_i \in M(A)$ and V_i is unitary because $(U_i - I)A$ and $(U_i^* - I)A$ are dense in A. One can check that $\sqrt{I - z_i^* z_i} = i|U_i - I|/2 = i(U_i^* - I)V_i/2$. Since $D = (U_1^* - I)(U_2^* - I)(I - Q_{1,1})A = (U_2^* - I)(U_1^* - I)(I - Q_{1,1})A$, it is sufficient to show now that $(U_i^* - I)(I - Q_{1,1})A$ is dense in A for $i = 1,2$. This follows by the same method as in the proof of Proposition 5 and completes the proof of the lemma and the proposition. □

□

Theorem 4. *The functor F_ψ is full.*

Proof. Let H_1, H_2 be two separable Hilbert spaces and $\pi_i \in \mathrm{Rep}(\mathfrak{B}_{\alpha,\beta}, H_i)$, $i = 1,2$. Then $\tilde{\pi}_i = id_{H_i} \otimes \pi_i$ is a representations of $CB(H_i) \otimes \mathfrak{B}_{\alpha,\beta}$, $i = 1,2$. Let C be a bounded operator intertwining representations $(\tilde{\pi}_1(X_1), \tilde{\pi}_1(X_2))$ and $(\tilde{\pi}_2(X_1), \tilde{\pi}_2(X_2))$ i.e.

$$C\tilde{\pi}_1(X_1) \subseteq \tilde{\pi}_2(X_1)C, \quad C\tilde{\pi}_1(X_2) \subseteq \tilde{\pi}_2(X_2)C \tag{4}$$

where $\tilde{\pi}_i$, $i = 1,2$ are the extensions to affiliated elements. (4) implies now

$$C\tilde{\pi}_1(U_i) = \tilde{\pi}_2(U_i)C, \quad C\tilde{\pi}_1(U_i)^* = \tilde{\pi}_2(U_i)^*C, \ i = 1,2.$$

We write C as an infinite matrix $(c_{n,m})_{n,m\in\mathbb{Z}}$ and $\tilde{\pi}_1(U_i)$, $\tilde{\pi}_2(U_i)$ as U_i, $\hat{U}_i$ respectively. We set also $v_n = v_1$, $\hat{v}_n = \hat{v}_1$ for $n < 1$ and $v_n = v_3$, $\hat{v}_n = \hat{v}_3$ for $n > 3$. Since $CU_1 = \hat{U}_1C$, we get $c_{n,m} = c_{n-1,m-1}$. From $CU_2 = \hat{U}_2C$ it follows that $c_{n,m}v_m = \hat{v}_n c_{n,m}$ and

$$c_{n,m}v_1 = c_{n-r,m-r}v_{m-r} = \hat{v}_{n-r}c_{n-r,m-r} = \hat{v}_1 c_{n-r,m-r} = \hat{v}_1 c_{n,m}$$

for $n - r \le 1$, $m - r \le 1$. Similarly,

$$c_{n,m}v_3 = \hat{v}_3 c_{n,m}, \quad \text{for any } n, m \in \mathbb{Z},$$
$$c_{n,m}v_2 = c_{n-(m-2),2}v_2 = \hat{v}_{n-(m-2)}c_{n-(m-2),2} = \hat{v}_1 c_{n,m} = c_{n,m}v_1 \quad \text{for any } n < m.$$

This gives $c_{n,m}v_1^k v_3^l (v_1 - v_2) = 0$, $k, l \in \mathbb{Z}$. It will cause no confusion if we use the same letter I to designate the identity operator in different Hilbert spaces. From the construction of the operators v_1, v_2, v_3 we obtain $v_1 - v_2 = vP_1 \otimes (I - w_1)$, $v_1^{-1}v_3(v_1 - v_2) = vP_1 \otimes w_2(I - w_1)$. If W denotes the Hilbert space where the operators w_1, w_2 act, one can easily check that $(I - w_1)W + w_2(I - w_1)W$ is dense in W and $(v_1 - v_2)H \otimes W + v_1^{-1}v_3(v_1 - v_2)H \otimes W$ is dense in $P_2H \otimes W$. Then, since $v_1^k P_2 H \otimes W = P_{k+2}H \otimes W$, we see that $l.s\ \{v_1^k((v_1 -$

