An Elementary Construction of Brownian Motion*

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Brownian motion on $[0,1]$ is a zero-mean Gaussian stochastic process $\{W(t)\}_{t\in[0,1]}$, that has covariance function $\mathbf{Cov}\{W(s),W(t)\} = s\wedge t = \min\{s,t\}$, and is continuous with probability 1. The purpose of this note is to give a short and self-contained proof of the existence of this process, making use of only the most elementary concepts in probability theory.

Let $\xi_1,\xi_2,\ldots$ be independent $\mathrm{N}(0,1)$-distributed random variables, and define

$$W(t) \equiv \sum_{k=0}^{\infty} \frac{\sqrt{2}}{\pi}\,\frac{2}{2k+1}\,\sin((2k+1)\pi t/2)\,\xi_k \qquad \text{for} \quad t\in[0,1]. \tag{1}$$

By the Cauchy criterion, this random series is well-defined as a mean-square limit

$$\mathbf{E}\bigg\{\bigg(\sum_{k=0}^{m} \frac{\sqrt{2}}{\pi}\,\frac{2}{2k+1}\,\sin((2k+1)\pi t/2)\,\xi_k - W(t)\bigg)^2\bigg\} \to 0 \qquad \text{as} \quad m\to\infty,$$

where $\mathbf{E}\{W(t)^2\}<\infty$, if and only if the partial sums form a Cauchy-sequence

$$\mathbf{E}\bigg\{\bigg(\sum_{k=0}^{m} \frac{\sqrt{2}}{\pi}\,\frac{2}{2k+1}\,\sin((2k+1)\pi t/2)\,\xi_k - \sum_{\ell=0}^{n} \frac{\sqrt{2}}{\pi}\,\frac{2}{2\ell+1}\,\sin((2\ell+1)\pi t/2)\,\xi_\ell\bigg)^2\bigg\} \to 0$$

as $m,n\to\infty$. However, this holds, since the mean on the left-hand side is

$$\mathbf{E}\bigg\{\bigg(\sum_{k=(m\wedge n)+1}^{(m\vee n)} \frac{\sqrt{2}}{\pi}\,\frac{2}{2k+1}\,\sin((2k+1)\pi t/2)\,\xi_k\bigg)^2\bigg\} = \sum_{k=(m\wedge n)+1}^{(m\vee n)} \frac{8\,\sin((2k+1)\pi t/2)^2}{\pi^2(2k+1)^2}.$$

By symmetry in (1), we have $\mathbf{E}\{W(t)\}=0$. The covariance function is given by

$$\begin{aligned}
&\mathbf{Cov}\{W(s),W(t)\}\\
&= \mathbf{Cov}\bigg\{\sum_{k=0}^{\infty} \frac{\sqrt{2}}{\pi}\,\frac{2}{2k+1}\,\sin((2k+1)\pi s/2)\,\xi_k,\ \sum_{\ell=0}^{\infty} \frac{\sqrt{2}}{\pi}\,\frac{2}{2\ell+1}\,\sin((2\ell+1)\pi t/2)\,\xi_\ell\bigg\}\\
&= \sum_{k=0}^{\infty} \frac{8\,\sin((2k+1)\pi s/2)\,\sin((2k+1)\pi t/2)}{\pi^2\,(2k+1)^2}\\
&= \sum_{k=0}^{\infty} \frac{4}{\pi^2\,(2k+1)^2}\Big(\cos((2k+1)\pi(s-t)/2) - \cos((2k+1)\pi(s+t)/2)\Big),
\end{aligned} \tag{2}$$

by the elementary trigonometric identity $2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y)$, and since $\mathbf{Cov}\{\cdot,\cdot\}$ commutes with mean-square limits. Here we have

$$\sum_{k=0}^{\infty} \frac{4}{\pi^2\,(2k+1)^2}\,\cos((2k+1)\pi t/2) = (1-|t|)/2 \qquad \text{for} \quad t\in[-2,2]: \tag{3}$$

