Coloring percolation clusters at random

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September 18, 2000

Abstract

We consider the random coloring of the vertices of a graph G, that arises by first performing i.i.d. bond percolation on G, and then assigning a random color, chosen according to some prescribed probability distribution on the finite set $\{0, \ldots, r-1\}$, to each of the connected components, independently for different components. We call this the **divide and color model**, and study its percolation and Gibbs (quasilocality) properties, with emphasis on the case $G = \mathbf{Z}^d$. These properties turn out to depend heavily on the parameters of the model. For $r = 2$, an FKG inequality is also obtained.

1 Introduction

The purpose of this paper is to introduce and study a simple and natural model for dependent colorings of the vertices of a (finite or infinite) graph G with vertex set V and edge set E. We allow $r \geq 2$ different colors, denoted $0, 1, \ldots, r-1$. Besides r, the model has the additional parameters p and $a_0, a_1, \ldots, a_{r-1}$, all taking values in $[0, 1]$, and satisfying $\sum_{i=0}^{r-1} a_i = 1$. The coloring is done according to the following two-step procedure.

Step 1. Assign each edge $e \in E$ value 1 (present) with probability p, and 0 (absent) with probability $1-p$, and do this independently for different edges. Denote the resulting $\{0, 1\}^E$-valued configuration by Y.

Step 2. For each connected component $\mathcal{C}$ of the subgraph of G obtained by removing all edges e with $Y(e) = 0$, assign the same color to all vertices of $\mathcal{C}$. This color is chosen according to the probability distribution $(a_0, a_1, \ldots, a_{r-1})$ on $\{0, 1, \ldots, r-1\}$, and independently for different connected components. The resulting colouring is denoted X, and takes values in $\{0, \ldots, r-1\}^V$.

For obvious reasons, we call this the **divide and color (DaC) model** for G with parameters p, r and $a_0, \ldots, a_{r-1}$. The resulting probability measure on $\{0, 1\}^V$ is called the **DaC measure** for G with parameters p, r and $a_0, \ldots, a_{r-1}$, and is denoted $\mu^G_{p,r,(a_0,\ldots,a_{r-1})}$. Note that the parameter a_0 is redundant; we therefore sometimes abbreviate $\mu_{p,2,(a_0,a_1)}$ as $\mu_{p,2,a_1}$.

We emphasize that it is the coloring X which is of primary interest in this paper; the edge configuration Y is merely viewed a auxiliary object in the construction of X. (Of

*Research supported by a grant from the Swedish Natural Science Research Council.

course, this is not to say that Y is uninteresting. To the contrary, it is a fundamental object in percolation theory, known as i.i.d. (or Bernoulli) bond percolation.)

An interesting special case of the DaC model is obtained with $r = 2$ and $(a_0, a_1) = (\frac{1}{2}, \frac{1}{2})$. The resulting model, and the DaC measure $\mu^G_{p,2,\frac{1}{2}}$, then resembles, to some extent, the Ising model in zero external field: The model is symmetric with respect to permutation of the single site state space $\{0, 1\}$ (which is often taken to be $\{-1, 1\}$ in the Ising model), and exhibits positive correlations between the values at different vertices. When $p = 0$ (corresponding to infinite temperature in the Ising model), the values at different sites are uncorrelated. When p increases (corresponding to lowering the temperature in the Ising model), the correlations increase as well, until at $p = 1$ (corresponding to the zero temperature limit in the Ising model) all sites are forced to take the same value. Further similarities (and also differences) between $\mu^G_{p,2,\frac{1}{2}}$ and the Ising model will be discussed in the coming sections.

We now give some motivations for studying the DaC model, arranged in approximately decreasing order of importance.

(M1) Ising and Potts model on randomly diluted lattices are of interest in the study of disordered materials, and have received a fair amount of attention in the statistical mechanics and probability literature; see, e.g., Georgii [15], Aizenman et al. [1], van Enter et al. [12] and Häggström, Schonmann and Steif [24]. For the important special case of i.i.d. edge dilution, the DaC model with $r = q$ and $(a_0, \ldots, a_{r-1}) = (\frac{1}{q}, \ldots, \frac{1}{q})$ arises as the zero temperature limit.

(M2) The DaC model may be used as an alternative to Ising and Potts models in the stochastic modelling of various spatial systems with positively correlated values at different vertices. A major advantage of the DaC model compared to Ising and Potts models is that it is easy to simulate: whereas Ising and Potts models require sophisticated Markov chain Monte Carlo (MCMC) algorithms for their simulation, the DaC model can be simulated directly using the two-step procedure indicated in its definition. Of course, it is important to know what properties of the system are assumed through a specific model choice, and this paper is an attempt to address such issues for the DaC model.

(M3) In Häggström [23], the so-called fractional fuzzy Potts model is introduced as a natural generalization of a hidden Markov random field known as the fuzzy Potts model. The DaC measure $\mu^G_{p,2,(a_0,a_1)}$ corresponds, in the terminology of [23], to the $(a_0 + a_1)$-state fractional fuzzy Potts model at inverse temperature $-\frac{1}{2}\log(1-p)$.

(M4) Amongst the most efficient MCMC procedures for simulating the q-state Potts model at temperature β is to first carry out an MCMC simulation of a certain dependent percolation model known as the random-cluster model, with parameters $p = 1 - e^{-2\beta}$ and q (see, e.g., Aizenman et al. [2] or Georgii, Häggström and Maes [17]), and then to obtain the desired spin configuration as in Step 2 of the construction of the DaC model, with $r = q$ and $(a_0, \ldots, a_{r-1}) = (\frac{1}{q}, \ldots, \frac{1}{q})$. This procedure is particularly suitable for combining with the coupling-from-the-past technique for perfect simulation; see Propp and Wilson [36]. One may ask what a naive user of this method, who (incorrectly) generates an i.i.d. percolation process instead of the random-cluster model, gets. In fact, what he gets is the DaC model.

(M5) A related, and widely used, algorithm for simulating Ising and Potts models is the Swendsen–Wang algorithm [37]. On a similar note as in (M4), the DaC model arises after a single iteration of the Swendsen–Wang algorithm, starting from a spin configuration with complete alignment between all vertices.

In the following sections, we shall study the DaC model, mainly on infinite graphs and in particular on the prototypical case where G is the cubic lattice $\mathbf{Z}^d$, from two different (but related) points of view:

In Section 2, we consider percolation properties of the model. That is, when do we see an infinite connected component of vertices with the same color? Our sharpest result in this direction (Theorem 2.2) concerns the two-dimensional case $G = \mathbf{Z}^2$ and the DaC measure $\mu^{\mathbf{Z}^2}_{p,2,\frac{1}{2}}$: Just as for the Ising model on $\mathbf{Z}^2$, the transition between non-percolation and percolation takes place at exactly the same point in the parameter space as where the large-scale symmetry between the spins (colors) is broken.

In Section 3, we begin with the result that the DaC model, unlike Ising and Potts models, is *not* a Markov random field, and in fact not an n-Markov random field for any n. We then go on to investigate whether the weaker property of being a Gibbs measure holds. This question turns out to have different answers in different regimes of the parameter space, and leads to considerations about quasilocality and almost sure quasilocality of single-site conditional distributions.

No proofs are given in Sections 2 and 3; these are deferred to Sections 4 and 5, respectively. One of the tools developed in Section 4 is of independent interest: Theorem 4.2, which states that the $r = 2$ DaC model satisfies positive correlations (also known as the FKG inequality). Perhaps surprisingly, the random-cluster model turns out to be a useful tool in Section 5.

2 Percolation properties

In this section, we consider the case where G is infinite, and ask under what conditions the DaC model yields an infinite connected component of vertices that are all of the same color. Of particular interest is the $\mathbf{Z}^d$ case for $d \geq 2$. With the usual abuse of notation, we write $\mathbf{Z}^d$ for the graph whose vertex set is $\mathbf{Z}^d$, with edges connecting vertices at Euclidean distance 1 from each other. We also write E_d for the edge set of this graph.

Throughout this section, we shall work exclusively with the DaC model with just $r = 2$ colors. This is natural, because the question of whether the DaC measure $\mu^G_{p,r,(a_0,\ldots,a_i,\ldots,a_{r-1})}$ produces an infinite connected component with color i, can immediately be reduced to that of whether $\mu^G_{p,2,(1-a_i,a_i)}$ produces an infinite connected component with color 1.

We first consider the case $G = \mathbf{Z}^2$, which (besides trees) is the one we understand best. After that, we shall move on to other cases: higher dimensions, trees, and other graph structures.

The planar case $\mathbf{Z}^2$ has some special features. It is a classical result of Coniglio et al. [9] that for the Ising model without external field on $\mathbf{Z}^2$, an infinite connected component of aligned spins occurs if and only if the temperature parameter is below the so-called Onsager critical value. This means that there is percolation of aligned spins if and only if there are multiple Gibbs measure, which in turn is equivalent to a symmetry breaking (i.e., the existence of a Gibbs measure where the large-scale proportion of 1's differs from $\frac{1}{2}$ with positive probability).

The DaC model with $r = 2$ and $(a_0, a_1) = (\frac{1}{2}, \frac{1}{2})$ turns out to exhibit a similar phenomenon: An infinite connected component of vertices with the same color occurs exactly for those values of p for which the limiting proportion of 1's in large boxes fails to be $\frac{1}{2}$. These things happen precisely when $p > \frac{1}{2}$, as stated in the following two results.

Proposition 2.1 *Pick $X \in \{0,1\}^{\mathbf{Z}^2}$ according to the DaC measure $\mu^{\mathbf{Z}^2}_{p,2,\frac{1}{2}}$, and let $b_n(X)$ be the number of vertices with color 1 in the box $\Lambda_n = \{-n, \ldots, n\}^2$. Then the limiting fraction $\lim_{n\to\infty} \frac{b_n(X)}{(2n+1)^2}$ of vertices with color 1, exists a.s. For $p \le \frac{1}{2}$, we have*

$$\lim_{n\to\infty} \frac{b_n(X)}{(2n+1)^2} = \frac{1}{2} \tag{1}$$

a.s., whereas for $p > \frac{1}{2}$,

$$\lim_{n\to\infty} \frac{b_n(X)}{(2n+1)^2} = \begin{cases} \frac{1+\theta(p)}{2} > \frac{1}{2} & \textit{with probability } \frac{1}{2} \\ \frac{1-\theta(p)}{2} < \frac{1}{2} & \textit{with probability } \frac{1}{2}\,, \end{cases} \tag{2}$$

where $\theta(p)$ is the probability that the origin is in an infinite cluster in Bernoulli($\frac{1}{2}$) bond percolation on $\mathbf{Z}^2$.

Theorem 2.2 *Pick $X \in \{0,1\}^{\mathbf{Z}^2}$ according to the DaC measure $\mu^{\mathbf{Z}^2}_{p,2,\frac{1}{2}}$. We have*

$$\mu^{\mathbf{Z}^2}_{p,2,\frac{1}{2}}(X \textit{ contains an infinite connected component of aligned spins}) = \begin{cases} 0 & \textit{if } p \le \frac{1}{2} \\ 1 & \textit{if } p > \frac{1}{2}\,. \end{cases}$$

These results, and all others in this section, will be proved in Section 4.

In $\mathbf{Z}^3$ and higher dimensions, the sharp equivalence for the Ising model on $\mathbf{Z}^2$ between symmetry breaking and the existence of infinite connected components, no longer holds (see, e.g., Campanino and Russo [7]). The situation turns out to be similar for the symmetric DaC measure $\mu^{\mathbf{Z}^d}_{p,2,\frac{1}{2}}$: In $d \ge 3$ dimensions, infinite connected components of aligned spins can occur in the absence of symmetry breaking. Note first that the proof of Proposition 2.1 makes no particular use of the two-dimensionality, so an analogous result holds in higher dimensions, with the critical value $\frac{1}{2}$ replaced by the critical value $p^{\mathbf{Z}^d}_{c,bond}$ for i.i.d. bond percolation on $\mathbf{Z}^d$. On the other hand, we have the following.