$v_2)H \otimes W + v_1^{-1}v_3(v_1 - v_2)H \otimes W), k \in \mathbb{Z}\}$ is dense in $H \otimes W$. From this it follows that $c_{n,m} = 0$ for $n < m$. Using the same arguments applied to the relations $CU_i^* = \hat{U}_i^* C$, $i = 1, 2$ we deduce that $c_{n,m} = 0$ for $n > m$. Hence, C is diagonal with $c_{n,n}$ on the diagonal and such that $c_{n,n}v_i = \hat{v}_i c_{n,n}$, $c_{n,n}v_i^* = \hat{v}_i^* c_{n,n}$ $i = 1, 2, 3$, i.e., $C = I \otimes C^1$, where C^1 is a bounded operator intertwining $(v_i, v_i^*)_{i=1}^3$ and $(\hat{v}_i, \hat{v}_i^*)_{i=1}^3$. We write C^1 as an infinite matrix with entries $c^1_{n,m}$, $n, m \in \mathbb{Z}$. From $C^1 v_1 = \hat{v}_1 C^1$ it follows that $c^1_{n,m} = c^1_{n-1,m-1}$, $n, m \in \mathbb{Z}$. From $C^1 v_2 = \hat{v}_2 C^1$ and $C^1 v_3 = \hat{v}_3 C^1$ we obtain

$$c^1_{2,j} w_1 = c^1_{2,j}, \quad j \neq 2, \quad c^1_{2,2} w_1 = \hat{w}_1 c^1_{2,2},$$
$$c^1_{3,j} w_2 = c^1_{3,j}, \quad j \neq 2, 3, \quad c^1_{3,3} w_2 = \hat{w}_2 c^1_{3,3}, \quad c^1_{3,2} w_2 = \hat{w}_1 c^1_{3,2}.$$

Taking into account that $c^1_{n,m} = c^1_{n-1,m-1}$, we have $c^1_{2,j}(w_1 - 1) = 0$, $j \neq 2$, $c^1_{2,j}(w_2 - 1) = 0$, $j \neq 1, 2$. It follows from the construction of w_1, w_2 that $(w_1 - 1)W + (w_2 - 1)W$ is dense in W. Hence, $c^1_{2,j} = 0$, $j \neq 1, 2$. Similarly, the relations $(C^1)^* \hat{v}_i = v_i (C^1)^*$, $i = 1, 2, 3$ give $c^1_{j,2} = 0$, $j \neq 1, 2$. Thus $c_{2,j} = 0$, $j \neq 2$ and finally $c_{n,m} = 0$ for $n \neq m$. This prove that $C^1 = I \otimes C^2$, where C^2 is a bounded operator intertwining (w_1, w_2, w_1^*, w_2^*) and $(\hat{w}_1, \hat{w}_2, \hat{w}_1^*, \hat{w}_2^*)$, i.e., $C^2 w_i = \hat{w}_i C^2$ and $C^2 w_i^* = \hat{w}_i^* C^2$, $i = 1, 2$. We write $C^2 = (c^2_{n,m})_{n,m=1,2}$. The relation $C^2 w_1 = \hat{w}_1 C^2$ implies $c^2_{1,2} = c^2_{2,1} = 0$ and $c^2_{1,1} q = \hat{q} c^2_{1,1}$. From this and $C^2 w_2 = \hat{w}_2 C^2$ it follows that $c^2_{1,1} p = \hat{p} c^2_{1,1}$ and $c^2_{1,1}(p - p^2)^{1/2} = (\hat{p} - \hat{p}^2)^{1/2} c^2_{2,2}$. Since $(p - p^2)^{1/2}$ is invertible, we obtain from the last two equalities that $c^2_{1,1} = c^2_{2,2}$, i.e., $C^2 = I \otimes C^3$, where C^3 is such that $C^3 p = \hat{p} C^3$ and $C^3 q = \hat{q} C^3$. We leave it to the reader to verify that the last two relations imply that $C^3 = \begin{pmatrix} C^4 & 0 & 0 \\ 0 & C^4 & 0 \\ 0 & 0 & C^4 \end{pmatrix}$. This finishes the proof. □

