The age of a Galton-Watson population with geometric offspring distribution*

F.C. Klebaner
Mathematics and Statistics
University of Melbourne
Parkville,
Victoria 3052
Australia

S. Sagitov
School of Mathematical
and Computing Sciences
Chalmers University
Gothenburg,
Sweden

July 23, 2001

Abstract

Motivated by the question of the age in a branching population we are trying to recreate the past by looking back from the observed now population size. We define a new backward Galton-Watson (GW) process and study the case of geometric offspring distribution with parameter p in details. The backward process is then the Galton-Watson process with immigration (GWI) again with a geometric offspring distribution but with parameter $1-p$, and it is also the dual to the original GW process. We give the asymptotic distribution of the age when the initial population size is large in supercritical and critical cases. To this end we give new asymptotic results on the GWI processes stopped at zero.

*Research supported by the Australian Research Council

1 Introduction

Here we address the question (amongst other questions) "how can one determine the age $T = T(N)$ of a branching population when only the current population size N is observed?" We know that in the Galton-Watson (GW) branching process the observed size is a result of the sum of the i.i.d. integer valued random variables. By "looking back" we reconstruct the process, and define the age as the first time the backward process attains the value 1 (see Section 4 for exact definitions).

The forward (direct) GW process can be seen as a result of the iteration of the summation of i.i.d. random variables, $Z_{n+1} = S_{Z_n}$, where S_n stands for the sum of of i.i.d. r.v.s. (each time n a new process $S^{(n)}$ is taken, but we prefer to suppress double indexing). The inverse process to summation is the renewal process conditioned on visiting a certain level, and the backward (time-reversed) GW process can be defined by the successive iterations of the renewal process. Unfortunately, such iterations are much less tractable than the direct branching process. For example, unlike in the direct process, the moments in general are known only asymptotically (by use of renewal theorems).

However, the case of the geometric offspring distribution turns out to be particularly nice and allows for more detailed analysis, as the backward process in this case is a GW branching process with Immigration (GWI) again with geometric offspring distribution but with the complimentary parameter. This implies, in particular, that the backward version of a supercritical GW process is a subcritical GWI process stopped at one.

This setup provides a motivation to study time to extinction in GWI processes stopped at zero. In Section 5 we give results on the tail of the time to extinction when the initial population size is large, for the subcritical and critical GWI processes stopped at zero with general offspring distributions. These general results are applied to our backward GW process in Section 4.

The main results of this paper, Theorem 4.1 and Theorem 4.2, should be compared to Levikson (1977), Pakes (1978) and Stigler (1970), Koteeswaran (1989), who considered similar questions. Levikson (1977) has introduced the age of a Markov chain with a single absorbing state 0 by using a reflecting barrier at 0. Under the assumption that the time to absorption has a finite mean, the new chain is constructed by an instantaneous return to 1 each time the original chain gets absorbed. Pakes (1978) modified the definition by considering return to 1 at the next step rather than instantly. The limiting age is the weak limit of the time $\min\{m : \tilde{X}_{n-m} = 0\}$ given that $\tilde{X}_n = j$ as $n \to \infty$. Our approach to the question of age via construction of the backward process is different to the approaches considered so far.

A different approach by Stigler (1970) (also applied in Koteeswaran (1989)) is based on the maximum likelihood method. Both approaches produced the population age estimates of order $\log N$ in the supercritical case and of order N in the critical case but with different limit distributions.

Our results regarding the age given in Section 4 can be described as follows. In the supercritical case when the average offspring size m is larger than 1, $T(N)$ is of order $\ln(N)$ (see Theorem 4.1), which is expected since the population grows exponentially.

In the critical reproduction case when $m = 1$ an unexpected population age is obtained. One would expect the population age to be of order N, since the size of a critical branching process Z_n, conditioned on nonextinction at time n, is of order n. But according to Theorem 4.2 for large N, $T(N) \approx N^W$, where the random variable $W \geq 1$ with $P(W > x) = 1/x$, $x \geq 1$. In particular, $T(N)/N \longrightarrow \infty$ in probability. Thus according to our model, the age of a critical GW population is much larger than the current population size, unlike the other two approaches Pakes (1978) and Stigler (1970), which gave the scaling N for the asymptotics of $T(N)$.