Theorem 2.3 *For any dimension $d \ge 3$, there exists an $\varepsilon > 0$ such that for all $p < \varepsilon$,*

$$\mu^{\mathbf{Z}^d}_{p,2,\frac{1}{2}}(X \textit{ contains an infinite connected component of aligned spins}) = 1\,. \tag{3}$$

Since $p^{\mathbf{Z}^d}_{c,bond} > 0$ (see, e.g., [19]), we thus have for small enough p that an infinite connected component exists, in the absence of a symmetry breaking in the sense of Proposition 2.1. This is in contrast to the $\mathbf{Z}^2$ case. Note also that by the 0-1 symmetry of $\mu^{\mathbf{Z}^d}_{p,2,\frac{1}{2}}$, Theorem 2.3 implies the a.s. coexistence, for $d \ge 3$ and p sufficiently small, of two infinite connected components of aligned spins: one of 0's and the other of 1's.

Theorem 2.3 tells us that for $d \geq 3$, the DaC measure $\mu^{\mathbf{Z}^d}_{p,2,\frac{1}{2}}$ produces an infinite connected component of aligned spins for p small enough. The same happens for p sufficiently close to 1 ($p > p^{\mathbf{Z}^d}_{c,site}$ is enough; see the proof of Theorem 2.2 in the supercritical case). A naive interpolation now suggests that we get an infinite connected component for all $p \in [0,1]$, and although we lack a proof, we believe this to be the case:

Conjecture 2.4 *For any dimension $d \geq 3$ and any $p \in [0,1]$, we have*

$$\mu^{\mathbf{Z}^d}_{p,2,\frac{1}{2}}(X \textit{ contains an infinite connected component of aligned spins}) = 1\,.$$

Let us move on to the asymmetric case where $(a_0, a_1) \neq (\frac{1}{2}, \frac{1}{2})$. An obvious coupling argument shows that for any G, p, a_1 and a_1' such that $a_1 \leq a_1'$, the DaC measure $\mu^G_{p,2,a_1}$ is stochastically dominated by $\mu^G_{p,2,a_1'}$. The following result is an immediate consequence.

Proposition 2.5 *For any graph G and any $p \in [0,1]$, there exists a critical value $a_c^{G,p} \in [0,1]$ such that*

$$\mu^G_{p,2,a_1}(X \textit{ contains an infinite connected component of } 1\textit{'s}) \begin{cases} > 0 & \textit{for } a_1 > a_c^{G,p} \\ = 0 & \textit{for } a_1 < a_c^{G,p}\,. \end{cases}$$

What do we know about the critical value $a_c^{G,p}$ when $G = \mathbf{Z}^d$? Well, we obviously have $a_c^{\mathbf{Z}^d,p} = 0$ when $p > p^{\mathbf{Z}^d}_{c,bond}$. For the $\mathbf{Z}^2$ case, we also know (see the proof of Theorem 2.2 in the critical case) that $a_c^{\mathbf{Z}^2,p} = 1$ for $p = p^{\mathbf{Z}^2}_{c,bond} = \frac{1}{2}$. For the subcritical case $p < p^{\mathbf{Z}^d}_{c,bond}$, all we can show so far is the following result, that $a_c^{\mathbf{Z}^d,p}$ is a nontrivial threshold in the sense that it lies strictly between 0 and 1.

Theorem 2.6 *For any dimension $d \geq 2$ and any $p < p^{\mathbf{Z}^d}_{c,bond}$, we have that*

$$0 < a_c^{\mathbf{Z}^d,p} < 1\,. \tag{4}$$

It would be of interest to add to the sparse knowledge of the behavior $a_c^{\mathbf{Z}^d,p}$ on the interval $p \in (0, p^{\mathbf{Z}^d}_{c,bond})$ provided by Theorem 2.6, such as obtaining continuity or monotonicity properties of $a_c^{\mathbf{Z}^d,p}$ as a function of p. For $d \geq 3$, we know nothing about $a_c^{\mathbf{Z}^d,p}$ at the critical point $p = p^{\mathbf{Z}^d}_{c,bond}$. It seems reasonable to expect that $a_c^{\mathbf{Z}^d,p} < 1$ in this case; this would give yet another contrast between two and higher dimensions. Perhaps we even have $a_c^{\mathbf{Z}^d,p} = 0$ at criticality in sufficiently high dimensions (as is the case on sufficiently large trees, see Proposition 2.10 below), making the constrast even more drastic.

We now leave the $\mathbf{Z}^d$ case and turn to other graph structures. It turns out that there is one class of graphs for which the critical value $a_c^{G,p}$ defined in Proposition 2.5 can be calculated for any p, namely trees. A tree is a connected graph without cycles. In order to state our result for trees, we need to recall what is meant by the **branching number** of a tree. Given any infinite locally finite tree Γ with vertex set V_Γ and a distinguished vertex $\rho \in V_\Gamma$ called the **root**, we call a finite set $\Pi \subset V_\Gamma$ a **cutset** for Γ if every infinite self-avoiding path starting at ρ has to intersect Π, and no proper subset of Π has this property. Informally, Π is a cutset if it is a minimal set which cuts off ρ from infinity. For $v \in V_\Gamma$, let $|v|$ denote the distance between ρ and v. For a sequence of cutsets $\Pi_1, \Pi_2, \ldots$, write $\Pi \to \infty$ if $\min\{|v| : v \in \Pi\} \to \infty$.

Definition 2.7 *The* **branching number** *of* Γ, *denoted* $\mathrm{br}(\Gamma)$, *is defined*

$$\mathrm{br}(\Gamma) = \inf\{\lambda > 0 : \liminf_{\Pi\to\infty} \sum_{v\in\Pi} \lambda^{-|v|} = 0\}\,.$$

Note that $\mathrm{br}(\Gamma)$ is independent of the choice of root $\rho \in V_\Gamma$. It has turned out that the branching number is of crucial relevance for a variety of different stochastic models on trees. In particular, Lyons [28] showed that the critical value for i.i.d. site (or bond) percolation on a tree Γ satisfies

$$p_{c,site}^{\Gamma} = \frac{1}{\mathrm{br}(\Gamma)}\,. \tag{5}$$

For instance, the infinite binary tree has $\mathrm{br}(\Gamma) = 2$ and $p_{c,site}^{\Gamma} = \frac{1}{2}$. More generally, an infinite regular tree in which every vertex has $n+1$ neighbours, has $\mathrm{br}(\Gamma) = n$ and $p_{c,site}^{\Gamma} = \frac{1}{n}$. Note also that every infinite tree has $\mathrm{br}(\Gamma) \geq 1$.

We shall prove the following two results.

Proposition 2.8 *For the DaC model on an infinite tree* Γ *with parameters* $p \in [0,1]$, $r = 2$ *and* (a_0, a_1), *we have*

$$\mu_{p,2,a_1}^{\Gamma}(X \textit{ contains an infinite connected component of } 1\textit{'s}) > 0$$

if and only if i.i.d. site percolation on Γ *with retention parameter* $1-(1-p)a_0$ *produces an infinite connected component of* 1*'s with positive probability.*

Proposition 2.9 *For the DaC model on an infinite tree* Γ *with parameters* $p \in (0,1)$, $r = 2$ *and* (a_0, a_1), *the* $\mu_{p,2,a_1}^{\Gamma}$*-probability of having an infinite connected component of* 1*'s is* 0 *or* 1.

Of course, the case $p = 1$ had to be excluded in Proposition 2.9. Note also that the asserted 0-1 law does not hold for the $\mathbf{Z}^d$ case.

Propositions 2.8 and 2.9, in conjunction with the formula (5) for $p_{c,site}^{\Gamma}$, have the following immediate consequences.

Corollary 2.10 *For the DaC model on an infinite tree* Γ *with parameters* $p \in (0,1)$, $r = 2$ *and* (a_0, a_1), *we have*

$$\mu_{p,2,a_1}^{\Gamma}(X \textit{ contains an infinite connected component of } 1\textit{'s}) = \begin{cases} 0 & \textit{if } a_1 < \frac{(\mathrm{br}(\Gamma))^{-1}-p}{1-p} \\ 1 & \textit{if } a_1 > \frac{(\mathrm{br}(\Gamma))^{-1}-p}{1-p}\,. \end{cases}$$

Hence, the critical value $a_c^{\Gamma,p}$ *(as defined in Proposition 2.5) satisfies*

$$a_c^{\Gamma,p} = \begin{cases} 0 & \textit{if } p > \frac{1}{\mathrm{br}(\Gamma)} \\ \frac{(\mathrm{br}(\Gamma))^{-1}-p}{1-p} & \textit{otherwise.} \end{cases}$$

Specializing to the symmetric case $(a_0, a_1) = (\frac{1}{2}, \frac{1}{2})$ gives the following.

Corollary 2.11 *For the DaC model on an infinite tree* Γ *with parameters* $p \in (0,1)$, $r = 2$ *and* (a_0, a_1), *we have*

$$\mu_{p,2,\frac{1}{2}}^{\Gamma}(X \textit{ contains an infinite connected component of } 1\textit{'s}) = \begin{cases} 0 & \textit{if } p > \frac{2}{\mathrm{br}(\Gamma)} - 1 \\ 1 & \textit{if } p < \frac{2}{\mathrm{br}(\Gamma)} - 1\,. \end{cases}$$

We remark that Lyons [29] has a criterion for which trees we get infinite connected components for i.i.d. site percolation at criticality. It is another simple consequence of Propositions 2.8 and 2.9, and (5), that it is exactly these trees that produce infinite connected components in the DaC model at the critical parameter values provided by Corollaries 2.10 and 2.11.

So much for the tree case. We have rather little to say at present regarding percolation properties of the DaC model on more general graph structures, and shall confine ourselves to a counterexample (Theorem 2.12 below) to a conjecture that might otherwise be tempting to make. Namely, in view of Theorem 2.2 and Corollary 2.11, one might be tempted to think that for every graph G, there is a critical value $p_{c,DaC} = p_{c,DaC}(G)$ such that

$$\mu^G_{p,2,\frac{1}{2}}(X \text{ contains an infinite connected component of aligned spins}) = \begin{cases} 0 & \text{if } p < p_{c,DaC} \\ 1 & \text{if } p > p_{c,DaC} \,. \end{cases}$$

The 0-1-law implicit in this statement does hold in this generality (see the proof of Proposition 2.9), but the monotonicity does not. In other words, there exists a graph G such that the $\mu^G_{p,2,\frac{1}{2}}$-probability of having an infinite connected component of aligned spins fails to be increasing in p. G can even be taken to be quasi-transitive, as stated in Theorem 2.12. An infinite graph $G = (V, E)$ is said to be **quasi-transitive** if V can be partitioned into finitely many sets $V_1, \ldots, V_k$ in such a way that for each $i \in \{1, \ldots, k\}$ and each $x, y, \in V_i$, there exists a graph automorphism of G mapping x to y. The class of quasi-transitive graphs have been shown to be well-behaved with respect to many percolation-theoretic and other probabilistic aspects; see, e.g., [4] and [30].

Theorem 2.12 *There exists an infinite quasi-transitive graph G, and $p_1, p_2 \in (0,1)$ with $p_1 < p_2$ such that*

$$\mu^G_{p_1,2,\frac{1}{2}}(X \textit{ contains an infinite connected component of aligned spins}) = 1$$

and

$$\mu^G_{p_2,2,\frac{1}{2}}(X \textit{ contains an infinite connected component of aligned spins}) = 0\,.$$

3 Markov and quasilocality properties

In this section, we consider the DaC model on $\mathbf{Z}^d$, and try to answer questions of the following kind: Is the DaC model a Markov random field? Is $\mu^{\mathbf{Z}^d}_{p,r,(a_0,\ldots,a_{r-1})}$ a Gibbs measure?

