On a kinetic approach to generalised curvature flows and its applications

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Abstract

We introduce a kinetic approach to approximation of generalised curvature flows. Basing on that a convolution-thresholding approximation scheme for the generalised mean curvature evolution is constructed. Conditions for the monotonicity of the scheme are found and the convergence of the approximations to the corresponding viscosity solution is proved. We also discuss some aspects of the numerical implementation of such schemes and present several numerical results.

Keywords: kinetic approach, curvature flow, convolution-thresholding scheme, viscosity solution, level-set equation

1 Background

Curvature flows of different types were during last 20 years and still are a popular topic both in pure and applied mathematics. By curvature flow we mean a family $\{\Gamma_t\}_{t\geq 0}$ of hyper-surfaces depending on time t with local normal velocity equal to the mean curvature or a function of it for generalised curvature flows.

In the three dimensional case a smooth initial surface can develop singularities after some finite time t_* (see [20], [33], Figure 1). After this moment, the classical motion by mean curvature is undefined. There have been several successful attempts to deal with singularities and topological complications: the varifold approach, the level-set approach and the phase field method.

A varifold generalisation of the problem was done by Brakke in [7] where an appropriate varifold subsolution for all $t > 0$ is constructed and regularity properties are established. However, as illustrated by Angenent, Ilmanen and Chopp in [2], even if the initial data is smooth, the evolution may possess non-uniqueness after the formation of a singularity.

An alternative approach to the extension of the surface evolution past singularities is the so called level-set approach. It was suggested in the physical literature [28] and was extensively developed for numerical purposes by Osher and Sethian [29]. The main idea of this method is to evolve some continuous function $u : \mathbb{R}^n \mapsto \mathbb{R}$ in such way that $\Gamma_t \subset \mathbb{R}^n$ would always be a level-set of $u(x,t)$ i.e. $\Gamma_t = \{x \in \mathbb{R}^n : u(x,t) = 0\}$ $\forall t \geq 0$. The evolution equation for u turns out to be

$$u_t = |Du|\, div \left(\frac{Du}{|Du|} \right). \qquad (1)$$

The evolution equation for a function u with each point of a level-set moving along the normal with velocity equal to some function G of the mean curvature is, so called generalised level-set equation, or generalised mean curvature evolution PDE

$$u_t = |Du|\, G \left(div \left(\frac{Du}{|Du|} \right) \right). \qquad (2)$$

Both equation (1) and (2) are invariant with respect to monotone transformations of u.

These equation are of degenerate parabolic type. The existence and uniqueness of generalised viscosity solutions (see [14]) to the Cauchy problem

$$\begin{cases} u_t = |Du|\, G \left(div \left(\frac{Du}{|Du|} \right) \right) & \text{in } \mathbb{R}^n \times (0,T) \\ u = g(x) \in BUC(\mathbb{R}^n) & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \qquad (3)$$

was investigated in [19], [13], [26].

Let us mention some applied problems, where curvature flow arise in a natural way. We begin with a fast reaction-slow diffusion problem

$$u_t = \epsilon \Delta u - \frac{1}{\epsilon} V_u(u) \quad u(x,0,\epsilon) = g(x) \quad \partial_\nu u = 0 \text{ on } \partial\Omega,$$

where the potential $V : \mathbb{R} \mapsto \mathbb{R}$ has several local minima $u_1, u_2, \ldots, u_k$. The formal asymptotic behaviour of u for small ϵ was studied in [31]. They showed that $u(x,t,\epsilon)$ tends to u_j at those points x where $g(x)$ is in the basin of attraction of u_j for the pure reaction equation $u_t = -\epsilon^{-1} V_u(u)$. The boundary

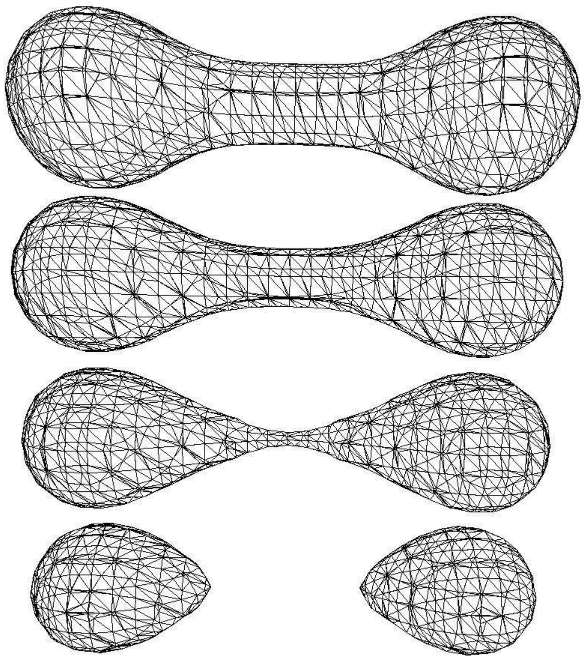

Figure 1: Example: topological changes during the evolution.