2 The backward GW process

The GW process Z_n is a simple population model assuming that individuals reproduce independently following the same distribution

$$p_j = P(\xi = j), \ j \geq 0$$

so that

$$Z_{n+1} = \xi_1 + \ldots + \xi_{Z_n}, \tag{1}$$

where ξ_i are i.i.d. random variables having the offspring distribution $\{p_j\}$. Here and elsewhere indices i, j, k, n are assumed to take interger values only.

The Markov chain Z_n describes the dynamics of the generation size of a GW population where n is the generation number. This dynamics is mostly determined by the average offspring size $m = E(\xi)$. Three different types of population dynamics are predicted by the model (see [1] Athreya and Ney, 1972):

subcritical case, $m < 1$: quick extinction due to a negative drift,

critical case, $m = 1$: slower extinction and linear growth if conditioned on non-extinction,

supercritical case, $m > 1$: either a quick extinction or exponential growth.

To construct a backward GW process X_n we start with the current population size $X_0 = N$ as the initial state of the Markov chain under construction. The previous generation size X_1 is a natural valued random variable such that (see (1))

$$\xi_1 + \ldots + \xi_{X_1} = N,$$

In a similar way the random variable X_2 is related to X_1.

To make the definition precise let $S_n = \sum_{i=1}^{n} \xi_i$ and $\tau_k = \inf\{n : S_n = k\}$ be the first time the random walk hits value k. If the upward random walk S_n misses the level k we put $\tau_k = \infty$.

Definition 2.1 *The backward GWP X_n is a Markov chain with the one step transition probabilities*

$$P(X_{n+1} = j | X_n = i) = P(\tau_i = j | \tau_i < \infty), \ i \geq 1, \ j \geq 1.$$

Theorem 2.1 *If the reproduction law is geometric $G(p)$ with probabilities*

$$p_j = pq^j, \ j \geq 0, \ 0 < p < 1, \ q = 1 - p, \tag{2}$$

then the backward GW process $\{X_n\}$ is a GWI process with the offspring distribution $G(q)$, $q = 1 - p$, and unit immigration. Moreover, the shifted process $Y_n := X_n - 1$ is a GWI process with offspring distribution $G(q)$ and immigration distribution $G(q)$.

PROOF:

The geometric distribution (2) models the number of failures before the first success in a sequence of independent Bernoulli trials with probability of success p. Therefore, S_n corresponds to the number of failures before the n-th success in Bernoulli trials. As an illustration consider a sequence of successes and failures

$$s_1 f_1 s_2 s_3 f_2 f_3 f_4 s_4 f_5 s_5 s_6 s_7 f_6 f_7 s_8 \ldots$$

In this case

$$S_1 = 0, S_2 = S_3 = 1, S_4 = 4, S_5 = S_6 = S_7 = 5, S_8 = 7, \ldots$$

and

$$\tau_1 = 2, \tau_2 = \tau_3 = \infty, \tau_4 = 4, \tau_5 = 5, \tau_6 = \infty, \tau_7 = 8, \ldots$$

Observe now that

$$P(\tau_i = j) = \binom{j+i-2}{j-1} p^j q^i = pP(R_i = j-1),$$

where R_i is distributed as the number of successes before the k-th failure (Negative Binomial distribution)

$$P(R_i = j) = \binom{j+i-1}{j} p^j q^i.$$

Hence $P(\tau_k < \infty) = p$ and $P(\tau_k = j | \tau_k < \infty) = P(R_k + 1 = j)$.