For a finite vertex set $W \subset \mathbf{Z}^d$, define the (outer) boundary ∂W of W as

$$\partial W = \{x \in \mathbf{Z}^d \setminus W : \exists y \in W \text{ such that } \|x - y\|_1 = 1\}$$

where $\|\cdot\|_1$ is the L^1 norm. More generally, for $n \in \{1, 2, \ldots\}$, we define

$$\partial_n W = \{x \in \mathbf{Z}^d \setminus W : \exists y \in W \text{ such that } \|x - y\|_1 \leq n\}\,.$$

Definition 3.1 *A probability measure ν on $\{0, \ldots, r-1\}^{\mathbf{Z}^d}$ is said to be a* **Markov random field** *if it admits conditional probabilities such that for all finite $W \subset \mathbf{Z}^d$, all*

$\xi \in \{0,1,\ldots,r-1\}^W$ *and all* $\zeta,\zeta' \in \{0,1,\ldots,r-1\}^{\mathbf{Z}^d\setminus W}$ *such that* $\zeta(\partial W) = \zeta'(\partial W)$, *we have*

$$\nu(X(W) = \xi \mid X(\mathbf{Z}^d \setminus W) = \zeta) = \nu(X(W) = \xi \mid X(\mathbf{Z}^d \setminus W) = \zeta') . \tag{6}$$

More generally, ν *is said to be an* n-**Markov random field** *if it admits conditional probabilities such that* (6) *holds for all* W *and* ξ *as above, and all* $\zeta,\zeta' \in \{0,1,\ldots,r-1\}^{\mathbf{Z}^d\setminus W}$ *such that* $\zeta(\partial_n W) = \zeta'(\partial_n W)$.

It is easy to see that the DaC model on $\mathbf{Z}^1$ with arbitrary parameters, is a Markov random field (the same holds, more generally, on an arbitrary tree, with $\|x-y\|_1$ replaced by graph-theoretic distance in the definition of ∂W). In contrast, we have the following.

Theorem 3.2 *The DaC measure* $\mu^{\mathbf{Z}^d}_{p,r,(a_0,\ldots,a_{r-1})}$ *with* $d \geq 2$, $p \in (0,1)$, *and* $a_0,\ldots,a_{r-1} \in (0,1)$, *is not an* n*-Markov random field for any* n.

This result, and the others in this section, will be proved in Section 5.

We now turn to the issue of whether $\mu^{\mathbf{Z}^d}_{p,r,(a_0,\ldots,a_{r-1})}$ satisfies the weaker property of being a Gibbs measure. Roughly speaking, a probability measure ν on $\{0,\ldots,r-1\}^{\mathbf{Z}^d}$ is a Gibbs measure if its conditional probabilities on finite sets $W \in \mathbf{Z}^d$ can be written as an exponential of an absolutely convergent sum of terms that each involve local events. It has been realized since the seminal paper by van Enter, Fernández and Sokal [11] that many examples of physical interest fail to be Gibbsian. This has triggered an intense activity in determining Gibbsianness or non-Gibbsianness of various measures; see, e.g., [35], [10], [32], [12] and [13]. Under the technical assumption of so-called uniform nonnullness (which holds for the DaC model; see Lemma 5.6) Gibbsianness is known to be equivalent to a property known as quasilocality (see, e.g., Georgii [16] or van Enter et. al. [11]). In our setting it is more natural to work with quasilocality than directly with Gibbs potentials, and therefore we shall formulate our results in terms of the former. Its definition is as follows.

Definition 3.3 *A probability measure* ν *on* $\{0,1,\ldots,r-1\}^{\mathbf{Z}^d}$ *is said to be* **quasilocal** *if it admits conditional probabilities such that for all finite* $W \subset \mathbf{Z}^d$, *all* $\xi \in \{0,1\ldots,r-1\}^W$ *and all* $\zeta \in \{0,1,\ldots,r-1\}^{\mathbf{Z}^d\setminus W}$, *we have*

$$\lim_{n\to\infty} \sup_{\substack{\zeta' \in \{0,1,\ldots,r-1\}^{\mathbf{Z}^d\setminus W} \\ \zeta'(\partial_n W) = \zeta(\partial_n W)}} \left| \nu(X(W) = \xi \mid X(\mathbf{Z}^d\setminus W) = \zeta) - \nu(X(W) = \xi \mid X(\mathbf{Z}^d\setminus W) = \zeta') \right| = 0 . \tag{7}$$

By compactness of $\{0,1,\ldots,r-1\}^{\mathbf{Z}^d}$ in the product topology, this is the same as requiring that, for all W and ξ as above,

$$\lim_{n\to\infty} \sup_{\substack{\zeta,\zeta' \in \{0,1,\ldots,r-1\}^{\mathbf{Z}^d\setminus W} \\ \zeta'(\partial_n W) = \zeta(\partial_n W)}} \left| \nu(X(W) = \xi \mid X(\mathbf{Z}^d\setminus W) = \zeta) - \nu(X(W) = \xi \mid X(\mathbf{Z}^d\setminus W) = \zeta') \right| = 0 .$$

In other words, quasilocality means that for any $\varepsilon > 0$, there exists an $n = n(\varepsilon)$ such that in order to determine the probability that $X(W) = \xi$ given $X(\mathbf{Z}^d \setminus W)$ to within an error of ε, it suffices to look at $X(\partial_n W)$. Note also that if ν is an m-Markov random field for some m, then the supremum in (7) is 0 for $n \geq m$, so that ν is quasilocal.

Our main result on quasilocality of DaC measures is the following.

Theorem 3.4 *For any $d \geq 2$, $r \geq 2$, and $a_0, \ldots, a_{r-1} \in (0,1)$ such that $\sum_{i=0}^{r-1} a_i = 1$, there exist p_1 and p_2 with $0 < p_1 \leq p_2 < 1$ such that*

(i) $\mu^{\mathbf{Z}^d}_{p,r,(a_0,\ldots,a_{r-1})}$ *is quasilocal if $p < p_1$, while*

(ii) $\mu^{\mathbf{Z}^d}_{p,r,(a_0,\ldots,a_{r-1})}$ *fails to be quasilocal if $p \in (p_2, 1)$.*

For concrete bounds, we may take

$$p_1 = \frac{\min_{i \in \{0,\ldots,r-1\}} a_i}{4d - 3 + \min_{i \in \{0,\ldots,r-1\}} a_i} \tag{8}$$

and $p_2 = \frac{1}{2}$.

Thus, quasilocality in the DaC model depends on the parameters in a more interesting way than the n-Markovianness property. We lack a proof of the monotonicity in p required to prove the following plausible improvement of Theorem 3.4.

Conjecture 3.5 *For any $d \geq 2$, $r \geq 2$, and $a_0, \ldots, a_{r-1} \in (0,1)$ such that $\sum_{i=0}^{r-1} a_i = 1$, there exists a critical value $p_c = p_c(d, r, a_0, \ldots, a_{r-1}) \in (0,1)$ such that*

$$\mu^{\mathbf{Z}^d}_{p,r,(a_0,\ldots,a_{r-1})} \text{ is } \begin{cases} \text{quasilocal} & \text{for } p \in [0, p_c) \\ \text{not quasilocal} & \text{for } p \in (p_c, 1)\,. \end{cases}$$

In recent years' work on non-Gibbsian measures, there has been a fair amount of interest in determining whether or not a weaker form of quasilocality, known as almost sure quasilocality, holds; see, e.g., [35], [21] and [12]. The relation between almost sure quasilocality and other weak forms of Gibbsianness is discussed in [32] and in [13]. For the DaC model, it is conceivable that $\mu^{\mathbf{Z}^d}_{p,r,(a_0,\ldots,a_{r-1})}$ satisfies almost sure quasilocality for all p, but all we have been able to show in this direction is Proposition 3.7 below, which is a fairly simple result.

Definition 3.6 *A probability measure ν on $\{0, 1, \ldots, r-1\}^{\mathbf{Z}^d}$ is said to satisfy* **almost sure quasilocality** *if it admits conditional probabilities such that for all finite $W \subset \mathbf{Z}^d$, all $\xi \in \{0, 1 \ldots, r-1\}^W$ and ν-almost all $\zeta \in \{0, 1, \ldots, r-1\}^{\mathbf{Z}^d \setminus W}$, we have*

$$\lim_{n \to \infty} \sup_{\substack{\zeta' \in \{0,1,\ldots,r-1\}^{\mathbf{Z}^d \setminus W} \\ \zeta'(\partial_n W) = \zeta(\partial_n W)}} \left| \nu(X(W) = \xi \,|\, X(\mathbf{Z}^d \setminus W) = \zeta) - \nu(X(W) = \xi \,|\, X(\mathbf{Z}^d \setminus W) = \zeta') \right| = 0\,. \tag{9}$$

Proposition 3.7 *Consider the DaC model on $\mathbf{Z}^d$, $d \geq 2$. If p, r and $(a_0, \ldots, a_{r-1})$ are chosen in such a way that*

$$\mu^{\mathbf{Z}^d}_{p,r,(a_0,\ldots,a_{r-1})}(X \text{ contains an infinite connected component of aligned spins}) = 0\,, \tag{10}$$

then $\mu^{\mathbf{Z}^d}_{p,r,(a_0,\ldots,a_{r-1})}$ satisfies almost sure quasilocality.

If we accept Conjecture 2.4, then the $r = 2$ instance of Proposition 3.7 is relevant only for $d = 2$. The $r \geq 3$ cases, however, are nonvacuous also in higher dimensions.

4 Proofs of percolation results

This section contains proofs of all results in Section 2, beginning with the two-dimensional results (Proposition 2.1 and Theorem 2.2).

Proof of Proposition 2.1: It is clear from the definition of the DaC model that $\mu^{\mathbf{Z}^2}_{p,2,\frac{1}{2}}$ is translation invariant. Therefore, the limit $\lim_{n\to\infty}\frac{b_n(X)}{(2n+1)^2}$ exists by the ergodic theorem.

Next, consider the random edge configuration $Y \in \{0,1\}^{E_2}$ obtained as in Step 1 in the definition of the DaC model. Furthermore let $W \in \{0,1\}^{\mathbf{Z}^2}$ be another auxiliary process, obtained by letting each vertex $v \in \mathbf{Z}^2$ take value 0 or 1 with probability $\frac{1}{2}$ each, and take Y and W to be independent of each other. Now obtain the random spin configuration $\tilde{X} \in \{0,1\}^{\mathbf{Z}^2}$ as follows: If a vertex $v \in \mathbf{Z}^2$ is in an infinite connected component of Y, then let $\tilde{X}(v) = 1$. Otherwise let $\tilde{X}(v) = W(w)$, where w is the first vertex, according to lexicographic ordering, of the (finite) connected component of Y containing v. Note that the distribution of $\tilde{X}$ equals that of X conditional on the event that all infinite clusters of Y are assigned value 1; there is no problem with the conditioning, because Y contains a.s. at most one infinite cluster (this is just the usual uniqueness-of-the-infinite-cluster result for percolation on $\mathbf{Z}^d$; see, e.g., Grimmett [19]).

Since $\tilde{X}$ is obtained in a stationary manner from an i.i.d. process, it is ergodic, with the spatial average $\lim_{n\to\infty}\frac{b_n(X)}{(2n+1)^2}$ equal to the expected value of the spin at the origin. But this expected value equals $\theta(p) + \frac{1-\theta(p)}{2} = \frac{1+\theta(p)}{2}$. Hence, (1) follows from the classical Harris–Kesten [25, 26] theorem, which states that the critical value for Bernoulli bond percolation on $\mathbf{Z}^2$ is $\frac{1}{2}$, with $\theta(p) > 0$ if and only if $p > \frac{1}{2}$. We similarly obtain (2), upon noting that the infinite cluster in Y is assigned value 0 or 1 with probability $\frac{1}{2}$ each, in Step 2 of the construction of the DaC model. □

We go on to prove Theorem 2.2. This task is natural to split in three parts: the subcritical case ($p < \frac{1}{2}$), the critical case ($p = \frac{1}{2}$), and the supercritical case ($p > \frac{1}{2}$). We do this in order of increasing difficulty, which turns out to be the reverse of the above order.

Proof of Theorem 2.2, supercritical case: For $p > \frac{1}{2}$, the bond process Y contians a.s. an infinite connected component. But then X contains an infinite connected component of aligned spins (containing the infinite connected component of Y). □

Proof of Theorem 2.2, critical case: Harris [25] showed that in the critical case $p = \frac{1}{2}$ we have a.s. the following situation: Y contains no infinite cluster, but it contains infinitely many finite clusters with the property that they contain a circuit "surrounding" the origin. Each of these clusters independently take value 0 or 1 with probability $\frac{1}{2}$ each, whence, by Borel–Cantelli, we have a.s. that at least one of them takes value 0. This prevents the origin from being in an infinite cluster of 1's in X. By the same argument, the event that the origin is in an infinite cluster of 0's in X, also has probability 0. By translation invariance, the corresponding statements are true with any vertex of $\mathbf{Z}^d$ in place of the origin. □

In order to prove Theorem 2.2 in the subcritical case, we need to recall a result of Gandolfi, Keane and Russo [14] concerning percolation models with positive correlations, and then to prove that the DaC model has positive correlations.