between two regions where the solution is equal to say u_i and u_j moves along the normal direction towards the region with greater u with the velocity that is proportional to $[V] = V(u_i) - V(u_i)$. In the situation when $[V] = 0$, the second order asymptotic analysis gives the velocity of the front equal to ϵk, where k is the mean curvature of the front. Connections between the reaction-diffusion problem and curvature flows have been rigorously investigated (see [3], [18], [22]). Moreover, such singular limits were used as a kind of definition for curvature flows past singularities in the phase-field approach [16], [8].

An important area, where an evolution of type (2) arises is image processing. It is common to represent a black and white image as a real function $u : \mathbb{R}^n \mapsto \mathbb{R}, n = 2, 3$. To reduce noise in an image one wishes to consider a family of smoothing operators $\{H_t\}_{t \geq 0}$, so called "scale space". The result by Alvarez et al. in [1] says, that if the scale space commutes with contrast changes and satisfies some stability and regularity properties, then $u(x,t) = H_t(u)$ is the viscosity solution of (3) for some nondecreasing continuous G and an initial image as g.

1.1 Approximate methods

In order to track the evolution of Γ_t or a function $u(x,t)$ defined by the Cauchy problem (3), one needs to construct numerical approximations $u_h(x,t)$ and prove, that these approximations converge to the unique viscosity solution $u(x,t)$ of (3).

In [29], the authors proposed a finite difference approximation scheme for the problem (3). Their technique is based on the use of a Hamilton- Jacobi formulation and the technology developed for the solution of hyperbolic conservation laws. This approach was further developed for a wide scale of applied problems (see [34]).

The convergence to viscosity solutions for a class of finite difference schemes was proved by Crandall and Lions in [15].

Another class of approximations, so called Matheron filters, comes from the image processing. Suppose $\mathbb{F} \in \mathbb{P}(\mathbb{R}^n)$, where $\mathbb{P}(\mathbb{R}^n)$ is a collection of all subsets of $\mathbb{R}^n$. Define

$$(Tu)(x) = \inf_{A\in\mathbb{F}} \sup_{y\in A} u(x+y). \tag{4}$$

The connection between such operators and the mean curvature evolution PDE (2) was established in [12]. The authors show how to choose $\mathbb{F}$ so, that suitably rescaled iterated Matheron filters would converge to the unique viscosity solution of (3). This result was then extended for a wider class of functions G in (3) by Guichard and Morel in [21] in the 2D case and by Cao in [9] in higher dimensions.

A class of geometric algorithms for tracking the evolution of a plane curve was proposed by Cao, Moisan and Lionel in [11] and [10] (for similar construction in higher dimensions see [24], [24]). This algorithm is general enough to resolve wide class of functions G.

Threshold dynamics models, introduced by Ishii, Pires and Souganidis in [25], lead to approximations of the solution of the Cauchy problem, where the right hand side can be interpreted as a general elliptic operator on a level set of the solution. This is a generalisation of the problem, but it does not entirely include (3) as a special case.

In the present work we construct a class of approximations of a convolution-thresholding type to the general curvature flows . We suggest here a flexible generalisation of this type of dynamics motivated by a kinetic approach that suites well for approximation of general curvature flows.

