VARIOUS CONSTRUCTIONS OF TAYLOR'S FUNCTIONAL CALCULUS FOR COMMUTING OPERATORS

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ABSTRACT. We discuss various constructions of Taylor's functional calculus for a tuple of commuting operators on a Banach space, and study their mutual relations. In particular, we provide a simple explicit link between the cohomological construction based on the Dolbeault complex, and the construction by means of the resolvent mapping (by abstract Cauchy-Fantappie-Leray formulas). The main idea is to introduce a kind of abstract weighted integral formulas. We also study the behaviour of the resolvent mapping under analytic mappings.

1. INTRODUCTION

Let $a_1, \ldots, a_n$ be an n-tuple of commuting operators on a Banach space X. For any polynomial or entire function $f(a)$ one can define an operator $f(a)$ on X, simply by replacing each z by a in the Taylor expansion for $f(z)$. One then gets an algebra homomorphism

$$\mathcal{O}(\mathbb{C}^n) \to \mathcal{L}(X), \quad \phi \mapsto \phi(a), \tag{1.1}$$

a functional calculus, where $\mathcal{L}(X)$ denotes the space of bounded operators on X. In order to find extensions of this functional calculus, one is led to the notion of joint spectrum. The relevant definition of spectrum of the n-tuple a was found by Taylor 1970, [16], and runs as follows. Let $e_1, \ldots, e_n$ be a basis (global frame) for a trivial bundle $E \to \mathbb{C}^n$ and let ΛE be the exterior algebra of E. If δ_{z-a} denotes interior multiplication by the operator-valued section

$$2\pi i \sum_j (z_j - a_j) e_j^*$$

to the dual bundle (e_j^* denoting the dual basis), then we get a complex

$$0 \leftarrow \Lambda^0 E_z \otimes X \leftarrow \Lambda^1 E_z \otimes X \leftarrow \cdots \leftarrow \Lambda^n E_z \otimes X \leftarrow 0 \tag{1.2}$$

for each $z \in \mathbb{C}^n$. The Taylor spectrum $\sigma(a)$ of the n-tuple a is, by definition, the set of all $z \in \mathbb{C}^n$ such that (1.2) is not exact. It turns

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out that the spectrum is a compact nonempty (unless $X = \{0\}$) subset of $\mathbb{C}^n$. The main result, due to Taylor, is

Theorem 1.1 (Taylor). *Let a be an n-tuple of commuting operators on the Banach space X. There is a continuous homomorphism from $\mathcal{O}(\sigma(a))$ into $L(X)$ that extends the functional calculus $\mathcal{O}(\mathbb{C}^n) \to X$. The image $f(a)$ of $f \in \mathcal{O}(\sigma(a))$ commutes with each $b \in L(X)$ that commutes with each a_j. If $f = (f_1, \dots, f_n)$ is an analytic mapping, $f_j \in \mathcal{O}(\sigma(a))$, and $f(a) = (f_1(a), \dots, f_n(a))$, then $\sigma(f(a)) = f(\sigma(a))$.*

It was proved by Putinar in [13] that furthermore the superposition property $g \circ f(a) = g(f(a))$ holds. Moreover, he proved in [14] that any two extensions of the functional calculus, which fulfill the properties stated in Theorem 1.1, coincide.

In the case of one single operator, the extension of the functional calculus from entire functions can be made by Cauchy's integral formula,

$$\phi(a)x = \int_{\partial D} \phi \omega_{z-a} x, \quad \phi \in \mathcal{O}(U), \ x \in X, \tag{1.3}$$

where $\sigma(a) \subset D \subset\subset U$, and $\omega_{z-a}x = (2\pi i)^{-1}(z-a)^{-1}x dz$. One thus expresses the function $\phi(z)$ as a superposition of simple rational functions, for each of which it is clear how to replace z by a.

Taylor's first construction of the multidimensional functional calculus, [17], was based on Cauchy-Weyl formulas. These are generalizations of Cauchy's product formula, and they were the most commonly used formulas for representing holomorphic functions of several variables at that time. Somewhat later Taylor made a construction of the functional calculus by homological methods, [18]. This approach has proved to be very useful for further results, such as the superposition property, the uniqueness mentioned above, and others, see [10]. It follows from the uniqueness result that the two mentioned constructions give the same functional calculus.

From the beginning of the 70's and on the Cauchy-Weyl formulas have been outclassed by Cauchy-Fantappie-Leray formulas in pluricomplex function theory. In [3] (and [4]) we made a construction of the functional calculus, based on such formulas, and proved the basic functorial properties. In [7] we even proved the superposition property along the same lines. The motivation for this approach is twofold. To begin with we think that it is more comprehensive, as the operator $\phi(a)$, acting on $x \in X$, is defined, just as in one variable, by an integral formula like (1.3), where $\omega_{z-a}x$ is a Dolbeault cohomology class that is represented by a $\bar\partial$-closed $(n, n-1)$-form in $U \setminus \sigma(a)$. We will refer the mapping $x \mapsto \omega_{z-a}x$ as the resolvent mapping. Furthermore, if D contains a Stein neighborhood of the spectrum, then the formulas obtained are nothing but usual Cauchy-Fantappie-Leray representation formulas for holomorphic functions where z is replaced by a, and this is a natural starting point for possible extensions, in various situations, to a nonholomorphic functional calculus, following the work of Dynkin [9] in the one-variable case. For some results in this direction, see [15]. For the case of real spectrum, see also [8] and [6].

The first aim of this note is to reveal explicit connections between various constructions of the functional calculus. To begin with we describe a (co)homological construction of the functional calculus, based on Dolbeault cohomology rather than Čech cohomology as in [18] and [10]. We then recall the construction in [3] and provide an explicit link between these two constructions. The crucial point here is to find a factorization of the mapping $A_U : \mathcal{O}(U,X) \to X$, obtained by integration of the resolvent, over a certain cohomology space that is isomorphic to X. Whereas the mapping A_U boils down to Cauchy-Fantappie-Leray formulas in concrete realizations, the extended mapping in a similar way corresponds to weighted representation formulas for holomorphic functions. This link provides a concrete realization of the homological construction, and it also sheds new light on the construction from [3]. We also include a discussion of the relation to Taylor's original constructions with Čech cohomology (Section 4).

In Section 5 we prove the wellknown functorial properties of the functional calculus by means of the resolvent mapping. The basic ideas come from [3], [4], [7], and [15], but the presentation is new and we also provide several completely new arguments.

In Section 6 we present new results about the behaviour of the resolvent mapping under analytic mappings. In particular, we show that the resolvent is coordinate invariant.

Throughout this paper X is a Banach space and e is the unit element in $\mathcal{L}(X)$. Moreover, if a is a commuting tuple in $\mathcal{L}(X)$, then $(a)'$ is the commutant, i.e., the subalgebra of $\mathcal{L}(X)$ consisting of all operators that commute with each a_j, and $(a)''$ is the bicommutant, i.e., the commutant of $(a)'$.

Contents

2. Construction by means of Dolbeault cohomology

First notice that any continuous algebra homomorphism $\Phi\colon \mathcal{O}(U) \to \mathcal{L}(X)$, U open subset of $\mathbb{C}^n$, gives rise to a unique continuous $\mathcal{O}(U)$-module structure $\mathcal{O}(U) \times X \to X$ on X, defined by $(\phi, x) \mapsto \Phi(\phi)x$. Conversely, any continuous $\mathcal{O}(U)$-module structure on X is obtained in this way from the homomorhism Φ, defined by letting $\Phi(\phi)x$ be the image of (ϕ, x) for $\phi \in \mathcal{O}(U)$ and $x \in X$. Let us assume from now on that we have a fixed tuple $a_1, \dots, a_n$ of commuting operators on our Banach space X. We then have the functional calculus (1.1), and extending to a functional calculus as in Theorem 1.1 thus amounts to extending the given $\mathcal{O}(\mathbb{C}^n)$-module structure of X to a $\mathcal{O}(U)$-module structure. To begin with, we have a mapping

$$A\colon \mathcal{O}(\mathbb{C}^n, X) \to X, \tag{2.1}$$

defined by $f(z) = \sum c_\alpha z^\alpha \mapsto f(a) = \sum c_\alpha a^\alpha$, $c_\alpha \in X$. One readily checks that $(\phi f)(a) = \phi(a) f(a)$ if $\phi \in \mathcal{O}(\mathbb{C}^n)$ and $f \in \mathcal{O}(\mathbb{C}^n, X)$, so A is a $\mathcal{O}(\mathbb{C}^n)$-module homomorphism. Furthermore, A commutes with $b \in (a)'$ and $Ax = x$. One can define A by means of integral formulas instead of power series, see Remark 6.

Let us identify E with the bundle $T^{1,0}$ of $(1,0)$-covectors; then δ_{z-a} means interior multiplication with the operator valued vector field $2\pi i \sum_j (z_j - a_j)(\partial/\partial z_j)$. Let $\mathcal{E}_{p,q}(V, X)$ denote the space of X-valued (p,q)-forms in the open set $V \subset \mathbb{C}^n$, and let $\mathcal{O}_p(V, X)$ be the subspace of holomorphic $(p,0)$-forms. Clearly δ_{z-a} induces continuous mappings $\mathcal{E}_{p,q}(V, X) \to \mathcal{E}_{p-1,q}(V, X)$, that anticommutes with $\bar\partial$, so we get a complex

$$0 \leftarrow \mathcal{E}_{0,q}(V, X) \xleftarrow{\delta_{z-a}} \mathcal{E}_{1,q}(V, X) \xleftarrow{\delta_{z-a}} \cdots . \tag{2.2}$$

for each q, and the complex

$$0 \leftarrow \mathcal{O}(V, X) \xleftarrow{\delta_{z-a}} \mathcal{O}_1(V, X) \xleftarrow{\delta_{z-a}} \dots . \tag{2.3}$$

It is wellknown, see, e.g., [10], that (2.2) is exact if $V \subset \mathbb{C}^n \setminus \sigma(a)$; in fact $V = \mathbb{C}^n \setminus \sigma(a)$ is the largest open set such that (2.2) is exact. (Moreover, (2.3) is exact if V is a Stein open subset of $\mathbb{C}^n \setminus \sigma(a)$, see Lemma 4.2 below).

Lemma 2.1. *The mapping* (2.1) *induces an* $\mathcal{O}(\mathbb{C}^n)$*-linear isomorphism*

$$\frac{\mathcal{O}(\mathbb{C}^n, X)}{\operatorname{Im}\left(\mathcal{O}_1(\mathbb{C}^n, X) \to \mathcal{O}(\mathbb{C}^n, X)\right)} \simeq X.$$

Proof. If $\delta_{z-a} u(z) = f(z)$, then it is clear that $f(a) = 0$. Conversely, for any entire f there is $u(z, w) \in \Lambda^1 \mathcal{O}(\mathbb{C}^n \times \mathbb{C}^n, X)$ such that $f(z) - f(w) = \delta_{z-w} u(z, w)$. Replacing w by a we get that $\delta_{z-a} u(z, a) = f(z)$ if $f(a) = 0$. □

Remark 1. One can prove, see [10], that if V is a Stein open set that contains $\sigma(a)$ then (2.3) is exact at each level but $k = 0$ and the cokernel there is X. However, we only need the special case contained in Lemma 2.1. See also Remark 3. □

For any V and $\ell \geq 0$ we have the X-valued Dolbeault complex

$$0 \to \mathcal{E}_{\ell,0}(V,X) \xrightarrow{\bar\partial} \mathcal{E}_{\ell,1}(V,X) \xrightarrow{\bar\partial}, \tag{2.4}$$

and the kernel of $\mathcal{E}_{\ell,0}(V,X) \to \mathcal{E}_{\ell,1}(V,X)$ is $\mathcal{O}_\ell(V,X)$.