Equip the set $\{0,1\}^V$, where V is finite or countable, with its coordinatewise partial order $\preceq$, defined by

$$\xi \preceq \xi' \quad \text{iff} \quad \xi(v) \leq \xi'(v) \text{ for all } v \in V\,.$$

A function $f : \{0,1\}^V \to \mathbf{R}$ is said to be **increasing** if $f(\xi) \leq f(\xi')$ whenever $\xi \preceq \xi'$. A probability measure π on $\{0,1\}^V$ is said to have **positive correlations** if

$$\int_{\{0,1\}^V} f \, d\pi \int_{\{0,1\}^V} g \, d\pi \leq \int_{\{0,1\}^V} fg \, d\pi$$

for all bounded increasing functions $f, g : \{0,1\}^V \to \mathbf{R}$. The well-known **Harris' inequality** [25] states that any product probability measure on $\{0,1\}^V$ satisfies positive correlations. The significance of positive correlations in percolation theory was demonstrated already in [25], and later, e.g., in the following result.

Theorem 4.1 (Gandolfi, Keane and Russo [14]) *Let π be a probability measure on $\{0,1\}^{\mathbf{Z}^2}$ which*

(i) *is translation invariant,*

(ii) *is invariant under permutations of coordinates and under reflections the coordinate axes,*

(iii) *is ergodic under horizontal and vertical translations (separately), and*

(iv) *has positive correlations.*

Then the π-probability of obtaining both an infinite connected component of 0's, and an infinite connected component of 1's, is 0.

The next result was proved jointly with O. Schramm, who has kindly given permission to publish it in this form.

Theorem 4.2 (Häggström and Schramm) *Let $G = (V,E)$ be any (finite or infinite) graph, and let $p \in [0,1]$ and $a_1 \in [0,1]$ be arbitrary. Then the DaC measure $\mu^G_{p,2,a_1}$ has positive correlations.*

Proof: An alternative way to obtain a $\{0,1\}^V$-valued random configuration X with distribution $\mu^G_{p,2,a_1}$, together with its auxiliary random bond configuration $Y \in \{0,1\}^E$ (so that the pair (X,Y) is distributed as in the definition of the DaC model), is as follows. Let $\{W(v)\}_{v\in V}$, $\{U_0(e)\}_{e\in E}$ and $\{U_1(e)\}_{e\in E}$ be independent $\{0,1\}$-valued random variables with

$$\begin{array}{ll} \mathbf{P}(W(v)=1) = a_1 & \text{for each } v \in V \\ \mathbf{P}(U_0(v)=1) = 1-p & \text{for each } e \in E \\ \mathbf{P}(U_1(v)=1) = p & \text{for each } e \in E\,. \end{array}$$

Let $(v_1, v_2, \ldots)$ be an arbitrary enumeration of V, and construct $(X,Y) \in \{0,1\}^V \times \{0,1\}^E$ in the following manner.

1. Let $i = 1$.

2. If $X(v_i)$ has not been determined earlier, then let $X(v_i) = W(v_i)$. Otherwise go to 4.

3. Consider the set of edges $e \in E$ satisfying

    (i) $U_{X(v_i)}(e) = X(v_i)$, and

    (ii) $Y(e)$ has not been determined earlier.

    Let $\mathcal{C}$ be the connected component of such edges, "containing" v_i. Let $Y(e) = 1$ for all edges in $\mathcal{C}$, and let $X(v) = X(v_i)$ for all vertices in $\mathcal{C}$. Also let $Y(e) = 0$ for all edges e that are adjacent to $\mathcal{C}$ and whose values have not been determined earlier.

4. Increase i by 1. If i exceeds the number of vertices in G, then stop, otherwise go to 2.

If G is finite, then the above "algorithm" (we use quotation marks, because step 3 may take an infinite number of operations to carry out) terminates in a finite number of iterations. Otherwise it does not, but note that any given vertex or edge is assigned a value in (X, Y) after a finite number of iterations. The connected component $\mathcal{C}$ in step 3 should be thought of as being obtained by a breadth-first search from v_i, and the "algorithm" can then be thought of as a sequential way to "discover" (X, Y). With this interpretation in mind, it is clear (or becomes, upon some thought) that (X, Y) obtained in this way has the desired distribution. In particular, X has distribution $\mu^G_{p,2,a_1}$.

For $i = 1, 2, \ldots$, let $(X_i, Y_i) \in \{0, \frac{1}{2}, 1\}^V \times \{0, \frac{1}{2}, 1\}^E$ be the state of the system after the i^{th} iteration of the main loop in the "algorithm"; here $\frac{1}{2}$ means "not yet determined". Note that $X(v) = \lim_{n\to\infty} X_i(v)$ and $Y(e) = \lim_{n\to\infty} Y_i(e)$ for all $v \in V$ and $e \in E$. By induction in i, it is easy to see that for each v, $X_i(v)$ is an increasing function of $(\{W(v)\}_{v\in V}, \{U_0(e)\}_{e\in E}, \{U_1(e)\}_{e\in E})$, and the same thing is therefore true for the limiting value $X(v)$. Hence, if $f, g : \{0,1\}^V \to \mathbf{R}$ are bounded increasing functions, then $f(X)$ and $g(X)$ are also bounded increasing functions of $(\{W(v)\}_{v\in V}, \{U_0(e)\}_{e\in E}, \{U_1(e)\}_{e\in E})$. Harris' inequality therefore implies that $f(X)$ and $g(X)$ are positively correlated. Since f and g were arbitrary, the distribution $\mu^G_{p,2,a_1}$ of X has positive correlations. □

Remark. Positive correlations was proved in [23] for the fractional fuzzy Potts model (recall motivation (M3) in Section 1) in a different regime of the parameter space. It seems that neither that proof, nor the above proof of Theorem 4.2, can be adapted to replace the other.

We are finally ready to finish the proof of Theorem 2.2.

Proof of Theorem 2.2, subcritical case: From its construction, it is clear that $\mu^{\mathbf{Z}^2}_{p,2,\frac{1}{2}}$ satisfies conditions (i) and (ii) of Theorem 4.1. Furthermore, from the construction of $\tilde{X}$ in the proof of Proposition 2.1, we have in the subcritical case $p < \frac{1}{2}$ that also (iii) is satisfied. Finally, (iv) holds due to Theorem 4.2. Hence, Theorem 4.1 applies to $\mu^{\mathbf{Z}^2}_{p,2,\frac{1}{2}}$ with $p < \frac{1}{2}$.

Suppose now for contradiction that the $\mu^{\mathbf{Z}^2}_{p,2,\frac{1}{2}}$-probability of getting an infinite connected component of 1's in X is positive. Then this probability is 1 by ergodicity. By the symmetry of the model, the probability of getting an infinite connected component of 0's must be the same, i.e., 1. This contradicts Theorem 4.1. □

We now move on to higher dimensions (Theorems 2.3 and 2.6). As a preparation for the proof of Theorem 2.3, we shall first recall some more definitions, and a result of Liggett, Schonmann and Stacey [27]. For V finite or countable, and two probability measures π and π' on $\{0,1\}^V$, we say that π **is stochastically dominated by** π' if

$$\int_{\{0,1\}^V} f\,d\pi \le \int_{\{0,1\}^V} f\,d\pi'$$

for every bounded increasing function $f : \{0,1\}^V \to \mathbf{R}$.

For $b = 1, 2, \ldots$, a probability measure π on $\{0,1\}^{\mathbf{Z}^d}$ is said to be **b-dependent** if for all disjoint vertex sets $V, V' \subset \mathbf{Z}^3$ with the property that no two vertices $x \in V$ and $y \in V'$ are within L^1-distance b from each other, we have that $\{X(x)\}_{x\in V}$ is independent of $\{X(x)\}_{x\in V'}$ whenever $X \in \{0,1\}^{\mathbf{Z}^d}$ has distribution π.

Theorem 4.3 (Liggett, Schonmann and Stacey [27]) *For any dimension d and any $b \in \{1,2,\ldots\}$, there exists a function $p^*_{b,d} : [0,1] \to [0,1]$ with*

$$\lim_{p\to 1} p^*_{b,d}(p) = 1$$

and the following property. For any $p \in [0,1]$ and any b-dependent probability measure π on $\{0,1\}^{\mathbf{Z}^d}$ which for all $x \in \mathbf{Z}^d$ assigns probability at least p to the event that x gets value 1, we have that π stochastically dominates i.i.d. site percolation on $\mathbf{Z}^d$ with parameter $p^*_{b,d}(p)$.

Remark. We find it convenient to refer to Theorem 4.3, even though it is a bit of overkill in the following application, because it can be replaced by the more elementary reasoning in Lyons and Schramm [31, Remark 6.2].

Proof of Theorem 2.3: Campanino and Russo [7] showed that the critical value $p_{c,site}^{\mathbf{Z}^3}$ for i.i.d. site percolation on $\mathbf{Z}^3$ satisfies $p_{c,site}^{\mathbf{Z}^3} < \frac{1}{2}$. We can therefore fix a $\delta > 0$ such that $p_{c,site}^{\mathbf{Z}^3} < \frac{1}{2} - \delta$. Since $p_{c,site}^{\mathbf{Z}^d} \le p_{c,site}^{\mathbf{Z}^3}$ for $d \ge 4$, we then also have $p_{c,site}^{\mathbf{Z}^d} < \frac{1}{2} - \delta$ for all $d \ge 3$.

Consider the auxiliary edge configuration $Y \in \{0,1\}^{E_d}$ given in the definition of the DaC model, and define the random site configuration $Z \in \{0,1\}^{\mathbf{Z}^d}$ by letting

$$Z(x) = \begin{cases} 1 & \text{if } Y(e) = 0 \text{ for all edges } e \text{ incident to } x \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in \mathbf{Z}^d$. In other words, $Z(x) = 1$ for exactly those vertices x that are isolated in Y. Note that

$$\lim_{p\to 0} \mathbf{P}(Z(x) = 1) = \lim_{p\to 0} (1-p)^{2d} = 1\,. \tag{11}$$

Furthermore, $Z = \{Z(x)\}_{x\in\mathbf{Z}^d}$ is 1-dependent, because if $V, V' \in \mathbf{Z}^d$ are as in the definition of b-dependence with $b = 1$, then $\{Z(x)\}_{x\in V}$ and $\{Z(x)\}_{x\in V'}$ are defined in terms of disjoint sets of edges in Y. Hence, Theorem 4.3 applies (in conjunction with (11)) to show that if we pick $p > 0$ small enough, then Z stochastically dominates i.i.d. site percolation with parameter $1 - 2\delta$. Fix such a p. Conditionally on Z, we have that each vertex x with $Z(x) = 1$ independently satisfies $X(x) = 1$ with probability $\frac{1}{2}$. Hence, X stochastically dominates i.i.d. site percolation with parameter $\frac{1}{2} - \delta$ (because it would do so even if we turned off all vertices in X that are not singleton connected components in Y), and by the choice of δ, we have that (3) holds for our choice of p (and, by the same argument, for all smaller values of p). □

Proof of Theorem 2.6: Fix d and $p < p_{c,bond}^{\mathbf{Z}^d}$. We shall use a renormalization argument. For $x \in \mathbf{Z}^d$ and $n \in \{1, 2, \ldots\}$, define $\Lambda_{n,x}$ to be the cubic block of vertices of side-length n, with nx in its "lower-left" corner, i.e.,

$$\Lambda_{n,x} = nx + \{0, 1, \ldots, n-1\}^d .$$

Given n, we define a renormalized process

$$W_n = \{W_n(x)\}_{x \in \mathbf{Z}^d} \in \{0,1\}^{\mathbf{Z}^d}$$

from the DaC configuration X, and its auxiliary edge process, Y, as follows. Declare the block $\Lambda_{n,x}$ to be **good** if

(C1) no connected component of Y intersecting $\Lambda_{n,x}$ contains a vertex at distance more than $n/3$ away from $\Lambda_{n,x}$, and

(C2) $X(y) = 1$ for all $y \in \Lambda_{n,x}$,

and declare it to be **bad** otherwise. Then set, for each $x \in \mathbf{Z}^d$,

$$W_n(x) = \begin{cases} 1 & \text{if } \Lambda_{n,x} \text{ is good} \\ 0 & \text{otherwise.} \end{cases}$$

Due to condition (C2), it is clear that if W_n contains an infinite connected component of 1's, then so does X. We will now show that this happens if n is first taken to be large, and then a_1 is taken to be close to 1.