By symmetry, it is enough to show (3) for $t\in[0,2]$. For such t, (3) holds since

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the left-hand side and right-hand side of (3) are continuous functions of t (by basic math), and, according to `Mathematica`, their one-sided Laplace transforms coincide:

```
In[1]:= Simplify[Sum[Integrate[4/(Pi*(2*k+1))^2*Cos[(2*k+1)*Pi*t/2]
          *Exp[-x*t], {t, 0, 2}], {k, 0, Infinity}]]
```

$$\text{Out[1]=}\ \frac{e^{-2x}\left(1+e^{2x}(-1+x)+x\right)}{2x^2}$$

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In[2]:= Simplify[Integrate[(1/2-t/2)*Exp[-x*t], {t, 0, 2}]]
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$$\text{Out[2]=}\ \frac{e^{-2x}\left(1+e^{2x}(-1+x)+x\right)}{2x^2}$$

From (2) and (3), we get the covariance function desired

$$\mathbf{Cov}\{W(s),W(t)\} = (1-|t-s|)/2 - (1-|t+s|)/2 = s\wedge t \qquad \text{for} \quad s,t\in[0,1].$$

Moreover, W is Gaussian, since each linear combination of process values is a mean-square limit of a sequence of univariate Gaussian random variables

$$\sum_{i=1}^{n} a_i W(t_i) \leftarrow \sum_{k=0}^{m} \frac{\sqrt{2}}{\pi}\,\frac{2}{2k+1}\left(\sum_{i=1}^{n} a_i \sin((2k+1)\pi t_i/2)\right)\xi_k \qquad \text{as} \quad m\to\infty$$

for $a_1,\ldots,a_n\in\mathbb{R}$ and $n\in\mathbb{N}$, so that the limit is also univariate Gaussian.

Finally, to prove that W is continuous with probability 1, we notice that

$$\begin{aligned}
&\mathbf{P}\left\{\sum_{k=0}^{\infty}\frac{\sqrt{2}}{\pi}\,\frac{2}{2k+1}\,\sin((2k+1)\pi t/2)\,\xi_k \ \text{ is continuous for } \ t\in[0,1]\right\}\\
&\geq \mathbf{P}\left\{\sum_{n=0}^{\infty}\sum_{k=2^n-1}^{2^{n+1}-2}\frac{\sqrt{2}}{\pi}\,\frac{2}{2k+1}\,\sin((2k+1)\pi t/2)\,\xi_k \ \text{ converges uniformly for } \ t\in[0,1]\right\}\\
&\geq 1-\mathbf{P}\left\{\sup\nolimits_{t\in[0,1]}|X_n(t)|>2^{-n/8} \ \text{ for infinitely many } \ n\right\},
\end{aligned} \tag{4}$$

where X_n is the zero-mean Gaussian process given by

$$X_n(t) = \sum_{k=2^n-1}^{2^{n+1}-2}\frac{\sqrt{2}}{\pi}\,\frac{2}{2k+1}\,\sin((2k+1)\pi t/2)\,\xi_k \qquad \text{for} \quad t\in[0,1].$$

By the elementary trigonometric identity $\sin(x)-\sin(y) = 2\cos(\frac{x+y}{2})\sin(\frac{x-y}{2})$, together with the fact that $|\sin(x)|\leq|x|$, we readily obtain

$$\begin{aligned}
\mathbf{E}\{(X_n(t)-X_n(s))^2\} &= \sum_{k=2^n-1}^{2^{n+1}-2}\frac{32\,\cos((2k+1)\pi(t+s)/4)^2\,\sin((2k+1)\pi(t-s)/4)^2}{\pi^2(2k+1)^2}\\
&\leq \sum_{k=2^n-1}^{2^{n+1}-2}\frac{16\,|t-s|^{1/2}}{\pi^{3/2}(2k+1)^{3/2}}\\
&\leq \frac{16\,2^n|t-s|^{1/2}}{\pi^{3/2}(2^{n+1}-1)^{3/2}}.
\end{aligned}$$

