HOPF ALGEBROIDS AND H-SEPARABLE EXTENSIONS

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Abstract. Since an H-separable extension $A|B$ is of depth two, we associate to it dual bialgebroids $S := \mathrm{End}\,{}_BA_B$ and $T := (A \otimes_B A)^B$ over the centralizer R as in [8, Kadison-Szlachányi]. We show that S has an antipode τ and is a Hopf algebroid. T^{op} is also Hopf algebroid under the condition that the centralizer R is an Azumaya algebra over the center Z of M. For depth two extension $A|B$, we show that $\mathrm{End}\,{}_AA \otimes_B A \cong T \ltimes \mathrm{End}\,{}_BA$.

1. Introduction and Preliminaries

There is a notion introduced in 1968 by Hirata [5] of an H-separable extension of noncommutative rings, which has been studied intensively in connection with simple rings, skew group rings and skew polynomial rings by S. Ikehata, K. Sugano, G. Szeto and others. In many ways H-separable extension has a theory parallel to that of depth two subfactors in von Neumann algebra theory, the explanation emerging that both are special cases of depth two ring extensions [8, Kadison-Szlachányi].

Hopf algebroids over noncommutative rings were introduced by Lu [9] in connection with quantization of Poisson groupoids in Poisson geometry. Examples of Hopf algebroids have also come from solutions to dynamical Yang-Baxter equations [3], although these are of a special self-dual type called weak Hopf algebras [1, 2]. A bialgebroid S, i.e., a Hopf algebroid without antipode, and its R-dual T has been associated with a depth two ring extension $A|B$ with centralizer R in [8]. S acts from the left on the over-ring A such that the right endomorphism ring is isomorphic to a smash product $A \rtimes S$ [8]. Moreover, T acts from the right on the left endomorphism ring $\mathcal{E}$ [8] such that the endomorphism ring $\mathrm{End}\,{}_AA \otimes_B A$ is similarly isomorphic to a smash product $T \ltimes \mathcal{E}$, as we show in this section.

In this paper we also show via the depth two theory in [8] that the bialgebroid S of an H-separable extension A over subring B has Hopf algebroid structure over R. If additionally R is Azumaya over the center of A, T is a Hopf algebroid as well. We summarize the results of the papers [7] and [8] together with this paper in the table below.

Let B be a unital subring of A, an associative noncommutative ring with unit. Recall that the ring extension $A|B$ is said to be of *depth two* if

$$A \otimes_B A \oplus * \cong \oplus^n A$$

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Depth Two Extension	*Centralizer*	*Dual Quantum Algebras with Galois Actions*
Biseparable Frobenius extension	Trivial	Semisimple Hopf algebra
Frobenius extension	Trivial	Hopf algebra
Frobenius extension	Separable	Weak Hopf algebra
H-separable extension	Azumaya	Hopf algebroid
Unrestricted	Unrestricted	Bialgebroid

as natural B-A and A-B-bimodules [8]. Equivalently, there are elements $\beta_i \in S := \mathrm{End}\,{}_BA_B$, $t_i \in T := (A \otimes_B A)^B$ (called a *left D2 quasibasis*) such that

$$a \otimes a' = \sum_i t_i \beta_i(a) a', \tag{1}$$

and a right D2 quasibasis $\gamma_j \in S$, $u_i \in T$ such that

$$a \otimes a' = \sum_j a \gamma_i(a') u_i.$$

We fix both D2 quasibases for our work below.

For example, an H-separable extension $A|B$ is of depth two since the condition above on the tensor-square holds even more strongly as natural A-A-bimodules [5]. Another example is a finitely generated (f.g.) projective algebra A over commutative ring B, since a left or right D2 quasibasis is easily constructed from a dual basis.

