On singular perturbations of order s, $s \leq 2$, of the free dynamics: Existence and completeness of the wave operators

J. Brasche[1]

Abstract For a large class of operators H a formula for the resolvent will be derived and sufficient conditions will be given in order that the resolvent difference $(H-z)^{-1}-(H_0-z)^{-1}$ is compact resp. the wave operators $\Omega^{\pm}(H, H_0)$ exist and are complete; here H_0 denotes the free quantum mechanical Hamiltonian.

The mentioned class of operators contains, a.o., the generator of a Brownian motion with killing, the generator of the superposition of a Brownian motion and a diffusion process on a submanifold and Hamiltonians describing the interaction of a quantum mechanical particle with a potential supported by a set with classical capacity zero.

1 Introduction

In several areas the Laplacian serves as the (infinitesimal) generator of the free dynamics; e.g., it is both the generator of a Brownian motion in $\mathbb{R}^d$ and the free quantum mechanical Hamiltonian. In a wide variety of models one wants to modify the free dynamics inside a closed subset Γ of $\mathbb{R}^d$ with Lebesgue measure zero; important examples are, in particular,

a) a Brownian particle which can be killed inside Γ,
b) the superposition of a Brownian motion in $\mathbb{R}^d$ and a diffusion process inside Γ,
c) the interaction of a "quantum mechanical particle" with a potential supported by Γ.

There has been published an enormous number of papers discussing the cases a) and c) and it is impossible to give a survey on the existing literature; [2], [3], [4], [6] and [10] discuss some aspects in great detail and give quite large lists of references on the topic.

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It is especially challenging to study c) if the set Γ is not only small in the sense that its Lebesgue measure equals zero but even in the stronger sense that its c_1 – capacity equals zero; cf. the next section for the notations. In this case the expectation value of the energy of the system in the state f equals the expectation value of the kinetic energy if the latter expectation value is finite and there exist states with infinite expectation value for the kinetic energy but finite expectation value for the total energy. This quite unusual fact implies that the Hamiltonian H of the system cannot be constructed via a form perturbation of the free Hamiltonian and it was necessary to develop new methods in order to construct and investigate such Hamiltonians. Various methods of contruction have already been found and for point interactions, i.e. discrete exceptional sets Γ, a rich theory, described in detail in the monograph [3], has been created. However, in the general case only little is known about the spectral properties of such Hamiltonians or the unitary groups generated by them. The present note is one of the first steps in order to fill this gap.

Let us briefly describe the organisation of this contribution. In the next section we shall explain the notation and recall a few well known facts about the general framework. In section 3, we shall describe certain classes of operators H, including the generators of the stochastic processes mentioned at a) and b), and give an explicit expression for the resolvents of these operators.

The operators treated in section 3 are obtained via a form perturbation of the Laplacian. In section 4 we shall present a method to construct Hamiltonians describing an interaction which takes places on a set with c_1 – capacity zero. As mentioned such operators cannot be obtained via a form perturbation of the Laplacian.

In section 5 we shall discuss both the operators H treated in section 3 and the ones from section 4. We shall give a condition which is sufficient in order that the wave operators $\Omega^{\pm}(H, H_0)$ exist and are complete and consequently H's absolutely continuous part is unitarily equivalent to the free quantum mechanical Hamiltonian H_0. Compactness of the resolvent difference $(H-z)^{-1} - (H_0 - z)^{-1}$ will also be discussed.

2 Notation and general framework

By definition, the free quantum mechanical Hamiltonian is an operator H_0 in the space $L^2(\mathbb{R}^d, dx)$ (dx being the Lebesgue measure) and

$$\begin{aligned} D(H_0) &:= H^2(\mathbb{R}^d), \quad f \in D(H_0), \\ H_0 f &:= -\Delta f, \quad f \in D(H_0). \end{aligned}$$

Here we use the standard notation

$$\begin{aligned} H^s(\mathbb{R}^d) &:= \{f \in L^2(\mathbb{R}^d, dx) : \int (1+p^2)^s \, |\hat{f}(p)|^2 \, dp\}, \\ \| f \|_{H^s} &:= (\int (1+p^2)^s \, |\hat{f}(p)|^2 \, dp)^{1/2}, \end{aligned}$$

for $s > 0$; $\hat{f}$ denotes the Fourier transform of f.