Let us consider a convolution generated motion of a hypersurface in $\mathbb{R}^n$. By this we mean the following. Assume, that initially the surface under consideration is a boundary of a compact set $C \in \mathbb{R}^n$. Take a compactly supported functions $\tilde{\rho}_i : \mathbb{R}_+ \longmapsto \mathbb{R}_+$ $i = 1, 2$ (in fact, one may also take $\tilde{\rho}_i$ with unbounded support, but fast decreasing for large x). We define $\rho_i : \mathbb{R}^n \mapsto \mathbb{R}_+$,

$$\rho_i(x) = \frac{1}{h^{n/2}} \tilde{\rho}_i \left(|x| / \sqrt{h} \right)$$

and introduce a convolution

$$M_i(C)(x,h) = \int_{\mathbb{R}^n} \chi_C(y) \rho_i(x-y) \, dy.$$

Now $M_i(C)(x,h)$ are function of x, and we can define a new position of the surface as a boundary of the set

$$\mathcal{H}_h C = \{ x \in \mathbb{R}^n : F(M_1(C)(x,h), M_2(C)(x,h)) \leq 0 \}, \tag{5}$$

where F is some (thresholding) function. The next step is to introduce an operator on the space of bounded functions $\mathbb{B}(\mathbb{R}^n)$: $H(h) : \mathbb{B}(\mathbb{R}^n) \mapsto \mathbb{B}(\mathbb{R}^n)$ by

$$[H(h)u](x) = \sup \{ \lambda \in \mathbb{R} : x \in \mathcal{H}_h [u \geq \lambda] \}. \tag{6}$$

The purpose of the present study is for a given function G in (3), to find a corresponding thresholding function F in (5), so that $H(t/m)^m g(x)$ would converge to the unique viscosity solution of (3) as $m \mapsto \infty$.

The solution of this problem in case when G is linear was proposed by Bence, Merriman and Osher in [5] (so called BMO method). A rigorous proof of the convergence of such approximations is due to Evans [17] and Ishii [23]. In this linear case it is enough to take a thresholding function depending only on one convolution.

Suppose, that G is non-linear. As we show in Section 3, in this case one has to use two convolutions M_1 and M_2 and a thresholding function depending on two variables $F(M_1, M_2)$. This is necessary to ensure that the operator H is consistent with the PDE in (3). We also show, how to choose convolution kernels in order to get a monotone H. These two conditions - monotonicity and consistency - are crucial for the convergence.

Using our approach we also suggest a construction of higher order schemes for the classical curvature flows. The numerical experiments with these schemes show a considerable improvement in the accuracy.

1.2 Kinetic generated schemes and convolution-thresholding dynamics

The following kinetic approximation scheme can serve as a useful physical interpretation of the convolution-thresholding approximation methods for generalised curvature flows that we suggest in the present paper.

Let $f(\tau, x, \xi)$ with $\xi \in \mathbb{R}^n$ be the distribution function having the sense of the amount of particles in the volume $dxdv$. The function $u(\tau, x) = \int f(\tau, x, \xi)\, d\xi$ has the sense of the mean density of the gas with distribution function f at position x at time τ. General mean values like $M(\tau, x) = \int f(\tau, x, \xi)\, \lambda(\xi)\, d\xi$ with a weight function $\lambda(\xi) \geq 0$ are ttraditionally interpreted as macro parameters for a gas with the distribution function f.

We consider the following BGK type model kinetic equation for $f(\tau, x, \xi)$ with collision term consisting of two qualitatively different terms

$$\frac{\partial f}{\partial \tau} + \xi \nabla_x f = \frac{1}{Kn}\left(u\rho(\xi) - f\right) + \frac{1}{\gamma} f(\xi)\, \mathcal{G}(u, M). \tag{7}$$

The first one is a usual relaxation term with normalised "Maxwellian" distribution $\rho(\xi)$, $\int \rho(\xi)\, d\xi = 1$. The second term describes a kind of chemical reaction that generates particles or eliminate them depending on the values of the density $u(\tau, x)$ and of some other macro-parameter $M(\tau, x)$.

The choice of the function $\mathcal{G}(\rho, M,)$ determines qualitative properties of solutions.

A standard method for solving an equation of kinetic type is the splitting method. Instead of solving equation (7) one can sequentially solve on small time steps $\Delta\tau$ the following simpler equations having a clear physical meaning:

$$\begin{aligned} \frac{\partial f_1}{\partial \tau} + \nabla_x \xi f_1 = 0, & \qquad - \text{collisionless flow,} \\ f_1|_{\tau=0} = F_0; & \\ \frac{\partial f_2}{\partial \tau} = \frac{1}{\gamma} f_2(\xi) \cdot \mathcal{G}(u, M), & \qquad - \text{ chemical reaction} \end{aligned} \tag{8}$$

$$
\begin{aligned}
f_2|_{\tau=\Delta\tau} &= f_1(\Delta\tau); \\
\frac{\partial f_3}{\partial \tau} = \frac{1}{Kn}(u\rho(\xi) - f_3), & \qquad - \text{ relaxation} \\
f_3|_{\tau=2\Delta\tau} &= f_2(2\Delta\tau);
\end{aligned}
$$

where the macro parameters u, M ingoing in the equations are calculated for the actual approximation, correspondingly f_1, f_2 , f_3.