In [8] a bialgebroid with action-and-smash-product structure was uncovered on the Jones tower of a depth two ring extension $A|B$. In more detail, if R denotes the centralizer of B in A, a left R-bialgebroid structure on S is given by the composition ring structure on S with the left regular representation $\lambda : R \to S$ and right regular representation $\rho : R^{\mathrm{op}} \to S$. Since these commute ($\lambda(r)\rho(r') = \rho(r')\lambda(r)$ for every $r, r' \in R$), we may induce an R-bimodule structure on S solely from the left by

$$r \cdot \alpha \cdot r' := \lambda(r)\rho(r')\alpha = r\alpha(-)r'.$$

Now an R-coring ("co-ring" [13]) structure (S, Δ, ε) is given by

$$\Delta(\alpha) := \sum_i \alpha(-t_i^1)t_i^2 \otimes_R \beta_i \tag{2}$$

for every $\alpha \in S$, denoting $t_i = t_i^1 \otimes t_i^2 \in B$ by suppressing a possible summation, and

$$\varepsilon(\alpha) = \alpha(1) \tag{3}$$

satisfying the additional axioms of a bialgebroid [8, Section 4], such as multiplicativity of Δ and a condition that makes sense of this requirement. We have the equivalent formula for the coproduct [8, Th'm 4.1]:

$$\Delta(\alpha) := \sum_j \gamma_j \otimes_R u_j^1 \alpha(u_j^2 -) \tag{4}$$

The left action of S on A given by evaluation, $\alpha \triangleright a = \alpha(a)$, has invariant subring (of elements $a \in A$ such that $\alpha \triangleright a = \varepsilon(\alpha)a$) equal precisely to B if the natural module A_B is balanced [8]. This action is measuring since $\alpha_{(1)}(a)\alpha_{(2)}(a') = \alpha(aa')$ by Eq. (1). The smash product $A \rtimes S$ is isomorphic as rings to $\mathrm{End}\, A_B$ via $a \otimes_R \alpha \mapsto \lambda_a \alpha$ [8].

In general $T = (A \otimes_B A)^B$ has a unital ring structure induced from End ${}_A(A \otimes_B A)_A \cong T$ via $F \mapsto F(1 \otimes 1)$, which is given by

$$tt' = {t'}^1 t^1 \otimes t^2 {t'}^2$$

for each $t, t' \in T$. There are obvious commuting mappings of R and $R^{\rm op}$ into T given by $r \mapsto 1 \otimes r$ and $r' \mapsto r' \otimes 1$, respectively. From the right, these two "source" and "target" mappings induce the R-R-bimodule structure ${}_R T_R$ given by

$$r \cdot t \cdot r' = (t^1 \otimes t^2)(r \otimes r') = rtr',$$

the ordinary bimodule structure on a tensor product.

There is a right R-bialgebroid structure on T with coring structure (T, Δ, ε) given by

$$\Delta(t) = \sum_i t_i \otimes_R (\beta_i(t^1) \otimes_B t^2) \tag{5}$$

$$\varepsilon(t) = t^1 t^2 \tag{6}$$

By [8, Th'm 5.2] Δ is multiplicative and the other axioms of a right bialgebroid are satisfied.

There is a right action of T on $\mathcal{E} :=$ End ${}_B A$ given by $f \triangleleft t = t^1 f(t^2 -)$ for $f \in \mathcal{E}$. This is a measuring action by Eq. (1) since

$$(f \triangleleft t_{(1)}) \circ (g \triangleleft t_{(2)}) = \sum_i t_i^1 f(t_i^2 \beta_i(t^1) g(t^2 -)) = fg \triangleleft t.$$

The subring of invariants in $\mathcal{E}$ is $\rho(A)$ [8]. We next show that, in analogy with End $A_B \cong A \rtimes S$, the smash product ring $T \ltimes \mathcal{E}$ is isomorphic to End ${}_A A \otimes_B A$ via Ψ given by

$$\Psi(t \otimes f)(a \otimes a') = at^1 \otimes_B t^2 f(a'). \tag{7}$$

Proposition 1.1. *Ψ is a ring isomorphism $T \ltimes \mathcal{E} \xrightarrow{\cong}$ End ${}_A A \otimes_B A$.*