Lemma 2.2. *If $V \subset \mathbb{C}^n$ is pseudoconvex, then the complex*

$$0 \to \mathcal{O}_\ell(V,X) \to \mathcal{E}_{\ell,0}(V,X) \to \mathcal{E}_{\ell,1}(V,X) \to$$

is exact.

This follows from the exactness of the usual scalar-valued Dolbeault complex, see [10]. Except for Section 4 we will only use the lemma for $V = \mathbb{C}^n$, in which case it can be proved simply by a weighted integral formula.

Since $\bar\partial \delta_{z-a} = -\delta_{z-a}\bar\partial$, we get for each open set $V \subset \mathbb{C}^n$ a double complex

$$\mathcal{L}^{\ell,k}(V) = \mathcal{E}_{-\ell,k}(V,X)$$

for $\ell \leq 0, k \geq 0$. This gives rise to the total complex

$$\xrightarrow{\delta_{z-a}-\bar\partial} \mathcal{L}^m(V) \xrightarrow{\delta_{z-a}-\bar\partial} \mathcal{L}^{m+1}(V) \xrightarrow{\delta_{z-a}-\bar\partial},$$

where $\mathcal{L}^m(V) = \oplus_{\ell+k=m}\mathcal{L}^{\ell,k}(V)$. All these sums are finite; such a double complex is said to have bounded diagonals. We will also consider the cohomology of the total complex. A class in $H^m(\mathcal{L}(V))$ is thus defined by an X-valued $(\delta_{z-a} - \bar\partial)$-closed form $v = \sum v_\ell$ in V, where v_ℓ has bidegree $(\ell, \ell + m)$, and two such forms v and v' define the same class if and only if there is an X-valued form w in V such that $(\delta_{z-a} - \bar\partial)w = v - v'$. There is a natural $\mathcal{O}(V)$-module structure on $H^m(\mathcal{L}(V))$, and the inclusion $\mathcal{O}(V,X) \to \mathcal{L}^0(V)$ induces a $\mathcal{O}(V)$-linear mapping $\mathcal{O}(V,X) \to H^0(\mathcal{L}(V))$.

We will use of the following standard result from homological algebra.

Lemma 2.3. *Let $\mathcal{L}^{\ell,k}$, $k \geq 0$, be a double complex with bounded diagonals.*
(i) If $\mathcal{L}^{\ell,k}$ has exact rows (columns), then $\mathcal{L}^\bullet$ is exact.
(ii) If $\mathcal{L}^{\ell,k}$ has exact columns except at $k = 0$ and

$$\mathcal{A}^\ell = \mathrm{Ker}\,(\mathcal{L}^{\ell,0} \to \mathcal{L}^{\ell,1}),$$

then $\ldots \mathcal{A}^\ell \to \mathcal{A}^{\ell+1} \to \cdots$ is a complex and there are canonical isomorphisms $H^m(\mathcal{A}^\bullet) \simeq H^m(\mathcal{L})$.

Proposition 2.4. *There is a $\mathcal{O}(\mathbb{C}^n)$-linear isomorphism*

$$T\colon H^0(\mathcal{L}(\mathbb{C}^n)) \simeq X,$$

such that the class defined by the constant function $x \in X$ in $H^0(\mathcal{L}(\mathbb{C}^n))$ is mapped to x.

Proof. In view of Lemma 2.2, the double complex $\mathcal{L}^{\ell,k}(\mathbb{C}^n)$ has exact columns except at $k = 0$, and the kernels there are $\mathcal{A}^\ell = \mathcal{O}_{-\ell}(\mathbb{C}^n, X)$. By Lemma 2.3 (ii) therefore the canonical mapping

$$\frac{\mathcal{O}(\mathbb{C}^n, X)}{\operatorname{Im}(\mathcal{O}_1(\mathbb{C}^n, X) \to \mathcal{O}(\mathbb{C}^n, X))} \to H^0(\mathcal{L}(\mathbb{C}^n))$$

induced by the $\mathcal{O}(\mathbb{C}^n)$-linear mapping $\mathcal{O}(\mathbb{C}^n, X) \to H^0(\mathcal{L}(\mathbb{C}^n))$ actually is an isomorphism. Combining with Lemma 2.1 we get the desired conclusion. □

Notice that we have natural mappings $H^m(\mathcal{L}(\mathbb{C}^n)) \to H^m(\mathcal{L}(U))$ induced by the restriction mapping $\mathcal{E}(\mathbb{C}^n, X) \to \mathcal{E}(U, X)$.

Proposition 2.5. *Suppose that $U \supset \sigma(a)$. Then the natural mappings*

$$H^m(\mathcal{L}(\mathbb{C}^n)) \to H^m(\mathcal{L}(U))$$

are $\mathcal{O}(\mathbb{C}^n)$-linear isomorphisms.

Proof. Let $W = \mathbb{C}^n \setminus \sigma(a)$ and consider the short sequence of double complexes

$$0 \to \mathcal{L}(\mathbb{C}^n) \to \mathcal{L}(U) \oplus \mathcal{L}(W) \to \mathcal{L}(U \cap W) \to 0, \tag{2.5}$$

where the first map is the one induced by the restriction mappings $\mathcal{E}_{n-\ell,k}(\mathbb{C}^n, X) \to \mathcal{E}_{n-\ell,k}(U, X)$ and $\mathcal{E}_{n-\ell,k}(\mathbb{C}^n, X) \to \mathcal{E}_{n-\ell,k}(W, X)$, respectively, and the second one is $(f, g) \mapsto f - g$. We claim that this sequence is exact. In fact, the first mapping is injective since $f \in \mathcal{E}_{0,k}(\mathbb{C}^n)$ is zero if it vanishes on both V and W, and the second map is surjective since $\mathcal{E}_{0,k}(\mathbb{C}^n, X)$ is a module over $\mathcal{E}(\mathbb{C}^n)$ which contains cutoff functions. By standard homological algebra, the serpent's lemma, we thus obtain a long exact sequence (of locally convex spaces; the ranges of the operators are not necessarily closed)

$$\begin{aligned} &\to H^{m-1}(\mathcal{L}(U \cap W)) \to H^m(\mathcal{L}(\mathbb{C}^n)) \to \\ &\qquad \to H^m(\mathcal{L}(U) \oplus \mathcal{L}(W)) \to H^m(\mathcal{L}(U \cap W)) \to . \end{aligned} \tag{2.6}$$

Since W and $U \cap W$ are contained in $\mathbb{C}^n \setminus \sigma(a)$, $\mathcal{L}(W)$ and $\mathcal{L}(U \cap W)$ have exact rows, and therefore, by Lemma 2.3 (i), $H^k(\mathcal{L}(W)) = H^k(\mathcal{L}(U \cap W)) = 0$ for all k. It now follows from (2.6) that $H^k(\mathcal{L}(\mathbb{C}^n)) \simeq H^k(\mathcal{L}(U))$. □

Combining Propositions 2.4 and 2.5 we obtain a $\mathcal{O}(\mathbb{C}^n)$-linear isomorphism

$$T_U\colon H^0(\mathcal{L}(U)) \simeq X \tag{2.7}$$

that maps the class of x to x. Since T_U is $\mathcal{O}(\mathbb{C}^n)$-linear, the $\mathcal{O}(U)$-module structure on $H^0(\mathcal{L}(U))$ gives rise to a $\mathcal{O}(U)$-module structure on X that extends the original $\mathcal{O}(\mathbb{C}^n)$-module structure. By the initial discussion above, we thus have obtained an extension of the functional calculus from $\mathcal{O}(\mathbb{C}^n)$ to $\mathcal{O}(U)$. One readily checks that each involved mapping commutes with the action of any $b \in (a)'$. Moreover, it is not hard to show, by similar arguments, that this construction of the extension to $\mathcal{O}(U)$ commutes with restrictions $\mathcal{O}(U) \to \mathcal{O}(U')$, if $U \supset U' \supset \sigma(a)$, but we skip the argument since this fact anyway will be a consequence of the concrete realizations of the extensions that we shall consider in the next section.

Remark 2. Summing up, if $\phi \in \mathcal{O}(U)$ and $x \in X$, then the element $\phi(a)x$ is obtained according to the sequence of mappings,

$$\phi x \in \mathcal{O}(U, X) \to H^0(\mathcal{L}(U)) \simeq H^0(\mathcal{L}(\mathbb{C}^n)) \simeq \\ \simeq \frac{\mathcal{O}(\mathbb{C}^n, X)}{\mathrm{Im}\,(\mathcal{O}_1(\mathbb{C}^n, X) \to \mathcal{O}(\mathbb{C}^n, X))} \simeq X.$$

□

Remark 3. If $V = \mathbb{C}^n$, then the complex (2.3) provides a resolution of the $\mathcal{O}(\mathbb{C}^n)$-module X, cf., Lemma 2.1 and Remark 1. Furthermore, each $\mathcal{O}(\mathbb{C}^n)$-module $\mathcal{O}_k(\mathbb{C}^n, X)$ is topologically free, i.e., of the form $\mathcal{O}(\mathbb{C}^n)\hat{\otimes}E_k$ for some topological vector space E. If we have another complex, consisting of topologically free $\mathcal{O}(\mathbb{C}^n)$-modules, that gives a resolution of X, then one can construct the extension to a $\mathcal{O}(U)$-module structure of X exactly as above, and any two such extensions will coincide. This follows by homological algebra for topological vector spaces, see [10]. For instance, one can use the Koszul complex with respect to another set of global coordinates; this case is studied explicitly in Section 6. □

3. Construction by means of the resolvent mapping

Again a is an n-tuple of commuting operators on X, and U is an open neighborhood of $\sigma(a)$. Let $H^{n,n-1}_{\bar\partial}(U \setminus \sigma(a), X)$ denote the Dolbeault cohomology group of bidegree $(n, n-1)$, i.e., the quotient of the spaces of $\bar\partial$-closed and $\bar\partial$-exact X-valued $(n, n-1)$-forms in $U \setminus \sigma(a)$. Notice that $H^{n,n-1}_{\bar\partial}(U \setminus \sigma(a), X)$ has a natural $\mathcal{O}(U)$-module structure. The construction in [3] was based on the continuous $\mathcal{O}(U)$-linear mapping

$$\omega_{z-a}\colon \mathcal{O}(U, X) \to H^{n,n-1}_{\bar\partial}(U \setminus \sigma(a), X), \quad f \mapsto \omega_{z-a} f, \tag{3.1}$$

which we will refer to as the resolvent mapping, and we shall now recall its definition. Since the double complex $\mathcal{L}(U \setminus \sigma(a))$ has exact rows,

$H^m(\mathcal{L}(U\setminus\sigma(a)))=0$ for all m, according to Lemma 2.3 (i) (or by a simple direct argument). Therefore, given $f\in\mathcal{O}(U,X)$, we can find $u\in\mathcal{L}^{-1}(U\setminus\sigma(a))$ such that $(\delta_{z-a}-\bar\partial)u=f$ in $U\setminus\sigma(a)$. Identifying bidegrees this means that

$$\delta_{z-a}u_1=\phi,\quad \delta_{z-a}u_{k+1}=\bar\partial u_k,\quad k\geq 1. \tag{3.2}$$

In particular, u_n is $\bar\partial$-closed. If u' is another solution, then $u-u'$ is $(\delta_{z-a}-\bar\partial)$-closed in $U\setminus\sigma(a)$, and therefore there is $w\in\mathcal{L}^{-2}(U\setminus\sigma(a))$ such that $(\delta_{z-a}-\bar\partial)w=u-u'$. For bidegree reasons again it follows that $\bar\partial w_n=u_n-u_n'$, and hence u_n defines a class in $H^{n,n-1}_{\bar\partial}(U\setminus\sigma(a))$ which we denote $\omega_{z-a}f$.