First note that, by the definition of good blocks, W_n is a 2-dependent process. Pick $\tilde{p} < 1$ close enough to 1 so that

$$p_{2,d}^*(\tilde{p}) > p_{c,site}^{\mathbf{Z}^d} , \tag{12}$$

where $p_{2,d}^*$ is defined as in Theorem 4.3.

It is a well-known result in percolation theory (see [19]) that for i.i.d. bond percolation on $\mathbf{Z}^d$ with $p < p_{c,bond}^{\mathbf{Z}^d}$, there exists a constant $c > 0$ (depending on p) such that the probability that a given vertex y is connected to some vertex at distance at least m away, is bounded by e^{-cm} for all m. Hence, we have for a given block $\Lambda_{n,x}$ that

$$\begin{aligned} \mathbf{P}(\text{condition (C1) holds for } \Lambda_{n,x}) &= 1 - \mathbf{P}(\text{condition (C1) does not hold for } \Lambda_{n,x}) \\ &\geq 1 - \mathbf{E}(\text{number of vertices } y \in \Lambda_{n,x} \text{ that have} \\ &\qquad \text{a path reaching at least } n/3 \text{ steps away}) \\ &\geq 1 - n^d e^{-cn/3} \end{aligned}$$

which tends to 1 as $n \to \infty$. We can therefore find an n large enough so that

$$\mathbf{P}(\text{condition (C1) holds for } \Lambda_{n,x}) > \frac{1+\tilde{p}}{2} . \tag{13}$$

If we now pick $a_1 \in ((\frac{1+\tilde{p}}{2})^{1/n^d}, 1)$ so that

$$a_1^{n^d} > \frac{1+\tilde{p}}{2} ,$$

then

$$\mathbf{P}(\text{condition (C2) holds for } \Lambda_{n,x}) > \frac{1+\tilde{p}}{2} \tag{14}$$

as well. By combining (13) and (14), we get that each block $\Lambda_{n,x}$ is good with probability greater than $\tilde{p}$. By Theorem 4.3 and the choice (12) of $\tilde{p}$, we have that W_n stochastically dominates some supercritical i.i.d. site percolation on $\mathbf{Z}^d$. Hence W_n contains an infinite connected compoenet of 1's, and so does X, for our choice of a_1. This proves the right hand inequality in (4).

To prove the left hand inequality, we use almost the same argument, with (12) replaced by

$$p^*_{2,d}(\tilde{p}) > 1 - p^{\mathbf{Z}^d}_{c,site} .$$

This ensures that W_n will not contain any infinite connected component of 0's, which in turn implies that X will not contain any infinite connected component of 0's, for a_1 sufficiently close to 1. But we can of course let 0's and 1's interchange roles in the DaC model (by replacing a_1 by $1 - a_1$), so this is then the same as saying that X will not contain any infinite connected component of 1's for a_1 close enough to 0. The left hand inequality in (4) is therefore established as well. □

Moving on to the tree case, our task is to prove Propositions 2.8 and 2.9.

Proof of Proposition 2.8: Consider a breadth-first search to investigate the connected component of 1's containing a given vertex v, where each time that a vertex with spin 0 is encountered, the corresponding branch of the tree is given up. When a vertex w is investigated in this search process, it has spin 1 if either

(i) the edge e leading to w has $Y(e) = 1$, or

(ii) the edge e leading to w has $Y(e) = 0$, and the (fresh new) connected component of Y containing w has spin 1.

The events in (i) and (ii) are mutually exclusive, and have respective probabilities p and $a_1(1-p)$, so that the probability of encountering spin 1 at w is $p+a_1(1-p) = 1-(1-p)a_0$. This is true also if we condition on the full search processs before encountering w. But this means that the search process has exactly the same distribution as it would have in the case of i.i.d. site percolation on Γ with retention parameter $1-(1-p)a_0$. Hence v has positive $\mu^{\Gamma}_{p,2,a_1}$-probability of being in an infinite connected component of 1's, if and only if the same event has positive probability under i.i.d. site percolation with parameter $1-(1-p)a_0$. The proposition follows. □

Remark. The proof shows that the connected component of 1's containing a given vertex is distributed as in i.i.d. site percolation. However, the full DaC process is not distributed as in i.i.d. site percolation (because, for instance, the spins at neighbouring vertices have strictly positive correlation).

Proof of Proposition 2.9: Consider first the case where the i.i.d. bond percolation process Y has probability 0 of producing an infinite connected component. Then X can be thought of as being obtained from the Y and W processes in the proof of Proposition 2.1. Any change in a finite number of the (independent) Y- and W-variables is unable to affect the outcome of the event that an infinite connected component of 1's exist. That event therefore has probability 0 or 1, by Kolmogorov's 0-1 law.

Next, consider the case that Y contains an infinite connected component with positive probability. It is well-known (see, e.g., Peres and Steif [34]) that it then has infinitely many infinite connected components, with probability 1. Of course, we then have probability 1 that at least one of these connected components gets spin 1. $\Box$

Our final task in this section will be to prove Theorem 2.12. The proof of will draw on ideas from Häggström [22]. The construction of the required counterexample G will use, as a building block, a finite graph which we denote D_k, and which is defined as follows. For $k \in \{1, 2, \ldots\}$, let D_k be a finite graph with vertex set $V_{D_k} = \{x, y, z_1, z_2, v_1, v_2, \ldots, v_k\}$ and edge set E_{D_k} consisting of all pairs containing exactly one of the vertices z_1 and z_2, and exactly one vertex in $\{x, y, v_1, v_2, \ldots, v_k\}$. In other words, D_k is a complete bipartite graph with the vertex set partitioned into $\{z_1, z_2\}$ and $\{x, y, v_1, v_2, \ldots, v_k\}$.

Let $(x \stackrel{X}{\longleftrightarrow} y)$ denote the event that x and y are in the same connected component of aligned spins in $X \in \{0,1\}^{D_k}$. Define

$$\theta_k(p) = \mu^{D_k}_{p,2,\frac{1}{2}}(x \stackrel{X}{\longleftrightarrow} y)\,.$$

Lemma 4.4 *For any k and any $p \in [0,1]$, we have*

$$\begin{aligned}\theta_k(p) \;&=\; (1-(1-p^2)^k)\left(1-\frac{(1-p)^2}{2}\right)^2 \\ &\quad +(1-p^2)^k\left(p^4+4p^3(1-p)+3p^2(1-p)^2+2p(1-p)^3+\frac{3(1-p)^4}{8}\right).\end{aligned}$$

Proof: Follows from a direct calculation, preferably by decomposing $\theta_k(p)$ as

$$\theta_k(p) = \mu^{D_k}_{p,2,\frac{1}{2}}(x \stackrel{X}{\longleftrightarrow} y) = P(A)P(x \stackrel{X}{\longleftrightarrow} y|A) + P(\neg A)P(x \stackrel{X}{\longleftrightarrow} y|\neg A)$$

where A is the event that the auxiliary configuration $Y \in \{0,1\}^{E_{D_k}}$ contains a path from z_1 to z_2 that does not go via x or y. For later purposes, we record that $P(A) = 1-(1-p^2)^k$, that $P(x \stackrel{X}{\longleftrightarrow} y|A) = (1-\frac{1}{2}(1-p)^2)^2$, and that

$$P(x \stackrel{X}{\longleftrightarrow} y|\neg A) = p^4+4p^3(1-p)+3p^2(1-p)^2+2p(1-p)^3+\frac{3(1-p)^4}{8}\,. \tag{15}$$

$\Box$

Proof of Theorem 2.12: The graph G is constructed in two steps. First we take Γ_3 to be the regular trinary tree, i.e., Γ_3 is the infinite tree in which every vertex has exactly 4 neighbours. We have $\mathrm{br}(\Gamma_3) = 3$ and, therefore, $p^{\Gamma_3}_{c,site} = \frac{1}{3}$.

Next, obtain G by replacing each edge e in Γ_3 by the graph structure D_k, with the vertices x and y at the endpoints of e. The choice of k in this construction will be determined below. Clearly, G is quasi-transitive.

Note that $\lim_{p\to 0}(1-\frac{1}{2}(1-p)^2)^2 = 0.25$. We can therefore fix a $p_2 > 0$ such that

$$P(x \stackrel{X}{\longleftrightarrow} y|A) = 1-\left(\frac{1}{2}(1-p_2)^2\right)^2 \le 0.26\,. \tag{16}$$

Note also that for $p > 0$ we have $\lim_{k\to\infty} P(A) = 1$. Hence, using (16), we can fix a k such that

$$\theta_k(p_2) \le 0.27$$

(this serves as our choice of k). We furthermore have that $\lim_{p\to 0} P(A) = 0$, and, using (15), that $\lim_{p\to 0} P(x \overset{X}{\longleftrightarrow} y | \neg A) = \frac{3}{8}$. Hence, $\lim_{p\to 0} \theta_k(p) = \frac{3}{8}$, so we can fix p_1 in such a way that $\theta_k(p_1) \ge 0.37$.

Finally, note that each pair (x, y) of vertices in G that were neighbours in Γ_3 in the first step of the construction of G, we have that x and y are in the same connected component of aligned spins in G with $\mu^G_{p,2,\frac{1}{2}}$-probability $\theta_k(p)$, and that these events are independent for all such choices of (x, y). Hence, the $\mu^G_{p,2,\frac{1}{2}}$-probability of having an infinite connected component of aligned spins is 1 if $\theta_k(p) > \frac{1}{3}$, and 0 otherwise. The theorem now follows with the given choices of G, p_1 and p_2. □

Remark. By combining the above ideas with those in [22], it is possible to show that the existence of an infinite cluster of aligned spins in the Ising model with inverse temperature β, fails to be increasing in β in the generality of quasi-transitive graphs.

5 Proofs of Markov and quasilocality resuts

A major part in our analysis of Markov and quasilocality properties of the DaC model will be played by the random-cluster model; see Definitions 5.1 and 5.3 below. The key relation between the DaC model and the random-cluster model is provided in Lemmas 5.2 and 5.4. Readers familiar with the random-cluster analysis of Ising and Potts models (see, e.g., [2] and [17]) that has played such a prominent role since the late 1980's, will notice certain similarities between those methods and ours. The following two differences are, however, worth noting:

1. Whereas random-cluster analysis of Ising and Potts models uses the random-cluster model with cluster parameter $q > 1$ (the FKG regime of the parameter space), our analysis uses the $q < 1$ (non-FKG regime) random-cluster model. To our knowledge, this is the first time that the $q < 1$ random-cluster model (other than the uniform spanning tree limit as $q \to 0$; see [20] and [3]) arises naturally in an application.

2. The vast majority of random-cluster studies of Ising and Potts models are confined to the zero external field case, corresponding to the symmetric DaC model with $(a_0, \ldots, a_{r-1}) = (\frac{1}{r}, \ldots, \frac{1}{r})$. This is because the random-cluster representation becomes much messier, and therefore more difficult to work with, in the absence of symmetry between the q different spins (although see [8] and [6] for some important recent steps towards overcoming these difficulties). In contrast, our analysis of the DaC model works just as easily in the nonsymmetric case as in the symmetric.

Definition 5.1 *Fix $p \in [0, 1]$, $q > 0$ and a finite graph $G = (V, E)$. The* **random-cluster measure** $\phi^G_{p,q}$ *is defined as the probability measure on $\{0, 1\}^E$ which to each $\eta \in \{0, 1\}^E$ assigns probability*

$$\phi^G_{p,q}(\eta) = \frac{1}{Z^G_{p,q}} q^{k(\eta)} \prod_{e\in E} p^{\eta(e)}(1-p)^{1-\eta(e)}$$

where $k(\eta)$ is the number of connected components in the random subgraph of G corresponding to η, and $Z^G_{p,q}$ is a normalizing constant.