We conclude that given the current state $X_n = i$ of the backward GW process, the next state X_{n+1} is distributed as $R_i + 1$. We interpret this as an immigrant counted together with the offspring of i individuals constituting the previous generation. The resulting GWI process has the offspring size distribution $p_j = qp^j$, the same as the number of successes before the first failure. The immigration process is deterministic: exactly one immigrant joins every generation of the population. In the framework of the forward GW process this immigrant might be treated as a guaranteed ancestor in the previous generation.

The shifted process Y_n which ignores the immigrant counted in X_n can again be vewed as a GWI process. The trick is to consider the daughters of the immigrant counted in X_{n-1} as the immigrants counted in Y_n. Obviously, the reproduction law of the new GWI process remains to be $G(q)$ while the immigration law ceases to be deterministic and becomes the same as the offspring size distribution $G(q)$.

□

Remark. A different definition of a reverse GW process X_n is presented in Esty (1975) in terms of the joint probability functions:

$$\begin{aligned} P(X_{n_1} &= i_1, \ldots, X_{n_k} = i_k | X_0 = i_0) \\ &= \lim_{n\to\infty} P(Z_{n-n_1} = i_1, \ldots, Z_{n-n_k} = i_k | Z_n = i_0, Z_{n-1} > 0) \end{aligned}$$

In the critical case this model predicts a linear growth of the population size when traced back in time.

3 Duality

In this section we establish the following property of duality (see Ligget, 1985) between a GW process with the geometric reproduction law and its backward counterpart.

Theorem 3.1 *If Z_n is the GW process with the geometric reproduction law (2) and X_n is its backward counterpart, then for all $j \geq 1$ and $i \geq 0$*

$$P(Z_n < j | Z_0 = i) = P(X_n > i | X_0 = j). \tag{3}$$

Furthermore, if $p < 1/2$, the backward GW process has a stationary geometric distribution

$$\pi_j = (1 - r) r^{j-1}, \; j \geq 1,$$

where $r = p/q$ is the extinction probability

$$r = \lim_{n \to \infty} P(Z_n = 0 | Z_0 = 1).$$

Remark. In terms of the shifted process $Y_n = X_n - 1$ relation (3) can be rewritten as

$$P(Z_n \leq j | Z_0 = i) = P(Y_n \geq i | Y_0 = j), \; i \geq 0, \; j \geq 0.$$

A similar duality relationship is known for two random walks on $\{0, 1, 2, \ldots\}$. The forward random walk makes either a unit step upward with probability p or a unit step downward with probability q and gets absorbed at 0 with probability one. The backward random walk goes up and down with the complimentary probabilities q and p and is reflected at 0 with probability q. The duality property indicates that one random walk is a proper time-reversal of the other random walk.

PROOF: of Theorem 3.1.

Using the Bernoulli trial description of the backward GWP process we can claim that

$$P(Z_1 < j | Z_0 = i) = P(X_1 > i | X_0 = j)$$

because both sides of the relation are equal to the probability of having the i-th success before the j-th failure. In terms of one-step transition matrices of the Markov chains Z_n and X_n the previous equality might be written as

$PH = HQ'$, where H is a matrix with elements $H_{ij} = I(i < j)$, $i \geq 0$, $j \geq 0$, and Q' stands for the transpose of Q. Therefore,

$$P^n H = P^{n-1} PH = P^{n-1} HQ' = \ldots = H(Q^n)'$$

which is equivalent to (3).

To show that in the case $p < 1/2$ the backward GW process has a stationary geometric distribution we use the following asymptotic property of supercritical GW process [1, Ch. 1]:

$$P(Z_n < j | Z_0 = i) \to r^i \text{ as } n \to \infty$$

valid for any $j \geq 1$ and $i \geq 0$. In view of (3) it follows that

$$\lim_{n \to \infty} P(X_n > i | X_0 = j) = r^i,$$

with the RHS giving the tail of a stationary geometric distribution.

□

To retain the duality property (3) in the general case, when the offspring distribution is not necessarily geometric, one can relax the Definition 2.1 by removing the requirement of the random walk S_n visiting a certain level.