Remark 4. It is worth to notice that any representative for the cohomology class $\omega_{z-a}f$ is u_n for some solution to $(\delta_{z-a}-\bar\partial)u=f$ (if $n>1$). In fact, if u is such a solution in $U\setminus\sigma(a)$ and $u_n'=u_n+\bar\partial w_n$, where w_n is any $(n,n-2)$-form in $U\setminus\sigma(a)$, then u_n' is the n-component of $u'=u-(\delta_{z-a}-\bar\partial)w_n$ and $(\delta_{z-a}-\bar\partial)u'=f$. □

Proposition 3.1. *The resolvent mapping ω_{z-a} is continuous and $\mathcal{O}(U)$-linear, it commutes with each $b\in(a)'$, and $(z_j-a_j)\omega_{z-a}f=0$. Moreover, ω_{z-a} commutes with the restriction mappings $\mathcal{O}(U,X)\to\mathcal{O}(U',X)$ and the induced mappings $H^{n-1}_{\bar\partial}(U\setminus\sigma(a))\to H^{n-1}_{\bar\partial}(U'\setminus\sigma(a))$ if $U\supset U'\supset\sigma(a)$,*

Proof. The continuity follows from the open mapping theorem for Frechet spaces. Suppose that $(\delta_{z-a}-\bar\partial)u=f$. If $\phi\in\mathcal{O}(U)$, then $(\delta_{z-a}-\bar\partial)\phi u=\phi f$ and hence $\phi\omega_{z-a}f=\omega_{z-a}\phi f$. In the same way, $b\omega_{z-a}f=\omega_{z-a}bf$ if $b\in(a)'$. If $f(z)=\delta_{z-a}u_1(z)$ for some holomorphic $u_1(z)$, then $(\delta_{z-a}-\bar\partial)u_1=f$ and hence $\omega_{z-a}f=0$. In particular, $(z_j-a_j)\omega_{z-a}f=\omega_{z-a}(z_j-a_j)f=0$. The commutation with restrictions is obvious from the construction. □

There is a natural mapping $H^{n,n-1}_{\bar\partial}(U\setminus\sigma(a),X)\to X$, obtained by integrating the cohomology class over some boundary ∂D, where $\sigma(a)\subset D\subset\subset U$. For each $U\supset\sigma(a)$ we thus have a mapping

$$A_U\colon\mathcal{O}(U,X)\to X,\qquad A_Uf=\int_{\partial D}\omega_{z-a}f. \tag{3.3}$$

Remark 5. Suppose that K is a compact subset of U that contains $\sigma(a)$ and $(\delta_{z-a}-\bar\partial)v=f$ in $U\setminus K$. Then, if $(\delta_{z-a}-\bar\partial)u=f$ in $U\setminus\sigma(a)$, it follows that v_n and u_n are cohomologous in $U\setminus K$. Thus we can compute A_Uf by integrating v_n over some appropriate boundary ∂D. □

Example 1. Assume that we can find an $(a)'$-valued $(1,0)$-form s in $U\setminus K$, $K\supset\sigma(a)$, such that $\delta_{z-a}s=e$. Then $u=\sum_1^n s\wedge(\bar\partial s)^{k-1}f$ solves $(\delta_{z-a}-\bar\partial)u=f$ in $U\setminus K$, if $f\in\mathcal{O}(U,X)$. Therefore, cf., Remark 5, $s \wedge (\bar\partial s)^{n-1} f$ represents $\omega_{z-a} f$ in $U \setminus K$ and hence, for an appropriate D, we get the Cauchy-Fantappie-Leray type formula

$$A_U f = \int_{\partial D} s \wedge (\bar\partial s)^{n-1} f(z).$$

For instance, if $\bar z \cdot a = \sum \bar z_j a_j$, then $|z|^2 e - \bar z \cdot a$ is invertible for z outside some large ball, and so we can choose

$$s = (|z|^2 e - \bar z \cdot a)^{-1} (2\pi i)^{-1} \sum \bar z_j dz_j$$

there. Since $\phi s \wedge (\bar\partial \phi s)^{n-1} = \phi^n s \wedge (\bar\partial s)^{n-1}$ if s is an operator-valued $(1,0)$-form (with commuting coefficients so that $s \wedge s = 0$; hence $s \wedge \bar\partial s = \bar\partial s \wedge s$) and ϕ is a (operator-valued) function that commutes with s, it follows that

$$s \wedge (\bar\partial s)^{n-1} = (|z|^2 e - \bar z \cdot a)^{-n} \partial |z|^2 \wedge (\bar\partial \partial |z|^2)^{n-1}.$$

If we choose D as a large ball with radius R, then for entire f we get the formula

$$Af = R \frac{(n-1)!}{2\pi^n} \int_{|z|=R} (R^2 e - \bar z \cdot a)^{-n} f(z) d\sigma(z), \tag{3.4}$$

where $d\sigma$ is the surface measure. If $f \equiv x$, then the integrand can be expressed as a uniformly convergent power series, and it follows that $Af = x$ since all terms but the first one will vanish for symmetry reasons. □

Notice that formula (3.4) is just the Szegő integral "evaluated at a".

Proposition 3.2. *The mapping A_U is a continuous $\mathcal{O}(\mathbb{C}^n)$-linear extension of the mapping A in* (2.1) *that commutes with any operator $b \in (a)'$. Moreover, if $f \in \mathcal{O}(U, X)$ and $U \supset U' \supset \sigma(a)$, then $A_U f = A_{U'} f$. In particular, $A_{\mathbb{C}^n} = A$ and $A_U x = x$ for $x \in X$.*

Proof. It follows from Proposition 3.1 that A_U is continuous, commuting with $b \in (a)'$, and with restrictions, and that $A_U(z_j f) = A_U(a_j f) = a_j A_U f$. From Example 1 we have that $A_U x = x$ for $x \in X$. Thus $A_U(a^\alpha x) = a^\alpha x$ and hence $A_U f = Af = f(a)$ for entire $f(z)$, cf., the definition of (2.1), by continuity. □

Remark 6. One can avoid the power series and define the mapping (2.1) by letting $Af = A_U f$ for entire X-valued f. To prove the homomorphism properties we must verify that

$$A(\phi f) = A(\phi A f). \tag{3.5}$$

We already know that (3.5) holds for polynomials ϕ. Writing $f(z) - f(\zeta) = \delta_{z-\zeta} u(z, \zeta)$, where u is holomorphic in both variables, for fixed z we therefore get that $f(z) - Af = \delta_{z-a} A u(z, \cdot)$. Since $Au(z, \cdot)$ is holomorphic, $\omega_{z-a}(f - Af) = 0$; hence $\omega_{z-a} \phi (f - Af) = 0$, which implies (3.5). □

The mapping A_U thus gives an extension of the functional calculus $\mathcal{O}(\mathbb{C}^n) \to \mathcal{L}(X)$ to a linear mapping $\mathcal{O}(U) \to \mathcal{L}(X)$ by defining $\phi(a)x = A_U(\phi x)$. However, so far we do not know that $\phi(a)\psi(a) = (\phi\psi)(a)$ (for this one needs the property (3.5) for A_U). In [3] the multiplicativity was derived by from a formula for pairs of commuting tuples, cf., Section 5, but here it will follow from the link to the homological approach. For a nice argument based on a generalization of the resolvent equality to the multivariable case, see [15].

The mapping A_U is factorized over $H^0(\mathcal{L}(U))$ via the natural mapping $j\colon \mathcal{O}(U,X) \to H^0(\mathcal{L}(U))$.

Proposition 3.3. *Suppose that $U \supset \sigma(a)$. There is a mapping*

$$B_U\colon H^0(\mathcal{L}(U)) \to X$$

such that

$$\begin{array}{ccc} \mathcal{O}(U,X) & \xrightarrow{j} & H^0(\mathcal{L}(U)) \\ \omega_{z-a} \downarrow & & \downarrow B_U \\ H^{n,n-1}_{\bar\partial}(U,X) & \longrightarrow & X \end{array}$$

is commutative, i.e., $A_U = B_U \circ j$.

Proof. We first extend the definition of A_U from $\mathcal{O}(U,X)$ to the space $\mathcal{Z}^0(U)$ of all $(\delta_{z-a} - \bar\partial)$-closed forms in $\mathcal{L}^0(U)$. If $f = f_0 + f_1 + \cdots + f_n$ is such a form, again using that $H^0(\mathcal{L}(U \setminus \sigma(a)))$ is vanishing, we can solve $(\delta_{z-a} - \bar\partial)u = f$ in $U \setminus \sigma(a)$. We now define $A_U f$ by

$$A_U f = \int_{\partial D} u + \int_D f, \tag{3.6}$$

where D is any smoothly bounded domain such that $U \supset\supset D \supset \sigma(a)$. For degree reasons only the components u_n and f_n come into play in (3.6); in particular this definition coincides with the previous one if $f = f_0$ is holomorphic. It follows from Stokes' theorem that (3.6) is independent of the particular choices of u and D.

It remains to check that $A_U f$ only depends on the class of f in $H^0(\mathcal{L}(U))$, and to this end it is enough to check that $A_U f = 0$ if $f = (\delta_{z-a} - \bar\partial)u$ for some u in U. However, then

$$A_U f = \int_{\partial D} u_n - \int_D f_n = \int_D \bar\partial u_n - \int_D f_n = 0,$$

by Stokes' theorem. □

Thus we have two different extensions to $\mathcal{O}(U)$ of the functional calculus, induced respectively by the mappings B_U and T_U, and it is now easy to see that they actually coincide. In fact, we know that T_U is invertible and that $T_U^{-1}x$ is the class in $H^0(\mathcal{L}(U))$ defined by the constant function x. Moreover, in view of Proposition 3.3, B_U maps this class to x, since $A_U x = x$. Therefore, $B_U \circ T_U^{-1}\colon X \to X$ is the identity on X. Thus we have proved

Theorem 3.4. *The mappings T_U and B_U coincide for all $U \supset \sigma(a)$.*

As an obvious corollary we get

Corollary 3.5. *The mapping $B_U\colon H^0(\mathcal{L}(U)) \to X$ is an $\mathcal{O}(\mathbb{C}^n)$-linear isomorphism.*

Remark 7. The corollary contains a new piece of information about the mapping B_U. Let us point out directly how this implies the multiplicativity for the induced functional calculus. Since $\phi(a)x = B_U(\phi x)$ by definition, and $\phi(a)x = B_U(\phi(a)x)$, we have that $B_U(\phi(a)x - \phi x) = 0$, and since B_U is an isomorphism it follows that $\phi(a)x - \phi x$ is zero in $H^0(\mathcal{L}(U))$. Since B_U is $\mathcal{O}(U)$-linear, we therefore have that $B_U(\psi\phi(a)x - \psi\phi x) = 0$, which precisely means that $\psi(a)\phi(a)x = (\psi\phi)(a)$.