Corollary 2. *There exists a $*$-homomorphism $\phi\colon \mathbb{C}[x_1, x_2] \to (CB(\mathcal{H}) \otimes C^*(\mathcal{F}_2))^\eta$ such that the corresponding functor F_ϕ is full.*

Proof. One can show easily that the C^*-algebra $\mathfrak{B}_{\alpha,\beta}$ is $*$-wild and there exists a $*$-homomorphism $\varphi\colon \mathfrak{B}_{\alpha,\beta} \to M_n(C^*(\mathcal{F}_2))$, $n > 0$, which generates the full functor F_φ from the category $\mathrm{Rep}(C^*(\mathcal{F}_2))$ into the category $\mathrm{Rep}(\mathfrak{B}_{\alpha,\beta})$. Setting $\phi = \varphi \circ \psi$, we obtain that ϕ is a $*$-homomorphism from $\mathbb{C}[x_1, x_2]$ to $(CB(\mathcal{H}) \otimes C^*(\mathcal{F}_2))^\eta$ and the corresponding functor is full. □

Remark 4. $\psi\colon C^*(\mathcal{F}_2) \ni u_i \to U_i \in M(CB(\mathcal{H}) \otimes \mathfrak{B}_{\alpha,\beta}) \subset (CB(\mathcal{H}) \otimes \mathfrak{B}_{\alpha,\beta})^\eta$, $i = 1, 2$ is a $*$-homomorphism and the corresponding functor F_ψ is full. However, U_1, U_2 do not generate $CB(\mathcal{H}) \otimes \mathfrak{B}_{\alpha,\beta}$, because $CB(\mathcal{H}) \otimes \mathfrak{B}_{\alpha,\beta}$ is a non-unital C^*-algebra.

Example 3. Consider the $*$-algebra $\mathfrak{A} = \langle x_1, x_2 \mid [x_1, [x_1, x_2]] = 0, x_i^* = x_i, i = 1, 2\rangle$. Clearly, any representation of the commutative algebra $\mathbb{C}[x_1, x_2]$ is a representation of the

$*$-algebra $\mathfrak{A}$. It follows from the preceding example that non-integrable representations can be complicated, i.e. the problem of unitary classification of such $*$-representations contains as a subproblem the problem of unitary classification of representations of the C^*-algebra $C^*(\mathcal{F}_2)$.

In this example we show that the class of representations π defined on a domain formed by analytic vectors for $\pi(x_1)$ and $\pi(x_2)$ is $*$-wild.

Let, as before, α, $\beta > 0$ and let $\mathfrak{B}_{\alpha,\beta} = C^*(a, b; a > 0, b > 0, ||a|| \leq \alpha, ||b|| \leq \beta)$. On the Hilbert space $H = L_2(\mathbb{R}, dx)$ we consider the multiplication operator q by x and the operator of differentiation $p = i\frac{d}{dx}$. Let a_1, a_2 denote the following elements in $M_3(\mathfrak{B}_{\alpha,\beta})$

$$a_1 = \begin{pmatrix} \lambda_1 e & 0 & 0 \\ 0 & \lambda_2 e & 0 \\ 0 & 0 & \lambda_3 e \end{pmatrix}, \qquad a_2 = \begin{pmatrix} 0 & a & b \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix}$$

where $\lambda_i \in \mathbb{R}$, with $\lambda_i \neq \lambda_j$, $i \neq j$, e is the unity of $\mathfrak{B}_{\alpha,\beta}$. Since q, $p \ \eta \ CB(H)$, there exist uniquely defined selfadjoint elements $X_1 = e_3 \otimes q$ and $X_2 = a_1 \otimes p + a_2 \otimes I_H$ such that $X_1, X_2 \ \eta \ M_3(\mathfrak{B}_{\alpha,\beta}) \otimes CB(H)$. Here e_3 is the unity of $M_3(\mathfrak{B}_{\alpha,\beta})$ and I_H is the identity operator on H.