We choose parameters γ and Kn small in comparison with $\Delta\tau$. The solutions of the second and the third equations of the splitting scheme will be after the time $\Delta\tau$ close to stationary solutions corresponding to zeroes of operators in the right hand sides of these equations. These zeroes determine geometric properties and the character of the whole process.

Zeroes of the linear operator $(u\rho(\xi) - f)$ are evidently all functions of the form $u(x)\rho(\xi)$. Depending on what kind of surface dynamics we want to model we will impose more concrete conditions for $\rho(\xi)$.

The integration of (8) with respect to ξ with weights 1 and $\lambda(\xi)$ gives the equations

$$
\begin{aligned}
\frac{\partial u}{\partial \tau} &= \frac{1}{\gamma} u \mathcal{G}(u, M) \\
\frac{\partial M}{\partial \tau} &= \frac{1}{\gamma} M \mathcal{G}(u, M)
\end{aligned}
$$

We choose the chemical reaction term $\mathcal{G}(u, M)$ such that this system of differential equations has two stable stationary points (u_1, M_1) and (u_2 , M_2) in the plane of macro parameters and one unstable stationary manifold determined by the equation $F(u, M) = 0$ and separating $\mathbb{R}^2$ in a part attracted to (u_1, M_1) and a part attracted to (u_2, M_2).

For $\gamma << 1$ the $u(\tau, x)$ and $M(\tau, x)$ reach after the finite time $\Delta\tau$ one of the stationary states depending on the initial data at the point x . Two sets with constant density values u_1, u_2 are formed with an interface set Γ that we shall use for the approximation of generalised curvature flows.

At the third step of the splitting method the distribution function evolves independently in all space points x tending to a locally equilibrium stage of the form depending on x only via the density $\rho(x)$. If the parameter $Kn << 1$, the equilibrium $\rho(x) M(\xi)$ is achieved after the time $\Delta\tau$.

Consider the family $\Gamma(\tau)$ of surfaces with initial state $\Gamma(0)$ such that $\Gamma(0) = \partial C_0$ is a boundary for a compact set C_0 . We use here the notation $\tau = \Delta\tau$.

Take the initial distribution function $f_1(0, x, \xi) = F_0(x, \xi)$ in the form

$$F_0(x, \xi) = u_0(x) \cdot \rho(\xi),$$

where u_0 is the characteristic function of the set C_0. We require here that the function $\rho(\xi)$ is non negative, dependent only on $|\xi|$ and has enough many moments.

We consider the function $f_1(\Delta\tau, x, \xi)$ and corresponding values of mean values ρ , M.

$$u_1(\tau, x) = \int_{R^n} u_0(x - \tau\xi)\, \rho(\xi)\, d\xi$$

The change of variables $y = x - \tau\xi$ gives

$$u_1(\tau, x) = \int_{R^n} u_0(y) \frac{1}{\tau^n} \rho((x - y)/\tau)\, dy = u_0 * m_\tau(x) \tag{9}$$

namely that the function $u_1(x)$ is a convolution of the initial density u_0 with the function $m_\tau(y) = \frac{1}{\tau^n}\rho(y/\tau)$.

The same proof shows similar convolution formulas for any weight $\lambda(\xi)$ with $m_\lambda(y) = \frac{1}{\tau^n}\lambda(y/\tau)\,\rho(y/\tau)$:

$$\begin{aligned} M_\lambda(\tau, x) &= \int_{R^n} \rho_0(x - \tau\xi)\, \lambda(\xi)\, \rho(\xi)\, d\xi = \\ &= \int_{R^n} \rho_0(y)\, \lambda((x - y)/\tau) \frac{1}{\tau^n} \rho((x - y)/\tau)\, dy = \\ &= (\rho_0 * m_{\lambda,\tau})(x) \end{aligned}$$

with $m_{\lambda,\tau} = \lambda(y/\tau) \cdot \dfrac{1}{\tau^n}\rho(y/\tau)$.