Definition 3.1 *Let $V(k) = \inf\{n : S_n \geq k\}$ be the first time the random walk S_n exceeds the value k. Define the relaxed backward GW process $\tilde{X}_n$ by the recursion $\tilde{X}_{n+1} = V(\tilde{X}_n)$, $\tilde{X}_0 = N$.*

Now, since obviously

$$P(Z_{n+1} < j | Z_n = i) = P(S_i < j) = P(V(j) > i) = P(\tilde{X}_{n+1} > i | \tilde{X}_n = j)$$

we conclude that the duality property (3) always holds for the relaxed backward GW process $\tilde{X}_n$. Note that in the case of geometric offspring distribution the backward process X_n and the relaxed backward process $\tilde{X}_n$ are equal in distribution.

We comment on the way of analysis of the general backward process $\tilde{X}_n$, but do not give details. Conditions for transience and recurrence are obtained by using results of Kersting (1986) and renewal theorems, but only in the non-critical case.

An alternative approach is possible by representing $\tilde{X}_n$ as a controlled GW process

$$\tilde{X}_{n+1} = \sum_{j=1}^{\phi(\tilde{X}_n)} \tilde{\xi}_j,$$

with the following geometric offspring distribution

$$P(\tilde{\xi} = i) = P^i(\xi = 0)P(\xi > 0),$$

and the control function $\phi(k)$ giving the number of positive jumps before the random walk S_n exceeds the value k. The theory of controlled GW processes was initiated in [12], [17] and developed for random control functions $\phi(k)$ in [15]. However, this representation also does not seem to handle the critical case in our model. Therefore we concentrate on studying processes with a geometric offspring distribution, where explicit results are possible to obtain.

4 The age of a GW population

Definition 4.1 *Given the current population size $X_0 = N$ the age of a GW population $T = T(N)$ is defined as a stopping time*

$$T = \inf\{n : X_n = 1\} = \inf\{n : Y_n = 0\}.$$

This definition neglects the possibility of the forward GW process revisiting the state 1. Strictly speaking our T underestimates the true age of the GW population.

The next two theorems present the main results of the paper concerning the asymptotic distribution of the population age $T(N)$ as $N \to \infty$. Theorem 4.1 treats the supercritical case when the offspring mean $m > 1$, and Theorem 4.2 the critical case $m = 1$.

Theorem 4.1 *Let $p < \frac{1}{2}$, so that $m = \frac{q}{p}$ is larger than 1. If $N \sim ym^n$, as $n \to \infty$, then for any interger k*

$$\lim_{N\to\infty} P(T(N) \le n + k) = \frac{m-1}{m} \sum_{i=0}^{\infty} w_i \exp(-y(m-1)m^{i-k-1}),$$

with a decreasing sequence w_i of positive numbers satisfying the recurrence relation (9) with coefficients

$$a_n = \frac{m^{n-1}}{(1 + \ldots + m^{n-1})(1 + \ldots + m^n)}. \quad (4)$$

Theorem 4.2 *If $p = \frac{1}{2}$, then for all $x \geq 1$*

$$\lim_{N\to\infty} P(T(N) > N^x) = 1/x.$$

We prove these theorems by using the fact that T is the extinction time of a certain GWI process stopped at zero, which follows by duality (Theorem 2.1) and by using general results on the extinction time of GWI processes stopped at zero. These results are given in the next section. Here and apply the results to the shifted backward GW process process Y_n.

PROOF: of Theorem 4.1 and Theorem 4.2.