More explicitly, there is a form $w \in \mathcal{L}^{-1}(U)$ such that $(\delta_{z-a} - \bar\partial)w = \phi(a)x - \phi x$, and this, in turn, means that the Dolbeault cohomology class $\omega_{z-a}(\phi(a) - \phi)x$ in $U \setminus \sigma(a)$ is the restriction, i.e., the image under the natural mapping $H^{n,n-1}_{\bar\partial}(U) \to H^{n,n-1}_{\bar\partial}(U \setminus \sigma(a))$, of the class in U defined by w_n. Thus

$$\int_{\partial D} \omega_{z-a}\psi\phi(a)x = \int_{\partial D} \omega_{z-a}\psi\phi x,$$

which means that $\psi(a)\phi(a)x = (\psi\phi)(a)x$. □

The observation used in the last argument is worth to point out separately.

Proposition 3.6. *If $f \in \mathcal{O}(U, X)$, then $A_U f = 0$ if and only if $\omega_{z-a} f$ in $U \setminus \sigma(a)$ is the image of a cohomology class in U.*

Remark 8. Let $f \in \mathcal{O}(U, X)$ and let χ be a be a cutoff function in U that is 1 in a neighborhood of $\sigma(a)$. If $(\delta_{z-a} - \bar\partial)u = f$ in $U \setminus \sigma(a)$, then $g = \chi f - \bar\partial\chi \wedge u$ defines a class in $H^0(\mathcal{L}(U))$. In fact, the same class as f, since $(\delta_{z-a} - \bar\partial)(1 - \chi)u = f - g$ in U. Hence, $A_U f = A_U g$ and by (3.6) therefore

$$A_U f = -\int \bar\partial\chi \wedge \omega_{z-a} f. \tag{3.7}$$

Of course, one can obtain (3.7) directly from (3.3) by Stokes' theorem. □

Let us conclude this section by short discussion about the resolvent. By definition a representing form u_n for the resolvent class $\omega_{z-a}x$ is obtained, in a neighborhood of ∂D, where $D \supset \sigma(a)$, by successively solving the equations (3.2). In the case when D is a ball, we have seen that this can be done in a very concrete way. Let us consider some other situations. If $\phi \in \mathcal{O}(U)$, $U \supset D \supset \sigma(a)$, and we can find an $(a)'$-valued $(1,0)$-form s in some neighborhood of ∂D such that $\delta_{z-a}s = e$, then, cf., Example 1,

$$\phi(a) = \int_{\partial D} \phi(z) s \wedge (\bar\partial s)^{n-1}.$$

For instance, such a representation is possible if ϕ is holomorphic in a neighborhood of a Stein compact K that contains $\sigma(a)$. In fact, then for each z outside K one can solve $\sum_j (z_j - w_j) v_j(w) \equiv 1$ holomorphically in a neighborhood of K and then, by the functional calculus, $s = \sum_j v_j(a) dz_j$ is in $(a)''$ and $\delta_{z-a} s = e$. It is easy to extend s to a smooth solution in a neighborhood of the given point z, and by a partition of unity we obtain a global solution in the complement of K.

Suppose now that $K \supset \sigma(a)$ and that there are linear homotopy operators $h_p\colon \mathcal{E}_{p,0} \to \mathcal{E}_{p+1,0}$ such that $h\delta_{z-a} + \delta_{z-a} h = Id$ outside K; if K is Stein as above, one can take h just as the wedge product with s. Then $h(\bar\partial h)^{n-1} x$ is a representing form for the class $\omega_{z-a} x$, and so we get the representation formula

$$\phi(a) x = \int_{\partial D} \phi h (\bar\partial h)^{n-1} x, \quad \phi \in \mathcal{O}(\overline{D}). \tag{3.8}$$

For instance, if X is a Hilbert space one can define hf pointwise in $\mathbb{C}^n \setminus \sigma(a)$ as the minimal solution to $\delta_{z-a} u = f$ if $\delta_{z-a} f = 0$ and $hf = 0$ if f is orthogonal to Ker δ_{z-a}. In the Hilbert space case therefore formulas like (3.8) always exist.

The complement of the set where such linear homotopy operators exist pointwise, is called the split spectrum. It is known to be strictly larger than $\sigma(a)$ in some cases, see [12]. Thus in general, close to the spectrum, we can only define a representative for $\omega_{z-a} x$ by relying on the existence of solutions to (3.2).

Remark 9. Let us now point out how the mapping B_U is related to weighted representation formulas. Let $X = \mathbb{C}$ so that a is just a fixed point in $\mathbb{C}^n$. We claim that if $f \in \mathcal{L}^0(U)$ is $\delta_{z-a} - \bar\partial$-closed, then $B_U f$, or rather $B_U[f]$, is equal to $f_0(a)$. Let $v \in \mathcal{L}^{-1}(U \setminus \{a\})$ be a solution to $(\delta_{z-a} - \bar\partial) v = 1$ in $U \setminus \{a\}$ and let $f = \phi g$, where $\phi \in \mathcal{O}(U)$ and $g_0(a) = 1$. Since $(\delta_{z-a} - \bar\partial)(v \wedge f) = f$ in $U \setminus \{a\}$, and by the definition of $B_U f$, we get the formula

$$\phi(a) = \int_{\partial D} \phi (v \wedge g)_n + \int_D \phi g_n. \tag{3.9}$$

All explicit weighted representation formulas we know of are special instances of this formula, see [5]. If $g = 1$ (3.9) is reduced to the (abstract) Cauchy-Fantappie-Leray formula.

To see the claim, notice that one can choose v above that is integrable over U and such that $(\delta_{z-a} - \bar\partial) v = 1 - [a]$ in the current sense, where $[a]$ denotes the (n,n)-current point evaluation at a. Moreover we may assume that $f_0(a) = 0$. Therefore, $(\delta_{z-a} - \bar\partial)(v \wedge f) = f - f_0(a)[a] = f$

in the current sense and hence it follows from Stokes' theorem that $B_U f = 0$.

With the same choice of v, and an arbitrary smooth function ϕ we have that $(\delta_{z-a} - \bar\partial)(\phi + \bar\partial\phi \wedge v) \wedge g = \bar\partial\phi \wedge [a] \wedge g = 0$ for bidegree reasons, and hence we get the more general formula

$$\phi(a) = \int_{\partial D} \phi(v \wedge g)_n - \int_D \bar\partial\phi \wedge (v \wedge g)_n + \int_D \phi g_n.$$

For details, and generalizations of these formulas, see [5]. □

4. Čech cohomology and Taylor's original constructions

The constructions in [17] and [18], as well as in [10], are made by means of Čech resolutions of $\mathcal{O}$. In this section we recall these constructions and show explicitly that they lead to the same functional calculus as we have considered in previous sections.

Let V be an open set in $\mathbb{C}^n$ and $\mathcal{V} = \{V_1, V_2, \dots\}$ a locally finite open cover of V, and let $C^k_\ell(\mathcal{V})$ denote the space of k-cochains with values in the sheaf $\mathcal{O}^X_\ell$ of X-valued holomorphic $(\ell, 0)$-forms. Let $\epsilon_1, \epsilon_2, \dots$ be a nonsense basis and consider formally the exterior algebra generated by this basis and dz_j. An element $f \in C^k_\ell(\mathcal{V})$ is represented as a formal sum $f = \sum_{|I|=k+1} f_I \wedge \epsilon_I$, where $f_I \in \mathcal{O}_\ell(V_I, X)$, $V_I = V_{I_0} \cap \ldots \cap V_{I_k}$, and $\epsilon_I = \epsilon_{I_0} \wedge \ldots \wedge \epsilon_{I_k}$. The coboundary operator $\rho\colon C^k_\ell(\mathcal{V}) \to C^{k+1}_\ell(\mathcal{V})$ is defined by $\rho f = \sum \epsilon_j \wedge f$, and for each fixed ℓ we thus get a complex

$$\to C^k_\ell(\mathcal{V}) \to C^{k+1}_\ell(\mathcal{V}) \to \tag{4.1}$$

We have a natural injective mapping $\mathcal{O}_\ell(V, X) \to C^k_\ell(\mathcal{V})$, defined by $f \mapsto \sum_j f|_{V_j} \epsilon_j$, which is precisely the kernel of $\rho\colon C^0_\ell(\mathcal{V}) \to C^1_\ell(\mathcal{V})$. If $\mathcal{V}$ is a Stein cover, i.e., each set V_j is $\mathcal{V}$ pseudoconvex, then by Lemma 2.2 each set V_I therefore has vanishing $\mathcal{O}^X_\ell$-cohomology, and by standard arguments it then follows that the cohomology groups of the complex (4.1), $\check{H}^k(\mathcal{O}^X_\ell, \mathcal{V})$, are isomorphic to the cohomology groups $H^{\ell,k}_{\bar\partial}(V, X)$ of the Dolbeault complex $\mathcal{E}_{\ell,\bullet}(V, X)$. In fact, these isomorphisms are realized by mappings like $\tau^{-\ell,k}\colon C^k_\ell(\mathcal{V}) \to \mathcal{E}_{\ell,k}(V, X)$ that are described below. In particular, we have

Lemma 4.1. *If $\mathcal{V}$ is a Stein cover of the pseudoconvex open set $V \subset \mathbb{C}^n$, then the complex*

$$0 \to \mathcal{O}_\ell(V, X) \to C^0_\ell(\mathcal{V}) \to C^1_\ell(\mathcal{V}) \to C^2_\ell(\mathcal{V}) \to$$

is exact.

The mapping δ_{z-a} extends naturally to a mapping $C^k_{\ell+1}(\mathcal{V}) \to C^k_\ell(\mathcal{V})$, and $\delta_{z-a}\rho = -\rho\delta_{z-a}$.

Lemma 4.2. *If $V \subset \mathbb{C}^n \setminus \sigma(a)$ is pseudoconvex, then the complex*

$$0 \leftarrow \mathcal{O}(V, X) \leftarrow \mathcal{O}_1(V, X) \leftarrow \mathcal{O}_2(V, X) \leftarrow,$$

induced by δ_{z-a}, is exact.

This is proved in [16] and [17]. It follows easily from the exactness of the X-valued Dolbeault complex in V, i.e., Lemma 2.2. Given our open cover $\mathcal{V}$ we define the double complex $\mathcal{M}(\mathcal{V})^{\ell,k} = C^k_{-\ell}(\mathcal{V})$.