The distribution function $f_3(3\Delta\tau, x, \xi)$ has the form $f_3(3\Delta\tau, x, \xi) = u_3(x)\,\rho(\xi)$ where u_3 is a characteristic function of a compact domain where the thresholding function $F(u, M) \geq 0$.

We see that the kinetic splitting scheme generates a kind of convolution thresholding dynamics of surfaces with thresholding depending on two (or more) convolutions.

This paper is organised as follows. After introducing the basic notions and stating some results for viscosity solutions in Section 2, we turn to our method of approximation for such solutions (Section 1.1). The main result of the present work is described in Section 3, where we show how to construct F in order to get the convergence of the convolution-thresholding approximation to the viscosity solution of (3) with a monotone continuous function G. More precisely, the following local uniform convergence is proved

$$((H(t/m))^m g)(x) \mapsto u(x,t), m \mapsto \infty,$$

where H defined by (6) and $u(x,t)$ is the viscosity solution of

$$\begin{cases} u_t = |Du| G\left(div\left(\frac{Du}{|Du|}\right)\right) & \text{in } \mathbb{R}^n \times (0,T) \\ u = g(x) \in BUC(\mathbb{R}^n) & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

We use this construction for numerical calculation for some cases of the general curvature flows in $\mathbb{R}^2$ and $\mathbb{R}^3$. Numerical results and two approaches to the implementation are described in Section 4.

2 The viscosity solution framework

Consider the non-linear equation (2) in an open set $\Omega \times (0,T)$ with function G continuous and nondecreasing. We perform a differentiation and rewrite it in the following form:

$$u_t = |Du| G\left(\frac{1}{|Du|} tr\left(\left(I - \frac{Du \otimes Du}{|Du|^2}\right) D^2 u\right)\right) \tag{10}$$

This is the second order equation with right hand side that is monotonous and degenerate elliptic (see [14]) provided that G is nondecreasing and $Du \neq 0$. Although it is not defined for $Du = 0$, one still can define a viscosity solution to this equation. This was done by Evans and Spruck in [19] and by Chen, Giga and Goto in [13]. In our presentation we will use a somewhat more general definition of viscosity solutions introduced by Ishii and Souganidis in [26] to allow a wider class of functions G in (2). For general degenerate elliptic equation they consider a special class of test functions and adapt the definition of viscosity solution for possible singularities of the right hand side.

Let us begin by introducing an auxiliary subclass of $C^2([0,\infty))$. We say that $f : [0,\infty) \mapsto \mathbb{R}$ lies in $\mathcal{F} \subset C^2$ if $f(0) = f'(0) = f''(0) = 0$, $f''(r) > 0$ for

$r > 0$ and the following limits hold

$$\lim \frac{f'(|p|)}{|p|} G(p, I) = \lim \frac{f'(|p|)}{|p|} G(p, -I) = 0.$$

As was shown in [26], this set of functions is a non-empty cone, provided that the hight hand side lies in $C((\mathbb{R}^n \setminus \{0\}) \times \mathbb{S}(n))$. The class of test functions $\mathcal{A}(G)$ depends on G and is defined as follows.

Definition 1. *A function ϕ is admissible if it is in $C^2(\mathbb{R}^n \times (0,T))$ and for each $\acute{z} = (\acute{x}, \acute{t})$ where $D\phi(\acute{z}) = 0$, there is $\delta > 0$, $f \in \mathcal{F}$ and $\omega \in C([0,\infty))$ such that $\omega = o(r)$ and for all $() \in B(\acute{z}, \delta)$*

$$\left|\phi(x,t) - \phi(\acute{z}) - \phi_t(\acute{z})\left(t - \acute{t}\right)\right| \le f(|x - \acute{x}|) + \omega\left(\left|t - \acute{t}\right|\right).$$

Let us also denote by u^* and u_* the upper and lower semi-continuous envelopes of u:

$$u^*(x,t) = \limsup_{(y,s)\mapsto(x,t)} u(y,s), \quad u_*(x,t) = \liminf_{(y,s)\mapsto(x,t)} u(y,s)$$

The definition of viscosity solution becomes

Definition 2. *$u : \mathcal{O} \subset \mathbb{R}^n \times (0,T) \mapsto \mathbb{R} \cup \{-\infty\}$ is a viscosity subsolution of (10) in $\mathcal{O}$ if $u^* < \infty$ and for all $\phi \in \mathcal{A}(G)$ and all local maximum points (z_0, t_0) of $u^* - \phi$,*

$$\begin{cases} \phi_t(z_0,t_0) \le |D\phi(z_0,t_0)|\, G\left(div \dfrac{D\phi(z_0,t_0)}{|D\phi(z_0,t_0)|}\right) & \text{if } D\phi(z) \ne 0 \\ \phi_t(z_0,t_0) \le 0 & \text{otherwise.} \end{cases}$$

