An analytical asymptotic solution to a conjugate heat transfer problem

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Abstract

In this paper an asymptotic solution to the conjugate heat transfer problem with a flush mounted heat source on the fluid-solid interface, in the case that the bottom of the solid is perfectly insulated and the velocity profile in the fluid is linear, is presented. The lowest order terms of the asymptotic solution can be naturally classified into contributions from pure convection, from the interaction of convection and the conduction in the solid and from the interaction of convection and the conduction in the fluid. It was found that downstream of the heat source the two leading order terms of the asymptotic expansion stem from pure convection, and that the leading term decays as $\mathcal{O}\left(x^{-2/3}\right)$, which confirms the result from the analysis by Liu *et al.*(1994) in the case of an adiabatic wall. The third term, however, is a contribution from the interaction of conduction in the solid and convection. If we furthermore neglect the conduction in the fluid we have been able to find the asymptotic solution upstream of the heat source as well, and in this case we find that the temperature decays exponentially with the distance from the heat source. Our results show good agreement with numerical solutions to the problem.

1 Introduction

Conjugate heat transfer is a term used to describe the class of fluid-solid problems where the heat exchange between the fluid and the solid is a priori unknown, yet significant. Typically, the desired unknown is the interface temperature, which should be determined from some knowledge of the heat source. Ever since Perelman (1961) coined the expression conjugate heat transfer, numerous studies have been devoted to this problem. The interest in the conjugate heat transfer problem stems not only from the fundamental nature of the problem, but also largely from the fact that the problem has a wide range of applications, such as for the measurement of wall-shear stress, for the cooling of electronic components, for aerodynamic heating and for heat exchangers. However, in spite of the all the attempts at analysing the problem over the last four decades, several of the mechanisms involved have remained poorly understood. By the development of novel asymptotic method we are in this paper able to provide an asymptotic solution to the full conjugate heat transfer problem in the case when the solid is an infinite slab, which is perfectly insulated at the bottom, the fluid has infinite extension with a linear velocity profile and the heat source is located in a portion of finite length of the fluid solid interface. By comparison with numerically obtained solutions, it is found that the asymptotic solution to be presented below accurately describes the interface temperature both upstream and downstream of the heat source, except, of course, in the immediate neighbourhood of the source. Furthermore, the asymptotic solution to be presented here can be divided into contributions from convection, from conduction in the solid as well as in the fluid, and from various coupled effects. In all cases, however, it is found that far downstream of the heat source the temperature distribution will resemble that in case of an adiabatic wall, which is due to the fact that in case the bottom wall of the solid is perfectly insulated, all heat will eventually reach the fluid.

In his 1961 paper, Perelman studied laminar flow over an internally heated flat plate with asymptotic expansions. He identified a parameter, that combined the ratio of the conductivities of the fluid and the solid, the Prandtl number and the Reynolds number, which he claimed was the essential parameter of the problem. This work was followed up by a number of researchers, e. g. Luikov (1974) who termed Perelman's parameter the Brun number, and used it in an approximate analysis involving a simplified thin-solid geometry with a linear temperature distribution across the solid, i.e. a one-dimensional model. Karvinen (1978) used the integral method to solve a problem similar to that considered by Luikov and achieved good agreement with measured data. In Rizk *et al.* (1991) a theoretical analysis was made with a uniform velocity profile (slug flow), and numerical solutions were made by Pop & Ingham (1993). In Ramadhyani *et al.* (1985), the conjugate heat transfer problem for a flush-mounted heat source on the fluid-solid interface was introduced. Numerical simulations on this problem were performed by Sugavanam *et al.* (1995). Using a suitable boundary element method Cole (1997) made a thorough numerical investigation of this case when the heat source distribution is uniform along the entire source. A fundamental contribution due to Cole (1997) is the introduction of a new dimensionless number, which he called "the conjugate Peclet-number", but which we will henceforth call the Cole number

$$\Lambda = \frac{k_f}{k_s} Pe^{1/3}, \tag{1}$$

where Pe is the Peclet number and k_f and k_s are the heat conductivities of the fluid and the solid respectively. In this paper we will see that, within the limitations of the mathematical model, Cole's parameter and the purely geometrical quantity L/D, where L is the length of the heat source, which is often used to non-dimensionalize all lengths, and D is the thickness of the slab of solid, contain *all* the information about the temperature on the interface, provided that the distribution of heat source density divided by k_s is known, and the streamwise conduction in the fluid is neglected. However, for the terms where streamwise conduction in the fluid gives a contribution the Peclet number Pe has an independent role as well.

Long before the work on conjugate heat transfer began, several studies into the cooling of a surface submerged in a fluid had been made, see e. g. Levéque (1928) or Fage & Falkner (1931). The major issue under consideration in these articles was the relation between the shear stress and the heat transfer into the fluid. This work was developed further with hot-film sensors in focus by e. g. Ludwieg (1950), Lighthill (1950), Liepmann & Skinner (1954), Bellhouse & Schultz (1966) and Menendez & Ramaprian (1985).

The conjugate heat transfer problem studied here is a reasonable model for a hot-film sensor, used for wall-shear stress measurements. Micro-Electro-Mechanical Systems (MEMS) have made it possible to manufacture very small such devices (between 0.1 mm and 1 micron) with large freedom in the choice of material that has been utilized to improve the insulation, see e. g. Löfdahl & Gad-el-Hak (1999). Unfortunately, not much progress has been made in attaining sufficient resolution for measurements using this technique. This failure has often been blamed on insufficient understanding of the heat transfer properties of the sensors, and hence the analytical solution to the full conjugate heat transfer problem presented here should have significant implications for hot-film sensor technology.

Recently, much interest has been focused on the case when the solid is silicon and the fluid is air. Indeed, this is the situation for a MEMS-sensor making air measurements as well as for a microelectronic circuit being cooled. However, with this combination of materials numerical simulation of the conjugate heat transfer problem, becomes difficult since a very large domain must be discretized due to the slow temperature decay on the interface. Therefore, there has been an increased interest in analytical methods, which have the added advantage of providing explicit parameter dependence. The most important analytic contribution to date is due to Liu *et al.* (1994), who analyzed the problem of a constant temperature heat source with the constraint that the solid is a perfect insulator outside the heat source, and managed to calculated the asymptotic solution in this case. However, since the conduction in the solid is irrelevant in this case, the problem treated is strictly speaking not a conjugate heat transfer one. The asymptotic technique used by Liu *et al.* (1994) is not easily modified to allow conjugate heat transfer, therefore we will in the present paper use a different asymptotic technique to make it possible to take into account the coupling between convection and conduction in the solid as well as the conduction in the fluid in the downstream direction. Our results give the surface temperature distribution in terms of the supplied energy, whereas Liu *et al.*'s results give it in terms of the temperature of the hot-film. If our results are combined, we obtain relations between supplied energy and increase in temperature, resembling that for example in Bellhouse & Schultz (1966).

2 The Thermal Principle

In Figure 1, the conjugate heat transfer problem under consideration in this article is illustrated. The heat source is flush-mounted on a substrate submerged in a fluid.

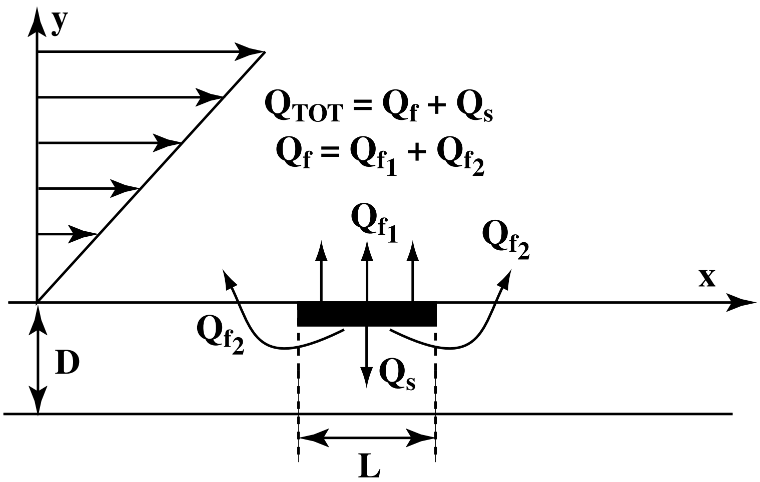

Figure 1: Illustration of the conjugate heat transfer problem under consideration

The ohmic (Joule) heating in the heat source Q_{TOT} is transferred both to the fluid and to the surrounding substrate, ($Q_{TOT} = Q_f + Q_s$) where Q_f represents the heat transferred to the fluid directly from the heated surface (Q_{f_1}) and indirectly through the heated portion of the substrate (Q_{f_2}). Q_s represents the heat lost irretrievably to the substrate. One important feature of our analytical asymptotic solution to this problem is that contributions to the interface temperature due to Q_{f_1} and Q_{f_2} are separated. In addition, it is shown that these contributions have different parameter dependence. However, to obtain our asymptotic solution, we must assume that the lower wall in Figure 1 is insulated. Note that assuming that the lower wall is insulated is not a priori the same as assuming that Q_s is zero, since heat may escape to infinity at the sides of the solid slab. However, as we will see in Section 6, Q_s will indeed be zero. For this reason, we will, when truncating the domain for the numerical computations, insulate the solid on the side walls as well. This introduces an error, but the method is asymptotically correct as the domain size increases.