Proof. We let $\mu : A \otimes_B A \to A$ denote the multiplication mapping defined on simple tensors by $a \otimes a' \mapsto aa'$. Letting $F \in$ End ${}_A A \otimes_B A$, define

$$\Phi(F) = \sum_i t_i \otimes_R \mu(\beta_i \otimes A) F(1 \otimes -).$$

We check that $\Phi \circ \Psi = \mathrm{id}$: given $t \otimes f \in T \otimes_R \mathcal{E}$,

$$\sum_i t_i \otimes_R \mu(\beta_i \otimes A)(t^1 \otimes t^2 f(-)) = \sum_i t_i \beta_i(t^1) t^2 \otimes f = t \otimes f.$$

Next, given $F \in$ End ${}_A A \otimes_B A$, let $F^1(a) \otimes F^2(a) := F(1 \otimes a)$ noting that $F(a \otimes a') = aF^1(a') \otimes F^2(a')$. We check that $\Psi\Phi = \mathrm{id}$:

$$\Psi\Phi(F)(a \otimes a') = at_i^1 \otimes t_i^2 \beta_i(F^1(a'))F^2(a') = aF^1(a') \otimes F^2(a') = F(a \otimes a').$$

Thus Ψ is bijective linear mapping.

Using Eq. (5), we check that Ψ is a ring isomorphism:

$$\begin{aligned} \Psi((t \ltimes f)(t' \ltimes f'))(a \otimes a') &= \Psi(tt'_{(1)} \ltimes (f \triangleleft t'_{(2)})f')(a \otimes a') \\ &= at_i^1 t^1 \otimes t^2 t_i^2 \beta_i(t'_{(1)}) f(t'_{(2)} f'(a')) \\ &= at'_{(1)} t_{(1)} \otimes t_{(2)} f(t'_{(2)} f'(a')) \\ &= \Psi(t \ltimes f)\Psi(t' \ltimes f')(a \otimes a'). \quad \square \end{aligned}$$

Sweedler [13] has defined the left and right R-dual rings of an R-coring. In the case of a left R-bialgebroid H with H_R and ${}_RH$ finitely generated projective, such as $(S, \lambda, \rho, \Delta, \varepsilon)$ above, the left and right Sweedler R-dual rings are extended to right bialgebroids H^* and ${}^*H$ in [8]. For example, H^* has a natural nondegenerate pairing with H denoted by $\langle h^*, h\rangle \in R$ for $h^* \in H^*, h \in H$. Then the R-bimodule structure on H^*, multiplication, and comultiplication are given below, respectively, where $R \xrightarrow{s} H \xleftarrow{t} R^{\rm op}$ denotes the commuting morphism set-up of the bialgebroid H:

$$\langle r \cdot h^* \cdot r', h\rangle := r\langle h^*, at(r')\rangle \tag{8}$$

$$\langle h^* g^*, h\rangle := \langle g^*, \langle h^*, h_{(1)}\rangle \cdot h_{(2)}\rangle \tag{9}$$

$$\langle\langle h^*, hh'\rangle := \langle h^*{}_{(1)} \cdot \langle h^*{}_{(2)}, h'\rangle, h\rangle \tag{10}$$

Of course, the unit of H^* is ε_H while the counit on H^* is $\varepsilon(h^*) = \langle h^*, 1_H\rangle$. Eq. (9) is the formula for multiplication [13, 3.2(b)].

2. Frobenius D2 extensions

This section is independent the rest of the paper, although it explains the interest in certain constructions on $B \subseteq A$ we considered above. If $A|B$ is a Frobenius D2 extension, such as a depth two subfactor of finite index [8, 4], the various objects and results above come together into a Jones tower as follows. In the Frobenius case we have $A \otimes_B A \cong \operatorname{End} A_B$ via