Likewise, $u : \mathcal{O} \mapsto \mathbb{R} \cup \{\infty\}$ is a viscosity supersolution in $\mathcal{O}$ if $u_* > -\infty$, and for all $\phi \in \mathcal{A}(G)$ and all local minimum points (z_0, t_0) of $u_* - \phi$,

$$\begin{cases} \phi_t(z_0,t_0) \ge |D\phi(z_0,t_0)|\, G\left(div \dfrac{D\phi(z_0,t_0)}{|D\phi(z_0,t_0)|}\right) & \text{if } D\phi(z) \ne 0 \\ \phi_t(z_0,t_0) \ge 0 & \text{otherwise.} \end{cases}$$

Consequently, a viscosity solution is a function that is sub- and supersolution simultaneously.

The result by Ishii and Souganidis presented in [26] can be restated in terms of the level-set equation (see [30]) as follows:

Theorem 1. *Assume, that G is continuous and nondecreasing. Then the initial value problem*

$$\begin{cases} u_t = |Du|\, G\left(div\left(\frac{Du}{|Du|}\right)\right) & in\ \mathbb{R}^n \times (0,T) \\ u = g(x) \in BUC(\mathbb{R}^n) & on\ \mathbb{R}^n \times \{0\} \end{cases}$$

has an unique viscosity solution $u \in BUC(\mathbb{R}^n \times (0,T))$.

In what follows, we also use another result by Ishii and Souganidis [26] concerning locally uniform perturbations of the right hand side of the equation. One can restate this result in case of (10) as follows (see [30]):

Theorem 2. *Assume, that G is continuous and nondecreasing. Suppose also, that $\{G_m\}_1^\infty$ is a sequence of continuous, nondecreasing functions on $\mathbb{R}$ and $G_m \mapsto G$ locally uniformly. Let for any m, $\mathcal{F}(G) \subset \mathcal{F}(G_m)$ and for any $f \in \mathcal{F}(G)$,*

$$\begin{aligned} \liminf_{p\mapsto 0,\ m\mapsto\infty} f'(|p|)\, G_m(1/p) &\geq 0 \\ (resp.\ \limsup_{p\mapsto 0,\ m\mapsto\infty} f'(|p|)\, G_m(-1/p) &\leq 0) \end{aligned}$$

Let u_m be a subsolution (resp. supersolution) of

$$\frac{\partial u_m}{\partial t} = |Du_m|\, G_m\left(div \frac{Du_m}{|Du_m|}\right) \ in\ \mathcal{O}.$$

Then

$$u^+(z) = \limsup_{r\mapsto 0} \{u_m(y),\, |y-z| \leq r,\ m > 1/m\} \tag{11}$$

$$(resp. u_+(z) = \liminf_{r\mapsto 0} \{u_m(y),\, |y-z| \leq r,\ m > 1/m\}) \tag{12}$$

is a subsolution (resp. supersolution) of (2) in $\mathcal{O}$ provided that $u^+ < \infty$ (resp. $u_+ > -\infty$).

3 A convolution-thresholding method for a generalised curvature flow

3.1 Convergence of approximation schemes

In what follows we make use of a theorem by Barles and Souganidis proved in [4]. In order to base the proof of our main result on this theorem, we follow

Pasquignon [30] and restate it in terms of the generalised mean curvature evolution PDE. Let $H(h)$ be the approximation operator i.e.

$$\begin{aligned} u_h(x,(n+1)h) &= H(h)\,u_h(x,nh) = H(h)^{n+1}\,u_0(x), \\ u_h(x,0) &= u_0(x). \end{aligned}$$

Definition 3.

1. *Consistency*
   An approximation operator $H(h)$, $h>0$ is consistent with

   $$\frac{\partial u}{\partial t} = |Du|\,G\left(div\,\frac{Du}{|Du|}\right),$$

   if for any $\phi \in C^{\infty}\left(\bar{\Omega}\right)$ and for any $x \in \bar{\Omega}$, the following holds,

   $$\frac{(H(h)\,\phi)(x) - \phi(x)}{h} = |D\phi|\,G\left(div\frac{Du}{|Du|}\right) + o_x(1) \ \ for \ \ D\phi \neq 0. \quad (13)$$

   If the convergence of $o_x(1)$ is locally uniform on sets, where $Du \neq 0$, then $H(h)$ is said to be uniformly consistent with the PDE.