3 Analysis

The problem we will analyze is, as is illustrated in figure 1, an infinite slab of source free solid of thickness D, which is insulated at the bottom and kept at a constant temperature at infinity (which we may take to be zero):

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \quad \left.\frac{\partial T}{\partial y}\right|_{y=-D} = 0, \quad T \to 0 \text{ as } |x| \to \infty \; . \tag{2}$$

Above the solid we will consider having a semi-infinite fluid, with thermal conductivity k_f, specific heat c_p and density ρ, which is kept at the same temperature at infinity as the solid. We assume that the thermal boundary layer will submerged within the viscous sublayer (not necessarily valid for all interesting combinations of parameters, see e. g. Cole (1997)) of the fluid, and for that reason we will assume that the velocity of the fluid is linear in y and constant in x, *i.e* that

$$U(x,y) = \left.\frac{\partial U}{\partial y}\right|_w y. \tag{3}$$

Consequently, the temperature distribution in the fluid satisfies the following relations

$$\left.\frac{\partial U}{\partial y}\right|_w y \frac{\partial T}{\partial x} = \frac{k_f}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \; , \quad T \to 0 \text{ as } x^2 + y^2 \to \infty \; . \tag{4}$$

As interface condition we assume that all heat sources are contained in the interval $|x| \le \frac{L}{2}$, i.e. that[1]

$$k_f \frac{\partial T}{\partial y}\left(x, 0^+\right) = k_s \frac{\partial T}{\partial y}\left(x, 0^-\right) , \text{ as } |x| > \frac{L}{2}. \tag{5}$$

Furthermore we assume that the heat source distribution is bounded and known for $|x| \le \frac{L}{2}$. This knowledge can either be explicit, as in for example Cole (1997), where the heat source distribution is assumed to be uniform,

[1] This condition is convenient, but not absolutely necessary, and it is conceivable that at least part of the analysis presented here can be extended to the case when the heat source distribution satisfies a weaker decay property for large $|x|$

or implicit, as is the case for example when the temperature along the heat source is constant. Hot-films used to measure wall shear stress are typically of the constant temperature type, and for that reason we choose this condition for our numerical simulations. However, the asymptotic analysis to be presented below, which is aimed at finding $T(x,0) \equiv T_w(x)$ for $|x|$ large, applies to the more general situation. Specifically, it covers the case with two heat sources considered in for example Ramadhyani *et al.* (1985); Sugavanam *et al.* (1995), however, it must be recalled that an asymptotic analysis can only be expected to be accurate far from all sources.

4 Numerical method

For the spatial discretization of (2) and (4), a standard finite volume method was used. In short, the domain was divided into control volumes (CV), and the governing equations were integrated over each CV, leading to a balance equation for the fluxes through the CV faces. The code was validated against an exact solution where the entire fluid-solid interface has constant temperature. The conclusion from this evaluation was that the error in the numerical scheme was at most 5%. A more detailed description of the code can be found in Buller (2000). When using the code it is apparent that the boundary conditions introduced where the domain is truncated is of crucial importance. Carelessly chosen boundary conditions will redirect the energy, causing large errors. It will be seen below that all the heat energy will be transported to infinity by the fluid, and hence the outflow condition in the fluid must allow for this transport of energy. Setting such a condition correctly is quite difficult and instead we chose to circumvent the problem by neglecting the streamwise conduction in the fluid term in the fluid equations, which makes the equation in the fluid parabolic rather than elliptic in the x-direction, and hence we must not specify any boundary condition on the outflow boundary. The neglect of streamwise conduction in the fluid term can be justified as in Cole (1997) or else by the fact established below that any contribution due to streamwise conduction in the fluid will decay much faster in the x-direction than the contributions due to convection or conduction in the solid.

There may be numerical methods better suited to solve this problem (see for example Cole (1997)), but for our purpose, which is to test the asymptotic solutions obtained below, this straight-forward method seems to be sufficiently accurate for the domains used.

5 Asymptotic solution to the problem

In this section we will seek an asymptotic solution to our conjugate heat transfer problem for $|x|$ large. Asymptotic solutions consist of a series of functions, with the property that as more terms of the series are added together the difference between the sum and the full solution decays increasingly quickly with $|x|$. Asymptotic series are often divergent, which means that to calculate the approximate value of the solution for a given value of x, the closest approximation is obtained after a finite number of terms, and adding more terms will only increase the error. Indeed, the point with asymptotic solutions is not to represent solutions exactly (which is almost always better done by for example a Taylor series), but rather to give a reasonable approximation of the solution with a very limited number of terms. For a more complete description of the properties of asymptotic series, the reader is refered to for example (Bleistein & Handelsman, 1986; Murray, 1984).

We begin the analysis by Fourier transforming the problem in the x-direction, using the definition

$$\hat{T}(\xi, y) = \int_{-\infty}^{+\infty} T(x, y)\, e^{-i\xi x} dx \ . \tag{6}$$

This transforms the problem in the lower half-plane to

$$\begin{aligned} \frac{\partial^2 \hat{T}}{\partial y^2} - \xi^2 \hat{T} &= 0 \ , & (7)\\ \hat{T}(\xi, 0) &= \hat{T}_w(\xi) \ , & (8)\\ \frac{\partial \hat{T}}{\partial y}(\xi, -D) &= 0, \quad \forall \xi \in \mathbf{R} \ . & (9) \end{aligned}$$

It is a straightforward task to calculate the solution to (7)-(9) which is given by

$$\hat{T}(\xi, y) = \hat{T}_w(\xi) \frac{\cosh(\xi(y + D))}{\cosh(\xi D)} \ . \tag{10}$$

The Fourier transformed problem in the upper half-plane is given by

$$\begin{aligned} i\xi \left.\frac{\partial U}{\partial y}\right|_w y\hat{T} &= \frac{k_f}{\rho c_p} \left(\frac{\partial^2 \hat{T}}{\partial y^2} - \xi^2 \hat{T} \right) \ , & (11)\\ \hat{T}(\xi, 0) &= \hat{T}_w(\xi) \ , & (12) \end{aligned}$$

Since all heat sources are located on the liquid-solid interface, we have that the total heat transfer across a plane $y = const$

$$E_T(y) \equiv k_f \int_{-\infty}^{\infty} \left| \frac{\partial T}{\partial y}(x, y) \right|^2 dx \tag{13}$$

is a non-increasing function of y. In addition $E_T(y) \leq E_T(0)$ is bounded since the heat source distribution is bounded and has compact support. According to Plancherel's (or Parseval's) formula we thus have that

$$E_T(y) = \frac{k_f}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\partial \hat{T}}{\partial y}(\xi, y) \right|^2 d\xi \ . \tag{14}$$

The equation (11) can be rewritten as

$$\frac{\partial^2 \hat{T}}{\partial y^2} - \left(\frac{i\xi \left.\frac{\partial U}{\partial y}\right|_w \rho c_p}{k_f} y + \xi^2 \right) \hat{T} = 0 \tag{15}$$

To simplify the notation, let us introduce the parameter $P \equiv Pe/L^2 \equiv \left.\frac{\partial U}{\partial y}\right|_w \rho c_p k_f^{-1}$, which contains the physical parameters of the liquid equation. It is now time to change the independent variable in (15) to

$$\nu = (iP\xi)^{1/3} y + (iP)^{-2/3} \xi^{4/3} \ , \tag{16}$$

which transforms (15) into the Airy equation

$$\frac{\partial^2 \hat{T}}{\partial \nu^2} - \nu \hat{T} = 0 \ . \tag{17}$$

The general solution to the Airy equation is given by (see e.g. (Abramowitz & Stegun, 1970))

$$\hat{T}(\xi, \nu) = C_1(\xi) \, Ai(\nu) + C_2(\xi) \, Bi(\nu) \tag{18}$$

where Ai and Bi are Airy functions of the first and second kind. (ν being complex is no complication since the Airy functions are entire.) Transforming back to y this is

$$\begin{aligned} \hat{T}(\xi, y) &= C_1(\xi) \, Ai\left((iP\xi)^{1/3} y + (iP)^{-2/3} \xi^{4/3} \right) + \\ &+ C_2(\xi) \, Bi\left((iP\xi)^{1/3} y + (iP)^{-2/3} \xi^{4/3} \right) \ . \end{aligned} \tag{19}$$

For large y, we have that $\nu \approx (iP\xi)^{1/3} y$. Now suppose that we choose the branch cut to be in the second quadrant, in fact for convenience later we will choose the branch cut to have the argument $\pi/2 + \epsilon$ where ϵ is an arbitrarily small positive number. In this case the argument of cube root increases from $-\pi/6$ to $\pi/6$ as ξ goes from $-\infty$ to $+\infty$, and then we have to leading order for large y (see e.g. (Abramowitz & Stegun, 1970))

$$Ai'(\nu) \sim -\frac{1}{2} \pi^{-1/2} \nu^{1/4} \exp\left(-\frac{2}{3} \nu^{3/2} \right) \tag{20}$$

$$Bi'(\nu) \sim \pi^{-1/2} \nu^{1/4} \exp\left(\frac{2}{3} \nu^{3/2} \right) \ . \tag{21}$$

We have that for large positive y, $\nu^{3/2} \approx (iP\xi)^{1/2} y^{3/2}$ which has a positive real part for $\xi \neq 0$, and thus we have that $Ai' \to 0$ as $y \to +\infty$ and $\left| Bi' \right| \to +\infty$ as $y \to +\infty$. However, from (14) and (19) we see that

$$E_T(y) = \frac{k_f}{2\pi} \int_{-\infty}^{\infty} (P\xi)^{2/3} \left| C_1(\xi) \, Ai'(\nu(\xi)) + C_2(\xi) \, Bi'(\nu(\xi)) \right|^2 d\xi. \tag{22}$$

Hence, $C_2(\xi) \equiv 0$ on all but a set of measure zero, or else E_T cannot be a non-increasing function of y. (If $C_2(\omega)$ attains non-zero values at a set of measure zero is immaterial, since these points have no impact on the behaviour of T in physical space.) If we now impose (12) we obtain the solution to (11)-(12) which is given by

$$\hat{T}(\xi, y) = \hat{T}_w(\xi) \frac{Ai\left((iP\xi)^{1/3} y + (iP)^{-2/3} \xi^{4/3}\right)}{Ai\left((iP)^{-2/3} \xi^{4/3}\right)} . \tag{23}$$

Note that $T(x, y)$ and T_w are real and thus

$$\hat{T}(-\xi, y) = \overline{\hat{T}(\xi, y)} \tag{24}$$

$$\hat{T}_w(-\xi) = \overline{\hat{T}_w(\xi)} . \tag{25}$$

From these relations we find by differentiation in y and taking $y = 0$ that if

$$s(\xi) = (iP\xi)^{1/3} \frac{Ai'\left((iP)^{-2/3} \xi^{4/3}\right)}{Ai\left((iP)^{-2/3} \xi^{4/3}\right)} \tag{26}$$

then

$$s(-\xi) = \overline{s(\xi)} . \tag{27}$$

Notice also that

$$Ls(\xi) = (iPeL\xi)^{1/3} \frac{Ai'\left((iPe)^{-2/3} (L\xi)^{4/3}\right)}{Ai\left((iPe)^{-2/3} (L\xi)^{4/3}\right)} \tag{28}$$

where $Pe = PL^2$ is the Peclet number.