For compact subsets K of $\mathbb{R}^d$ the c_s – capacity of K is defined by

$$c_s(K) := \inf\{\| f \|^2_{H^s} \colon f \in C_0^\infty(\mathbb{R}^d), \, \forall x \in K : f(x) \geq 1\},$$

and for arbitrary Borel sets B by

$$c_s(B) := \sup c_s(K)$$

where the supremum is taken over all compact subsets K of B. As a special case of a general result by Maz'ja and Havin [12] the following implications hold for closed subsets Γ of $\mathbb{R}^d$:

$$\dim(\Gamma) > d - 2s \Longrightarrow c_s(\Gamma) > 0 \Longrightarrow \dim(\Gamma) \geq d - 2s.$$

Moreover for C^1–submanifolds of $\mathbb{R}^d$

$$c_s(\Gamma) > d - 2s \Longleftrightarrow \dim(\Gamma) > d - 2s.$$

A function $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ is called quasi – continuous w.r.t. the c_s – capacity if and only if for every $\varepsilon > 0$ there exists an open set D_ε such that the restriction $f_{|\mathbb{R}^d \setminus D_\varepsilon}$ of f to the complement of D_ε is continuous and the c_s – capacity of D_ε is less than ε. It is known that every $f \in H^s(\mathbb{R}^d)$ has a representative which is quasi – continuous w.r.t. the c_s – capacity and that $\tilde{f} = f^\circ$ c_s – q.e. for any two quasi – continuous representatives $\tilde{f}$ and f° of

$f \in H^s(\mathbb{R}^d)$, i.e. $\tilde{f}$ and f° differ only on a set with c_s – capacity zero. In what follows $\tilde{f}$ will denote a representative of $f \in H^s(\mathbb{R}^d)$ which is quasi – continuous w.r.t. the c_s –capacity; we do not refer to s since it will be clear from the context which c_s – capacity is meant.

If a "quantum mechanical particle" only interacts with a potential which vanishes outside a given closed subset Γ of $\mathbb{R}^d$ with Lebesgue measure zero, then the Hamiltonian H of the system has to satisfy the following requirements:
(i) H is a self – adjoint operator in $L^2(\mathbb{R}^d, dx)$,
(ii) $C_0^\infty(\mathbb{R}^d \setminus \Gamma) \subset D(H)$,
(iii) $Hf = H_0 f = -\Delta f$ for all $f \in C_0^\infty(\mathbb{R}^d \setminus \Gamma)$,
(iv) $H \neq H_0$.
We denote by $\mathcal{A}_\Gamma$ the set of all operators H satisfying these four conditions.

Obviously $\mathcal{A}_\Gamma$ is not – empty if and only if the space $C_0^\infty(\mathbb{R}^d \setminus \Gamma)$ of smooth functions with compact support in the complement of Γ is dense in the Sobolev space $H^2(\mathbb{R}^d)$. Hedberg has given a characterization of the closures of $C_0^\infty(\mathbb{R}^d \setminus \Gamma)$ in the Sobolev spaces $W^{m,p}(\mathbb{R}^d)$ [9]. As a trivial consequence of his celebrated Theorem on the Spectral Synthesis in Sobolev Spaces we get that the space $C_0^\infty(\mathbb{R}^d \setminus \Gamma)$ is dense in $H^m(\mathbb{R}^d)$, $m = 1, 2, \ldots$, if and only if the c_m – capacity of Γ is strictly positive.