$$a \otimes a' \mapsto \lambda_a \circ E \circ \lambda_{a'}$$

for some "Frobenius homomorphism" $E \in \operatorname{Hom}_{B-B}(A, B)$. Its inverse sends the identity into $\sum_i x_i \otimes y_i$ where $\{x_i\}$, $\{y_i\}$ are "dual bases" for $E : A \to B$ [6]. (This may be more familiar in the case B is a subring of the center of A.) If we denote $A_1 := A \otimes_B A$ with the induced "E-multiplication" from composition of endomorphisms, we see that A_1 has cyclic A-bimodule generator $e_1 := 1 \otimes 1$, and the tower of rings induced by $\lambda : A \hookrightarrow \operatorname{End} A_B \cong A_1$,

$$B \hookrightarrow A \hookrightarrow A_1$$

is isomorphic to

$$B \hookrightarrow A \hookrightarrow A \rtimes S$$

where the latter mapping is $a \mapsto a \rtimes 1$, via the isomorphism $A \rtimes S \xrightarrow{\cong} A_1$ given by

$$a \rtimes \alpha \mapsto \sum_i a\alpha(x_i)e_1 y_i.$$

It is a basic fact in Frobenius extension theory that the extension $A_1|A$ above is Frobenius as well — with canonical Frobenius homomorphism and dual bases $E_M := \mu : ae_1a' \mapsto aa'$, $\{x_ie_1\}$, $\{e_1y_i\}$ (several sources, e.g. [11]). If we now iterate the endomorphism ring construction above using the left, instead of right, endomorphism ring, we have an anti-isomorphism instead

$$A_2 := A_1e_2A_1 \to \operatorname{End}\, {}_AA_1$$

where $e_2 = 1 \otimes 1 \in A_1 \otimes_A A_1$, via $ze_2w \mapsto \rho(w)E_M\rho(z)$ for $z, w \in A_1$, with inverse given by $f \mapsto \sum_i x_ie_1e_2f(e_1y_i)$. Similarly we have an anti-isomorphism inducing $A_1^{\rm op} \cong \mathcal{E}$. Again we consider a tower of rings induced by the monomorphism $\lambda : A_1 \hookrightarrow A_2$, $w \mapsto w^1e_1w_21_{A_2} = w^1e_1e_2e_1w^2$:

$$A \hookrightarrow A_1 \hookrightarrow A_2$$

which is isomorphic to the tower induced by the anti-monomorphism $w \mapsto 1_T \ltimes w$,

$$A \hookrightarrow A_1 \hookrightarrow T \ltimes A_1^{\rm op}$$

via the anti-isomorphism $T \ltimes A_1^{\rm op} \cong A_2$ given by

$$t \ltimes w \mapsto w^1 e_1 e_2 t^1 e_1 t^2 w^2$$

where $w = w^1 e_1 w^2 \in A_1$ and $t \in T$.

3. Hopf algebroids and Lu's examples

With an adaptation to algebras over commutative ground rings, Lu's Hopf algebroid and examples are the following. Let K be a commutative ring. We briefly review the definition of *antipode* τ for a left bialgebroid $(H, R, \tilde{s}, \tilde{t}, \Delta, \varepsilon)$, where H and R are K-algebras and all four maps are K-linear. (H, R, τ) is a *Hopf algebroid* if $\tau : H \to H$ is an algebra anti-automorphism such that

1. $\tau\tilde{t} = \tilde{s}$;
2. $\mu(\tau \otimes H)\Delta = \tilde{t}\varepsilon\tau$;
3. there is a linear section $\eta : H \otimes_R H \to H \otimes_K H$ to the natural projection $H \otimes_K H \to H \otimes_R H$ such that:

$$\mu(H \otimes \tau)\eta\Delta = \tilde{s}\varepsilon.$$

Cf. [9, Section 4].

Lu's examples of bialgebroids and Hopf algebroids are the following. Given an algebra C over commutative ground ring K such that C is finitely generated projective as K-module, the following two are left bialgebroids over C (with $\otimes = \otimes_K$):

Example 3.1. The endomorphism algebra $E := \mathrm{End}_K C$ with $\tilde{s}(c) = \lambda(c)$, $\tilde{t}(c') = \rho(c')$, coproduct $\Delta(f)(c \otimes c') = f(cc')$ for $f \in \mathrm{End}_K C$ after noting that $E \otimes_C E \cong \mathrm{Hom}_K(C \otimes C, C)$ via $f \otimes g \mapsto (c \otimes c' \mapsto f(c)g(c'))$. The counit is given by $\varepsilon(f) = f(1)$. We see that this is the left bialgebroid S above when $B = K$, a subring in the center of A.