Proposition 7. *$\psi\colon \mathfrak{A} \ni x_i \to X_i \in (M_3(\mathfrak{B}_{\alpha,\beta}) \otimes CB(H))^\eta$ is a $*$-homomorphism.*

Proof. Let G be the Heisenberg group, i.e., the group of matrices of the form

$$g = g(t, s, r) = \begin{pmatrix} 1 & t & r \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}, \ t, s, r \in \mathbb{R}.$$

Then $u\colon G \to B(H)$ defined by

$$u(g(t, 0, 0)) = e^{itq}, \ u(g(0, s, 0)) = e^{isp}, \ u(g(0, 0, r)) = e^{ir} I$$

is a unitary representation of G on $CB(H)$. By [WN], given a unitary representation u of a real Lie group in a C^*-algebra A, there always exists a dense in A domain Φ which is invariant with respect to operators of infinitesimal representation of the Lie algebra and is their essential domain. Let $D = M_3(\mathfrak{B}_{\alpha,\beta}) \odot \Phi$. Clearly D satisfies all the required conditions: D is dense in $M_3(\mathfrak{B}_{\alpha,\beta}) \otimes CB(H)$, D is a core for X_1, X_2 and $[X_1, [X_1, X_2]]a = 0$ for any $a \in D$. □

We denote now by R the set of all representations π of $\mathfrak{A}$ on a Hilbert space H_π defined on a dense invariant domain consisting of analytic vectors for $\pi(x_1)$, $\pi(x_2)$.

Theorem 5. *R is a $*$-wild class of representations.*

Proof. Let id be the identity representation of $M_3(CB(H))$ on $H \oplus H \oplus H$. Given a representation π of $\mathfrak{B}_{\alpha,\beta}$, $((\pi \otimes id)(X_1), (\pi \otimes id)(X_2))$ defines a representation from the class R, where $\pi \otimes id$ is the unique extension to affiliated elements of $\mathfrak{B}_{\alpha,\beta} \otimes M_3(CB(H)) = M_3(\mathfrak{B}_{\alpha,\beta}) \otimes CB(H)$.

To prove that R is $*$-wild it is sufficient to show that $X_1,\ X_2\ \eta\ M_3(\mathfrak{B}_{\alpha,\beta}) \otimes CB(H)$ generate the C^*-algebra $M_3(\mathfrak{B}_{\alpha,\beta}) \otimes CB(H)$. By Theorem 2, the statement will be proved once we prove that X_1, X_2 separates representations of $M_3(\mathfrak{B}_{\alpha,\beta}) \otimes CB(H)$ and that $(I + X_2^2)^{-1}(I + X_1^2)^{-1}(I + X_2^2)^{-1} \in M_3(\mathfrak{B}_{\alpha,\beta}) \otimes CB(H)$.

We realize $\mathfrak{B}_{\alpha,\beta}$ as an algebra of operators in a Hilbert space $\mathcal{H}$. Let F be the Fourier transform operator in $H = L_2(\mathbb{R}, dx)$. Then $\mathcal{F} = I^3_{\mathcal{H}} \otimes F$ is a bounded operator acting on the space $(\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}) \otimes H$ and such that $\mathcal{F} X_1 \mathcal{F}^{-1} = I^3_{\mathcal{H}} \otimes p$, $\mathcal{F} X_1 \mathcal{F}^{-1} = a_1 \otimes q + a_2 \otimes I_H$ (here $I^3_{\mathcal{H}}$ is the identity operator on $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$). The operator $(1+q^2)^{-1}(1+p^2)^{-1}(1+q^2)^{-1}$ acting on H is integral with the kernel $K(x, y) = \frac{1}{2}(1 + x^2)e^{-|x-y|}(1 + y^2)^{-1}$ as is easy to check. Moreover, this operator is positive with finite trace which implies that it is compact. Therefore, $r = I^3_{\mathcal{H}} \otimes (I_H + q^2)^{-1}(I_H + p^2)^{-1}(I_H + q^2)^{-1} \in M_3(\mathfrak{B}_{\alpha,\beta}) \otimes CB(H)$. Let

$$s = (I + (a_1 \otimes q + a_2 \otimes I)^2)^{-1}(I + I \otimes q^2).$$

Clearly, s is bounded. Moreover, s is affiliated with $M_3(\mathfrak{B}_{\alpha,\beta}) \otimes CB(H)$. This is due to the following lemma.