Now, let U be a fixed neighborhood of $\sigma(a)$, let $\mathcal{U}$ be a locally finite Stein open cover of U, and let $\mathcal{W}$ a locally finite Stein open cover of $W = \mathbb{C}^n \setminus \sigma(a)$. We may assume as well that the union $\mathcal{U} \cup \mathcal{W}$ is locally finite (if necessary we can decrease U a little). Moreover, we can arrange so that the indices of the cover $\mathcal{U}$ run over the even natural numbers and the indices of $\mathcal{W}$ over the odd ones. Then $\mathcal{U} \cup \mathcal{W}$ is a cover of $\mathbb{C}^n$, with indices running over the the natural numbers. We are going to copy the construction in Section 2 but it appears a little but different, since the Mayer-Vietoris step will be included in the setup. The $\mathcal{O}(\mathbb{C}^n)$-linear mappings $r^{\ell,k}\colon \mathcal{M}^{\ell,k}(\mathcal{U} \cup \mathcal{W}) \to \mathcal{M}^{\ell,k}(\mathcal{U})$ are surjective and hence we can define the complex $\mathcal{K}^{\ell,k}$ so that

$$0 \to \mathcal{K}^{\ell,k} \to \mathcal{M}^{\ell,k}(\mathcal{U} \cup \mathcal{W}) \to \mathcal{M}^{\ell,k}(\mathcal{U}) \to 0$$

is a short exact sequence of double complexes. This means that a section to $\mathcal{K}^{\ell,k}$ is like $\sum_I f_I \epsilon_I$, where at least one index I_i is odd for each I. Therefore, $\mathcal{K}$ has exact rows by Lemma 4.2, so $H^m(\mathcal{K}) = 0$ for all m according to Lemma 2.3. By the serpent's lemma we get the long exact sequence

$$\to H^m(\mathcal{K}) \to H^m(\mathcal{M}(\mathcal{U} \cup \mathcal{W})) \to$$
$$\to H^m(\mathcal{M}(\mathcal{U})) \to H^{m+1}(\mathcal{K}) \to,$$

and thus we have in particular the $\mathcal{O}(\mathbb{C}^n)$-linear isomorphisms

$$H^0(\mathcal{M}(\mathcal{U} \cup \mathcal{W})) \simeq H^0(\mathcal{M}(\mathcal{U})). \tag{4.2}$$

induced by the mappings $r^{\ell,k}$. Notice that the class in $H^0(\mathcal{M}(\mathcal{U} \cup \mathcal{W}))$ defined by the constant function $x \in X$, is mapped to the corresponding class in $H^0(\mathcal{M}(\mathcal{U}))$. On the other hand, since $\mathbb{C}^n$ is pseudoconvex, Lemma 4.1 implies that the complex $\mathcal{M}(\mathcal{U} \cup \mathcal{W})$ has exact columns except at $k = 0$, and that the kernels are $\Lambda_{-\ell}\mathcal{O}(\mathbb{C}^n, X)$, so again by Lemma 2.3 we get a $\mathcal{O}(\mathbb{C}^n)$-linear isomorphism

$$X \simeq H^0(\mathcal{M}(\mathcal{U} \cup \mathcal{W})). \tag{4.3}$$

Again notice that the class in $H^0(\mathcal{M}(\mathcal{U} \cup \mathcal{W}))$ defined by the constant function x is mapped to x by this isomorphism. As before, we can combine the isomorphisms (4.2) and (4.3) to an $\mathcal{O}(\mathbb{C}^n)$-linear isomorphism $X \simeq H^0(\mathcal{M}(\mathcal{U}))$. Since $H^0(\mathcal{M}(\mathcal{U}))$ has an $\mathcal{O}(U)$-module structure, that extends its $\mathcal{O}(\mathbb{C}^n)$-module structure, we obtain an extension of the $\mathcal{O}(\mathbb{C}^n)$-module structure of X to an $\mathcal{O}(U)$-module structure. We shall now see that this $\mathcal{O}(U)$-module structure on X is the same as the one defined in Section 2. To this end we shall define an explicit $\mathcal{O}(U)$-module isomorphism $H^0(\mathcal{M}(\mathcal{U})) \simeq H^0(\mathcal{L}(U))$, where $\mathcal{L}^{\ell,k}(U)$ is the double complex from Section 2.

Let ϕ_j be a partition of unity subordinate the cover $\mathcal{U}$, and let ϵ_j^* denote the formal dual basis of ϵ_j. For any cochain f with values in the sheaf $\mathcal{E}_{p,q}^X$ of X-valued (p,q)-forms, we can define Φf as interior multiplication by $\sum_j \phi_j \epsilon_j^*$, and as usual we have the relation

$$\rho\Phi + \Phi\rho = I. \tag{4.4}$$

Let $\bar\partial\Phi$ denote interior multiplication with $\sum_j \bar\partial\phi_j e_j^*$. Then $\bar\partial(\Phi f) = (\bar\partial\Phi)f - \Phi\bar\partial f$. We define $\tau^{\ell,k}\colon \mathcal{M}^{\ell,k}(\mathcal{U}) \to \mathcal{L}^{\ell,k}(U)$, by

$$\tau^{\ell,k} f = (-1)^\ell \Phi(\bar\partial\Phi)^k f.$$

Notice that if $f \in \mathcal{O}(U,X)$ and $\tilde f$ is its image in $\in \mathcal{M}^{0,0}(\mathcal{U})$, then $\tau^{0,0}\tilde f = f$.

Proposition 4.3. *We have the relations*

$$\bar\partial\tau = \tau\rho, \qquad \delta_{z-a}\tau = \tau\delta_{z-a}, \qquad \tau\phi = \phi\tau, \tag{4.5}$$

where ϕ means multiplication by the function $\phi \in \mathcal{O}(U)$.

Proof. It is clear that τ commutes with $\phi \in \mathcal{O}(U)$. We claim that

$$\rho\bar\partial\Phi = \bar\partial\Phi\rho. \tag{4.6}$$

In fact, $u = \rho\Phi u + \Phi\rho u$ and so $\bar\partial u = -\rho\bar\partial\Phi u + \rho\Phi\bar\partial u + \bar\partial\Phi\rho u + \Phi\rho\bar\partial u$, and in view of (4.4) we get (4.6).

Now suppose f has degree (ℓ,k). Then

$$\delta\tau f = \delta_{z-a}(-1)^\ell\Phi(\bar\partial\Phi)^k f = (-1)^{\ell+1}\Phi(\bar\partial\Phi)^k\delta_{z-a} f = \tau\delta,$$

since δf has degree $(\ell+1,k)$. Furthermore,

$$\begin{aligned}\tau\rho f = (-1)^\ell\Phi(\bar\partial\Phi)^{k+1}\rho f = (-1)^\ell\Phi\rho(\bar\partial\Phi)^{k+1} = \\ = (-1)^\ell(\bar\partial\Phi)^{k+1} = \bar\partial(-1)^\ell\Phi(\bar\partial\Phi)^k = \bar\partial\tau f.\end{aligned}$$

□

The proposition means that $\tau\colon \mathcal{M}(\mathcal{U}) \to \mathcal{L}(U)$ is a $\mathcal{O}(U)$-linear morphism of double complexes (chain mapping), and hence that τ induces an $\mathcal{O}(U)$-linear homomorphism

$$\tilde\tau\colon H^0(\mathcal{M}(\mathcal{U})) \to H^0(\mathcal{L}(U)). \tag{4.7}$$

The class in $H^0(\mathcal{M}(\mathcal{U}))$ defined by $f \in \mathcal{O}(U,X)$ is mapped to the corresponding class in $H^0(\mathcal{L}(U))$ since $\tau\tilde f = f$ as noted above. In particular, the constant function x is mapped to x. Since both spaces in (4.7) are isomorphic to X, such that the classes of x are mapped to x, we conclude that $\tilde\tau$ actually is an $\mathcal{O}(U)$-linear isomorphism. Therefore the two constructions of the $\mathcal{O}(U)$-module structure of X are isomorphic.

Finally let us describe Taylor's original construction by means of Cauchy-Weyl formulas. Choose a Stein cover $\mathcal{V}$ of $V = U \setminus \sigma(a)$. Given a function $f \in \mathcal{O}(U, X)$ we consider it as an element in $\mathcal{M}^0(\mathcal{V})$ and solve the equation $(\delta - \rho)u = f$, which is possible by virtue of Lemma 4.2. Then u_n is a Čech cocycle of degree $n-1$ of holomorphic $(n,0)$-forms, and if u' is another solution to $(\delta - \rho)u = f$, then there is a solution to $(\delta - \rho)w = u - u'$, and so $\rho w_n = u_n - u'_n$. Thus we obtain a well-defined Čech cohomology class $\check{\omega}_{z-a} f$ in $\check{H}^{n-1}(\mathcal{O}^X_n, \mathcal{V})$, and one can prove an analogue to Proposition 3.1 for the mapping $\check{\omega}_{z-a}\colon \mathcal{O}(U, X) \to \check{H}^{n-1}(\mathcal{O}^X_n, \mathcal{V})$. The analog $\check{A}_U f$ of $A_U f$ is obtained by taking the abstract Cauchy-Weyl integral of the cohomology class $\check{\omega}_{z-a} f$. This can be done by integrating its canonical image in $H^{n,n-1}_{\bar\partial}(V)$ over an appropriate boundary ∂D. However, the image is obtained by a mapping like τ above, but for the cover $\mathcal{V}$ instead of $\mathcal{U}$. If $(\delta_{z-a} - \rho)u = f$, then by Proposition 4.3, $(\delta_{z-a} - \bar\partial)\tau u = \tau f = f$ and therefore $\tau\check{\omega}_{z-a} f = \omega_{z-a} f$, which shows that $\check{A}_U f = A_U f$. This proves that this original construction of Taylor gives is equivalent to the other ones we have considered.

5. Functorial properties

Let a be an n-tuple and b an m-tuple of operators on X which all commute. We let $dz_1, \ldots, dz_n, dw_1, \ldots, dw_n$ be a basis for the corresponding Koszul complex, and let $\delta = \delta_a + \delta_b$. It is quite easy to see that the whole tuple (a, b) is nonsingular if a is (nonsingular here means that the corresponding Koszul complex is exact). However, we can say even more.