Let us now consider the function

$$f(x) = k_f \frac{\partial T}{\partial y}(x, 0^+) - k_s \frac{\partial T}{\partial y}(x, 0^-) . \tag{29}$$

Because of the interface conditions f must be zero for all x such that $|x| > L/2$, *i.e.* f has compact support. The Paley-Wiener theorem (see e.g. (Hörmander, 1983)) thus implies that $\hat{f}$, the Fourier transform of f, is an entire function, and thus it can be represented by a Maclaurin series

$$\hat{f}(\xi) = \sum_{k=0}^{\infty} a_k \xi^k . \tag{30}$$

On the other hand, $\hat{f}$ can be calculated from our solutions in the half-planes to yield

$$\hat{f}(\xi) = \hat{T}_w(\xi) \left\{k_f s(\xi) - k_s \xi \tanh D\xi\right\} . \tag{31}$$

Hence we can obtain an expression for the Fourier transform of our unknown function T_w:

$$\hat{T}_w(\xi) = \frac{\sum_{k=0}^{\infty} a_k \xi^k}{k_f s(\xi) - k_s \xi \tanh D\xi}\ . \tag{32}$$

Notice that we can use (28) to rewrite this expression as

$$\hat{T}_w(\xi) = L \frac{\hat{f}(\xi)/k_s}{\Lambda (iL\xi)^{1/3} \frac{Ai'\left((iPe)^{-2/3}(L\xi)^{4/3}\right)}{Ai\left((iPe)^{-2/3}(L\xi)^{4/3}\right)} - (L\xi)\tanh\frac{D}{L}(L\xi)}, \tag{33}$$

where Λ is defined in (1). Since this is a function of $L\xi$ multiplied by L we can replace x by x/L below if we also replace P by Pe and $\hat{f}$ by $\hat{f}/k_s$. If the streamwise conduction in the fluid (the $\partial^2 T/\partial x^2$ -term) is neglected then the calculations above hold, the only difference being that the argument of the Airy function and its derivative becomes exactly zero. In this case it is evident that only parameters affecting (33) are the Fourier transform heat source density distribution divided by the conductivity of the solid, $\hat{f}/k_s$, the ratio of the length scales L/D and Cole's parameter Λ. This establishes that these are the only parameters affecting the interface temperature distribution if the streamwise conduction in the fluid is neglected. However, it is also evident from (33) that when the streamwise conduction in the fluid is retained we must add the Peclet number Pe to the list of parameters affecting the problem. Therefore, in this case, not much is gained by working with dimensionless parameters, and we therefore prefer to perform the analysis directly on the dimensional parameters.

In order to estimate T_w for large $|x|$, we will study the regularity properties of $\hat{T}_w(\xi)$. However, it turns out that due to the convection, the behaviour for positive and negative values of x is very different. Since the distribution for negative x, in general, is of minor importance we will defer this analysis until subsection 5.4. For positive x the decay is algebraic, and our next task is to show that the singularity at the origin contains all the information about the terms dominating the behaviour of T_w for large $x > 0$. In order to do this we establish two technical auxiliary results, Lemma 1 and 2, the proof of which are rather long and therefore deferred Section 7.

To ensure that we have no contributions from poles along the real axis we first formulate the following lemma.

Lemma 1 *For $\xi \neq 0$ the denominator in (32) is different from zero regardless of the values of the parameters.*

Now let $h(\xi)$ and $1 - h(\xi)$ be a C^∞ partition of unity, *i.e.* non-negative C^∞ functions such that $h(\xi) + (1 - h(\xi)) \equiv 1$. Furthermore, let $h(\xi) = 0$ for all $|\xi| > 2\xi_0$ and $h(\xi) = 1$ for all $|\xi| < \xi_0$. We thus have that

$$T_w(x) = \mathcal{F}^{-1}\left(\hat{T}_w(\xi)\, h(\xi)\right) + \mathcal{F}^{-1}\left(\hat{T}_w(\xi)\,(1 - h(\xi))\right) \tag{34}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

Our next second lemma tells us that for any choice of ξ_0 the second term decays quickly with x.

Lemma 2 *Suppose that f is bounded. Then for any value of ξ_0 and any positive integer n we have that*

$$\mathcal{F}^{-1}\left(\hat{T}_w(\xi)(1-h(\xi))\right)(x) = \mathcal{O}\left(|x|^{-n}\right) \tag{35}$$

for large $|x|$.

Henceforth, we will therefore exclusively consider $\mathcal{F}^{-1}\left(\hat{T}_w(\xi)h(\xi)\right)$ for a sufficiently small value of ξ_0. Now since

$$\lim_{\xi\to 0}\frac{k_s\xi\tanh D\xi}{s(\xi)\xi^p} = 0, \tag{36}$$

for all p, $0 \leq p < \frac{5}{3}$, there exists an ξ^* such that for all $\xi < \xi^*$ we have that

$$\frac{1}{k_f s(\xi) - k_s\xi\tanh D\xi} = \frac{1}{k_f s(\xi)}\sum_{k=0}^{\infty}\left(\frac{k_s\xi\tanh D\xi}{k_f s(\xi)}\right)^k, \tag{37}$$

where we have used the well known formula for the sum of a geometric series. Hence, we have that

$$\hat{T}_w(\xi)h(\xi) = \frac{\left(\sum_{k=0}^{\infty}a_k\xi^k\right)\left(\sum_{k=0}^{\infty}\left(\frac{k_s\xi\tanh D\xi}{k_f s(\xi)}\right)^k\right)h(\xi)}{k_f s(\xi)} \tag{38}$$

However, we can also expand $s(\xi)$ and $\tanh D\xi$ for small ξ:

$$\begin{aligned} s(\xi) &= (iP)^{1/3}\frac{Ai'(0)}{Ai(0)}\xi^{1/3}\left(1 - \frac{Ai'(0)}{Ai(0)}(iP)^{-2/3}\xi^{4/3} + \right.\\ &+ \left.\frac{1}{2}\left(\frac{Ai(0)}{Ai'(0)} + 2\left(\frac{Ai'(0)}{Ai(0)}\right)^2\right)(iP)^{-4/3}\xi^{8/3} + \cdots\right) \quad (39)\\ \tanh D\xi &= D\xi - \frac{D^3\xi^3}{3} + \cdots \quad (40) \end{aligned}$$

For the $s(\xi)$ appearing in the denominators in (38) we can invert the expression in the big brackets, which leaves us with just one term in the denominators. Consequently, we have achieved a small ξ expansion of $\hat{T}_w(\xi)h(\xi)$. This general expression, however, does look very complicated, so rather than writing it out in full we will list the four terms of lowest order. In this list of terms we will find those terms which give the highest order asymptotic contributions to $T_w(x)$ for large $|x|$. The precise contribution of each of these terms will be calculated in the subsequent subsections.

The four terms of lowest order of $\hat{T}_w(\xi)h(\xi)$ for small values of $|\xi|$ are:

1.

$$\hat{T}_{w,1}(\xi) = \frac{a_0 Ai(0)}{k_f (iP)^{1/3} Ai'(0)} h(\xi) \frac{1}{\xi^{1/3}} \ . \tag{41}$$

From (37) and (40), this, the lowest order term, is seen to arise from the convection in the fluid.

2.

$$\hat{T}_{w,2}(\xi) = \frac{a_1 Ai(0)}{k_f (iP)^{1/3} Ai'(0)} h(\xi) \xi^{2/3} \ . \tag{42}$$

This term also arises from the convection in the fluid.

3.

$$\hat{T}_{w,3}(\xi) = \frac{a_0}{k_f iP} h(\xi) \xi \ . \tag{43}$$

This term is the lowest order contribution from the conduction in the fluid. Note, however, that it is coupled with effects of the convection.

4.

$$\hat{T}_{w,4}(\xi) = \frac{a_0 k_s Ai^2(0)}{k_f^2 (iP)^{2/3} (Ai'(0))^2} h(\xi) D\xi^{4/3} \tag{44}$$

This term is lowest order contribution from conduction of heat through the solid. Once again, it should be noted that this contribution is coupled with effects of the convection.

It should be mentioned that the higher order terms include several terms, which take into effects of the coupling of all three of the involved mechanisms of heat transfer. The higher order terms are all on the form $C\xi^p$ with $p > 4/3$ and hence their inverse Fourier transforms will be on the form $\tilde{C}x^{-p-1}$. Consequently, they are of limited importance for large positive values of x.