Example 3.2. The ordinary tensor algebra $C \otimes C^{\rm op}$ with $\tilde{s}(c) = c \otimes 1$, $\tilde{t}(c') = 1 \otimes c'$ with bimodule structure $c \cdot c' \otimes c'' \cdot c''' = cc' \otimes c''c'''$. Coproduct $\Delta(c \otimes c') = c \otimes 1 \otimes c'$ after a simple identification, with counit $\varepsilon(c \otimes c') = cc'$ for $c, c' \in C$. $C \otimes C^{\rm op}$ is a left C-bialgebroid by arguing as in [9], or [8, $N = K$] since $C|K$ is D2. In addition, $\tau : C \otimes C^{\rm op} \to C \otimes C^{\rm op}$ defined as the twist $\tau(c \otimes c') = c' \otimes c$ is an antipode satisfying the axioms of a Hopf algebroid (in addition, $\tau^2 = \mathrm{id}$, an *involutive* antipode).

A bialgebroid homomorphism from $(H_1, R_1, s_1, t_1, \Delta_1, \varepsilon_1)$ into $(H_2, R_2, s_2, t_2, \Delta_2, \varepsilon_2)$ consists of a pair of algebra homomorphisms, $F : H_1 \to H_2$ and $f : R_1 \to R_2$, such that four squares commute: $Fs_1 = s_2 f$, $Ft_1 = t_2 f$, $\Delta_2 F = p(F \otimes F)\Delta_1$ and $\varepsilon_2 F = f\varepsilon_1$, where f induces an R_1-R_1-bimodule structure on H_2 via "restriction of scalars," $p : H_2 \otimes_{R_1} H_2 \to H_2 \otimes_{R_2} H_2$ is the canonical mapping and $F : {}_{R_1}H_{1R_1} \to {}_{R_1}H_{2R_1}$ is a bimodule homomorphism since

$$F(r \cdot h \cdot r') = F(s_1(r)t_1(r')h) = s_2(f(r))t_2(f(r'))F(h) = r \cdot_f F(h) \cdot_f r'.$$

Proposition 3.3. *If $F : H_1 \to H_2$ and $f : R_1 \to R_2$ are ring isomorphisms and τ_1 is an antipode for H_1, then $\tau_2 := F\tau_1 F^{-1}$ is an antipode for H_2.*

The proof is an easy checking for τ_2 of the axioms above and therefore omitted. As an example of bialgebroid homomorphism with fixed base ring, let C be the algebra introduced above and $\tilde{F} : C \otimes C^{\text{op}} \to \text{End}_K C$ be defined by $\tilde{F}(c \otimes c')(c'') = cc''c'$. The following is consequence of the well-known Azumaya theorem (cf. [9, 3.8], also [6, 5.9]).

Proposition 3.4. *$F : C \otimes_K C^{\text{op}} \to \text{End}_K C$ is a bialgebroid isomorphism if C is an Azumaya K-algebra.*

4. H-separable extensions

Again let B be a subring of A with centralizer subring R, endomorphism ring $S = \text{End}\,{}_B A_B$ and ring $T = (A \otimes_B A)^B$. The definition and proposition below are due to [5, Hirata].

Lemma & Definition 4.1. *$A|B$ is H-separable if $A \otimes_B A \oplus * \cong \oplus^n A$ as A-A-bimodules. Equivalently, $A|B$ is H-separable if there are element $e_i \in (A \otimes_B A)^A$ and $r_i \in R$ (a so-called H-separability system) such that*

$$1 \otimes 1 = \sum_i r_i e_i. \tag{11}$$

For example, an Azumaya algebra $A|Z$ is H-separable [5]. We note that $e_i \in T$, and for $a, a' \in A$

$$a \otimes a' = \sum_i e_i \rho_{r_i}(a) a' = \sum_i a \lambda_{r_i}(a') e_i,$$

whence e_i, λ_{r_i} is right D2 quasibasis and e_i, ρ_{r_i} is left D2 quasibasis for $A|B$.