Let μ be a positive Radon measure on $\mathbb{R}^d$ such that $\mu(B) = 0$ for every set with zero c_2 – capacity and

$$\int |\tilde{f}|^2 d\mu < \infty, \quad f \in H^2(\mathbb{R}^d).$$

We define the mapping $J_\mu : H^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d, \mu)$ by

$$J_\mu f := \tilde{f}, \quad f \in H^2(\mathbb{R}^d).$$

Assume now that $\mu(B) = 0$ even for every Borel set B with c_1 –capacity zero. For every $\alpha > 0$ we denote by $J_{\mu,\alpha}$ the operator from $H^1(\mathbb{R}^d)$, equipped with the norm $(\int (p^2 + \alpha)\, |\hat{f}(p)|^2 dp)^{1/2}$, to $L^2(\mathbb{R}^d, \mu)$ which is defined as follows:

$$\begin{aligned} D(J_{\mu,\alpha}) &:= \{f \in H^1(\mathbb{R}^d) : \int |\tilde{f}|^2 d\mu < \infty\}, \\ J_{\mu,\alpha} f &:= \tilde{f}, \quad f \in D(J_{\mu,\alpha}). \end{aligned}$$

While the operators $J_{\mu,\alpha}$ equal each other their adjoints $J^*_{\mu,\alpha}$ may be different due to the different interpretation of the operators.

3 Form perturbations

Througout this section μ denotes a positive Radon measure on $\mathbb{R}^d$ such that $\mu(B) = 0$ for every Borel set B with c_1 – capacity zero and

$$\int |\tilde{f}|^2 d\mu < \infty, \quad f \in H^2(\mathbb{R}^d).$$

At various places additional requirements will be made.

There exists a unique self – adjoint operator H in $L^2(\mathbb{R}^d, dx)$ such that

$$\begin{aligned} D(H) &\subset \{f \in H^1(\mathbb{R}^d) : \int |\tilde{f}|^2 d\mu < \infty\}, \\ (f, Hf) &= \int |\nabla f|^2 dx + \int |\tilde{f}|^2 d\mu, \quad f \in D(H). \end{aligned}$$

This operator will be denoted by $-\Delta + \mu$ and plays a great role both in stochastics and quantum mechanics: It is the generator of a Brownian motion with killing ([8], § 4.5) and a Schrödinger operator with potential μ.

The resolvent of the operator $-\Delta + \mu$ can be described with the aid of the operators $J_{\mu,\alpha}$ and J_μ defined in the previous section:

Lemma 1 *(cf. [6], Lemma 3) For every $\alpha > 0$ the resolvent of the operator $-\Delta + \mu$ at $-\alpha$ is given by*

$$(-\Delta + \mu + \alpha)^{-1} = G_\alpha - (J_\mu G_\alpha)^*[1 + J_{\mu,\alpha} J^*_{\mu,\alpha}]^{-1} J_\mu G_\alpha.$$

Under the additional requirement that there exists an $\alpha > 0$ such that the operator norm $\| J_{\mu,\alpha} \|$ of the operator $J_{\mu,\alpha}$ is strictly less than one the operator $-\Delta - \mu$ can be defined in the analogous way and for $\alpha > 0$ satisfying $\| J_{\mu,\alpha} \| < 1$ the resolvent is given by

$$(-\Delta - \mu + \alpha)^{-1} = G_\alpha + (J_\mu G_\alpha)^*[1 - J_{\mu,\alpha} J^*_{\mu,\alpha}]^{-1} J_\mu G_\alpha,$$

cf. [6] (27).

Let $\rho : \mathbb{R}^{d-1} \longrightarrow \mathbb{R}$ be a bounded strictly positive continuous function. The quadratic form $\mathcal{E}_\rho$ on $L^2(\mathbb{R}^{d-1}, \rho\, d\xi)$, defined by

$$\begin{aligned} D(\mathcal{E}_{\rho,0}) &:= C_0^\infty(\mathbb{R}^d), \\ \mathcal{E}_{\rho,0}(u,u) &:= \int |\nabla u(\xi)|^2 \rho(\xi) d\xi, \end{aligned}$$

is closable and the self – adjoint operator L_ρ associated with its closure $\mathcal{E}_\rho$ generates a diffusion process on $\mathbb{R}^{d-1}$ ([8], § 4.5).