5.1 The asymptotic contribution to $T_w(x)$ for large $x > 0$ arising from convective terms

Convection acts only to the right, *i.e.* we should only get a contribution for positive values of x. At this stage remember that we have chosen the branch line to have the argument $\pi/2 + \epsilon$. It turns out, however, that the presence of the ϵ does not affect any of the calculations below, and hence we suppress it in the calculations below in order to simplify the calculations somewhat.

Our aim is to calculate the asymptotic behaviour of

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{a_j Ai(0)}{k_f (iP)^{1/3} Ai'(0)} h(\xi) \xi^{j-1/3} e^{i\xi x} d\xi \ , \tag{45}$$

for $j = 0, 1$.

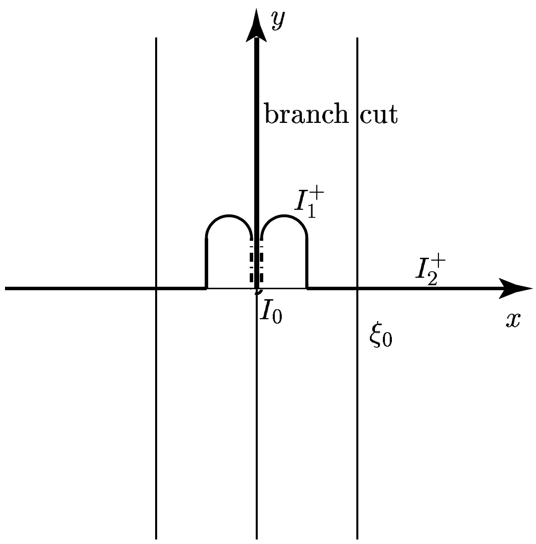

Figure 2: A typical choice of the integration contour. I_0 is the dot-dashed part of the contour, *i.e.* the part in the strip on both sides of the imaginary axis as well as the small circle around the origin. I_2^+ is the part of the contour in the right half plane which is along the real axis. I_1^+ is the part of the contour in the right half plane that begins with a semicircle connecting the two other parts. I_1^- and I_2^- are defined analogously in the left half plane.

The way one typically calculates the asymptotic expansion of integrals like (45) is well described for example in Murray (1984). When $x > 0$ one deforms the integral contour so that it runs along both sides of the branch line (the imaginary axis). The two half-contours are then joined by a small circle around the singularity. Had we done this for $x < 0$ the same procedure would have been repeated in the lower half-plane where there is no branch line to avoid, and hence there would have been no asymptotic contributions. All the deformations needed are then justified using Cauchy's theorem. Unfortunately, however, this requires the integrand to be analytic in some neighbourhood of the upper half plane, and in our case this is not so due to the presence of $h(\xi)$, which cannot be chosen to be real analytic.

Nevertheless, we must deform our contour and we must devote some effort into arguing why we can do so. Firstly, if $|\xi| < \xi_0$ we have that $h(\xi) \equiv 1$, and hence it may be extended analytically to be identically one in any strip $|\Re\xi| \leq \xi_1 < \xi_0$, and thus we may deform the contour in this strip. In fact, we will deform it to the contour shown in Fig. 2.

In section 7 we will prove the following lemma, which shows the suitability of this choice of integration contour.

Lemma 3 *Suppose that $p(\xi)$ is an analytic function in a neighbourhood of $I_1^+ \cup I_1^- \cup I_2^+ \cup I_2^-$. With the contour chosen as in Fig. 2 we have for any integer n that*

$$\frac{1}{2\pi}\int_{I_1^+\cup I_1^-\cup I_2^+\cup I_2^-} h(\xi)\,p(\xi)\,e^{i\xi x}d\xi \sim \mathcal{O}\left(x^{-n}\right) \tag{46}$$

It is clear that this lemma is applicable for any function $p(\xi) = \xi^p$ since they are analytic except along the branch cut, which is disjoint from $I_1^+ \cup I_1^- \cup I_2^+ \cup I_2^-$.

All that remains is thus to calculate the integral along I_0, but along this contour $h(\xi) \equiv 1$ and we have the situation in Murray (1984) except that the integration is only along a contour of finite length. However, it is a straightforward task to carry out Murray's analysis in this case since the critical element is Watson's lemma (see e.g. (Murray, 1984)), which shows that the asymptotic contributions all arise in the vicinity of the origin.

Hence, for $x > 0$ we have that for any integer n

$$\mathcal{F}^{-1}\left(\hat{T}_{w,1}\right)(x) \sim -\frac{a_0\exp(i\pi/6)}{\pi k_f \exp(i\pi/6) P^{1/3}}\frac{\Gamma(1/3)\,3^{1/2}}{2\cdot 3^{1/3}}\frac{1}{x^{2/3}}+\mathcal{O}(x^{-n}) \tag{47}$$

$$\sim -\frac{a_0}{\pi k_f P^{1/3}}\frac{\Gamma(1/3)\,3^{1/6}}{2}\frac{1}{x^{2/3}} + \mathcal{O}\left(x^{-n}\right) \tag{48}$$

$$\mathcal{F}^{-1}\left(\hat{T}_{w,2}\right)(x) \sim \frac{ia_1}{\pi k_f P^{1/3}}\frac{\Gamma(1/3)}{3^{5/6}}\frac{1}{x^{5/3}} + \mathcal{O}\left(x^{-n}\right) \tag{49}$$

where we have used that

$$\frac{Ai(0)}{Ai'(0)} = -\frac{\Gamma(1/3)}{3^{1/3}\Gamma(2/3)}\ . \tag{50}$$

To interpret these results we must calculate a_0 and a_1 in terms physical quantities. However, from (29) and (30) we have that

$$a_0 = \hat{f}(0) \tag{51}$$

$$= \int_{-\infty}^{\infty} f(x)\,dx \tag{52}$$

$$= \int_{-\infty}^{\infty}\left(k_f\frac{\partial T}{\partial y}(x,0^+) - k_s\frac{\partial T}{\partial y}(x,0^-)\right)dx \tag{53}$$

$$\equiv -Q_{TOT} \tag{54}$$

$$ia_1 = i\hat{f}'(0) \tag{55}$$

$$= \int_{-\infty}^{\infty} x f(x)\,dx \tag{56}$$

$$= \int_{-\infty}^{\infty} x\left(k_f\frac{\partial T}{\partial y}(x,0^+) - k_s\frac{\partial T}{\partial y}(x,0^-)\right)dx \tag{57}$$

$$= -Q_{m1}, \tag{58}$$

where Q_{TOT} denotes the total amount of heat energy released by the hot film, and Q_{m1} is the first moment of the heat source distribution. Hence, if the heat production is symmetric with respect to the centre of the hot film, and we choose that centre to be the origin, we find that $a_1 = 0$.

To conclude this subsection, let us recall the main finding of this subsection is, which undoubtedly is that we have found that the leading order asymptotic contribution of convection to the heat distribution along the surface for $x > 0$ is given by

$$\begin{aligned} T_{w,conv}(x) \sim & \frac{Q_{TOT}}{\pi k_f^{2/3}\left(\left.\frac{\partial U}{\partial y}\right|_w \rho c_p\right)^{1/3}} \frac{\Gamma(1/3)\, 3^{1/6}}{2} \frac{1}{x^{2/3}} \\ & - \frac{Q_{m1}}{\pi k_f^{2/3}\left(\left.\frac{\partial U}{\partial y}\right|_w \rho c_p\right)^{1/3}} \frac{\Gamma(1/3)}{3^{5/6}} \frac{1}{x^{5/3}} + \mathcal{O}\left(\frac{1}{x^{8/3}}\right). \end{aligned} \tag{59}$$

If we use Cole's parameter Λ we can rewrite this as

$$\begin{aligned} T_{w,conv}(x) \sim & \frac{\Gamma(1/3)\, 3^{1/6}}{2\pi} \frac{Q_{TOT}}{k_s \Lambda} \left(\frac{x}{L}\right)^{-2/3} \\ & - \frac{\Gamma(1/3)}{3^{5/6}\pi} \frac{Q_{m1}}{k_s \Lambda} \left(\frac{x}{L}\right)^{-5/3} + \mathcal{O}\left(\left(\frac{x}{L}\right)^{-8/3}\right). \end{aligned} \tag{60}$$

5.2 The asymptotic contribution to $T_w(x)$ for large $x > 0$ arising from terms due to conduction in the fluid

Unlike convection, conduction in the fluid and conduction in the solid act in both directions, the asymptotic effects of these terms are, however, strongly coupled with the convection, since the latter is the most efficient way to transfer heat to infinity. For this reason, which mathematically manifests itself in shape of the branch line in the upper half plane, there will not be any algebraic asymptotic contributions from these terms for negative x.

The first term arising from terms due to conduction in the fluid is

$$\hat{T}_{w,3}(\xi) = \frac{a_0}{k_f i P} h(\xi)\, \xi\ , \tag{61}$$

which is a C_0^∞ function, and hence it belongs to the Schwartz class of functions $\mathcal{S}$, which consists of C^∞ functions for which the function and all its derivatives decay faster than any polynomial. However, the Fourier transform maps this class of functions into itself (see e.g. (Hörmander, 1983)), and thus this term will be of $\mathcal{O}(x^{-n})$ for any positive integer n.

There are, however, asymptotically more significant effects of conduction in the fluid. In fact, there is a term of order $\mathcal{O}\left(x^{-10/3}\right)$.

5.3 The asymptotic contribution to $T_w(x)$ for large $x > 0$ arising from terms due to conduction in the solid

For the same reasons as for the conduction in the fluid above, there will not be any algebraic asymptotic contribution for negative x in this case either. However, if we would have replaced the slab of solid by a semi-infinite solid kept at temperature zero at infinity, the conduction in the fluid-solid interaction weakens sufficiently to enable the conduction in the solid to give an algebraic decay for the temperature for negative x.