Note that $q = 1$ gives ordinary i.i.d. bond percolation, whereas other choices of q result in dependencies between edges. The basic connection between the DaC model and the random-cluster model is the following.

Lemma 5.2 *Fix $p \in [0,1]$, $r \in \{1, 2, \ldots\}$, $a_0, \ldots, a_{r-1} \in (0,1)$ satisfying $\sum_{i=0}^{r-1} a_i = 1$, and a finite graph $G = (V, E)$. Suppose that we pick $(X, Y) \in \{0, \ldots, r-1\}^V \times \{0,1\}^E$ as in the two-step procedure in Section 1. The conditional distribution of Y given X is then given as follows:*

(I) *All edges $\langle u, v\rangle$ with $X(u) \neq X(v)$ take value* 0.

(II) *The edge configuration on a spin component $\mathcal{D}$ of X (i.e., on the set of edges $\langle u, v\rangle$ with $u, v \in \mathcal{D}$, and $\mathcal{D}$ is a maximal connected component of vertices in G that take the same spin value in X) is conditionally independent of the edge configuration on all other spin components.*

(III) *If the vertices on a spin component $\mathcal{D}$ take value $i \in \{0, \ldots, r-1\}$, then the conditional distribution of the edge configuration on $\mathcal{D}$ is given by the random-cluster measure $\phi^{\mathcal{D}}_{p,a_i}$.*

Proof: (I) is immediate from the construction, so we go on to prove (II) and (III). The (unconditional) joint distribution of (X, Y) assigns probability

$$\mathbf{P}(\xi, \eta) = \prod_{e \in E} p^{\eta(e)}(1-p)^{1-\eta(e)} \prod_{\mathcal{C} \in \{\mathcal{C}_1, \ldots, \mathcal{C}_l\}} a_{\xi(\mathcal{C})} \tag{17}$$

to each $(\xi, \eta) \in \{0, \ldots, r-1\}^V \times \{0,1\}^E$ such that (I) holds; here the second product ranges over the set $\{\mathcal{C}_1, \ldots, \mathcal{C}_l\}$ of connected components of the edge configuration η, and $\xi(\mathcal{C})$ is the common spin value in ξ of the vertices in $\mathcal{C}$. Now let $\mathcal{D}_1, \ldots \mathcal{D}_m$ denote the spin components in ξ, and let $E(\mathcal{D}_1), \ldots, E(\mathcal{D}_m)$ be the corresponding edge sets (defined as in (II)). Note that the factors in (17) can be reorganized as

$$\mathbf{P}(\xi, \eta) = \prod_{\mathcal{D} \in \{\mathcal{D}_1, \ldots \mathcal{D}_m\}} \left[a^{k(\mathcal{D})}_{\xi(\mathcal{D})} \prod_{e \in E(\mathcal{D})} p^{\eta(e)}(1-p)^{1-\eta(e)} \right]$$

where $k(\mathcal{D})$ is the number of connected components $\mathcal{C} \in \{\mathcal{C}_1, \ldots, \mathcal{C}_l\}$ of η that are contained in $\mathcal{D}$. Conditioning on the event $\{X = \xi\}$ gives

$$\mathbf{P}(Y = \eta \,|\, X = \xi) = \frac{1}{\mathbf{P}(X = \xi)} \prod_{\mathcal{D} \in \{\mathcal{D}_1, \ldots \mathcal{D}_m\}} \left[a^{k(\mathcal{D})}_{\xi(\mathcal{C})} \prod_{e \in E(\mathcal{D})} p^{\eta(e)}(1-p)^{1-\eta(e)} \right]$$

and parts (II) and (III) of the lemma follow. □

We now proceed to extend Definition 5.1 and Lemma 5.2 to the case of infinite graphs. The following is a single-edge version of the usual DLR definition of random-cluster measures for infinite graphs, introduced by Grimmett [18]. To see that it is equivalent to the usual definition, consult, e.g., [17, Lemma 6.18].

Definition 5.3 *Fix $p \in [0,1]$, $q > 0$, and a (possibly infinite) graph $G = (V,E)$. A probability measure ϕ on $\{0,1\}^E$ is said to be a* **random-cluster measure for G with parameters p and q** *if it admits conditional probabilities such that for any $e = \langle u,v\rangle \in E$ and any $\eta \in \{0,1\}^{E\setminus\{e\}}$ we have*

$$\phi(Y(e) = 1 \,|\, Y(E \setminus \{e\}) = \eta) = \begin{cases} p & \text{if } u \stackrel{\eta}{\longleftrightarrow} v \\ \frac{p}{p+(1-p)q} & \text{otherwise,} \end{cases} \tag{18}$$

where $u \stackrel{\eta}{\longleftrightarrow} v$ is the event that η contains an open path from u to v.

It is easy to see (and a standard fact) that this is consistent with Definition 5.1 in the case where G is finite.

Lemma 5.4 *Fix p, r, and $a_0, \ldots, a_{r-1}$ as in Lemma 5.2, and let G be a (possibly infinite) graph. Suppose that we pick $(X,Y) \in \{0,\ldots,r-1\}^V \times \{0,1\}^E$ as in the two-step procedure in Section 1. (A version of) the conditional distribution of Y given X is then given as in Lemma 5.2, with* (III) *replaced by*

(III') *If the vertices on a spin component $\mathcal{D}$ take value $i \in \{0,\ldots,r-1\}$, then the conditional distribution of the edge configuration on $\mathcal{D}$ is given by some random-cluster measure for $\mathcal{D}$ with parameters p and $q = a_i$.*

For the proof of this result (and others), it is useful to have a construction of the DaC model on an infinite graph as a limit of the DaC model on a sequence of finite graphs. Let $G = (V,E)$ be infinite, and let $(v_1, v_2, \ldots)$ be an arbitrary enumeration of V. For $n = 1, 2, \ldots$, define the vertex set

$$V_n = \{v_1, \ldots, v_n\},$$

the edge set

$$E_n = \{e = \langle x,y\rangle \in E : \, u,v \in V_n\},$$

and the graph

$$G_n = (V_n, E_n).$$

Fix $p \in [0,1]$, $r \in \{2,3,\ldots,\}$ and $(a_0,\ldots,a_{r-1})$ such that $\sum_{i=0}^{r-1} a_i = 1$. Let $\{Z(v)\}_{v\in V}$ be i.i.d. $\{0,\ldots,r-1\}$-valued random variables with distribution $(a_0,\ldots,a_{r-1})$. Independently of these, let $\{Y(e)\}_{e\in E}$ be i.i.d. $\{0,1\}$-valued random variables with distribution $(1-p,p)$. For each $n \in \{1,2,\ldots\}$, we define the $\{0,\ldots,r-1\}^{V_n} \times \{0,1\}^{E_n}$-valued random object (X_n, Y_n) by setting $Y_n(e) = Y(e)$ for each $e \in E_n$, and

$$X_n(v) = Z(v_i) \quad \text{where } i = \min\{k : \, v \stackrel{Y_n}{\longleftrightarrow} v_k\} \tag{19}$$

for each $v \in V_n$. Finally, define, for each $v \in V$,

$$X(v) = Z(v_i) \quad \text{where } i = \min\{k : \, v \stackrel{Y}{\longleftrightarrow} v_k\}.$$

Clearly, for each n, the pair (X_n, Y_n) is distributed according to the DaC model on G_n (with parameters p, r and $(a_0,\ldots,a_{r-1})$) together with its auxiliary edge configuration. The same is true for the pair (X,Y) with respect to the DaC model on G.

Note that

$$Y(e) = \lim_{n\to\infty} Y_n(e) \tag{20}$$

for each $e \in E$ (trivially), and that

$$X(v) = \lim_{n\to\infty} X_n(v) \tag{21}$$

for each $v \in V$ (this is because the expression $\min\{k : v \overset{Y_n}{\longleftrightarrow} v_k\}$ in (19) is decreasing in n, and can therefore change value only finitely many times as $n \to \infty$). Hence, we have obtained the DaC model, together with its auxiliary edge configuration, on an infinite graph G, as a pointwise limit of the corresponding objects on finite subgraphs.

Proof of Lemma 5.4: As in Lemma 5.2, (I) is immediate from the construction, so we go on to prove (II) and (III'). For this, it suffices to show that the underlying probability measure $\mathbf{P}$ admits conditional probabilities such that, for every $e = \langle u, v\rangle \in E$, every $i \in \{0, \dots, r-1\}$, every $\xi \in \{0, \dots, r-1\}^V$ such that $X(u) = X(v) = i$, and every $\eta \in \{0,1\}^E$, we have

$$\mathbf{P}\Big(Y(e) = 1 \,\Big|\, X = \xi, Y(E \setminus \{e\}) = \eta\Big) = \begin{cases} p & \text{if } u \overset{\eta}{\longleftrightarrow} v \\ \frac{p}{p+(1-p)a_i} & \text{otherwise.} \end{cases} \tag{22}$$

For the case where G is finite, (22) is immediate from Lemma 5.2. To go from the finite case to the infinite, we just appeal to the pointwise limiting construction in (20) and (21), upon noting that

- if there is an open path from u to v in $Y(E \setminus \{e\})$, then the same is true for $Y_n(E \setminus \{e\})$ for sufficiently large n,

and, conversely, that

- if there is no open path from u to v in $Y(E \setminus \{e\})$, then the same is true for $Y_n(E \setminus \{e\})$ for sufficiently large n (in fact, for all n).

□

The next lemma looks a bit specialized, but is useful for the proofs of Theorems 3.2 and 3.4.

Lemma 5.5 *Consider the DaC model with parameters $p \in (0,1)$, $r \in \{2,3,\dots\}$ and $(a_0, \dots, a_{r-1})$ with $a_0, a_1 \in (0,1)$ on a (possibly infinite) graph $G = (V, E)$ with a distinguished vertex $u \in V$ which is the endpoint of exactly four edges $e_1 = \langle u, v_1\rangle$, $e_2 = \langle u, v_2\rangle$, $e_3 = \langle u, v_3\rangle$ and $e_4 = \langle u, v_4\rangle$. Suppose that $(X, Y) \in \{0, \dots, r-1\}^V \times \{0,1\}^E$ is picked as in the two-step procedure in Section 1. Let $\xi \in \{0, \dots, r-1\}^{V\setminus\{u\}}$ be a spin configuration with the properties that*

- $\xi(v_1) = \xi(v_2) = 0$,
- $\xi(v_3) = \xi(v_4) = 1$, *and*
- *G contains no path from v_1 to v_2 which does not go through u and not through any vertex w with $\xi(w) \neq 0$.*

Furthermore, let $\eta \in \{0,1\}^{E\setminus\{e_1,e_2,e_3,e_4\}}$ be an edge configuration which is consistent with ξ. Let A be the event that η contains an open path from v_3 to v_4. We then have, *on the event A, that*

$$\frac{\mathbf{P}\Big(X(u)=0\,\Big|\,X(V\setminus\{u\})=\xi, Y(E\setminus\{e_1,e_2,e_3,e_4\})=\eta\Big)}{\mathbf{P}\Big(X(u)=1\,\Big|\,X(V\setminus\{u\})=\xi, Y(E\setminus\{e_1,e_2,e_3,e_4\})=\eta\Big)} = \frac{(1-p)^2a_0^2a_1+2p(1-p)a_0a_1+p^2a_1}{(1-p)^2a_0a_1^2+2p(1-p)a_0a_1+p^2a_0a_1}\,. \tag{23}$$

On $\neg A$, we instead have

$$\frac{\mathbf{P}\Big(X(u)=0\,\Big|\,X(V\setminus\{u\})=\xi, Y(E\setminus\{e_1,e_2,e_3,e_4\})=\eta\Big)}{\mathbf{P}\Big(X(u)=1\,\Big|\,X(V\setminus\{u\})=\xi, Y(E\setminus\{e_1,e_2,e_3,e_4\})=\eta\Big)} = \frac{(1-p)^2a_0^2a_1+2p(1-p)a_0a_1+p^2a_1}{(1-p)^2a_0a_1^2+2p(1-p)a_0a_1+p^2a_0}\,. \tag{24}$$

Note in particular that the right hand side in (23) *is strictly greater than the right hand side in* (24).