We can identify $\mathbb{R}^{d-1}$ with the hyperplane $\{0\} \times \mathbb{R}^{d-1}$ and $L^2(\mathbb{R}^{d-1}, \rho\, d\xi)$ with $L^2(\mathbb{R}^d, \mu)$, where $\mu := \delta_0 \otimes \rho\, d\xi$. Define the operator J^ρ from $H^1(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d, \mu)$ by

$$\begin{aligned} D(J^\rho) &:= \{f \in H^1(\mathbb{R}^d) : J_{\mu,\alpha} f \in D(\mathcal{E}_\rho)\}, \\ J^\rho f &:= \sqrt{L_\rho} J_{\mu,\alpha} f, \quad f \in D(J^\rho). \end{aligned}$$

By using the fact, that for every $f \in H^1(\mathbb{R}^d)$ the mappings $\eta \mapsto \tilde{f}(\eta, \xi)$ belong to $H^1(\mathbb{R})$ for $d\xi$ – a.e. $\xi \in \mathbb{R}^{d-1}$, we can show that the operator J^ρ from $H^1(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d, \mu)$ is closed and $D(J^\rho) \supset H^2(\mathbb{R}^d)$. This implies that there exists a unique self – adjoint operator H^ρ in $L^2(\mathbb{R}^d, dx)$ such that

$$\begin{aligned} D(H^\rho) &\subset D(J^\rho), \\ (f, H^\rho f) &= \int |\nabla f|^2 dx + \int |J^\rho f|^2 d\mu, \quad f \in D(H^\rho). \end{aligned}$$

By applying [6], Lemma 3, we get the following

Lemma 2 *For every $\alpha > 0$ the resolvent of the operator H^ρ is given by*

$$(H^\rho + \alpha)^{-1} = G_\alpha - (J_\mu G_\alpha)^* \sqrt{L_\rho}[1 + \sqrt{L_\rho} J_{\mu,\alpha} (\sqrt{L_\rho} J_{\mu,\alpha})^*]^{-1} \sqrt{L_\rho} J_\mu G_\alpha.$$

Remark: (i) Closability of the Markov form $\mathcal{E}_{\rho,0}$ defined as above, holds under much weaker conditions, cf. [1], § 4, [5], [8], § 3.1, [11], § II,2, and it is an interesting problem how to extend the last lemma.

(ii) The operators H^ρ are generators of the superposition of the Brownian motion in $\mathbb{R}^d$ and a diffusion process on the hyperplane $M = \{0\} \times \mathbb{R}^{d-1}$ ([8], §§ 3.1,4.5). It is noteworthy that these operators are obtained by a perturbation of H_0 by a second order differential operator on M.

4 A resolvent formula

Throughout this section Γ denotes a closed subset of $\mathbb{R}^d$ such that $c_2(\Gamma) > 0$. As mentioned in section 2 the set $\mathcal{A}_\Gamma$ is not empty under (and only under) this condition. Obviously the operators $-\Delta \pm \mu$ belong to $\mathcal{A}_\Gamma$ if $\mu \neq 0$ and $\mu(\mathbb{R}^d \setminus \Gamma) = 0$. Also the operators H^ρ discussed in the last lemma belong to $\mathcal{A}_M$ where $M = \{0\} \times \mathbb{R}^{d-1}$.

It is, however, not possible to construct operators in $\mathcal{A}_\Gamma$ via a form perturbation of H_0, as we did it in the last section, if the set Γ is so small that even its c_1 – capacity equals zero ([1]). There has been proposed many other methods of construction in this case. Here we will give a method which has the advantage that one gets from the very beginning important additional information about the operator.