However, our aim in this subsection is to calculate the asymptotic contribution for large positive x of

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{a_0 k_s D Ai^2(0)}{k_f^2 (iP)^{2/3}(Ai'(0))^2}h(\xi)\,\xi^{4/3}e^{i\xi x}d\xi \ . \tag{62}$$

For this integral we can use the same technique as for the convective contributions. After following the same contour, invoking Lemma 3 and using the result in Murray (1984) we will find that

$$\mathcal{F}^{-1}\left(\hat{T}_{w,4}\right)(x) \sim -\frac{a_0 k_s}{\pi k_f^2 P^{2/3}}\frac{2\Gamma(1/3)^3}{3^{13/6}\Gamma(2/3)^2}\frac{D}{x^{7/3}} + \mathcal{O}\left(x^{-n}\right) \tag{63}$$

$$\sim \frac{Q_{TOT} k_s}{\pi k_f^2 P^{2/3}}\frac{2\Gamma(1/3)^3}{3^{13/6}\Gamma(2/3)^2}\frac{D}{x^{7/3}} + \mathcal{O}\left(x^{-n}\right). \tag{64}$$

This result can be expressed in terms of Cole's parameter Λ and D/L:

$$\mathcal{F}^{-1}\left(\hat{T}_{w,4}\right)(x) \sim \frac{2\Gamma(1/3)^3}{3^{13/6}\pi\Gamma(2/3)^2}\frac{Q_{TOT}}{k_s\Lambda^2}\frac{D}{L}\left(\frac{x}{L}\right)^{-7/3} + \mathcal{O}\left(x^{-n}\right). \tag{65}$$

5.4 The temperature distribution for large negative values of x

None of the terms considered above gives any asymptotic contribution for large negative x. If we substitute $-i\eta$ for ξ in the denominator in (32) it becomes

$$k_f(\eta P)^{1/3}\frac{Ai'\left(-P^{-2/3}\eta^{4/3}\right)}{Ai\left(-P^{-2/3}\eta^{4/3}\right)} + k_s\eta\tan D\eta \ . \tag{66}$$

It is clear that this expression has a zero, η_0, somewhere between zero and π/D. This pole will give an asymptotic contribution to the interface temperature of the order $\exp(-\eta_0 x)$. Unfortunately, in this general case we have not been able to extend Lemma 1 to prove that there are no poles for (32) in lower half-plane outside the negative imaginary axis, and therefore it could well be that the asymptotic behaviour is dominated by some other pole. However, if we ignore the streamwise conduction in the fluid, *i.e.* the $\partial^2 T/\partial x^2$ term, we will instead of $s(\xi)$ obtain

$$s_0(\xi) = (iP\xi)^{1/3} \frac{Ai'(0)}{Ai(0)} \tag{67}$$

and in this case we are able to prove

Lemma 4 *All the zeros of the function $k_f s_0(\xi) - k_s \xi \tanh \xi$ in the lower half-plane are on the negative imaginary axis.*

Proof

$$k_f s_0(-i\eta) + k_s i\eta \tanh i\eta = k_f (\eta P)^{1/3} \frac{Ai'(0)}{Ai(0)} + k_s \eta \tan D\eta\ . \tag{68}$$

This function is zero if and only if

$$\tan D\eta = -\frac{k_f}{k_s} P^{1/3} \frac{Ai'(0)}{Ai(0)} \eta^{-2/3}\ . \tag{69}$$

For this function to be zero the difference of the arguments of the left and right hand sides must be a multiple of 2π. Suppose that there is a zero η^* with argument $p \neq 0$. In this case the right hand side has the argument $-2p/3$ (note that $Ai'(0)/Ai(0) < 0$). The argument of the left hand side is given by

$$\arg \tan D\eta = \arctan\left(\frac{\sinh 2\Im(D\eta)}{\sin 2\Re(D\eta)}\right) \text{ if } \Re(D\eta) \geq 0 \tag{70}$$

$$\arg \tan D\eta = \arctan\left(\frac{\sinh 2\Im(D\eta)}{\sin 2\Re(D\eta)}\right) + \pi \operatorname{sign}(\Im(D\eta)) \text{ otherwise.} \tag{71}$$

From this expression we see that the argument for fixed p varies with $D|\eta^*|$, however, we also see that

$$|p| \leq |\arg \tan D\eta| \leq \frac{\pi}{2} \text{ if } |p| \leq \frac{\pi}{2} \tag{72}$$

$$\frac{\pi}{2} \leq |\arg \tan D\eta| \leq |p| \text{ if } |p| > \frac{\pi}{2} \tag{73}$$

Hence for the difference of the arguments of the left and right hand sides to be a multiple of 2π we clearly cannot have that $|p| \leq \pi/2$ **qed**.

The poles on the imaginary axis are give by the solutions to the transcendental equation

$$\tan D\eta = \frac{k_f}{k_s} P^{1/3} \frac{3^{1/3}\Gamma(2/3)}{\Gamma(1/3)} \eta^{-2/3}\ . \tag{74}$$

There are countably many such zeros η_j. If we use this information together with the previous lemma we have that for negative x

$$T_w(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{T_w}(\xi)\, e^{i\xi x} d\xi \qquad (75)$$

$$= i\sum_{j=1}^{\infty} \mathrm{Res}_{\xi=-i\eta_j}\left[\hat{T_w}(\xi)\, e^{i\xi x}\right] . \qquad (76)$$

If we neglect all but the lowest order contribution we find that for large negative x we only have to calculate the residue for the first zero. This can be done by standard techniques, and we will only present the result in the case when

$$\frac{k_f}{k_s} P^{1/3} \frac{2^{2/3}3^{1/3}\Gamma(2/3)}{\pi^{2/3}\Gamma(1/3)} D^{2/3} >> 1 \qquad (77)$$

In this case the first zero occurs approximately for $\eta_1 \approx \pi/(2D)$, and it can be seen that if we include the highest order expression in (77) the residue becomes

$$\mathrm{Res}_{\xi=-i\eta_1} \approx \frac{k_s \pi^{1/3}\Gamma(1/3)^2}{ik_f^2 P^{2/3} D^{4/3} 3^{2/3} 2^{1/3} \Gamma(2/3)^2} \hat{f}\left(-i\frac{\pi}{2D}\right) \exp\left(-\frac{\pi x}{2D}\right) , \qquad (78)$$

and consequently if (77) holds we have for large negative x

$$T_w(x) \sim \frac{k_s \pi^{1/3}\Gamma(1/3)^2}{k_f^2 P^{2/3} D^{4/3} 3^{2/3} 2^{1/3} \Gamma(2/3)^2} \hat{f}\left(-i\frac{\pi}{2D}\right) \exp\left(-\frac{\pi x}{2D}\right). \qquad (79)$$

If $L/D << 1$ then $\hat{f}(-i\pi/(2D)) \approx \hat{f}(0) = Q_{TOT}$. In this case, (79) can be expressed in terms of Cole's parameter and D/L as

$$T_w(x) \sim \frac{\pi^{1/3}\Gamma(1/3)^2}{3^{2/3}2^{1/3}\Gamma(2/3)^2} \frac{Q_{TOT}}{k_s \Lambda^2} \left(\frac{D}{L}\right)^{-4/3} \exp\left(-\frac{\pi}{2}\frac{L}{D}\frac{x}{L}\right). \qquad (80)$$

6 Interpretations and Comparisons

6.1 Some features of the solution

In the previous section we found that the asymptotically leading order term was directly proportional to the total heat production and that it was due to convection. The first comment we will make here is that in the present problem we cannot distinguish between total heat production and the portion of the heat which enters a fluid. In fact, since convection is the most efficient mechanism to transfer heat to infinity all heat will eventually escape that way. To see this formally, let us consider the total net heat flux to the solid Q_s

$$
\begin{aligned}
Q_s &= -\int_{-\infty}^{\infty} k_s \frac{\partial T}{\partial y}\left(x, 0^-\right) dx && (81)\\
&= -k_s \lim_{\xi \to 0} \hat{T}_w\left(\xi\right) \xi \tanh D\xi && (82)\\
&= 0 && (83)
\end{aligned}
$$

where the last inequality is a direct consequence of (41)-(44). Needless to say if the solid has high thermal conductivity compared to the fluid the length scale at which almost all heat has escaped to the fluid can be very large.

Another feature of our solution worth mentioning is absence of both a temperature and a length scale. For the temperature it is of course convenient to use the linearity of the system in temperature and let the temperature at a certain fixed point be a reference, but our formulae are non-dimensional and could thus be used to predict absolute values as well. The absence of a length scale is perhaps more striking, since the problem has a distinct candidate for such a scale, *i.e.* the length of the hot film, L. However, since our temperature is proportional to the total heat flux, it. i.e. the integral of the heat flux distribution over the length of the film, we have in a sense hidden the dependence of the length of the film. We could, of course, have formulated the result in terms of $\overline{Q}L$, where $\overline{Q}$ is the average total heat production along the hot film.