2. *Monotonicity*
   An operator $H(h)$, $h>0$ is locally monotone if there exists $r>0$ such that for any functions $u,v \in \mathbb{B}\left(\bar{\Omega}\right)$ with $u \geq v$ on $B(x,r)\setminus\{x\}$, it holds

   $$H(h)\,u(x) \geq H(h)\,v(x) + o(h).$$

3. *Stability*
   An approximation scheme $H(h)$ is stable if $H(h)^n\,u \in \mathbb{B}\left(\bar{\Omega}\right)$ for every $u \in B\left(\bar{\Omega}\right), n \in \mathbb{N}, h>0$.

In this setting the result of Barles and Souganidis reads:

Theorem 3. *Consider a monotone, stable approximation operator $H(h)$ that commutes with additions of constants (i.e $H(h)(u+C) = H(h)\,u + C$, $\forall C \in \mathbb{R}$) and is consistent with (2). Suppose also, that*

$$\lim_{h \mapsto 0} \frac{H(h)\,(f(|x-x_0|))\,(x_0)}{h} = 0$$

for any $f \in \mathcal{F}(G)$. Then $u_h(x,nh)$ converges locally uniformly to the unique viscosity solution $u(x,t)$ of (2) as $nh \mapsto t$.

3.2 Properties of $\mathcal{H}$

Let us turn back to our main problem posed in Section 1.1. We consider a convolution generated motion of a hypersurface in $\mathbb{R}^n$ defined by (5) and the corresponding evolution of an initially bounded function $g : \mathbb{R}^n \mapsto \mathbb{R}$ defined by (6). Consider also the initial value problem (3) with given G and g. We are looking for such a thresholding function F in (5), that $H^m_{t/m} g(x)$ would converge (in some sense) to the unique viscosity solution of (3).

For example, set $F(M_1, M_2) = M_1 - 1/2$ and $\tilde{\rho}_1(x) = \frac{1}{(4\pi)^{n/2}} e^{-x^2/4}$ to get corresponding operators $\mathcal{H}_h$ and $H(h)$ by (5) and (6). Then we get the BMO procedure and the main result of [17] applies, and $H(h)^n u_0$ converges locally uniformly to the unique viscosity solution of (3) with $G(k) = k$.

We will see that, in fact, one has to use two convolutions M_1 and M_2 with different kernels and construct a thresholding function depending on two variables respectively to resolve this problem when G is not linear.

Let us now consider an operator $H(h)$ defined by (6) with help (5) of an operator $\mathcal{H}_h$ with an arbitrary thresholding function. We check whether such $H(h)$ satisfies the conditions of the Theorem 3. More precisely, we look for requirements on F sufficient to fulfil the conditions of Theorem 3.

1. Stability
   Suppose $u(x) \in \mathbb{B}(\mathbb{R}^n)$. We want to show, that $H(h)u \in \mathbb{B}(\mathbb{R}^n)$. Intuitively, we require
   $$\mathcal{H}_h \mathbb{R}^n = \mathbb{R}^n, \tag{14}$$
   $$\mathcal{H}_h \emptyset = \emptyset, \tag{15}$$
   and denote $A = \max|u|$. With these settings $[u \leq A] = \mathbb{R}^n$ and
   $$-A \leq H(h)u(x) = \inf\{\lambda \in \mathbb{R} : x \in \mathcal{H}_h[u \leq \lambda]\} \leq A.$$
   It remains to find out for which F the conditions (14) and (15) are satisfied. To do this, we substitute the corresponding sets into the definition of $\mathcal{H}$
   $$\begin{aligned} \mathcal{H}_h \mathbb{R}^n &= \{x \in \mathbb{R}^n : F(M_1\mathbb{R}^n(x,h), M_2\mathbb{R}^n(x,h)) \geq 0\} = \\ &= \left\{x \in \mathbb{R}^n : F\left(\int_{\mathbb{R}^n} \rho_1 dx, \int_{\mathbb{R}^n} \rho_2 dx\right) \geq 0\right\} = \mathbb{R}^n \\ \mathcal{H}_h \emptyset &= \{x \in \mathbb{R}^n : F(M_1\emptyset(x,h), M_2\emptyset(x,h)) \geq 0\} = \\ &= \{x \in \mathbb{R}^n : F(0,0) \geq 0\} = \emptyset. \end{aligned}$$