Lemma 3 *Let μ be a positive Radon measure on $\mathbb{R}^d$ such that $\mu(B) = 0$ for every Borel set B with c_2 – capacity zero and*

$$\int |\tilde{f}|^2 d\mu < \infty, \quad f \in H^2(\mathbb{R}^d).$$

Let A be a non – negative self – adjoint operator in $L^2(\mathbb{R}^d, \mu)$ such that

$$D(A) \supset ran(J_\mu) \quad and \quad AJ_\mu \neq 0.$$

Let $\alpha > 0$. Then there exists a unique self – adjoint operator H in $L^2(\mathbb{R}^d, dx)$ such that $-\alpha$ is in the resolvent set of H and

$$(H + \alpha)^{-1} = G_\alpha + (J_\mu G_\alpha)^* A J_\mu G_\alpha.$$

H is different from H_0 and the infimum of the spectrum of H is strictly larger than $-\alpha$. If in addition $\mu(\mathbb{R}^d \setminus \Gamma) = 0$ then the operator H belongs to the set $\mathcal{A}_\Gamma$ defined in section 2.

Proof: The operator J_μ is closed since every sequence $\{f_n\}$ converging in $H^2(\mathbb{R}^d)$ has a subsequence $\{f_{n_j}\}$ such that $\{\tilde{f}_{n_j}\}$ converges c_2 – q.e. By the closed graph theorem, this implies that the operator $J_\mu G_\alpha$ is bounded. Since the operator A is self – adjoint and the domain of $AJ_\mu G_\alpha$ equals the whole space $L^2(\mathbb{R}^d, dx)$ also the operators $AJ_\mu G_\alpha$, $(J_\mu G_\alpha)^*$ and $(J_\mu G_\alpha)^* A$ are bounded.

Since

$$((J_\mu G_\alpha)^* A J_\mu G_\alpha f, f) = \| \sqrt{A} J_\mu G_\alpha f \|^2 \geq 0$$

for all $f \in L^2(\mathbb{R}^d, dx)$ the operator $(J_\mu G_\alpha)^* A J_\mu G_\alpha$ on $L^2(\mathbb{R}^d, dx)$ is non – negative, bounded and self – adjoint. Since $(f, G_\alpha f) > 0$ for all $f \neq 0$ it follows that the operator

$$R_\alpha := G_\alpha + (J_\mu G_\alpha)^* A J_\mu G_\alpha$$

on $L^2(\mathbb{R}^d, dx)$ is non – negative, bounded, self – adjoint and invertible. Thus the operator H in $L^2(\mathbb{R}^d, dx)$, defined by

$$H := R_\alpha^{-1} - \alpha,$$

is self – adjoint, $-\alpha$ belongs to the resolvent set of H, $(H+\alpha)^{-1} = R_\alpha$ and $H + \alpha$ is non – negative. Since 0 belongs to the resolvent set of $H + \alpha$ and $H + \alpha$ is non – negative the infimum of the spectrum of H is strictly larger than $-\alpha$.

Since $AJ_\mu \neq 0$ also $\sqrt{A} J_\mu \neq 0$. Thus there exist f such that

$$((J_\mu G_\alpha)^* A J_\mu G_\alpha f, f) > 0.$$

Thus $H \neq H_0$.

Suppose now in addition that $\mu(\mathbb{R}^d \setminus \Gamma) = 0$ for some closed set Γ. Then

$$J_\mu G_\alpha (H_0 + \alpha) f = 0$$

for all $f \in C_0^\infty(\mathbb{R}^d \setminus \Gamma)$. Thus

$$(H+\alpha)^{-1}(H_0 + \alpha) f = f$$

for all $f \in C_0^\infty(\mathbb{R}^d \setminus \Gamma)$. Thus $C_0^\infty(\mathbb{R}^d \setminus \Gamma) \subset D(H)$ and $Hf = H_0 f$ for all $f \in C_0^\infty(\mathbb{R}^d \setminus \Gamma)$. □