Lemma 2. *Let A be a C^*-algebra, $S \eta A$, $v \in M(A)$. Assume that $[v, z_S] = [v, z_S^*] = 0$. Then there exists $T \eta A$ such that*

$$Ta = vSa, \qquad \text{for any } a \in D(S).$$

Proof. We shall use [Wor2][Theorem 2.3]. Let

$$a = (I - z_S z_S^*)^{1/2},\ b = v z_S,\ c = v z_S,\ d = (I - z_S^* z_S)^{1/2}.$$

One can easily see that a, b, c, $d \in M(A)$, $ab = cd$, the sets $a^*A = aA$, $dA = d^*A$ are dense in A.

For $Q = \begin{pmatrix} d & -c^* \\ b & a \end{pmatrix}$ we have $Q^*Q = \begin{pmatrix} I - z_S^* z_S + v^* v z_S^* z_S & 0 \\ 0 & I - z_S z_S^* + v v^* z_S z_S^* \end{pmatrix}$.

Let π be an irreducible representation of A on a Hilbert space H_π, id the canonical representation of $M_2(\mathbb{C})$ on $\mathbb{C}^2$. Since $(I - z_S^* z_S)^{1/2}A$, $(I - z_S z_S^*)^{1/2}A$ are dense in A and $v^* v z_S^* z_S$, $v v^* z_S z_S^* \geq 0$, one can easily deduce that the range of $(id \otimes \pi)(Q^*Q)$ is dense in H_π which implies that $(id \otimes \pi)(Q)$ is dense in H_π. Using [Wor2][Proposition 2.5] we see that $Q(A \oplus A)$ is dense in $A \oplus A$. By [Wor2][Theorem 2.3] there exists an element $T \; \eta \; A$ such that $dA = (I - z_S^* z_S)^{1/2} A$ is a core of T and $T(I - z_S^* z_S)^{1/2} x = v z_S x$ for any $x \in A$. Since $(I - z_S^* z_S)^{1/2} A = D(S)$, $Ta = vSa$ for any $a \in D(S)$. □

It is known that multipliers are the only bounded elements affiliated with a C^*-algebra. Therefore $s \in M(M_3(\mathfrak{B}_{\alpha,\beta} \otimes CB(H))$ and $srs \in M_3(\mathfrak{B}_{\alpha,\beta} \otimes CB(H))$. On the other hand,

$$srs = \mathcal{F}(I + X_2^2)^{-1}(I + X_1^2)^{-1}(I + X_2^2)^{-1}\mathcal{F}^{-1}$$

which yields $(I+X_2^2)^{-1}(I+X_1^2)^{-1}(I+X_2^2)^{-1} \in M_3(\mathfrak{B}_{\alpha,\beta}) \otimes CB(H)$. Since $\mathfrak{B}_{\alpha,\beta}$ is a $*$-wild C^*-algebra, we conclude that R is $*$-wild class of representations due to Proposition 3. The fact that X_1, X_2 separate representations follows from [NT][Theorem 3] and the fact that any representation of $M_3(\mathfrak{B}_{\alpha,\beta}) \otimes CB(H)$ is of the form $V^{-1}(\pi \otimes id)V$, where V is a unitary operator, π is a representation of $\mathfrak{B}_{\alpha,\beta}$ and id is the identity representation of $M_3(CB(H))$. The proof is done. □

Example 4. Let B be an algebra and let p_1, p_2, p_3, p_4 be idempotents in B such that $p_1 + p_2 + p_3 + p_4 = 0$. Idempotents with this property were studied in [BES1]. They arise, in particular, in the study of logarithmic residues in Banach algebras (see also [BES2]). In [BES1] it is shown that non-trivial zero sums of four idempotents do not exist in Banach algebras, however, there are unbounded idempotents in a Hilbert space having this property. Unbounded representations of a $*$-algebra generated by idempotents p_1, p_2, p_3, p_4 and p_1^*, p_2^*, p_3^*, p_4^* satisfying $p_1 + p_2 + p_3 + p_4 = 0$ were discussed in [ST2]. In this example we shall see that the class of representations defined in [ST2] is $*$-wild. Let $p = (p_1 + p_2)/2$, $q = (p_3 + p_4)/2$, $r = (p_1 - p_2)/2$, $s = (p_3 - p_4)/2$ (we have $p_1 = p + r$, $p_2 = p - r$, $p_3 = q + s$, $p_4 = q - s$). Direct computation shows that they satisfy the following relations:

$$\begin{array}{ll} pr = r(1-p), & ps = s(-1-p), \\ r^2 = p(1-p), & s^2 = -p(p+1). \end{array} \tag{5}$$