Lemma 5.1. *Suppose that a is nonsingular. Let ξ be an X-valued form of degree at most p in dw, and suppose that $(\delta_a + \delta_b)\xi = 0$. Then we can find a solution u to $(\delta_a + \delta_b)u = \xi$ such that u has degree at most p in dw.*

Proof. If $p = 0$ it is obviously true, since then we have that $\delta_a \xi = 0$. Now suppose it is proved for $p-1$. Since $\delta_a\delta_b\xi = \delta_a(\delta_a + \delta_b)\xi = 0$, we can solve $\delta_a v = \delta_b \xi$ such that v has degree at most $p-1$ in dw. Now, $\delta_a(\xi + v) = \delta_a\xi + \delta_a v = -\delta_b\xi + \delta_b\xi = 0$ so we can solve $\delta_a u = \xi + v$ with u of degree at most p in dw. Now $v + \delta_b u$ is $(\delta_a + \delta_b)$-closed and has degree at most $p-1$, so by the induction hypothesis we have η of degree at most $p-1$ such that $(\delta_a + \delta_b)\eta = v + \delta_b u$. Finally, we have $(\delta_a + \delta_b)(u - \eta) = \xi$. □

As a consequence we have the inclusion

$$\sigma(a, b) \subset \sigma(a) \times \sigma(b). \tag{5.1}$$

Remark 10. In particular it follows that the Taylor spectrum is bounded, since this holds for each single operator. The fact that $\sigma(a)$ is closed in $\mathbb{C}^n$ follows form the open mapping theorem (if a complex depending on a continuous parameter is exact at some point, then it is also exact in a small neighborhood, see [16]). □

Proposition 5.2. *Let a and b be commuting, and let π be the natural projection $\pi\colon \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n$. Then $\pi(\sigma(a,b)) = \sigma(a)$.*

Sketch of proof. From (5.1) it follows that $\pi(\sigma(a,b)) \subset \sigma(a)$, so it is enough to prove the converse inclusion. Assume for simplicity that $0 \notin \pi(\sigma(a,b))$, and let ξ be a δ_a-closed X-valued form in dz. Then it is also $(\delta_a + \delta_{w-b})$-closed for each $w \in \mathbb{C}^m$, and so we can find a smooth solution $u(w)$ to $(\delta_a + \delta_{w-b})u(w) = \xi$. Since the Dolbeault complex is exact in $\mathbb{C}^m$ we can assume that $u(w)$ is holomorphic. Then $\delta_a u(b) = \xi$. Thus $0 \notin \sigma(a)$. □

Let a and b be commuting tuples of lengths n and m, respectively, let $\Omega_a = (\mathbb{C}^n \setminus \sigma(a)) \times \mathbb{C}^m$ and $\Omega_b = \mathbb{C}^n \times (\mathbb{C}^m \setminus \sigma(b))$. We say that a form α (in both z and w) has degree (k,ℓ) if it has degree k in dz and ℓ in dw, disregarding degrees in $d\bar{z}$ and $d\bar{w}$. As before, α_k denotes the component of degree k in dz, dw. We say that α has total degree ℓ if the difference between barred and unbarred degrees for each component is ℓ.

Lemma 5.3. *Let ξ a smooth form in $V \times V' \subset \Omega_a$ of degree at most p in dw, and suppose that $(\delta_{z-a} + \delta_{w-b} - \bar{\partial})\xi = 0$. Then there is a solution to $(\delta_{z-a} + \delta_{w-b} - \bar{\partial})u = \xi$ of degree at most p in dw. If ξ has total degree ℓ, then we may assume that u has total degree $\ell - 1$.*

Proof. There is a smooth solution to $\delta_{z-a}\alpha = \beta$ in $V \times V'$ if β is smooth and $\delta_{z-a}\beta = 0$; see [10]. Following (the proof of) Lemma 5.1 we can then successively solve

$$(\delta_{z-a} + \delta_{w-b})u^{1,*} = \xi^{0,*}, \quad (\delta_{z-a} + \delta_{w-b})u^{k+1,*} = \bar{\partial}u^{k,*}$$

so that each $u^{k,*}$ has degree at most p in dw. Finally we take $u = u^{1,*} + \ldots + u^{n,*}$. □

For $x \in X$ shall define a cohomology class

$$\omega_{w-b} \wedge \omega_{z-a} x \tag{5.2}$$

in $\Omega_a \cap \Omega_b$. First we solve $(\delta_{z-a} - \bar{\partial})u = x$ in Ω_a and $(\delta_{w-b} - \bar{\partial})v = x$ in Ω_b, so that u and v has degrees $(*,0)$ and $(0,*)$, respectively, and total degree -1. Now $(\delta_{z-a} + \delta_{w-b} - \bar{\partial})(u - v) = 0$ in $\Omega_a \cap \Omega_b$ and in view of Lemma 5.3 we can therefore find a solution to

$$(\delta_{z-a} + \delta_{w-b} - \bar{\partial})c = u - v \tag{5.3}$$

of total degree -2. It follows that $\bar{\partial}c_{n+m} = 0$. We let c_{n+m} define the cohomology class (5.2). Suppose that we had other choices u', v' and c'. Then there are α and β of degrees $(*,0)$ and $(0,*)$, respectively, such that $(\delta_{z-a} - \bar{\partial})\alpha = u - u'$ and $(\delta_{w-b} - \bar{\partial})\beta = v - v'$. Then

$(\delta_{z-a}+\delta_{w-b}-\bar\partial)(c-c'-\alpha+\beta)=0$ so we can solve $(\delta_{z-a}+\delta_{w-b}-\bar\partial)\xi=(c-c'-\alpha+\beta)$ in $\Omega_a\cap\Omega_b$. For degree reasons again it follows that $\bar\partial\xi_{n+m}=c_{n+m}-c'_{n+m}$, and thus (5.2) is indeed welldefined. From the definition it is clear that $\omega_{w-b}\wedge\omega_{z-a}x=-\omega_{z-a}\wedge\omega_{w-b}x$. Moreover, we have

Proposition 5.4. *Let χ be a cutoff function such that $\{\chi,1-\chi\}$ is a partition of unity subordinate the cover $\{\Omega_a,\Omega_b\}$ of $\mathbb{C}^{n+m}\setminus\sigma(a)\times\sigma(b)=\Omega_a\cap\Omega_b$. Then*

$$\bar\partial\chi\wedge\omega_{w-b}\wedge\omega_{z-a}x=\omega_{w-b,z-a}x$$

as cohomology classes in $\mathbb{C}^{n+m}\setminus\sigma(a)\times\sigma(b)$.

Proof. By the notation above we have that

$$(\delta_{z-a}+\delta_{w-b}-\bar\partial)(\chi u+(1-\chi)v+\bar\partial\chi\wedge c)=x$$

in $\mathbb{C}^{n+m}\setminus\sigma(a)\times\sigma(b)$, and hence the $(n+m)$-component of $(\chi u+(1-\chi)v+\bar\partial\chi\wedge c)$ represents $\omega_{w-b,z-a}x$. □

Proposition 5.5. *Let G be holomorphic in a neighborhood U of $\sigma(a)\times\sigma(b)$, and let $\phi(z)$ and $\psi(w)$ be cutoff functions such that $\phi\otimes\psi$ has compact support in U and is 1 in a neighborhood of $\sigma(a)\times\sigma(b)$. Then*

$$G(a,b)x=\int G(z,w)\bar\partial\psi(w)\wedge\bar\partial\phi(z)\wedge\omega_{w-b}\wedge\omega_{z-a}x. \tag{5.4}$$

Moreover, if $G=g_1\otimes g_2$, then

$$G(a,b)=g_2(b)g_1(a). \tag{5.5}$$

Proof. Let α be a form in $\Omega_a\cap\Omega_b$ that represents $\omega_{w-b}\wedge\omega_{z-a}x$. By the definition of $G(a,b)x$, cf., (3.7), and Proposition 5.4 we have that

$$G(a,b)x=-\int_w\int_z G\bar\partial(\phi\otimes\psi)\wedge\bar\partial\chi\wedge\alpha. \tag{5.6}$$

Now

$$\beta=G(\bar\partial\phi\otimes\psi\wedge\chi-\phi\otimes\bar\partial\psi\wedge(1-\chi))\wedge\alpha$$

is a welldefined compactly supported form in $\mathbb{C}^{n+m}$. Hence the integral of $\bar\partial\beta$ vanishes, and this yields the equality between the right hand sides of (5.4) and (5.6).

Let u and v be as before and let $(\delta_{w-b}-\bar\partial)\xi=u_n$ in $\Omega_a\cap\Omega_b$, such that ξ has degree $(n,*)$. Then, for fixed z, ξ_{n+m} represents $\omega_{w-b}u_n(z)$. We claim that furthermore ξ_{n+m} represents $\omega_{w-b}\wedge\omega_{z-a}x$. In fact, $u-v-\delta_{z-a}\xi-(\delta_{w-b}-\bar\partial)\xi$ is $(\delta_{z-a}+\delta_{w-b}-\bar\partial)$-closed and has degree at most $n-1$ in dz, so according to Lemma 5.3 there is a solution c' to $(\delta_{z-a}+\delta_{w-b}-\bar\partial)c'=u-v-\delta_{z-a}\xi-(\delta_{w-b}-\bar\partial)\xi$ of degree at most $n-1$ in dz. If $c=\xi+c'$, therefore $c_{n+m}=\xi_{n+m}$. Moreover, c solves (5.3), so c_{n+m} represents (5.2), and hence the claim follows. By (5.4) we therefore have that

$$G(a,b)x=\int_z\int_w g_1(z)g_2(w)\bar\partial\psi(w)\wedge\bar\partial\phi(z)\wedge f_{n+m}$$

but since f_{n+m} represents $\omega_{w-b}u_n(z)$, after evaluating the inner integral we get

$$-\int_z g_1(z)\bar\partial\phi(z)\wedge g_2(b)u_n(z)$$

which is equal to $g_2(b)g_1(a)x$. □

Remark 11. In the same way one can define $\omega_{w-b}\wedge\omega_{z-a}f$ for any $f\in\mathcal{O}(U,X)$ if $U\supset\sigma(a)\times\sigma(b)$ and Proposition 5.4 holds (in $U\backslash\sigma(a)\times\sigma(b)$) for f instead of x. Also (5.4) has a natural counterpart. Morover, if $G(z,w)=g_2(w)g_1(z)$ where $g_2\in\mathcal{O}(U')$ and $g_1\in\mathcal{O}(U'',X)$, then (5.5) still holds. For the proof, first solve $(\delta_{z-a}-\bar\partial)u=g_1$. One can assume then that u is independent of w. Therefore, g_2u_n is holomorphic in w for fixed z, and one can proceed as before. □

Proposition 5.6. *If $T\colon\mathbb{C}^n_z\to\mathbb{C}^m_w$ is linear and $b=Ta$, then*

$$T\sigma(a)=\sigma(Ta), \tag{5.7}$$

and for $g\in\mathcal{O}(\sigma(Ta))$ we have

$$g(b)=(T^*g)(a). \tag{5.8}$$

Proof. Since $T(z-a)=w-b$, for any X-valued form in dw at the point $w=Tz$ we have that

$$\delta_{z-a}T^*\xi=T^*\delta_{w-b}\xi. \tag{5.9}$$

If T is an isomorphism it follows that $z-a$ is singular if and only of $w-b$ is, and hence (5.7) follows. Moreover, if $(\delta_{w-b}-\bar\partial)v=x$ it follows that $(\delta_{z-a}-\bar\partial)T^*v=T^*x=x$, and hence

$$\omega_{z-a}x=T^*\omega_{w-b}x, \tag{5.10}$$

which implies (5.8).

If T is the projection $(z,w)\mapsto z$, then (5.7) is just Proposition 5.2 and if $g\in\mathcal{O}(\sigma(a))$, then $(T^*g)(a,b)=g\otimes 1(a,b)=g(a)=g(T(a,b))$ according to (5.5).

If T is the injection $z\mapsto(z,0)$, then $T(a)=(a,0)$ and hence $\sigma(T(a))\subset\sigma(a)\times\{0\}=T(\sigma(a))$. Because of Proposition 5.2, the inclusion is actually an equality. If $s=(2\pi i)^{-1}\partial|w|^2/|w|^2$, then $s\wedge(\bar\partial s)^{m-1}$ represents ω_w, and hence $s\wedge(\bar\partial s)^{m-1}\wedge u_n$ represents $\omega_w\wedge\omega_{z-a}x$ if u_n represents $\omega_{z-a}x$. Thus, by (5.4),

$$g(Ta)=\int\int g(z,w)\bar\partial\phi\wedge\bar\partial\psi\wedge s\wedge(\bar\partial s)^{m-1}\wedge u_n=$$
$$=-\int g(z,0)\bar\partial\phi\wedge u_n=T^*g(a).$$

Since any linear mapping is a composition of these types, the proposition is proved. □

Proposition 5.7. *Suppose that $a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_m$ are commuting operators, and that f is holomorphic in a neighborhood of $\sigma(a) \cup \sigma(b)$. If ξ is an X-valued form such that $(\delta_{b-a} - \delta_c)\xi = 0$, then one can solve $(\delta_{b-a} - \delta_c)u = (f(a) - f(b))\xi$.*

Since $f(a), f(b)$ and $\delta_{b-a} - \delta_c$ commute, it is clear that $f(a)$ and $f(b)$ induce mappings on each homology group $H^m((b-a,c),X)$. The theorem states that $f(a) - f(b)$ acts as the zero operator on $H^m((b-a,c),X)$.