5 The wave operators

The resolvents of the operators H discussed in the previous sections "have the same structure" and one only needs some additional information about the measure μ appearing in the respective resolvent formulas in order to prove that the essential spectrum $\sigma_{ess}(H)$ of H equals $[0, \infty)$:

Theorem 4 *Let H be a self – adjoint operator in $L^2(\mathbb{R}^d, dx)$. Suppose that there exist an $\alpha > 0$, a positive Radon measure μ on $\mathbb{R}^d$ and a self – adjoint operator A in $L^2(\mathbb{R}^d, \mu)$ such that the following holds:*
(i) $-\alpha$ belongs to the resolvent set of H.
(ii) There exists an $s < 2$ such that $\mu(B) = 0$ for every Borel set B with c_s – capacity zero and

$$\int |\tilde{f}|^2 d\mu < \infty, \quad f \in H^s(\mathbb{R}^d).$$

(iii) A is a self – adjoint operator in $L^2(\mathbb{R}^d, \mu)$.
(iv)

$$(H + \alpha)^{-1} = G_\alpha + (J_\mu G_\alpha)^* A J_\mu G_\alpha.$$

(v)

$$\mu(\{y \in \mathbb{R}^d : |x - y| \leq 1\}) \longrightarrow 0, \quad as \quad |x| \longrightarrow \infty.$$

Then for every z which belongs both to the resolvent set of H and H_0 the operator $(H - z)^{-1} - (H_0 - z)^{-1}$ is compact and, in particular, $\sigma_{ess}(H) = [0, \infty)$.

Proof: In the proof of Lemma 3, we have shown that the operator $(J_\mu G_\alpha)^* A$ is bounded. Thus we have only to show that the operator $J_\mu G_\alpha$ is compact. By [6], Lemma 1, this holds true provided the operator $J_\mu G_\alpha^{s/2}$ is bounded and $J_\mu G_\alpha^l$ is compact for some $l \in \mathbb{N}$.

Since every convergent sequence in $H^s(\mathbb{R}^d)$ has a subsequence converging c_s – q.e., the hypothesis about the measure μ implies that J_μ is a densely defined closable operator from $H^s(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d, \mu)$. Since $G_\alpha^{s/2}$ is a bounded linear bijection from $L^2(\mathbb{R}^d, dx)$ to $H^s(\mathbb{R}^d)$ it follows that the operator $J_\mu G_\alpha^{s/2}$ from $L^2(\mathbb{R}^d, dx)$ to $L^2(\mathbb{R}^d, \mu)$ is bounded.

We choose any $l > d/2$. It is well known that G_α^l is an integral operator on $L^2(\mathbb{R}^d, dx)$ with a kernel $g_\alpha^{(l)}(x - y)$ where the function $g_\alpha^{(l)}$ decreases exponentially at infinity and, due to the choice $l > d/2$, is continuous. It follows that $J_\mu G_\alpha^l$ is an integral operator from $L^2(\mathbb{R}^d, dx)$ to $L^2(\mathbb{R}^d, \mu)$ with the same kernel. Now some standard argument yields compactness of the operator $J_\mu G_\alpha^l$. □

Under stronger conditions about the self – adjoint operator A and the behaviour of the measure μ at infinity we can even prove existence and completeness of the wave operators:

Theorem 5 *Let H be an operator satisfying the hypothesis of the previous theorem. Suppose in addition that the infimum of the spectrum of the operator H is strictly larger than $-\alpha$, the operator A is bounded and the measure μ is finite (here we use the notation from the previous theorem). Then the wave operators $\Omega^{\pm}(H, H_0)$ exist and are complete and, in particular the absolutely continuous part of H is unitarily equivalent to H_0.*

Proof: By [7], condition 2.6, we have only to prove that there exists an $N \in \mathbb{N}$ such that the operator

$$D_N := (H + \alpha)^{-N} - (H_0 + \alpha)^{-N}.$$

is compact and

$$(H + \alpha)^{-N} D_N (H_0 + \alpha)^{-N}$$

belongs to the trace class. By the previous theorem, the operator D_N is compact for every $N \in \mathbb{N}$.