We assume additionally that

$$pr^* = rp, \;\; ps^* = sp, \;\; p = p^*, \tag{6}$$

and denote by $\mathfrak{A}$ the $*$-algebra generated by p, q, r, s and relations (5)–(6). We define $R \subset \mathrm{Rep}_{unb}(\mathfrak{A})$ as follows: a family of closed operators (P, Q, R, S) on a Hilbert space H belongs to R iff there exists a linear dense subset $\Phi \subseteq H$ such that $\Phi \subseteq \mathcal{H}_w(P, Q, R^*R, S^*S)$, Φ is a core for the operators R, R^*, S and S^* and relations (5)–(6) hold on Φ. Here $H_w(P, Q, R, S)$ denotes the set of analytical vectors for P, Q, R, S. It was proved in [ST1] that a subclass $\tilde{R}$ of R defined by the condition $\ker P \neq \{0\}$ is manageable, i.e.

there exists a C^*-algebra A and elements p, q, r, s η A generating A and such that $\tilde{R} = \{(\pi(p), \pi(q), \pi(r), \pi(s)) \mid \pi \in \text{Rep}(A)\}$. Moreover, all such representations were classified up to a unitary equivalence.

Let now $\mathcal{A}_{\alpha,\beta} = C^*(a, b : ||a|| \leq \alpha, ||b|| \leq \beta\}$. Let H be a separable infinite dimensional Hilbert space with an orthonormal basis $\{e_k\}_{k\in\mathbb{Z}}$, let P_k be the orthoprojection onto $\mathbb{C}\langle e_k\rangle$, $k \in \mathbb{Z}$. We consider operators v, w defined by $ve_k = e_{k+1}$, $ve_{k+1} = e_k$ if k is even and $we_k = e_{k+1}$, $we_{k+1} = e_k$ if k is odd. Clearly, $(P_{2k}+P_{2k+1})H$ (respectively $(P_{2k+1}+P_{2k+2})H$) is invariant with respect to v (respectively w).

Let now

$$\tilde{p} = \textstyle\sum_{k\neq 0}(-1)^{k+1}kP_k \otimes \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, \ \tilde{q} = \textstyle\sum_{k\neq 0}(-1)^{k}kP_k \otimes \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix},$$

$$\tilde{r} = (\textstyle\sum_{k\neq 0}(2k+1)vP_{2k} - 2kvP_{2k+1}) \otimes \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} + vP_0 \otimes \begin{pmatrix} e & 0 & 0 \\ 0 & 2e & 0 \end{pmatrix},$$

$$\tilde{s} = (\textstyle\sum_{k\neq 0}(2k+1)wP_{2k+2} - (2k+2)P_{2k+1}) \otimes \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} + wP_0 \otimes \begin{pmatrix} e & e & a+ib \\ 0 & e & e \end{pmatrix}.$$

Here e is the identity element in $\mathcal{A}_{\alpha,\beta}$. We write $\mathcal{H}$ for the Hilbert space generated by the basis elements $f_n = e_n \oplus e_n$, $n \in \mathbb{Z}$, $n \neq 0$, $f_0 = e_0 \oplus e_0 \oplus e_0$. Direct verification shows that P, Q, R, S are affiliated with $CB(\mathcal{H}) \otimes \mathcal{A}_{\alpha,\beta}$ and separate representations of $CB(\mathcal{H}) \otimes \mathcal{A}_{\alpha,\beta}$. Moreover, since $(I+\tilde{p}^2)^{-1} = \sum_{k\neq 0}(1+k^2)^{-1}P_k \otimes \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$, $(I+\tilde{p}^2)^{-1} \in CB(\mathcal{H}) \otimes \mathcal{A}_{\alpha,\beta}$. Therefore, by Theorem 2, P, Q, R, S generate the C^*-algebra $CB(\mathcal{H}) \otimes \mathcal{A}_{\alpha,\beta}$.