Proof. First assume that there are no c_j at all. Let ξ be a δ_{b-a}-closed X-valued form in dw of degree p and let $[\xi]$ denote its homology class. We claim that if u is a solution to

$$(\delta_{z-a} + \delta_{b-a} - \bar\partial)u = \xi, \tag{5.11}$$

of total degree $-p-1$ with respect to dz, dw, then $[f(a)\xi]$ is represented by

$$\int_{\partial D} f(z)u^{n,p}, \tag{5.12}$$

where $u^{n,p}$ is the component of degree n in dz and p in dw. First notice that $\delta_{b-a}u^{n,p} = \bar\partial u^{n,p-1}$; therefore (5.12) is δ_{b-a}-closed by Stokes' theorem (the integral over ∂D of $f\bar\partial_w u^{n,p-1}$ vanishes since it has bidegree $(n, n-2)$ in dz). Moreover, $\bar\partial u^{n,p} = \delta_{b-a}u^{n,p+1}$, and therefore, the homology class of (5.12) is independent of the choice of ∂D, again by Stokes' theorem (notice that $f\bar\partial_w u^{n,p}$ has bidegree $(n, n-1)$ in dz). Moreover, if v is another solution to (5.11), then $(\delta_{z-a} + \delta_{b-a} - \bar\partial)\alpha = u - v$ for some α (of total degree $-p-2$), and hence $u^{n,p} - v^{n,p} = \bar\partial\alpha^{n,p} + \delta_{b-a}\alpha^{n,p+1}$. This means that the homology class of (5.12) is independent of the solution u to (5.11).

Let us now choose a solution to $(\delta_{z-a} - \bar\partial)v = \xi$ of degree p in dw. Then clearly $v^{n,p}$ represents the cohomology class $\omega_{z-a} \wedge \xi$, and hence $f(a)\xi$ is obtained by integrating $f(z)v^{n,p}$ over ∂D. The form $\delta_{b-a}v$ has degree $p-1$ in dw and

$$(\delta_{z-a} + \delta_{b-a} - \bar\partial)\delta_{b-a}v = -\delta_{b-a}\xi = 0,$$

so by Lemma 5.3 we can find a solution to

$$(\delta_{z-a} + \delta_{b-a} - \bar\partial)v' = -\delta_{b-a}v$$

of degree at most $p-1$ in dw. Hence $v+v'$ solves (5.11) and $(v+v')^{n,p} = v^{n,p}$, and so the claim follows.

Now, let u be a solution to (5.11) and let T be the linear mapping $(z,w) \mapsto (z+w, w)$. Then $(z,0) \mapsto (z,0)$ and $(b, a-b) \mapsto (a, a-b)$. If $v = T^*u$, then by (5.9),

$$(\delta_{z-b} + \delta_{b-a} - \bar\partial)v = T^*(\delta_{z-a} + \delta_{b-a} - \bar\partial)u = T^*\xi = \xi.$$

Thus, $[f(b)\xi]$ is represented by the integral of $f(z)v^{n,p}$. However, it is easy to verify that $v^{n,p} = u^{n,p}$, and hence $[f(b)\xi] = [f(a)\xi]$.

The case with c now follows by applying the result already proved to $a' = (a, c)$, $b' = (b, 0)$, and $f'(z, \zeta) = f(z)$, and using that $f'(a') = f(a)$ and $f'(b') = f(b)$, cf., (5.5). □

Theorem 5.8 (Spectral mapping and the superposition property)**.** *Let $f = (f_1, \dots, f_m)$ be holomorphic in a neighborhood of $\sigma(a)$. Then*

$$\sigma(f(a)) = f(\sigma(a)). \tag{5.13}$$

If furthermore g is holomorphic in a neighborhood of $\sigma(f(a))$, then

$$g \circ f(a) = g(f(a)).$$

Proof. The first aim is to prove the inclusion

$$\sigma(a, f(a)) \subset \{(z, w);\ w = f(z)\}. \tag{5.14}$$

Let (z, w) be a fixed point outside the graph, i.e., such that $w \neq f(z)$ (in view of Lemma 5.1 we may assume that $z \in \sigma(a)$). Moreover, let ξ be a fixed X-valued form such that $(\delta_{z-a} + \delta_{w-f(a)})\xi = 0$. From Theorem 5.7, with $a = a$, $b = z$, and $c = w - f(a)$ we get forms u_j such that

$$(\delta_{z-a} + \delta_{w-f(a)})u_j = (f_j(z) - f_j(a))\xi, \quad j = 1, \dots, m.$$

Now,

$$v = |f(z) - w|^{-2} \sum_1^m \overline{(f_j(z) - w_j)}(u_j - dw_j \wedge \xi)$$

is a welldefined X-valued form, and $(\delta_{z-a} + \delta_{w-f(a)})v = \xi$. Thus (z, w) is a point outside $\sigma(a, f(a))$ by definition, and hence (5.14) is settled. From (5.14) and Proposition 5.2 it follows that

$$\sigma(a, f(a)) = \{(z, f(z);\ z \in \sigma(a)\},$$

and taking the projection onto $\mathbb{C}^m_w$, using Proposition 5.2 again, we get (5.13). The superposition property can now be obtained by means of Propositions 5.4 and 5.5, but we omit the arguments and refer to p. 3 in [7]. □

Suppose we have a given $\mathcal{O}(\mathbb{C}^n)$-module structure on X. Let z be global coordinates on $\mathbb{C}^n$ and let a be the tuple connected with z. If $w = \psi(z)$ is another choice of global coordinates, then $b = \psi(a)$ is the tuple connected with w, and it follows from the spectral mapping theorem that $\sigma(b) = \sigma(a)$. Hence the spectrum σ of the given $\mathcal{O}(\mathbb{C}^n)$-module structure is invariantly defined. Moreover, because of the superposition property, the extension to a $\mathcal{O}(\sigma)$-module structure is also independent of the choice of coordinates.

Remark 12. In the previous conclusion, $\mathbb{C}^n$ can be replaced by any Stein manifold Ω. Thus, if we have given an $\mathcal{O}(\Omega)$-module structure on X, then one can define, in an invariant way, the spectrum σ of this structure, and furthermore there is a canonical extension to an $\mathcal{O}(\sigma)$-module structure of X, see [10]. □

6. Behaviour of the resolvent under analytic mappings

Let a be a commuting tuple, let $\psi\colon \Omega \to \Omega'$ be a biholomorphism in a neighborhood Ω of $\sigma(a)$, and let $b = \psi(a)$. The superposition property holds even when the outer function is X-valued (with the same proof), and hence we have for $f \in \mathcal{O}(\sigma(b), X)$ that

$$(6.1)\qquad \int_{\partial D} \omega_{z-a}\psi^* f(z) = \psi^* f(a) = f(b) = \\ = \int_{\partial\psi(D)} \omega_{w-b} f(w) = \int_{\partial D} \psi^*(\omega_{w-b} f).$$

Thus $\omega_{z-a}\psi^* f$ and $\psi^*(\omega_{w-b}f)$ have the same integral for all $f \in \mathcal{O}(\sigma(b), X)$. If Ω is pseudoconvex we can say even more.

Theorem 6.1. *Suppose that Ω is a pseudoconvex neighborhood of $\sigma(a)$ and $\psi\colon \Omega \to \Omega'$ is a biholomorphic mapping, $w = \psi(z)$, and $b = \psi(a)$. If $\sigma(a) \subset U \subset \Omega$, then $\omega_{z-a}(\psi^* f) = \psi^*(\omega_{w-b} f)$ in $U \setminus \sigma(a)$ for $f \in \mathcal{O}(\psi(U), X)$ if $n > 1$. If $n = 1$ then the difference of the holomorphic forms $\omega_{z-a}(\psi^* f)$ and $\psi^*(\omega_{w-b} f)$ has a holomorphic extension over the spectrum.*

The theorem can be rephrased in the following way: Suppose we have a given $\mathcal{O}(\mathbb{C}^n)$-module structure on X, and let Ω be a pseudoconvex neighborhood of the spectrum σ. If z and w are holomorphic coordinates in Ω, then the cohomology classes $\omega_{z-a} f$ and $\omega_{w-b} f$ in $U \setminus \sigma(a)$ coincide if $n > 1$. If $n = 1$ then the difference has a holomorphic extension over the spectrum.

Proof. Since Ω is pseudoconvex, Hefer's theorem yields functions $H_{jk}(z,\zeta) \in \mathcal{O}(\Omega \times \Omega)$ such that

$$\sum_k H_{jk}(z,\zeta)(z_k - \zeta_k) = \psi_j(z) - \psi_j(\zeta).$$

Therefore, $h_{jk}(z) = H_{jk}(z, a)$ satisfy

$$(6.2)\qquad \sum_k h_{jk}(z)(z_k - a_k) = w_j - b_j.$$

For each z we define a mapping $\alpha^*(z)\colon T^*_{\psi(z)} \otimes X \to T^*_z \otimes X$ by

$$\alpha^*(dw_j|_w) = \sum_k h_{jk}(z) dz_k|_z, \quad \alpha^*(d\bar{w}_j|_w) = \psi^*(d\bar{w}_j|_w).$$

Then, $\delta_{z-a}(\alpha^* dw_j|_w) = w_j - b_j$. Therefore,

$$(6.3)\qquad \delta_{z-a}\alpha^*\xi = \alpha^*\delta_{w-b}\xi$$

if ξ is an X-valued 1-form (if ξ is a $(0,1)$-form, then both sides vanish) and hence (6.3) follows for general X-valued forms by induction. Moreover, $\bar\partial$ commutes with α^*, and therefore

$$(6.4)\qquad (\delta_{z-a} - \bar\partial)\alpha^* = \alpha^*(\delta_{w-b} - \bar\partial).$$