Using that X_n is continuous and symmetric, with $X_n(0)=0$, it follows that

$$
\begin{aligned}
s_n &\equiv \mathbf{P}\{\sup_{t\in[0,1]} |X_n(t)| > 2^{-n/8}\} \\
&\le 2\,\mathbf{P}\Big\{\bigcup_{k=0}^{\infty}\bigcup_{\ell=0}^{2^k-1}\{X_n(2^{-k}\ell) > 2^{-n/8}\}\Big\} \\
&\le 2\,\mathbf{P}\Big\{\bigcup_{k=0}^{\infty}\bigcup_{\ell=0}^{2^k-1}\Big\{X_n(2^{-k}\ell) > 2^{-n/8-1}\Big(1+(1-2^{-1/8})\textstyle\sum_{j=0}^{k} 2^{-j/8}\Big)\Big\}\Big\} \\
&= 2\,\mathbf{P}\Big\{X_n(0) > 2^{-n/8-1}\Big(1+(1-2^{-1/8})\Big)\Big\} \\
&\qquad +2\sum_{k=1}^{\infty}\mathbf{P}\Big\{\bigcup_{\ell=0}^{2^k-1}\Big\{X_n(2^{-k}\ell) > 2^{-n/8-1}\Big(1+(1-2^{-1/8})\textstyle\sum_{j=0}^{k} 2^{-j/8}\Big)\Big\}, \\
&\qquad\qquad \bigcap_{m=0}^{k-1}\bigcup_{\ell=0}^{2^m-1}\Big\{X_n(2^{-m}\ell) \le 2^{-n/8-1}\Big(1+(1-2^{-1/8})\textstyle\sum_{j=0}^{m} 2^{-j/8}\Big)\Big\}\Big\} \\
&\le 0 + 2\sum_{k=1}^{\infty}\sum_{\ell=0}^{2^{k-1}-1}\mathbf{P}\Big\{X_n(2^{-k}(2\ell+1)) > 2^{-n/8-1}\Big(1+(1-2^{-1/8})\textstyle\sum_{j=0}^{k} 2^{-j/8}\Big), \\
&\qquad\qquad X_n(2^{-k+1}\ell) \le 2^{-n/8-1}\Big(1+(1-2^{-1/8})\textstyle\sum_{j=0}^{k-1} 2^{-j/8}\Big)\Big\} \\
&\le 2\sum_{k=1}^{\infty}\sum_{\ell=0}^{2^{k-1}-1}\mathbf{P}\Big\{X_n(2^{-k}(2\ell+1)) - X_n(2^{-k}2\ell) > 2^{-n/8-1}(1-2^{-1/8})2^{-k/8}\Big\} \\
&= 2\sum_{k=1}^{\infty}\sum_{\ell=0}^{2^{k-1}-1}\mathbf{P}\Bigg\{\mathrm{N}(0,1) > \frac{2^{-n/8-1}(1-2^{-1/8})2^{-k/8}}{\sqrt{\mathbf{E}\{(X_n(2^{-k}(2\ell+1)) - X_n(2^{-k}2\ell))^2\}}}\Bigg\} \\
&\le \sum_{k=1}^{\infty} 2^k\,\mathbf{P}\Bigg\{\mathrm{N}(0,1) > \frac{2^{-n/8-1}(1-2^{-1/8})2^{-k/8}\,\pi^{3/4}\,(2^{n+1}-1)^{3/4}}{4\,2^{n/2}\,2^{-k/4}}\Bigg\}.
\end{aligned}
$$

Since $\sum_{n=0}^{\infty} s_n < \infty$, the right-hand side of (4) is 1 by the Borel-Cantelli lemma. □