According to Theorem 2.1 the shifted backward GW process Y_n is a GWI process with the reproduction probability generating function (p.g.f.), average offspring size, and the immigration p.g.f. given by

$$F(s) = \frac{q}{1-ps},\ \mu = \frac{p}{q},\ B(s) = F(s).$$

In the geometric reproduction case the p.g.f. iterations

$$F_1(s) = F(s),\ F_{n+1}(s) = F(F_n(s)),\ n \geq 1 \tag{5}$$

are calculated explicitly [1, p. 7]:

$$F_n(s) = 1 - \frac{\mu^n(\mu-1)}{\mu^{n+1}-1} + \frac{\mu^n(\frac{\mu-1}{\mu^{n+1}-1})^2 s}{1-(\frac{\mu^n-1}{\mu^{n+1}-1})\mu s},\ \text{if } p \neq \tfrac{1}{2}$$

and

$$F_n(s) = \frac{n-(n-1)s}{n+1-ns},\ \text{if } p = \tfrac{1}{2}.$$

It follows that

$$F_n(0) = \frac{1-\mu^n}{1-\mu^{n+1}},\ \text{if } p < \tfrac{1}{2},$$

and

$$F_n(0) = \frac{n}{n+1},\ \text{if } p = \tfrac{1}{2}.$$

If $p < \frac{1}{2}$, then $\mu < 1$ and we can apply Theorem 5.1. To see that it implies Theorem 4.1 observe that $\mu = \frac{1}{m}$,

$$1 - F_n(0) = \frac{\mu^n(1-\mu)}{1-\mu^{n+1}} \sim c\mu^n,\ \text{as } n \to \infty$$

with $c = 1 - \mu = \frac{m-1}{m}$. Furthermore, the sequence defined by (6) equals

$$P_n(0) = \prod_{i=1}^{n} F_i(0) = \frac{1-\mu}{1-\mu^{n+1}}$$

and converges to $\alpha = 1 - \mu = \frac{m-1}{m}$ as $n \to \infty$. It remains to check that formula (4) gives the sequence (8) appearing in (9).

If $p = \frac{1}{2}$, then

$$\mu = 1,\ \beta = B'(1) = 1,\ \gamma = F''(1)/2 = 1,\ \sigma = \beta/\gamma = 1$$

and we use Theorem 5.3. It suffices to notice that

$$P_n(0) = \frac{1}{n+1} \text{ and } \sum_{i=1}^{n} P_i(0) \sim \ln n, \text{ as } n \to \infty.$$

□

5 GWI process stopped at zero

First results on the GWI processes stopped at zero were obtained in [16] and [14]. We introduce a GWI process stopped at zero following [11] and [5]. Let $\{\eta_i\}$, be a sequence of i.i.d. integer valued nonnegative random variables with p.g.f. $F(s)$, $\{I_n\}$ is a sequence of i.i.d. integer valued nonnegative random variables with p.g.f. $B(s)$, and define the GWI process by taking $Y_0 = N$ to be a positive integer, and provided $Y_n > 0$,

$$Y_{n+1} = \sum_{i=1}^{Y_n} \eta_i + I_{n+1}.$$

The process stops first time $Y_n = 0$. Let T be the time to extinction, i.e. $T = \inf\{n : Y_n = 0\}$. For each n a different iid sequence of η_i's is taken, so that we avoid double indices.

Let $u_n = P(T(N) > n)$ and put

$$\mu = F'(1-),\ \beta = B'(1-),\ \gamma = F''(1-)/2,\ \sigma = \beta/\gamma.$$

The purpose of this section is to give some results on the tail of T, or asymptotic behaviour of u_n for large N and $n = n(N)$ assuming that $P(T < \infty) = 1$. The latter obviously holds in the subcritical case $\mu < 1$ and is violated in the superctitical case $\mu > 1$. The true threshold lies in the critical domain $\mu = 1$

with $P(T < \infty) = 1$ being valid iff $\sigma \leq 1$ (see [5, p.224]). If $\mu = 1$ and $\sigma > 1$, the immigration inflow at rate $\beta = E(I_n)$ might overcome the extinction trend due to the reproduction uncertanty $\gamma = Var(\eta_i)$.