Proof: It suffices to prove the lemma when G is a finite graph, because the infinite case then follows from a similar appeal to the pointwise limit in (20) and (21) as in the proof of Lemma 5.4. Note that

$$\begin{aligned}&\mathbf{P}\Big(X(u)=0\,\Big|\,X(V\setminus\{u\})=\xi, Y(E\setminus\{e_1,\ldots,e_4\})=\eta\Big)\\ &= \sum_{\substack{b\in\{0000,0100,\\1000,1100\}}} \mathbf{P}\Big(X(u)=0, Y(e_1,\ldots,e_4)=b\,\Big|\,X(V\setminus\{u\})=\xi, Y(E\setminus\{e_1,\ldots,e_4\})=\eta\Big)\end{aligned}$$

and that

$$\begin{aligned}&\mathbf{P}\Big(X(u)=1\,\Big|\,X(V\setminus\{u\})=\xi, Y(E\setminus\{e_1,\ldots,e_4\})=\eta\Big)\\ &= \sum_{\substack{b\in\{0000,0001,\\0010,0011\}}} \mathbf{P}\Big(X(u)=0, Y(e_1,\ldots,e_4)=b\,\Big|\,X(V\setminus\{u\})=\xi, Y(E\setminus\{e_1,\ldots,e_4\})=\eta\Big)\,.\end{aligned}$$

Using these decompositions, we get, on the event A, that

$$\begin{aligned}&\frac{\mathbf{P}\Big(X(u)=0\,\Big|\,X(V\setminus\{u\})=\xi, Y(E\setminus\{e_1,e_2,e_3,e_4\})=\eta\Big)}{\mathbf{P}\Big(X(u)=1\,\Big|\,X(V\setminus\{u\})=\xi, Y(E\setminus\{e_1,e_2,e_3,e_4\})=\eta\Big)}\\ &= \frac{(1-p)^4a_0^3a_1+p(1-p)^3a_0^2a_1+p(1-p)^3a_0^2a_1+p^2(1-p)^2a_0a_1}{(1-p)^4a_0^2a_1^2+p(1-p)^3a_0^2a_1+p(1-p)^3a_0^2a_1+p^2(1-p)^2a_0^2a_1}\end{aligned}$$

which simplifies into (23). We similarly obtain (24) on $\neg A$. □

The proofs of Theorems 3.2 and 3.4 also need the following simple lemma, which establishes a strong form of the so-called finite energy condition (Newman and Schulman [33]) for the DaC model on $\mathbf{Z}^d$. Another term which is sometimes (e.g., in [11]) used for the property proved in the lemma, is "uniformly nonnull".

Lemma 5.6 *Consider the DaC model on $\mathbf{Z}^d$ with parameters $p\in(0,1)$, $r\in\{2,3,\ldots\}$ and $(a_0,\ldots,a_{r-1})$ with $a_i>0$ for each i. There exists an $\varepsilon>0$ (depending on p,*

r and $(a_0, \ldots, a_{r-1})$) such that the following holds. The DaC measure $\mu^{\mathbf{Z}^d}_{p,q,(a_0,\ldots,a_{r-1})}$ admits conditional probabilities such that for any $x \in \mathbf{Z}^d$, any $i \in \{0, \ldots, r-1\}$ and any $\xi \in \{0, \ldots, r-1\}^{\mathbf{Z}^d \setminus \{x\}}$ we have

$$\mu^{\mathbf{Z}^d}_{p,q,(a_0,\ldots,a_{r-1})}(X(x) = i \,|\, X(\mathbf{Z}^d \setminus \{x\} = \xi) \geq \varepsilon \,.$$

Proof: Suppose that (X, Y) is obtained as in the two-step procedure in Section 1. By Lemma 5.4 and (18), we have for any edge $e \in E_d$, any $\zeta \in \{0, \ldots, r-1\}^{\mathbf{Z}^d}$ and any $\eta \in \{0,1\}^{E_d \setminus \{e\}}$ that

$$\mathbf{P}(Y(e) = 1 \,|\, X = \zeta, Y(E_d \setminus \{e\}) = \eta) \leq \frac{p}{p + (1-p) \min_{i \in \{0,\ldots,r-1\}} a_i} \,,$$

where we note that the right hand side is stricly less than 1. Hence, letting $e_1, \ldots, e_{2d}$ denote the $2d$ edges incident to x, we get, by averaging over all possible values of $X(x)$ and $Y(E_d \setminus \{e_1, \ldots, e_{2d}\})$, that

$$\mathbf{P}(Y(e_1) = \ldots = Y(e_{2d}) = 0 \,|\, X(\mathbf{Z}^d \setminus \{x\}) = \xi) \geq \left(1 - \frac{p}{p + (1-p) \min_{i \in \{0,\ldots,r-1\}} a_i}\right)^{2d} .$$

By the construction of (X, Y), we also have

$$\mathbf{P}(X(x) = i \,|\, X(\mathbf{Z}^d \setminus \{x\} = \xi, Y(e_1) = \cdots = Y(e_{2d}) = 0) = a_i \,.$$

Hence, the lemma holds with

$$\varepsilon = \min_{i \in \{0,\ldots,r-1\}} a_i \left(1 - \frac{p}{p + (1-p) \min_{i \in \{0,\ldots,r-1\}} a_i}\right)^{2d} .$$

□

Proof of Theorem 3.2: We restrict to the case of $\mathbf{Z}^2$, as the generalization to higher dimensions is straightforward (and requires an equally straightforward generalization of Lemma 5.5).

Let (X, Y) be as in the two-step procedure in Section 1. Write $\mathbf{0}$ for the origin $(0,0)$ in $\mathbf{Z}^2$. We shall consider a configuration $\xi \in \{0, 1, \ldots, r-1\}^{\mathbf{Z}^2 \setminus \{\mathbf{0}\}}$ which will serve as a "point of discontinuity" (here and in the proof of Theorem 3.4 (ii)) for the conditional distribution of $X(\mathbf{0})$ given $X(\mathbf{Z}^2 \setminus \{\mathbf{0}\})$. We define ξ by letting

$$\xi(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = 0 \text{ and } |x_2| = 1 \\ & \text{or if } x_1 = -1 \text{ and } |x_2| \geq 2 \\ 1 & \text{otherwise,} \end{cases}$$

see Figure 1.

Fix an arbitrary n, and let Λ_n denote the box $\{-n, \ldots, n\}^2 \subset \mathbf{Z}^2$. Consider the two configurations $\zeta_n^0, \zeta_n^1 \in \{0, \ldots, r-1\}^{\Lambda_{2n} \setminus \{\mathbf{0}\}}$ defined by

$$\zeta_n^0(x) = \begin{cases} \xi(x) & \text{for } x \in \Lambda_n \setminus \{\mathbf{0}\} \\ 0 & \text{for } x \in \Lambda_{2n} \setminus \Lambda_n \end{cases}$$