Using a model with no streamwise conduction in the fluid and an adiabatic wall, Liu *et al.* (1994) calculated the asymptotic temperature distribution of a constant temperature hot-film. Although it seems difficult to extend their method to include conduction in the solid, our solutions should agree to the highest order, since this term only involves convective effects. If we place the origin at the trailing edge of the hot-film and compare our coefficients we obtain the following formula

$$
\begin{aligned}
\frac{3^{3/2}}{4\pi} L^{2/3} \Delta T &= \frac{Q_{TOT}}{\pi k_f^{2/3} \left(\left.\frac{\partial U}{\partial y}\right|_w \rho c_p \right)^{1/3}} \frac{\Gamma\left(1/3\right) 3^{1/6}}{2} && (84)\\
\Longleftrightarrow \quad \left.\frac{\partial U}{\partial y}\right|_w &= \frac{8}{81} \Gamma\left(1/3\right)^3 \frac{1}{L^2 k_f^2 \rho c_p} \left(\frac{Q_{TOT}}{\Delta T}\right)^3 && (85)\\
&= \frac{8}{81} \Gamma\left(1/3\right)^3 \frac{L}{k_f^2 \rho c_p} \left(\frac{\overline{Q}}{\Delta T}\right)^3 . && (86)
\end{aligned}
$$

This result resembles the part of a formula presented e.g. in Bellhouse & Schultz (1966), that is the contribution from the linear term of the velocity profile. It should be stressed, however, that Bellhouse & Schultz (1966) obtained their formula with a completely different technique, and with somewhat different assumptions.

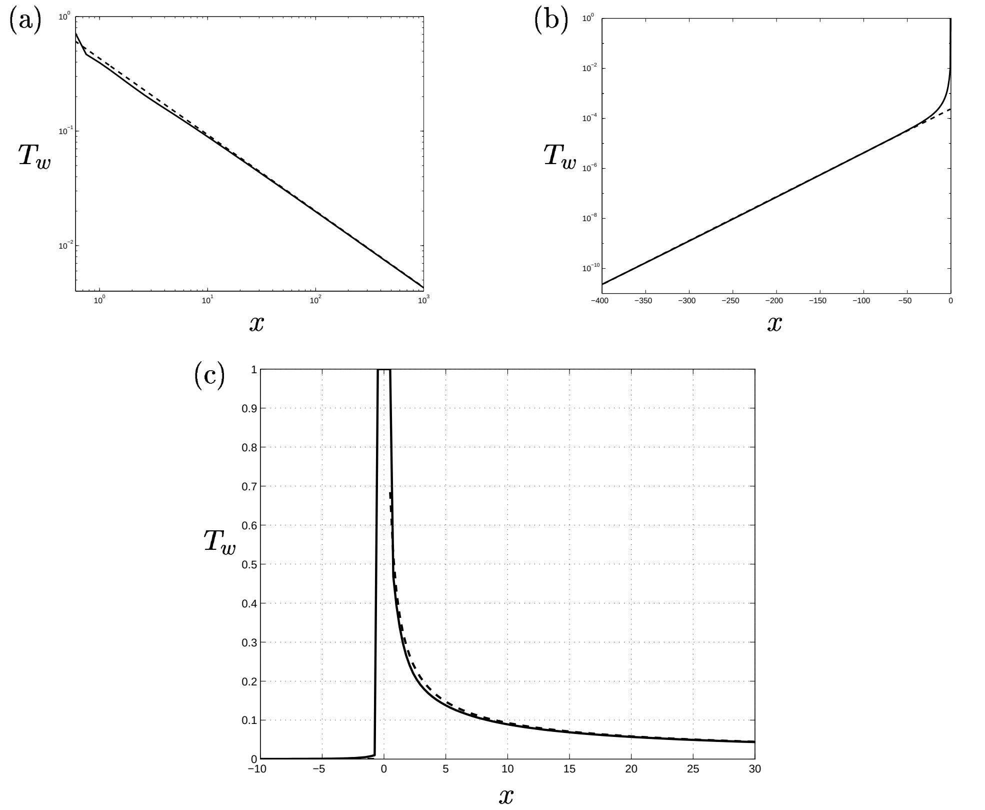

Figure 3: A comparison between the numerical (the solid lines) and the highest order asymptotic (the dashed lines) solutions for a case when $\Lambda \approx 58$ and $D/L \approx 39$. In (a) the solutions are compared downstream of the heat source, in (b) they are compared upstream of the heat source and in (c) the temperature profiles on both sides of the source are shown.

6.2 Numerical comparisons

To test as many features as possible of the asymptotic solution, we chose to verify them for a case with large Cole number Λ, in which case the importance of conduction in the solid is comparatively small. Specifically, the case we chose for extensive calculations had $\Lambda \approx 58$. This would resemble the situation for a laminar water flow over a silicon surface. The size of the domain was $700 \times L$ in the upstream direction and $1000 \times L$ downstream. The thickness of the solid was chosen to $D \approx 39 \times L$. In Fig. 3 the result of this comparison between the asymptotic and numerical solution is shown. To emphasize the algebraic decay of the temperature profile downstream of the heat source, we present this comparison in Fig. 3(a) in a log-log graph, and as is seen the agreement is good. Upstream of the heat source the temperature profile decays exponentially, and for that reason the comparison

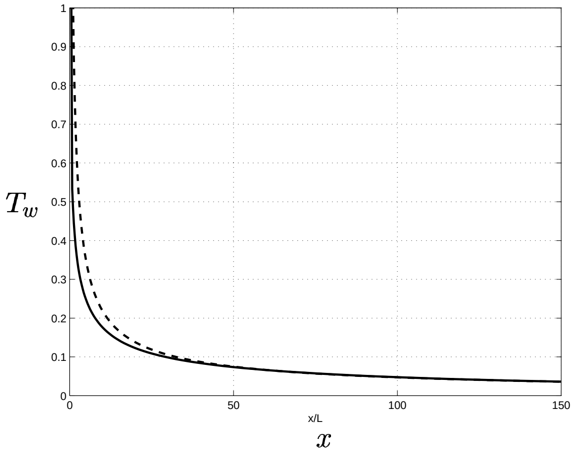

Figure 4: A comparison between the numerical (the solid lines) and the highest order asymptotic (the dashed lines) solutions downstream of the heat source for a case when $\Lambda \approx 0.22$

in this region as shown in Fig. 3(b) is presented in a lin-log graph. Once again the agreement between the asymptotic and numerical solutions is good. In Fig. 3(c) the result upstream and downstream of the heat source is shown in one lin-lin graph. In this figure we have only shown the region in the vicinity of the heat source and here we can notice a slight deviation due to higher order contributions.

To test the asymptotic formulae in a more demanding case we chose to perform computations for $\Lambda \approx 0.22$. In this case, we see for the downstream temperature profile, which is presented in Fig. 4 that some distance away from the heat source, the agreement between the asymptotic and numerical solution is satisfactory. That the distance before which the leading order asymptotic solution describes the solution accurately increases with decreasing Cole number is quite evident from the analysis above.

7 Proofs of technical lemmata

Proof of Lemma 1
We will prove that the expression $s(\xi)$ is never real for non-zero values of ξ. Since all other terms are real this will clearly suffice to prove the lemma.

Because of (27) it suffices to consider $\xi > 0$. In this case we have that

$$\arg (iP\xi)^{1/3} = \frac{\pi}{6} \quad (87)$$

$$\arg \left((iP)^{-2/3} \xi^{4/3} \right) = -\frac{\pi}{3} \,. \quad (88)$$

For every positive ξ there are positive, real numbers x and q such that

$$s(\xi) = q \exp\left(i\frac{\pi}{6}\right) \frac{Ai'\left(x \exp\left(-i\frac{\pi}{3}\right)\right)}{Ai\left(x \exp\left(-i\frac{\pi}{3}\right)\right)} \tag{89}$$

By successively using well known formulae for Airy and Bessel functions (see e.g. (Abramowitz & Stegun, 1970)) we can make the following transformations:

$$\begin{aligned} s(\xi) &= -q\sqrt{x} \frac{K_{2/3}\left(\frac{2}{3}x \exp\left(-i\frac{\pi}{2}\right)\right)}{K_{1/3}\left(\frac{2}{3}x \exp\left(-i\frac{\pi}{2}\right)\right)} & (90) \\ &= -q\sqrt{x} \exp\left(i\frac{\pi}{6}\right) \frac{H^{(1)}_{2/3}\left(\frac{2}{3}x\right)}{H^{(1)}_{1/3}\left(\frac{2}{3}x\right)} & (91) \\ &= -q\sqrt{x} \frac{\exp\left(-i\frac{\pi}{3}\right) J_{2/3}\left(\frac{2}{3}x\right) - \exp\left(i\frac{\pi}{3}\right) J_{-2/3}\left(\frac{2}{3}x\right)}{\exp\left(-i\frac{\pi}{6}\right) J_{1/3}\left(\frac{2}{3}x\right) - \exp\left(i\frac{\pi}{6}\right) J_{-1/3}\left(\frac{2}{3}x\right)} & (92) \end{aligned}$$

However, we know that the Bessel function of the first kind is real valued for real valued arguments, and hence we have that the expression is real if and only if the quotient of the imaginary and real part of the nominator is the same as that of the denominator. Hence, all that remains to prove is that for all positive values of x

$$\tan\left(\frac{\pi}{3}\right) \frac{J_{2/3}\left(\frac{2}{3}x\right) + J_{-2/3}\left(\frac{2}{3}x\right)}{J_{2/3}\left(\frac{2}{3}x\right) - J_{-2/3}\left(\frac{2}{3}x\right)} \neq \tan\left(\frac{\pi}{6}\right) \frac{J_{1/3}\left(\frac{2}{3}x\right) + J_{-1/3}\left(\frac{2}{3}x\right)}{J_{1/3}\left(\frac{2}{3}x\right) - J_{-1/3}\left(\frac{2}{3}x\right)} \tag{93}$$

All that remains now is therefore to prove that there are no real and positive solutions to (93). By simple algebra it is shown that this is equivalent to proving the absence of real and positive solutions to

$$\begin{aligned} J_{2/3}(s) J_{-1/3}(s) &- J_{-2/3}(s) J_{1/3}(s) \\ &+ \frac{1}{2}\left(J_{-2/3}(s) J_{-1/3}(s) - J_{2/3}(s) J_{1/3}(s)\right) = 0\,. \end{aligned} \tag{94}$$

However, if we now go back to expressing this in terms of Airy functions and their derivatives we find that (94) has no real and positive solutions if there are no real negative solutions to

$$3\left(Ai(s) Ai'(s) + Bi(s) Bi'(s)\right) = 0\ , \tag{95}$$

i.e. that the derivative of the modulus of the Airy functions is non-zero on the negative half-axis (this is trivially true on the positive half-axis, but this fact has no bearing on our proof).