An important quantity in the analysis of the GWI process is the probability of hitting zero by a GWI process with stationary immigration

$$P_n(0) = \prod_{i=0}^{n-1} B(F_i(0)), \tag{6}$$

where p.g.f. are defined by (5). The sequence of interest u_n satisfies the following renewal equation (see [11, p.13])

$$u_n = c_n + \sum_{i=1}^{n} a_i u_{n-i}, \tag{7}$$

where

$$a_n = P_{n-1}(0) - P_n(0), \; c_n = P_n(0)(1 - F_n(0)^N). \tag{8}$$

The corresponding renewal function is given by the recursion

$$w_n = \sum_{i=1}^{n} a_i w_{n-i}, \quad w_0 = 1. \tag{9}$$

It is known that in the subcritical case, $\mu < 1$, (see [11, p. 14])

$$P_n(0) \to \alpha \in (0,1), \quad n \to \infty \tag{10}$$

and (see [4])

$$1 - F_n(0) \sim c\mu^n, \text{ for some } 0 < c \leq 1, \quad n \to \infty. \tag{11}$$

Theorem 5.1 *If $\mu < 1$, and $N \sim ym^{-n}$ as $n \to \infty$, then for all integer k*

$$\lim_{N\to\infty} P(T(N) \leq n + k) = \alpha \sum_{i=0}^{\infty} w_i \exp(-cym^{k-i}), \tag{12}$$

with α, c, and w_i given by (10), (11), and (9).

Theorem 5.2 *If $\mu = 1$ and $\sigma < 1$, then*

$$\lim_{N\to\infty} P(T(N) > xN) = \int_0^x (x - v)^{-\sigma}(1 - \exp(-\frac{1}{\gamma(x - v)}))dv^{\sigma}.$$

Theorem 5.3 *If $\mu = 1$ and $\sigma = 1$ then for any $x \geq 1$*

$$\lim_{N\to\infty} P(l(T(N)) > xl(N)) = 1/x,$$

where $l(n) := \sum_{i=1}^{n} P_i(0)$ is known (see [5, p. 225]) to be a slowly varying at infinity function.

5.1 Proofs of stopped GWI results

We prove first that the sequence w_n defined by (9) is decreasing. We have

$$\begin{aligned} w_n &= a_1 w_{n-1} + a_2 w_{n-2} + \ldots + a_{n-1} w_1 + a_n \\ &= w_{n-1} - P_1(0) w_{n-1} + P_1(0) w_{n-2} - P_2(0) w_{n-2} + \ldots \\ &\quad + P_{n-2}(0) w_1 - P_{n-1}(0) w_1 + P_{n-1}(0) - P_n(0) \end{aligned}$$

It follows that the difference $\Delta_n = w_{n-1} - w_n$ satisfies the recurrence

$$\Delta_n + P_1(0)\Delta_{n-1} + P_2(0)\Delta_{n-2} + \ldots + P_{n-1}(0)\Delta_1 = P_n(0).$$

Write the same for $n-1$,

$$\Delta_{n-1} + P_1(0)\Delta_{n-2} + \ldots + P_{n-2}(0)\Delta_1 = P_{n-1}(0),$$

and use $\frac{P_n(0)}{P_{n-1}(0)} = B(F_{n-1}(0))$ to obtain

$$\begin{aligned} \Delta_n &+ P_1(0)\Delta_{n-1} + P_2(0)\Delta_{n-2} + \ldots + P_{n-1}(0)\Delta_1 \\ &= B(F_{n-1}(0))(\Delta_{n-1} + P_1(0)\Delta_{n-2} + \ldots + P_{n-2}(0)\Delta_1) \\ &= \frac{P_t(0)}{P_{t-1}(0)}\Delta_{t-1} + \frac{P_t(0)P_1(0)}{P_{t-1}(0)}\Delta_{t-2} + \ldots. \end{aligned}$$

It now follows that $\Delta_n \geq 0$, since

$$B(F_{n-1}(0))P_i(0) = \frac{B(F_{n-1}(0))}{B(F_i(0))} P_{i+1}(0) \geq P_{i+1}(0)$$

for all $0 \leq i \leq n-1$. Thus the sequence w_n is monotone, the fact we use in the proof of Theorem 5.3.

PROOF: of Theorem 5.1.