Let Q_n be the orthoprojection onto a subspace of $\mathcal{H}$ generated by $\{f_k\}_{k=-n}^{n}$ and let $D = l.s.\ \{a \otimes b \mid a \in CB(\mathcal{H}), a = Q_naQ_n$ for some $n \in \mathbb{N}, b \in \mathcal{A}_{\alpha,\beta}\}$. Then D is dense in $CB(\mathcal{H}) \otimes \mathcal{A}_{\alpha,\beta}$ and invariant with respect to $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, $\tilde{s}$, D is a core for the elements $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, $\tilde{s}$ and $\tilde{p}$, $\tilde{q}$, $\tilde{r}$, $\tilde{s}$ satisfy relations (5)–(6) on D. Moreover, any representation $(\pi(\psi)(p))$, $\pi(\psi(q))$, $\pi(\psi(r)$, $\pi(\psi(s)))$ belongs to R, where π is a representation of $CB(\mathcal{H}) \otimes \mathcal{A}_{\alpha,\beta}$. From this it follows that the class R is $*$-wild.

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References

[BES1] H.Bart, T.Ehrhardt and B.Silbermann, *Zero sums of idempotents in Banach algebras*, Integr. Equat. Oper. Th. **19** (1994), 125-134.

[BES2] H.Bart, T.Ehrhardt and B.Silbermann, *Logarithmic residues in Banach algebras*, Integr. Equat. Oper. Th. **19** (1994), 135-152.

[D] *J.Dixmier*, C^*-algebras, North-Holland Publ. Comp., Amsterdam, 1982.

[KS] S. A. Kruglyak and Yu. S. Samoĭlenko, *On unitary equivalence of collections of self-adjoint operators*, Funct. Anal. i Prilozhen. **14** (1980), no. 1, 60 – 62, (Russian).

[KW] P. Kruszyński and S. L. Woronowicz, *A noncommutative Gelfand Naimark theorem*, J. Operator Theory **8** (1982), 361–389.

[N] *E. Nelson*, Ann. Math. **70** 572-615, (1959) .

[NT] L. P. Nizhnik and L. B. Turowska, *Representations of double commutator by matrix-differential operators*, Methods Funct. Anal. Topol. **3** (1997), no. 3, 75–80.

[OS] *Ostrovskiĭ V., Samoĭlenko Yu.*, Introduction to the Theory Representation of Finitely Presented $*-$algebras. 1. Representations by bounded operators. The Gordon and Breach Publ. Group, London,1999.

[Ped] *G. K. Pedersen*, "C^*-algebras and their Automorphism Groups," Academic Press, London, 1979.

[PS] A. Yu. Piryatinskaya and Yu. S. Samoĭlenko, *Wild problems in representation theory of $*$-algebras with generators and relations*, Ukr. Mat. Zhurn. **47** (1995), no. 1, 70–78.

[ST1] Yu. S. Samoĭlenko and L. B. Turowska, *On representations of $*$-algebras by unbounded operators*, Funkt. Anal. Prilozh. **31** (1997), no. 4, 80–83.

[ST2] Yu. S. Samoĭlenko and L. B. Turowska, *On four idempotents with zero sum*, in preparation.

[Schm] *K. Schmüdgen*, Unbounded operator algebras and representation theory, Akademie-Verlag, Berlin, 1990.

[Wor1] S. L. Woronowicz, *Duality in the C^*-algebra theory*, Proc. Internat. Congr. Mathematicians Warsaw 1983, 1347-1356, PWN Warszawa.

[Wor2] S. L. Woronowicz, *Unbounded elements affiliated with C^*-algebras and non-compact quantum groups*, Commun. Math. Phys. **136** (1991), 399–432.

[WN] S. L. Woronowicz and K. Napiorkowski, *Operator theory in the C^*-algebra framework*, Rep. Math. Phys. **31** (1992), 353-371.

[Wor3] S. L. Woronowicz, *Operator equalities related to the quantum $E(2)$ group*, Commun. Math. Phys. (1992), 417–428.

[Wor4] S. L. Woronowicz, *C^*-algebras generated by unbounded elements*, Rewiews in Mathematical Physics **7** (1995), no. 3, 481–521.