Now, suppose that $f \in \mathcal{O}(\Omega', X)$ and $v \in \mathcal{L}^{-1}(\Omega' \setminus \sigma(b))$ solves $(\delta_{w-b} - \bar\partial)v = f$. Then $(\delta_{z-a} - \bar\partial)\alpha^* v = \alpha^* f = \psi^* f$. Thus $\widetilde{\omega}_{w-b} f = v_n$ is a representative for the class $\omega_{w-b} f$ and

$$\widetilde{\omega}_{z-a}\psi^* f = (\alpha^* v)_n = \alpha^* v_n = \alpha^* \widetilde{\omega}_{w-b} f$$

is a representative for $\omega_{z-a}\psi^* f$. So far we have only used that ψ is a holomorphic mapping. Now assume that ψ is a biholomorphism and let $\phi = \psi^{-1}$. On $(0,q)$-forms, $\alpha^* = \psi^*$ and hence $\phi^*\alpha^*$ is the identity. Since $\widetilde{\omega}_{w-b} f$ has full degree in dw it follows that $\phi^*\alpha^*\widetilde{\omega}_{w-b} f = g(w)\widetilde{\omega}_{w-b} f$, where

$$g(w) = \det \frac{\partial \phi}{\partial w}(w) \det(h_{jk} \circ \phi(w)).$$

From (6.2) we have that

$$dw_j = \sum_k h_{jk}(z) + \mathcal{O}(z-a),$$

where $\mathcal{O}(z-a)$ means holomorphic terms that vanish if $z = a$. Therefore,

$$\det \frac{\partial w}{\partial z}(z) = \det(h_{jk}(z)) + \mathcal{O}(z-a),$$

and thus $\det(\partial w/\partial z)(a) = \det(h_{jk})(a)$. Moreover, from the superposition property (Theorem 5.8) we get that $\phi(b) = a$ and that $e = \det(\partial w/\partial z)(a)\det(\partial z/\partial w)(b)$; therefore, $g(b) = e$. Thus

$$\widetilde{\omega}_{w-b} f - \phi^*\widetilde{\omega}_{z-a}\psi^* f = h(w)\widetilde{\omega}_{w-b} f,$$

where $(h(w) = 1 - g(w)$ and$)$ $h(b) = 0$. By a simple argument, cf., the proof of Proposition 3.1, it follows that $h\omega_{w-b} f = \omega_{w-b} h f$. Since Ω is pseudoconvex we can find a holomorphic solution to $\delta_{w-\xi} u(w,\xi) = h(w) - h(\xi)$. Therefore, $\delta_{w-b} u(w,b) = h(w)$, and hence $(\delta_{w-b} - \bar\partial)u(w,b)f = h(w)f$ in $U' = \psi(U)$. If $n > 1$ this means that $\omega_{w-b} h f = 0$ (since then the n-component of a solution in $U' \setminus \sigma$ is zero), and if $n = 1$ it means that $(w-b)^{-1}h(w)f$ is holomorphic in U'. From this Theorem 6.1 follows. □

The adjoint $\alpha(z)\colon T_z \to T_{\psi(z)}$ of α^* satisfies that

$$\alpha \sum (z_j - a_j)(\partial/\partial z_j)|_z = \sum (w_j - b_j)(\partial/\partial w_j)|_{\psi(z)},$$

and this immediately implies (6.3) (formally).

Remark 13. If Ω is an arbitrary neighborhood of $\sigma(a)$ it might be the case that $\omega_{z-a}\psi^* f - \psi^*(\omega_{w-b} f)$ is at least the image of a Dolbeault cohomology class in Ω but we have no proof. □

Remark 14. Assume that $\Omega \supset \sigma(a)$ is pseudoconvex and let a and $w = \psi(z)$ be holomorphic coordinates in Ω. Let $\mathcal{L}_{z-a}$ be the double complex from Section 2 and let $\mathcal{L}_{w-b}$ be the corresponding complex with respect to δ_{w-b}. From the proof of Theorem 6.1 above it follows that $\alpha^*\colon \mathcal{L}_{w-b}(U) \to \mathcal{L}_{z-a}(U)$ is a morphism for each $U \subset \Omega$, and hence it induces a mapping

$$\tilde{\alpha}^*\colon H^0(\mathcal{L}_{w-b}(U)) \to H^0(\mathcal{L}_{z-a}(U)). \tag{6.5}$$

However, if $U \supset \sigma(a)$, then both these spaces are $\mathcal{O}(\Omega)$-linearly isomorphic to X (by the same proof as for $\Omega = \mathbb{C}^n$ in Section 2) and the class of a function $f \in \mathcal{O}(U, X)$ in $H^0(\mathcal{L}_{w-b}(U))$ is mapped to the class of f in $H^0(\mathcal{L}_{z-a}(U))$. It follows that (6.5) is an isomorphism and that the induced $\mathcal{O}(U)$-module structures on X coincide; cf., Remark 3. □

Recall that if $\psi\colon \Omega \to \mathbb{C}^m_w$ is a proper mapping and ω is (p,q)-form, then the push-forward $\psi_*\omega$ is the $(m-n+p, m-n+p)$-current in $\mathbb{C}^m$ defined by the relation

$$\int_w \xi \wedge \psi_*\omega = \int_z \psi^*\xi \wedge \omega, \tag{6.6}$$

for test forms ξ of bidegree $(n-p, n-q)$. Since ψ_* commutes with $\bar\partial$, it induces a mapping on Dolbeault cohomology (for currents).

So far we have only discussed smooth representatives of the Dolbeault cohomology class $\omega_{z-a}f$ in $U \setminus \sigma(a)$. We say that an X-valued current u_n in $U \setminus \sigma(a)$ represents $\omega_{z-a}f$ if there is a smooth representative v_n in $U \setminus \sigma(a)$ and a current w such that $\bar\partial w = u_n - v_n$. It is clear then that $f(a)$ can be computed by the formula (3.7) with the representative u_n for $\omega_{z-a}f$.

Let U be a neighborhood of $\sigma(a)$ and let $\psi\colon U \to U'$ be a proper holomorphic mapping, and let $b = \psi(a)$. By the spectral mapping property $\sigma(a)$ is mapped to the compact subset $\sigma(b)$ of U'. If furthermore ψ is injective, then it follows that also $\psi\colon U \setminus \sigma(a) \to U' \setminus \sigma(b)$ is proper, and hence ψ_* maps $\omega_{z-a}x$ to a cohomology class in $U' \setminus \sigma(b)$. One can then ask whether this class coincides with ω_{w-b} in $U' \setminus \sigma(b)$.

Proposition 6.2. *If $T\colon \mathbb{C}^n_z \to \mathbb{C}^m_w$ is an injective linear mapping, then*

$$T_*\omega_{z-a}x = \omega_{w-Ta}x.$$

Proof. If T is invertible, then $T_* = (T^{-1})^*$, so this case follows from (5.10). Since the pushforward is functorial, it is enough to check the case when T is the natural inclusion $\mathbb{C}^n \ni z \mapsto (z, 0) \in \mathbb{C}^n \times \mathbb{C}^m$. Let $\widetilde{\omega}_{z-a}x$ be a fixed representative of the class $\omega_{z-a}x$ in $\Omega \setminus \sigma(a)$ and let $[w=0]$ denote the (m,m)-current of integration over the submanifold $w = 0$. It is easily checked that

$$T_*\widetilde{\omega}_{z-a}x = -[w=0] \wedge \widetilde{\omega}_{z-a}x.$$

Let χ be a function in $\Omega \times \mathbb{C}^m \setminus \sigma(a, 0)$ which is 1 in $\Omega \setminus \sigma(a)$ when $w = 0$ and 0 for $z \in \sigma(a)$ when $w \neq 0$. Moreover, let $\widetilde{\omega}_w$ be a locally integrable solution to $\bar\partial_w \widetilde{\omega}_w = [w=0]$ in $\mathbb{C}^m_w$, which is smooth outside

$w = 0$. Now,

$$(6.7) \quad -[w=0]\wedge\widetilde{\omega}_{z-a}x = -\chi[w=0]\wedge\widetilde{\omega}_{z-a}x = -\chi\bar{\partial}\widetilde{\omega}_w\wedge\widetilde{\omega}_{z-a}x = \\ = -\bar{\partial}(\chi\widetilde{\omega}_w\wedge\widetilde{\omega}_{z-a}x) + \bar{\partial}\chi\wedge\widetilde{\omega}_w\wedge\widetilde{\omega}_{z-a}x.$$

Observe that $\chi\widetilde{\omega}_w\wedge\widetilde{\omega}_{z-a}x$ is welldefined in $(\Omega\times\mathbb{C}^m)\setminus\sigma(a,0)$, and that the last term in (6.7) defines $\omega_{z-a,w}x$ according to Proposition 5.4. Hence $T_*\widetilde{\omega}_{z-a}x$ defines the cohomology class $\omega_{z-a,w}x$. □

Theorem 6.3. *Assume that $w=\psi(z)$ is a holomorphic mapping in a pseudoconvex neighborhood Ω of $\sigma(a)$, $b=\psi(a)$, and $n>1$.*
(i) If ψ is a biholomorphism, then $\psi_*\omega_{z-a}x = \omega_{w-b}x$.
(ii) If $\Psi(z)=(z,\psi(z))$, then $\Psi\colon\Omega\to\Omega\times\mathbb{C}^m$ is proper and injective, and

$$(6.8) \qquad \Psi_*\omega_{z-a}x = \omega_{z-a,w-b}x, \quad x\in X.$$

Proof. Part (i) is just a reformulation of Theorem 6.1. The mapping $\Psi(z)=(z,\psi(z))$ is the composition of the mappings $z\mapsto(z,0)$ and the biholomorphic mapping $G\colon\Omega\times\mathbb{C}^m\to\Omega\times\mathbb{C}^m$ defined by $G(z,w)=(z,\psi(z)-w)$. Hence part (ii) follows from part (i) and Proposition 6.2. □

Example 2. From Theorem 6.3 one can get (back) the superposition property (in the pseudoconvex case). In fact, let $g(w)$ be holomorphic in a neighborhood of $\sigma(\psi(a))$, and let χ be a cutoff function which is 1 in a neighborhood of $\sigma(a,\psi(a))$. Then if $G(z,w)=g(w)$ we have by (5.5) that

$$g(\psi(a))x = G(a,\psi(a))x = -\int\bar{\partial}\chi\wedge G\omega_{z-a,w-\psi(a)}x = \\ = -\int\bar{\partial}\Psi^*\chi\wedge(\Psi^*G)\omega_{z-a}x = g\circ\psi(a)x,$$

since $\Psi^*\chi$ is a cutoff function with support in Ω which is 1 in a neighborhood of $\sigma(a)$ and $\Psi^*G(z)=g\circ\psi(z)$. □

Example 3. Again let g be holomorphic in a neighborhood U' of $\sigma(\psi(a))$. Then

$$\alpha = (g(w)-g(\psi(z))\omega_{z-a,w-\psi(a)}x$$

is a welldefined cohomology class in $U\times U'\setminus\sigma(a,\psi(a))$ if $U=\psi^{-1}(U')$. Actually this class is zero, since it is the image under Φ_* of a class (in $U\setminus\sigma(a)$), and $G(z,w)=g(w)-g(\psi(z))$ vanishes on the image of Ψ.

If ψ is defined in an arbitrary, not necessarily pseudoconvex, neighborhood Ω of $\sigma(a)$, then at least the class α has a $\bar{\partial}$-closed extension over $\sigma(a,\psi(a))$ (i.e., α is the image of a class in a neighborhood of $\sigma(a,\psi(a))$.) In fact, if $G(z,w)=g\circ\psi(z)-g(w)$ and $b=\psi(a)$, then $G(a,b)=g\circ\psi\circ\pi_1(a,b)-g\circ\pi_2(a,b)=g\circ\psi(a)-g(b)=0$ by the superposition property. The claim now follows from Proposition 3.6.

Conversely, the fact that α is the image of a class over the spectrum immediately implies that $g(b) = g \circ \psi(a)$ in view of (5.5). □

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