Expressing the solution to the renewal equation (7) in terms of the renewal function (9)

$$u_n = \sum_{i=1}^{n} c_i w_{n-i} \tag{13}$$

we obtain

$$P(T > n+k) = \sum_{i=0}^{n+k-1} w_i c_{n+k-i}, \ k = 0, \pm 1, \ldots$$

It follows by (8), (10), and (11) that under the conditions of the theorem,

$$c_{n+k-i} \to \alpha(1 - \exp(-cym^{k-i})).$$

Hence the result.

□

PROOF: of Theorem 5.2.

Writing (13) in the integral form

$$u_n = \int_0^n c_{n-v} dU(v), \;\; U(n) := \sum_{i=0}^{n} w_i$$

and performing substitution, we get

$$u_n = \int_0^1 c_{n(1-v)} dU(nv). \tag{14}$$

Now, it is well-known (see [1, p. 19]), that in the critical case, $\mu = 1$,

$$1 - F_n(0) \sim \frac{1}{\gamma n}, \quad n \to \infty. \tag{15}$$

On the other hand if $\sigma \leq 1$, then (see [16])

$$P_n(0) \sim n^{-\sigma} L(n), \quad n \to \infty, \tag{16}$$

where $L(t)$ is a slowly varying at infinity function. So by (8) for any fixed $0 < t \leq 1$

$$\begin{aligned} c_{nt} &\sim (nt)^{-\sigma} L(nt)(1 - e^{-\frac{1}{\gamma t}}) \\ &\sim t^{-\sigma}(1 - e^{-\frac{1}{\gamma t}}) n^{-\sigma} L(n), \text{ as } n \to \infty. \end{aligned} \tag{17}$$

Finally, according to the renewal theory (see [3], p. 471) conditions (16) and $\sigma < 1$ imply

$$U(nv) \sim v^{\sigma} n^{\sigma} / L(n), \text{ as } n \to \infty, \; v > 0,$$

hence from (14) and (17) it follows

$$\lim_{N \to \infty} u_{xN} = \int_0^x (x - v)^{-\sigma} (1 - \exp(-\frac{1}{\gamma(x - v)})) dv^{\sigma}, \; x > 0.$$

□

PROOF: of Theorem 5.3.

Assume that $m = 1$ and $\sigma = 1$ and fix some $x > 1$. In view of

$$P(l(T) > xl(N)) = P(T > l_{-1}(xl(N))) = u_{\tau},$$

we have to verify that

$$u_\tau \to 1/x, \;\; N \to \infty, \tag{18}$$

where $\tau := l_{-1}(xl(N))$ is such that $\tau/N \to \infty$. Our proof of (18) hinges upon the asymptotic relation

$$w_n \sim 1/l(n), \;\; n \to \infty \tag{19}$$

which follows from the relation between different p.g.f.

$$\sum_{i=0}^{\infty} w_i s^i = \frac{1}{1 - \sum_{i=1}^{\infty} a_i s^i} = \frac{1}{(1-s)\sum_{i=0}^{\infty} P_i(0) s^i},$$

monotonicity of w_n, and a Tauberian Theorem 5 from [3, p. 447].

To arrive at (18) split the sum from (13) in three parts

$$u_\tau = \sum_{i=1}^{\tau} P_i(0)(1 - F_i^N(0)) w_{\tau-i} = \sum_{i \le \epsilon N} + \sum_{\epsilon N < i \le KN} + \sum_{i > KN}, \tag{20}$$

where ϵ and K are a small and a large constant respectively. Since $F_i(0)$ is monotone increasing,

$$1 - F_{\epsilon N}^N(0) \le 1 - F_i^N(0) < 1, \;\; i \le \epsilon N$$

so that we obtain for the first sum in (20)

$$(1 - F_{\epsilon N}^N(0)) l(\epsilon N) w_{\tau - \epsilon N} \le \sum_{i \le \epsilon N} P_i(0)(1 - F_i^N(0)) w_{\tau-i} < l(\epsilon N) w_\tau.$$