We fix an integer $l > \max(d/2, 2)$. By the proof of Lemma 3, the operators $AJ_\mu G_\alpha$ and $(J_\mu G_\alpha)^* A$ are bounded. This implies that for arbitrary positive integer m and every $N > l(m+1)$ the operator D_N can be written as the sum of $2^N - 1$ operators T_j where for every j
(i) $T_j = R_j J_\mu G_\alpha^l S_j$ and R_j and S_j are bounded or
(ii) $T_j = R_j (J_\mu G_\alpha^l)^* S_j$ and R_j and S_j are bounded or
(iii)

$$T_j = R_1 R_2 \dots R_k$$

for some $k > m$ and some bounded operators $R_1, R_2, \dots, R_k$ where at least m of the operators $R_1, R_2, \dots, R_k$ equal $J_\mu G_\alpha$ or $(J_\mu G_\alpha)^*$.

As shown in the proof of the previous theorem, $J_\mu G_\alpha^l$ is an integral operator from $L^2(\mathbb{R}^d, dx)$ to $L^2(\mathbb{R}^d, \mu)$ with a kernel $g_\alpha^{(l)}(x - y)$ where the function $g_\alpha^{(l)}$ decreases exponentially at infinity and is continuous. Since the measure μ is finite and the function $g_\alpha^{(l)}$ square integrable w.r.t. the Lebesgue measure this implies that the operator $J_\mu G_\alpha^l$ belongs to the Hilbert – Schmidt class. Thus the operators T_j satisfying (i) or (ii) belong to the Hilbert – Schmidt class, too.

We have shown in the proof of the previous theorem that the operator $J_\mu G_\alpha^{s/2}$ from $L^2(\mathbb{R}^d, dx)$ to $L^2(\mathbb{R}^d, \mu)$ is bounded. This together with the fact that

$J_\mu G_\alpha^l$ is a Hilbert – Schmidt operator implies that the operator $J_\mu G_\alpha$ belongs to a Schatten class of finite order p (actually we can chose

$$p = 2 + 2\,\frac{l-1}{1-s/2},$$

cf. [6], Lemma 2, where even an upper bound for the p – th Schatten norm is given). Thus, by chosing m sufficiently large ($m \geq 2 + 2(l-1)/(1-s/2)$ is sufficient), we get that also the operators T_j satisfying the condition (iii) belong to the Hilbert – Schmidt class. Thus for sufficiently large N the operator D_N belongs to the Hilbert – Schmidt class, too.

We have

$$(H+\alpha)^{-N} D_N (H_0+\alpha)^{-N} = (D_N + G_\alpha^N) D_N G_\alpha^N .$$

We chose N such that both D_N and $J_\mu G_\alpha^N$ are Hilbert – Schmidt operators. Then $D_N D_N G_\alpha^N$ belongs to the trace class since D_N is a Hilbert – Schmidt operator. The operator $G_\alpha^N D_N G_\alpha^N$ can be written as the sum of $2^N - 1$ operators of the form

$$G_\alpha^N (J_\mu G_\alpha)^* S J_\mu G_\alpha G_\alpha^N$$

where the operator S is bounded. Since $J_\mu G_\alpha^N$ belongs to the Hilbert – Schmidt class each of these $2^N - 1$ operators belongs to the trace class. Thus $(H+\alpha)^{-N} D_N (H_0+\alpha)^{-N}$ belongs to the trace class, too. □

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Authors address and e-mail address:

J. Brasche
Department of Mathematics
Chalmers University of Technology
Eklandagatan 86
41296 Göteborg
Sweden
brasche@math.chalmers.se