However, according to Reid (1995) one has the following expression for the modulus of the Airy functions on the negative half-axis. ($z > 0$)

$$Ai^2(-z) + Bi^2(-z) = \frac{1}{\pi^{3/2}} \int_0^\infty t^{-1/2} \exp\left(-zt - \frac{1}{12}t^3\right) dt \; . \tag{96}$$

Hence we have that

$$Ai(-z)Ai^{'}(-z) + Bi(-z)Bi^{'}(-z) = \frac{1}{2\pi^{3/2}} \int_0^\infty t^{1/2} \exp\left(-zt - \frac{1}{12}t^3\right) dt. \tag{97}$$

However, the integrand is positive for all positive values of z, and thus we have shown that for all real and negative values of s

$$Ai(s)\, Ai^{'}(s) + Bi(s)\, Bi^{'}(s) > 0 \; , \tag{98}$$

which rules out the possibility of having real and negative solutions to (95), and consequently the lemma is established. **qed**

Proof of Lemma 2

Since f is bounded and has compact support we have that $x^p f(x) \in L^2(\mathbf{R})$ for all positive values of p. Hence we have that

$$(-i)^p \left(\frac{d}{d\xi}\right)^p \hat{f}(\xi) \in L^2(\mathbf{R}) \cap A(\mathbf{C}) \, , \tag{99}$$

where $A(\mathbf{C})$ denotes the space of entire functions.

We have that

$$\hat{T}_w(\xi)(1 - h(\xi)) = \frac{\hat{f}(\xi)(1 - h(\xi))}{k_f s(\xi) - k_s \xi \tanh D\xi}. \tag{100}$$

Hence, Leibniz' rule implies that for any non-negative integer q we have that

$$\left(\frac{d}{d\xi}\right)^q \left(\hat{T}_w(\xi)(1 - h(\xi))\right) = $$
$$= \sum_{k=0}^{q} \binom{q}{k} \left(\frac{d}{d\xi}\right)^k \hat{f}(\xi) \left(\frac{d}{d\xi}\right)^{q-k} \left(\frac{1 - h(\xi)}{k_f s(\xi) - k_s \xi \tanh D\xi}\right). \tag{101}$$

We now claim that we have for any non-negative integer m that

$$\left(\frac{d}{d\xi}\right)^m \left(\frac{1 - h(\xi)}{k_f s(\xi) - k_s \xi \tanh D\xi}\right) \in L^2(\mathbf{R}) \; . \tag{102}$$

If we assume that this claim is true it directly follows from (99), Hölder's inequality (see e.g. (Folland, 1984)) and the linearity of the L^p spaces that

$$\left(\frac{d}{d\xi}\right)^q \left(\hat{T}_w(\xi)(1 - h(\xi))\right) \in L^1(\mathbf{R}) \; , \tag{103}$$

for any non-negative integer q. According to a well-known theorem in Fourier Analysis (see e.g. (Arsac, 1966)) this implies the lemma. Consequently, it suffices for us to prove the claim (102).

To this end, let us start by noting that the expression in (102) is bounded. For non-zero values of ξ this follows from Lemma 1 and the fact that $h_{\xi_0,B(\xi)}$, $s(\xi)$ and $\xi \tanh D\xi$ all are C^∞ functions outside the origin. Moreover, since $(1-h(\xi)) \equiv 0$ in a neighbourhood of the origin it follows that the expression in (102) is bounded there as well. As a consequence, the claim exclusively concerns decay at infinity. Remember that $(1-h(\xi)) \equiv 1$ for all $\xi > 2\xi_0$. This implies that for these values of ξ we have that

$$\left(\frac{d}{d\xi}\right)^m \left(\frac{1-h(\xi)}{k_f s(\xi) - k_s \xi \tanh D\xi}\right) = \left(\frac{d}{d\xi}\right)^m \left(\frac{1}{g(\xi)}\right), \tag{104}$$

where $g(\xi) = k_f s(\xi) - k_s \xi \tanh D\xi$.

The following asymptotic property of $s(\xi)$ will be established in a separate lemma.

Lemma 5 *We have that*

$$s(\xi) \sim -|\xi| + \mathcal{O}\left(\xi^{-1}\right) \tag{105}$$

$$s'(\xi) \sim -sign(\xi) + \mathcal{O}\left(\xi^{-2}\right) \tag{106}$$

$$s^{(m)}(\xi) \sim (-1)^{m+1}\frac{iP}{4}m!\xi^{-m-1} + \mathcal{O}\left(\xi^{-m-3}\right), \text{ for } m \geq 2, \tag{107}$$

which should be interpreted as saying that the absolute value of the difference between the left hand side and the first term on the right hand side is bounded by a constant times the function inside the big ordo for large values of $|\xi|$.

Note that the rate of decay of the leading order term increases as m increases, and in particular notice the extra increase in decay rate as m increases from one to two.

In addition to this result we need the simple fact that for any integer $N > 0$ we have that

$$\frac{d^m}{d\xi^m}\xi \tanh D\xi \sim \begin{cases} |\xi| + \mathcal{O}\left(\xi^{-N}\right), \text{ for } m = 0 \\ \text{sign}(\xi) + \mathcal{O}\left(\xi^{-N}\right), \text{ for } m = 1 \\ \mathcal{O}\left(\xi^{-N}\right), \text{ for } m \geq 2 \end{cases} \tag{108}$$

If we combine Lemma 5 and (108) we find that

$$g^{(m)}(\xi) \sim \begin{cases} -(k_f + k_s)|\xi| + \mathcal{O}(\xi^{-1}), \text{ for } m = 0 \\ -(k_f + k_s)\,\text{sign}(\xi) + \mathcal{O}(\xi^{-2}), \text{ for } m = 1 \\ (-1)^{m+1}\frac{iP}{4}m!\xi^{-m-1} + \mathcal{O}(\xi^{-m-3}), \text{ for } m \geq 2 \end{cases}. \tag{109}$$

By induction it is readily established that

$$\frac{d^m}{d\xi^m}\left(\frac{1}{g(\xi)}\right) = \frac{\sum_{\mathbf{p}} a_{\mathbf{p},m} \prod_{j=1}^{m} g^{(p_j)}(\xi)}{(g(\xi))^{m+1}} , \tag{110}$$

where the $a_{\mathbf{p},m}$ are coefficients and $\mathbf{p} = (p_1, \ldots, p_m)$ is a vector of non-negative integers such that $0 \leq p_1 \leq \cdots \leq p_m \leq m$, and $\sum_{j=1}^{m} p_j = m$. Since $g^{(m)}$ decreases as ξ^{-m-1} for $m \geq 2$, but only as ξ^{1-m} for $m = 0, 1$, the decay rate of the nominator in (110) will be minimal if all the p_i's are one. Hence, for all $\mathbf{p}$ satisfying the conditions above we have that

$$\left|\prod_{j=1}^{m} g^{(p_j)}(\xi)\right| \leq (k_f + k_s)^m + \mathcal{O}\left(\xi^{-2}\right) . \tag{111}$$

This implies that for each m we can pick a value C_m such that there exists an ξ_m such that

$$\left|\sum_{\mathbf{p}} a_{\mathbf{p},m} \prod_{j=1}^{m} g^{(p_j)}\right| \leq C_m, \text{ for } |\xi| > \xi_m . \tag{112}$$

However, from this, (110) and (109) we have that

$$\left|\frac{d^m}{d\xi^m}\left(\frac{1}{g(\xi)}\right)\right| \leq \frac{C_m}{(k_f + k_s)^{m+1} |\xi|^{m+1}}, \text{ for } |\xi| > \xi_m . \tag{113}$$

This result combined with the fact that expression in (102) is bounded proves that the claim in (102) is true, provided that we can prove Lemma 5 **qed**

Proof of Lemma 5

By differentiation of $s(\xi)$ and utilization of $Ai''(z) = zAi(z)$ we find that $s(\xi)$ satisfies the ODE

$$s'(\xi) = \frac{4}{3iP}\left(\xi^2 - s^2(\xi)\right) + \frac{s(\xi)}{3\xi} . \tag{114}$$

From (Abramowitz & Stegun, 1970) we find that for large $|z|$, $|\arg z| \leq \pi/3$ we have that

$$Ai(z) \sim \frac{1}{2}\pi^{-1/2} z^{-1/4} e^{-\zeta} \sum_{k=0}^{\infty} (-1)^k c_k \zeta^{-k} \tag{115}$$

$$Ai'(z) \sim \frac{1}{2}\pi^{-1/2} z^{1/4} e^{-\zeta} \sum_{k=0}^{\infty} (-1)^k d_k \zeta^{-k} \tag{116}$$

where $\zeta = \frac{2}{3} z^{3/2}$, $c_0 = d_0 = 1$,

$$c_k = \frac{\Gamma(3k + 1/2)}{54^k k! \Gamma(k + 1/2)}$$

and $d_k = -(6k+1)/(6k-1)\,c_k$. These expressions mean that the absolute value of the difference between the function and a partial sum is smaller than the first omitted term (Hörmander, 1983), but below we will only require that the absolute value of this difference is bounded by the first omitted term multiplied by some positive constant.