Thus we obtain the main contribution in u_τ in (20) by taking limits as $N \to \infty$ and using (15) together with (19)

$$(1 - e^{-\frac{1}{\epsilon\gamma}})\frac{1}{x} \le \lim_{N\to\infty} \sum_{i \le \epsilon N} P_i(0)(1 - F_i^N(0)) w_{\tau-i} \le \frac{1}{x}. \tag{21}$$

Since ϵ is arbitrary small, we conclude that

$$\lim_{N\to\infty} \sum_{i \le \epsilon N} P_i(0)(1 - F_i^N(0)) w_{\tau-i} = \frac{1}{x}. \tag{22}$$

It is easy to see that the contribution of the other two sums in (20) is nil. Indeed,

$$\limsup_{N\to\infty} \sum_{\epsilon N < i \le KN} P_i(0)(1 - F_i^N(0)) w_{\tau-i} \le \limsup_{N\to\infty} \frac{l(KN) - l(\epsilon N)}{xl(N)} = 0, \tag{23}$$

by the property of slow variation of l. For the third sum in (20) we write

$$\limsup_{N\to\infty} \sum_{KN<i\leq\tau} P_i(0)(1-F_i^N(0))w_{\tau-i} \leq \lim_{N\to\infty}(1-F_{KN}^N(0)) = 1-e^{-\frac{1}{\gamma K}}.$$

Now, since K is arbitrary large, we conclude that the third sum in (20) also gives zero contribution to u_τ.

□

6 References

1. Athreya K.B., Ney P.E. *Branching Processes*, Springer-Verlag, 1972.

2. Esty W.W. (1975) The Reverse Galton-Watson Process. *J. Appl. Prob.* 12, 574-580.

3. Feller W. *An Introduction to Probability Theory and Its Applications. Vol. 2,* Wiley, 1971.

4. Heathcote C.R., Seneta E., and Vere-Jones D. (1967) A Refinement of Two Theorems in the Theory of Branching Processes. *Theory Probab. Appl.* 12, 297-301.

5. Ivanoff G.B. and Seneta E. (1985) The Critical Branching Process with Immigration Stopped at Zero. *J. Appl. Prob.* 22, 223-227.

6. Kersting G. (1986) On Recurrence and Transience of Growth Models. *J. Appl. Prob.* 23, 614-625.

7. Koteeswaran, P. (1989) Estimating the Age of a Galton-Watson Process with Binomial Offspring Distribution. *Stoch. Anal. Appl.* 7, 413-423.

8. Levikson B. (1977) The Age Distribution of Markov Process. *J. Appl. Prob.* 14, 492-506.

9. Ligget T.M (1985) Interacting Particle Systems. *Springer.*

10. Pakes A. G. (1978) On the Age Distribution of a Markov Chain. *J. Appl. Prob.* 15, 65-77.

11. Seneta E. and Tavare S. (1983) A Note on the Models Using the Branching Process with Immigration Stopped at Zero. *J. Appl. Prob.* 22, 11-18.

12. Sevaststyanov B. A. and Zubkov A. M. (1974) Controlled Branching Processes. *Theory Probab. Appl.* 19, 1, 14-24.

13. Stigler, S.M. (1970) Estimating the Age of a Galton-Watson Branching Process. *Biometrika* 57, 505-512.

14. Vatutin V.A. (1977) A Conditional Limit Theorem for a Critical Branching Process with Immigration. *Mat. Zametki* 21, 727-736.

15. Yanev N. M. (1975) Conditions for Degeneracy of ϕ-Branching Processes with Random ϕ. *Theory Probab. Appl.* 20,2, 421-427.

16. Zubkov A. M. (1972) Life-periods of a Branching Process with Immigration. *Theory Probab. Appl.* 17, 174-183.

17. Zubkov A. M. (1974) Analogies Between Galton-Watson Processes and ϕ-Branching Processes. *Theory Probab. Appl.* 19, 2, 309-331.