and

$$\zeta_n^1(x) = \begin{cases} \xi(x) & \text{for } x \in \Lambda_n \setminus \{\mathbf{0}\} \\ 1 & \text{for } x \in \Lambda_{2n} \setminus \Lambda_n \,. \end{cases}$$

```
1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 ? 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1
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Figure 1: The configuration ξ restricted to the box Λ_8. The "?" in the middle, is the origin $\mathbf{0}$. A key property of ξ is that, for any given n, it is not enough to know the restriction of ξ to Λ_n in order to figure out whether ξ contains a path of 1's connecting the vertex $(-1,0)$ to the vertex $(1,0)$.

Let A be the event that the auxiliary edge configuration Y contains a path from $(-1,0)$ to $(1,0)$ not going through $\mathbf{0}$. Let π_A and $\pi_{\neg A}$ denote the right hand sides of (23) and (24), respectively, and recall that $\pi_A > \pi_{\neg A}$.

Note that by Lemma 5.6, the events $\{X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^0\}$ and $\{X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^1\}$ both have positive probability. Hence, to show that X is not an n-Markov random field, it is enough to show that

$$\mathbf{P}(X(\mathbf{0}) = i \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^0) \neq \mathbf{P}(X(\mathbf{0}) = i \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^1)$$

for some $i \in \{0, \ldots, r-1\}$. We may assume that

$$\mathbf{P}(X(\mathbf{0}) = 1 \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^0) = \mathbf{P}(X(\mathbf{0}) = 1 \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^1) \qquad (25)$$

because otherwise we are done. Write γ for the left (or right) hand side in (25). By Lemma 5.5, we have

$$\mathbf{P}(X(\mathbf{0}) = 0 \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^0) = \gamma \pi_{\neg A} \qquad (26)$$

because the event $X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^0$ precludes the event A. Lemma 5.5 also gives

$$\begin{aligned} &\mathbf{P}(X(\mathbf{0}) = 0 \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^1) \\ &= \gamma(\pi_A \mathbf{P}(A \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^1) + \pi_{\neg A} \mathbf{P}(\neg A \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^1)) \\ &= \gamma(\pi_{\neg A} + (\pi_A - \pi_{\neg A}) \mathbf{P}(A \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^1)) \end{aligned}$$

which is strictly greater than the right hand side of (26), because $\mathbf{P}(A \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^1) > 0$ due to Lemma 5.4. Hence,

$$\mathbf{P}(X(\mathbf{0}) = 0 \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^0) < \mathbf{P}(X(\mathbf{0}) = 0 \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^1)$$

as desired. $\square$

Our next task is to prove Theorem 3.4. Part (ii) of that theorem will be proved using a refined version of the above proof of Theorem 3.2. The following lemma will be needed.

Lemma 5.7 *Consider i.i.d. bond percolation on* $\mathbf{Z}^2$ *with parameter* $p' > p_{c,bond}^{\mathbf{Z}^2} (= \frac{1}{2})$. *Write* Y *for the corresponding random bond configuration, and* $P_{p'}$ *for its distribution on* $\{0,1\}^{E_2}$. *For* $n \in \{1, 2, \ldots\}$, *define the following events:*

$$\begin{aligned}
B_1 &= \{Y(\langle(-2,0),(-1,0)\rangle) = 1\} \\
B_2^{(n)} &= \{Y \textit{ has an open path, contained in the half-plane } (-\infty,-2] \times \mathbf{R}, \\
&\qquad \textit{from } (-2,0) \textit{ to some vertex in } \Lambda_{2n} \setminus \Lambda_n\} \\
B_3^{(n)} &= \{Y \textit{ has an open path, contained in the half-plane } [1,\infty) \times \mathbf{R}, \\
&\qquad \textit{from } (1,0) \textit{ to some vertex in } \Lambda_{2n} \setminus \Lambda_n\} \\
B_4^{(n)} &= \{Y \textit{ has an open circuit that is contained in } \Lambda_{2n} \setminus \Lambda_n \\
&\qquad \textit{and that ``surrounds'' } \Lambda_n\} \\
B^{(n)} &= B_1 \cap B_2^{(n)} \cap B_3^{(n)} \cap B_4^{(n)}
\end{aligned}$$

Then $\inf_n P_{p'}(B^{(n)}) > 0$.

Proof: Follows by combining a number of standard facts from percolation theory: First, define the additional events

$$\begin{aligned}
B_2 &= \{Y \text{ has an infinite open path starting at } (-2,0) \\
&\qquad \text{contained in the half-plane } (-\infty,-2] \times \mathbf{R}\} \\
B_3 &= \{Y \text{ has an infinite open path starting at } (1,0) \\
&\qquad \text{contained in the half-plane } [1,\infty) \times \mathbf{R}\}
\end{aligned}$$

and note that

$$\mathbf{P}(B_2) = \mathbf{P}(B_3) > 0$$

by the fact that supercritical percolation in $\mathbf{Z}^d$ also creates an infinite cluster in half-space (see [19]). But B_2 implies $B_2^{(n)}$, and B_3 implies $B_3^{(n)}$. Hence,

$$\mathbf{P}(B_2^{(n)}) > 0 \quad \text{and} \quad \mathbf{P}(B_3^{(n)}) > 0\,.$$

Next, note that $\inf_n P_{p'}(B_4^{(n)}) > 0$ by the Russo–Seymour–Welsh Theorem (see [19] again). Finally, Harris' inequality gives

$$P_{p'}(B^{(n)}) \geq \mathbf{P}(B_1)\mathbf{P}(B_2^{(n)})\mathbf{P}(B_3^{(n)})\mathbf{P}(B_4^{(n)})$$

so that

$$\inf_n P_{p'}(B^{(n)}) \geq p'\mathbf{P}(B_2^{(n)})\mathbf{P}(B_3^{(n)}) \inf_n \mathbf{P}(B_4^{(n)}) > 0\,.$$

$\Box$

Proof of Theorem 3.4 (ii): Again, we give the proof for the $\mathbf{Z}^2$ case only, omitting the straightforward generalization to higher dimensions.

Let the configurations ξ, ζ_n^0 and ζ_n^1, and the event A, be as in the proof of Theorem 3.2. The required non-quasilocality is established if we can show that

$$\max_{i \in \{0,\ldots,r-1\}} \left| \mathbf{P}(X(\mathbf{0}) = i \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^0) - \mathbf{P}(X(\mathbf{0}) = i \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^1) \right|$$

is bounded away from 0 as $n \to \infty$. By Lemma 5.6, this follows if we can show that

$$\frac{\mathbf{P}(X(\mathbf{0}) = 0 \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^0)}{\mathbf{P}(X(\mathbf{0}) = 1 \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^0)} - \frac{\mathbf{P}(X(\mathbf{0}) = 0 \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^1)}{\mathbf{P}(X(\mathbf{0}) = 1 \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^1)} \tag{27}$$

is bounded away from 0 as $n \to \infty$. What we did in the proof of Theorem 3.2 was to show that the expression in (27) is strictly positive (for any n), using the observation that $\mathbf{P}(A \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^1) > 0$ for any n. By similar reasoning, it is easy to see that if $\mathbf{P}(A \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\})$ is bounded away from 0 as $n \to \infty$, then so is the expression in (27). Our task is therefore reduced to showing that

$$\inf_n \mathbf{P}(A \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^1) > 0\,. \tag{28}$$

To this end, we shall use Lemma 5.7. The event $B^{(n)}$ in the lemma was chosen carefully so as to only depend on edges whose two endvertices both take value 1 in ζ_n^1. Let us denote this edge set by $E_{\zeta_n^1}$. Note also that $B^{(n)}$ implies the existence of a path of open edges from $(-2, 0)$ to $(1, 0)$ not going through $\mathbf{0}$, i.e., it implies A. By Lemma 5.4 and (18), we have that the conditional distribution of $Y(E_{\zeta_n^1})$ stochastically dominates i.i.d. percolation on $E_{\zeta_n^1}$ with retention parameter p. Combining these observations, we get

$$\mathbf{P}(A \,|\, X(\Lambda_{2n} \setminus \{\mathbf{0}\}) = \zeta_n^1) \geq P_p(B^{(n)})\,.$$

Thus, we may use Lemma 5.7 to deduce that (28) holds whenever $p > p_{c,bond}^{\mathbf{Z}^2} = \frac{1}{2}$. □

For the proof of the remaining part (i) of Theorem 3.4, the following lemma, in which D plays the role of a "cutset", is useful.

Lemma 5.8 *Let $G = (V, E)$ be a (possibly infinite) graph, and let D be a finite subset of E. Define*

$$V_D = \{v \in V : \textit{ any infinite path in } G \textit{ starting at } v \textit{ contains at least one edge in } D\}\,,$$

$$E_D = \{e = \langle x, y\rangle \in E : \, u, v \in V_D\}\,,$$

and

$$G_D = (V_D, E_D)\,.$$

Fix, as usual, the DaC model parameters p, r, and $(a_0, \ldots, a_{r-1})$, and construct $(X, Y) \in \{0, \ldots, r-1\}^V \times \{0, 1\}^E$ as in the two-step procedure in Section 1. Let C denote the event that $Y(e) = 0$ for all $e \in D$. Conditional on the event C and on any additional information about $X(V \setminus V_D)$ and $Y(E \setminus E_D)$, we have that $X(V_D)$ has distribution

$$\mu^{G_D}_{p,r,(a_0,\ldots,a_{r-1})}\,.$$

Proof: Immediate from the two-step construction in Section 1. □

As a warmup for the proof of Theorem 3.4 (ii), let us first consider an easier application of Lemma 5.8, namely, Proposition 3.7.

Proof of Proposition 3.7: Fix a finite set $W \subset \mathbf{Z}^d$ and a configuration $\zeta \in \{0, \ldots, r-1\}^{\mathbf{Z}^d \setminus W}$ with the property that

$$\zeta \text{ contains no infinite connected component of aligned spins.} \tag{29}$$

Let $W' \subset \mathbf{Z}^d \setminus W$ be the union of all the spin-components in ζ intersecting ∂W. Due to (29), we have that W' is finite. Furthermore, given ζ, the auxiliary edge configuration Y is forced to take value $Y(e) = 0$ for all edges $e = \langle x, y \rangle$ with $x \in W'$ and $y \in \mathbf{Z}^d \setminus (W \cup W')$. Fix n large enough so that $W' \subseteq \partial_{n-1} W$, and let $\zeta' \in \{0, \ldots, r-1\}^{\mathbf{Z}^d \setminus W}$ be any configuration satisfying $\zeta'(\partial_n W) = \zeta(\partial_n W)$. Let G^* denote the graph with vertex set $V(G^*) = W \cup W'$ and edge set

$$E(G^*) = \{e = \langle x, y \rangle \in E : x, y \in W \cup W'\}.$$

Lemma 5.8 implies that the conditional distribution of $X(W \cup W')$ given that $X(\mathbf{Z}^d \setminus W) = \zeta$, is simply the DaC measure $\mu^{G^*}_{p,r,(a_0,\ldots,a_{r-1})}$ conditioned on taking values $\zeta(W')$ on W'. The exact same argument gives that the conditional distribution of $X(W \cup W')$ given that $X(\mathbf{Z}^d \setminus W) = \zeta'$, is the DaC measure $\mu^{G^*}_{p,r,(a_0,\ldots,a_{r-1})}$ conditioned on taking values $\zeta'(W')$ on W'. But since $\zeta'(W') = \zeta(W')$ and the set of configurations in (29) has full $\mu^{\mathbf{Z}^d}_{p,r,(a_0,\ldots,a_{r-1})}$-measure (due to the assumption (10)), the desired almost sure quasilocality follows. □

We are finally ready to complete the proof of Theorem 3.4.

Proof of Theorem 3.4: Take $p < p_1$, where p_1 is as in (8), so that

$$\frac{2p}{p + (1-p)\min_i a_i} < \frac{1}{2d-1} \le p^{\mathbf{Z}^d}_{c,bond} \tag{30}$$

where the second inequality is a standard result in percolation theory (see, e.g., [19]). We shall show that $\mu^{\mathbf{Z}^d}_{p,r,(a_0,a_{r-1})}$ is quasilocal using Lemma 5.8 backed up by a coupling trick which is similar to the one that was introduced by van den Berg [5] and that is known as disagreement percolation.

We need to show that for any finite $W \subset \mathbf{Z}^d$, any $\xi \in \{0, \ldots, r-1\}^W$ and any $\varepsilon > 0$, there exists an $n < \infty$ such that

$$\sup_{\substack{\zeta,\zeta' \in \{0,1,\ldots,r-1\}^{\mathbf{Z}^d \setminus W} \\ \zeta'(\partial_n W) = \zeta(\partial_n W)}} \Big| \mu^{\mathbf{Z}^d}_{p,r,(a_0,\ldots,a_{r-1})}(X(W) = \xi \,|\, X(\mathbf{Z}^d \setminus W) = \zeta) - \mu^{\mathbf{Z}^d}_{p,r,(a_0,\ldots,a_{r-1})}(X(W) = \xi \,|\, X(\mathbf{Z}^d \setminus W) = \zeta') \Big| \le \varepsilon. \tag{31}$$

Because of (30), we can find an $n < \infty$ with the property that if we perform i.i.d. bond percolation on $\mathbf{Z}^d$ with retention parameter $\frac{2p}{p+(1-p)\min_i a_i}$, then the probability that there exist $x \in W$ and $y \in \mathbf{Z}^d \setminus \partial_n W$ such that x and y are connected by an open path, is at most ε. Fix such an n, and two arbitrary configurations $\zeta, \zeta' \in \{0, 1, \ldots, r-1\}^{\mathbf{Z}^d \setminus W}$ satisfying $\zeta'(\partial_n W) = \zeta(\partial_n W)$.

We shall now construct two pairs $(X, Y), (X', Y') \in \{0, 1, \ldots, r-1\}^{\mathbf{Z}^d} \times \{0, 1\}^{\mathbf{E}_d}$ distributed according to conditional distributions for the DaC model and its auxiliary bond percolation, given $X(\mathbf{Z}^d \setminus W) = \zeta$ and $X(\mathbf{Z}^d \setminus W) = \zeta'$, respectively, satisfying Lemmas 5.4 and 5.8. We take (X, Y) and (X', Y') to be independent of each other. It follows from Lemma 5.4 and (18) that $\{Y(e)\}_{e \in E}$ is stochastically dominated by i.i.d. bond percolation with parameter $\frac{p}{p+(1-p)\min_i a_i}$. Of course, the same thing also holds with Y' in place of Y. Now define $Y'' \in \{0,1\}^{E_d}$ by setting

$$Y''(e) = \begin{cases} 0 & \text{if } Y(e) = Y'(e) = 0 \\ 1 & \text{otherwise.} \end{cases}$$

By the assumed independence between Y and Y', we have that $\{Y''(e)\}_{e\in E_d}$ is stochastically dominated by i.i.d. bond percolation with parameter $\frac{2p}{p+(1-p)\min_i a_i}$. By the choice of n, we therefore have

$$\mathbf{P}(\exists x \in \mathbf{Z}^d \backslash \partial_n W \text{ and } y \in W \text{ such that } x \text{ and } y \text{ are connected by an open path in } Y'') \leq \varepsilon\,. \tag{32}$$

Define the random edge set D as

$$\begin{aligned} D &= \{e \in E : \text{exactly one of the endvertices of } e \text{ is connected to} \\ &\qquad \text{some } x \in \mathbf{Z}^d \setminus \partial_n W \text{ by an open path in } Y''\} \end{aligned}$$

and note that $Y(e) = Y(e') = 0$ for all $e \in D$. Define V_D, E_D and G_D as in Lemma 5.8. We have

$$V_D \subseteq \partial_n W \tag{33}$$

by the definition of D. By (32), we also have

$$\mathbf{P}(W \subseteq V_D) \geq 1 - \varepsilon\,. \tag{34}$$

A crucial observation is now that D is defined in such a way that D is not affected if we alter the status of any edges in E_D. The value of the set-valued random object D thereore gives us no other information than the one allowed in Lemma 5.8. Therefore, Lemma 5.8 applies to show that the conditional distribution of $X(V_D)$ given D equals $\mu^{G_D}_{p,r,(a_0,\ldots,a_{r-1})}$ conditioned on taking values $\zeta(V_D \setminus W)$ on $V_D \setminus W$. The same holds, by the same argument, with $X'(V_D)$ in place of $X(V_D)$, because $\zeta'(V_D \setminus W) = \zeta(V_D \setminus W)$ due to (33). Thus, (34) implies

$$\begin{aligned} \left|\mu^{\mathbf{Z}^d}_{p,r,(a_0,\ldots,a_{r-1})}(X(W) = \xi \,|\, X(\mathbf{Z}^d \setminus W) = \zeta) - \mu^{\mathbf{Z}^d}_{p,r,(a_0,\ldots,a_{r-1})}(X'(W) = \xi \,|\, X'(\mathbf{Z}^d \setminus W) = \zeta')\right| \\ \leq 1 - \mathbf{P}(W \subseteq V_D) \\ \leq \varepsilon\,. \end{aligned}$$

Since ζ and ζ' were arbitrary, we have (31), so the proof is complete. □

Acknowledgement. I am grateful to Oded Schramm for the permission to include Theorem 4.2 and its proof. This project was triggered by a question that Jeff Steif asked me (and which I partially answer in Theorems 3.2 and 3.4) several years ago in the context of motivation (M4) in the introduction.

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