We next let Z denote the center of A.

Proposition 4.2. *If $A|B$ is an H-separable extension, then*

1. *R is f.g. projective Z-module;*
2. *$R \otimes_Z R^{\text{op}} \cong S$ via $r \otimes r' \mapsto \lambda_r \rho_{r'}$;*
3. *$T^{\text{op}} \cong \text{End}_Z R$ via $t \mapsto \lambda(t^1)\rho(t^2)$.*

Proof. We offer some short alternative proofs to these facts. R_Z is f.g. projective since for each $r \in R$, we note that $r = r_i e_i^1 r e_i^2$ where summation over i is understood and for each i, $r \mapsto e_i^1 r e_i^2$ defines a map in $\text{Hom}_Z(R, Z)$.

The inverse $S \to R \otimes_Z R^{\text{op}}$ to the mapping above is given by $\alpha \mapsto \alpha(e_i^1)e_i^2 \otimes r_i$, since $\alpha(e_i^1)e_i^2 a r_i = \alpha(a e_i^1)e_i^2 r_i = \alpha(a)$ $(a \in A)$, while $r e_i^1 r' e_i^2 \otimes_Z r_i = r \otimes e_i^1 r' e_i^2 r_i = r \otimes r'$ for $r, r' \in R$.

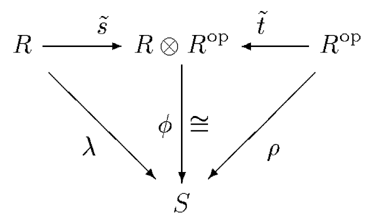
The inverse $\text{End}_Z R \to T^{\text{op}}$ to the second mapping above is given by $g \mapsto g(r_i)e_i$, since for each $t = t^1 \otimes t^2 \in T$, $t^1 r_i t^2 e_i = t^1 r_i e_i t^2 = t$, while for each $r \in R$ $g \in \text{End}_Z R$, $g(r_i)e_i^1 r e_i^2 = g(r_i e_i^1 r e_i^2) = g(r)$. □

5. When S and T are Hopf algebroids

Putting together Proposition 4.2.2 with Example 3.2 and the fact from Section 1 that $S = \text{End}\,{}_B A_B$ is a left bialgebroid, we are led to the following.

Theorem 5.1. *If $A|B$ is H-separable, then the isomorphism $\phi : R \otimes_Z R^{\text{op}} \to S$ given in Proposition 4.2.2 is an isomorphism of bialgebroids; whence S is a Hopf algebroid.*

Proof. It suffices by Proposition 3.3 to check the commutativity of four diagrams in the definition of bialgebroid homomorphism $\phi : (R \otimes R^{\rm op}, R, \tilde{s}, \tilde{t}, \Delta, \varepsilon) \to (S, R, \lambda, \rho, \Delta', \varepsilon')$. First, given $r \in R$, $\phi(\tilde{s}(r)) = \phi(r \otimes 1) = \lambda(r)$, and $\phi(\tilde{t}(r)) = \phi(1 \otimes r) = \rho(r)$.



Second, for $r, r' \in R$ we have $\varepsilon' \phi(r \otimes r') = rr' = \varepsilon(r \otimes r')$. Finally, we use the left D2 quasibasis $t_i \in T, \beta_i \in S$ with Eq. (2), then a canonical isomorphism to compute:

$$\begin{aligned}
(\phi^{-1} \otimes \phi^{-1})\Delta' \phi(r \otimes r') &= (\phi^{-1} \otimes \phi^{-1}) \sum_i r(-t_i^1) r' t_i^2 \otimes \beta_i \\
&= \sum_{i,j,k} r \underbrace{e_j^1 t_i^1 r' t_i^2 e_j^2}_{\text{in } Z} \otimes_Z r_j \otimes_R \beta_i(e_k^1) e_k^2 \otimes_Z r_k \\
&\xrightarrow{\cong} \sum_{i,j,k} r \otimes_Z e_j^1 t_i^1 r' t_i^2 e_j^2 r_j \beta_i(e_k^1) e_k^2 \otimes_Z r_k \\
&= \sum_{i,k} r \otimes t_i^1 r' t_i^2 \beta_i(e_k^1) e_k^2 \otimes r_k \\
&= \sum_k r \otimes e_k^1 r' e_k^2 \otimes r_k \\
&= \sum_k r \otimes 1 \otimes e_k^1 r' e_k^2 r_k = r \otimes 1 \otimes r'. \quad \square
\end{aligned}$$

It follows that the involutive antipode τ on S is given by $(a \in A, \alpha \in S)$

$$\tau(\alpha)(a) = \sum_i r_i a \alpha(e_i^1) e_i^2, \tag{12}$$

where r_i, e_i is an H-separability system. The antipode τ does not automatically transfer to an antipode on its R-dual right bialgebroid T. However, under the additional hypothesis that R is Azumaya over the center Z of A,[1] the isomorphisms in Proposition 3.4 and Proposition 4.2.3 lead us to the following.

Theorem 5.2. *If $A|B$ is H-separable with centralizer R an Azumaya Z-algebra, then $T^{\rm op}$ is a Hopf algebroid over R.*

Proof. We note that $(T^{\rm op}, R, \tilde{s}, \tilde{t}, \Delta', \varepsilon')$ is a left bialgebroid where the product on $T^{\rm op}$ is given by $tt' = t^1 t'^1 \otimes t'^2 t^2$, $\tilde{s}(r) = r \otimes 1$, $\tilde{t}(r) = 1 \otimes r$, which together induce from the left the ordinary R-bimodule structure on $(A \otimes_B A)^B$, Δ' given by Eq. (5) and ε' given by Eq. (6).

Since R is f.g. projective Z-algebra by Proposition 4.2, we have a left bialgebroid $(E := \mathrm{End}_Z R, R, \lambda, \rho, \Delta, \varepsilon)$ as in Example 3.1. But $R \otimes R^{\rm op} \cong E$ as bialgebroids by Proposition 3.4, whence E is a Hopf algebroid over R (with antipode induced by the

[1] It is interesting to note that these are the conditions on $A|B$ that ensure that $\mathrm{End}\, A_B | \lambda(A)$ is H-separable [12].

twist on $R \otimes R^{\mathrm{op}}$). It suffices to show that the algebra isomorphism $\psi : T^{\mathrm{op}} \to E$ given by $t \mapsto (r \mapsto t^1 r t^2)$ is a bialgebroid homomorphism (w.r.t. id_R).

It is clear that $\psi\tilde{s} = \lambda$, $\psi\tilde{t} = \rho$, and $\varepsilon\psi = \varepsilon'$. For the final computation, we note that $1 \otimes r_i \in R \otimes R$, $\eta_i : r \mapsto e_i^1 r e_i^2$ is a right D2 quasibasis for $R|Z$ where e_i, r_i is an H-separability system for $A|B$. Moreover, $e_i \in T$, $\rho_{r_i} \in S$ is a left D2 quasibasis for $A|B$ as noted in Section 4. Then by Eq. (4)

$$\begin{aligned} (\psi \otimes \psi)\Delta'(t) &= (\psi \otimes \psi)(\sum_i e_i \otimes_R (\rho_{r_i}(t^1) \otimes_B t^2)) \\ &= \sum_i e_i^1(-)e_i^2 \otimes t^1 r_i(-)t^2 \\ &= \Delta(\psi(t)). \quad \square \end{aligned}$$

If Z also coincides with the center of B, T^{op} possesses a weak Hopf Z-algebra structure [8, Prop. 9.4]. As a closing remark, it is not known on this date if the bialgebroids S or T^{op} associated to a Frobenius D2 extension are Hopf algebroids.

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