Let us now restrict ourselves to the case $\xi > 0$ (the negative case can be treated analogously. In fact only some signs have to be changed for the subsequent analysis to hold in this case as well). When these asymptotic expansions are substituted into (26) and the formula for the sum of a geometric series is used we find that there exists coefficients a_k such that

$$s(\xi) \sim \sum_{k=-1}^{\infty} a_k \xi^{-1-2k} , \tag{117}$$

where $a_{-1} = -1$. In principle, we could calculate all coefficients directly, but the procedure is quite cumbersome. We should note that the asymptotic property is preserved in the division since both the nominator (except for a factor of ξ^{-1}) and the denominator are expanded in terms of the asymptotic sequence $\{\xi^{-2p}\}_{p=0}^{\infty}$ and since the denominator has a non-zero constant term. (see e.g. the Corollary to Theorem 1.7.4 in Bleistein & Handelsman (1986)). Hence the statement for $s(\xi)$ has been established.

Let us substitute this expression formally into (114) to find that

$$s'(\xi) \sim \sum_{k=-1}^{\infty} \left(\frac{a_k}{3} - \frac{4}{3iP} \sum_{p=-1}^{k+1} a_r a_{k-r} \right) \xi^{-2-2k} \tag{118}$$

where the fact that $a_{-1} = -1$ has cancelled out the ξ^2-term. The asymptotic properties are retained since we have only performed simple algebraic operations (addition, multiplication and division by ξ) of sequences expanded in terms of the common asymptotic sequence, $\{\xi^{-2p}\}_{p=0}^{\infty}$. The only problem is that if the rest term of $s(\xi)$ is $\mathcal{O}\left(\xi^{-N}\right)$ then the corresponding term in the asymptotic expansion for $s'(\xi)$ will be $\mathcal{O}\left(\xi^{1-N}\right)$ since $s(\xi)$ is of $\mathcal{O}(\xi)$, but this is of little consequence since N can be chosen arbitrarily.

Consider now $s(\xi) + \xi$. From (117) we have that $s(\xi) + \xi$ can be expanded asymptotically in terms of the asymptotic sequence $\left\{\xi^{-1-2k}\right\}_{k=0}^{\infty}$. The terms of this sequence are differentiable and they all vanish at infinity. Furthermore, these properties are shared by the sequence

$$\left\{(-1-2k)\,\xi^{-2-2k}\right\}_{k=0}^{\infty} = \left\{\frac{d}{d\xi}\xi^{-1-2k}\right\}_{k=0}^{\infty} .$$

In addition, the terms of this second sequence are all negative for $\xi > 0$, and since (118) tells us that s' can be expanded in terms of this sequence (Bleistein & Handelsman, 1986, Theorem 1.7.7) tells us that we may differentiate the asymptotic series term-by-term, *i.e.* that

$$s^{'}(\xi) \sim \sum_{k=-1}^{\infty} a_k (-1-2k)\, \xi^{-2-2k} \; . \tag{119}$$

This implies the statement for $s^{'}(\xi)$ and in addition that the following recursive relation must hold for the coefficients a_k

$$a_k(-1-2k) = \left(\frac{a_k}{3} - \frac{4}{3iP} \sum_{r=-1}^{k+1} a_r a_{k-r} \right) \; . \tag{120}$$

If we use the fact that $a_{-1} = -1$ we find that this can be rearranged to yield

$$a_{k+1} = \frac{1}{2} \sum_{r=0}^{k} a_r a_{k-r} - \frac{iP}{4} (3k+2)\, a_k \; . \tag{121}$$

This is a far quicker way to calculate the a_k's than to use the expansion for the Airy functions. For example, this may be used to calculate $a_0 = -iP/4$.

In order to conclude the proof of the lemma we must now extend these results to higher derivatives of s, in fact we will prove that for $m \geq 2$ we have that

$$s^{(m)}(\xi) \sim \sum_{k=0}^{\infty} a_k (-1)^n \frac{(m+2k)!}{(2k)!} \xi^{-1-2k-m} \tag{122}$$

from which the lemma follows since $a_0 = -iP/4$.

The only things which could cause difficulties are the ξ^2 term in (114) and the presence of the first terms in the expansions for $s(\xi)$ and $s^{'}(\xi)$. However, these difficulties cancel each other. To see this let us assume that the lemma holds for $s,\ s^{'}, \ldots,\ s^{(m)}$. Consequently, there exists an asymptotic expansion for the mth derivative of $\xi^2 - s^2(\xi)$. However, if we use the fact that $a_{-1} = -1$ we find that

$$\xi^2 - s^2(\xi) \sim 2a_{-1}a_0 + \left(a_0^2 + 2a_{-1}a_1\right) \xi^{-2} + \cdots \; . \tag{123}$$

Now if $m \geq 1$ there are b_k such that

$$\frac{d^m}{d\xi^m} \left(\xi^2 - s^2(\xi)\right) \sim \sum_{k=0}^{\infty} b_k \xi^{-m-2-2k}. \tag{124}$$

Similarly, the asymptotic expansion

$$\frac{s(\xi)}{\xi} \sim a_{-1} + a_0 \xi^{-2} + \cdots \tag{125}$$

implies that the mth derivative of this expression has the expansion

$$\frac{d^m}{d\xi^m} \left(\frac{s(\xi)}{\xi} \right) \sim \sum_{k=0}^{\infty} c_k \xi^{-m-2-2k}, \tag{126}$$

for some c_k. By differentiating (114) m times we find that (124) and (126) imply that $s^{(m+1)}(\xi)$ has an asymptotic expansion of the desired form. Hence, by (Bleistein & Handelsman, 1986, Theorem 1.7.7) we find that the asymptotic expansion for $s^{(m+1)}(\xi)$ is given by term-by-term differentiation of the expansion for $s^{(m)}(\xi)$. Consequently, (122) follows by induction. This concludes the proof of the lemma. **qed**

Proof of Lemma 3

We will prove that

$$\frac{1}{2\pi}\int_{I_1^+\cup I_2^+} h(\xi)\,p(\xi)\,e^{i\xi x}d\xi \sim \mathcal{O}\left(x^{-n}\right) \ , \tag{127}$$

and the same result in the left half-plane is established analogously.

We are confined to the real axis along I_2^+ and therefore we will use the method of stationary phase there. On I_1^+, however, we will use the method of steepest descent, which requires analyticity.

Suppose that the point of intersection between I_1^+ and I_2^+ is $\xi^\dagger$. Since the integrand is C^∞ and since $h(\xi)$ vanishes for all $\xi > 2\xi_0$ we have that (see e.g. (Bleistein & Handelsman, 1986, Example 3.2.2)) for any integer N

$$\frac{1}{2\pi}\int_{I_2^+} h(\xi)p(\xi)\,e^{i\xi x}d\xi \sim \sum_{n=0}^{N}\frac{(-1)^{n+1}}{(ix)^{n+1}}p^{(n)}\left(\xi^\dagger\right)e^{ix\xi^\dagger}+\mathcal{O}\left(x^{-N}\right), \tag{128}$$

where we have also used the fact that $h(\xi)\equiv 1$ in a neighbourhood of $\xi^\dagger$.

When applying the method of steepest descent (see e.g. (Murray, 1984; Bleistein & Handelsman, 1986)) to the integral along I_1^+ we find that there are no other critical points than the end points, $\xi^\dagger$ and ξ^A (the intersection of I_1^+ and I_0). Furthermore, our contour already pass through these points parallel to the lines of steepest descent. Hence we may apply Watson's lemma in the neighbourhoods of the end points to obtain for any integer N

$$\begin{aligned}\frac{1}{2\pi}\int_{I_1^+} h(\xi)\,p(\xi)\,e^{i\xi x}d\xi &\sim -e^{ix\xi^\dagger}\sum_{n=0}^{N}\frac{(-1)^{n+1}g^{(n)}\left(\xi^\dagger\right)}{(ix)^{n+1}}+\\ &+\ e^{ix\xi^A}\sum_{n=0}^{N}\frac{(-1)^{n+1}g^{(n)}\left(\xi^A\right)}{(ix)^{n+1}}+\mathcal{O}\left(x^{-N}\right)\end{aligned} \tag{129}$$

If we sum (128) and (129) we obtain

$$\frac{1}{2\pi}\int_{I_1^+\cup I_2^+} h(\xi)p(\xi)e^{i\xi x}d\xi \sim e^{ix\xi^A}\sum_{n=0}^{N}\frac{(-1)^{n+1}g^{(n)}\left(\xi^A\right)}{(ix)^{n+1}}+\mathcal{O}\left(x^{-N}\right). \tag{130}$$

for any integer N. However, A has a positive imaginary part and hence the first terms on the right hand side are decreasing exponentially with x. This concludes the proof of the lemma. **qed**

8 Conclusions

In this paper we have presented an asymptotic solution to the conjugate heat transfer problem with a flush mounted heat source on the fluid-solid interface, in the case that the bottom of the solid is perfectly insulated and the velocity profile in the fluid is linear. The lowest order terms of the asymptotic solution can be naturally classified into contributions from pure convection, from the interaction of convection and the conduction in the solid and from the interaction of convection and the conduction in the fluid. It was found that downstream of the heat source the two leading terms of the asymptotic expansion come from pure convection, and that the leading term decays as $\mathcal{O}\left(x^{-2/3}\right)$, which confirms the result from the analysis by Liu *et al.* (1994) in the case of an adiabatic wall. However, we also seen that the third term in the asymptotic expansion of the interface temperature involves conduction in the solid, and hence this contribution could not have been captured by the technique used in Liu *et al.* (1994). If we neglect the conduction in the fluid we have been able to find the asymptotic solution upstream of the heat source as well, and in this case we find that the temperature decays exponentially with the distance from the heat source. Our results have been compared with numerical solutions to the problem with good agreement.

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