Thus, the requirements on F become

$$\begin{aligned} F\left(\int_{\mathbb{R}^n}\rho_1 dx, \int_{\mathbb{R}^n}\rho_2 dx\right) &\geq 0, \\ F(0,0) &< 0. \end{aligned}$$

2. Monotonicity
   Let us now show, that if $\mathcal{H}_h$ satisfies the so called inclusion principle, then H_h is monotonous.

   Lemma 1. *Assume, that $\mathcal{H}_h$ satisfies the inclusion principle i.e.*

   $$\forall C_1, C_2 \subseteq \mathbb{R}^n : C_1 \subseteq C_2 \text{ we have } \mathcal{H}_h C_1 \subseteq \mathcal{H}_h C_2,$$

   then H_h is monotone, that is

   $$\forall u, v \in \mathbb{C}(\mathbb{R}^n) : v \leq u \text{ we have } H_h(v) \leq H_h(u).$$

   Proof. Suppose, there exists x_0 s.t. $H(h)u(x_0) < H(h)v(x_0)$. We denote $\lambda_1 = H(h)u(x_0)$, $\lambda_2 = H(h)v(x_0)$ and $\epsilon = \frac{\lambda_2-\lambda_1}{2} > 0$. Since $\lambda_1 + \epsilon < \inf\{\lambda \in \mathbb{R} : x_0 \in \mathcal{H}_h[v \leq \lambda]\}$, $x_0 \notin \mathcal{H}_h[v \leq \lambda_1 + \epsilon]$, but $\mathcal{H}_h[v \leq \lambda_1 + \epsilon] \supseteq \mathcal{H}_h[u \leq \lambda_1 + \epsilon]$. Therefore $x_0 \notin \mathcal{H}_h[u \leq \lambda_1 + \epsilon]$, which is in contradiction with the definition of λ_1. □

3. Consistency
   We sum up some calculations in the following

   Lemma 2. *Let $\phi \in C^\infty(\mathbb{R}^n)$ $\phi(0) = 0$ and $D\phi(0) = (0, 0, \ldots, \beta)$. Then the consistency of an operator $H(h)$ with (3) is equivalent to*

   $$\gamma(0) = hG(-\Delta\gamma(0)) + o(h), \tag{16}$$

   where $x_n = \gamma(\acute{x})$ is a parametrisation of the surface

   $$\{x \in \mathbb{R}^n : u(x) = H(h)u(0)\}$$

   near $\acute{x} = 0$.

   We observe that in these settings $-\Delta\gamma(0) = k$ is the $(n-1)$ times the mean curvature of the graph of γ at the point $(0, \gamma(0))$.

Proof. Without loss of generality, one can consider the consistency condition (13) only for ϕ as in the statement. We rewrite (13) in a more convenient form

$$(H(h)\phi)(0) = h|D\phi(0)|\,G\left(\mathrm{div}\frac{Du}{|Du|}(0)\right) + o(h). \qquad (17)$$

Looking closer at $\mathrm{div}\frac{Du}{|Du|}$, we write

$$div\left(\frac{D\phi}{|D\phi|}\right) = \frac{1}{|D\phi|}\sum_{i,j=1}^{n}\left(\delta_{i,j} - \frac{\phi_{x_i}\phi_{x_j}}{|D\phi|^2}\right)\phi_{x_ix_j}.$$

Since $\phi(0) = 0$ and $\phi_{x_i}(0) = \delta_{ni}\beta$,

$$\begin{aligned}\mathrm{div}\frac{D\phi}{|D\phi|}\bigg|_{x=0} &= \frac{1}{\beta}\left[\sum_{i=1}^{n}\phi_{x_ix_i}(0) - \frac{\phi_{x_n}(0)\,\phi_{x_n}(0)}{\beta^2}\phi_{x_nx_n}(0)\right] = \\ &= \frac{1}{\beta}\Delta'\phi(0). \qquad (18)\end{aligned}$$

Here $\Delta'\phi = \sum_{i=1}^{n-1}\phi_{x_ix_i}$. Our next step is to take small $\acute{x}$, namely $|\acute{x}| < Rh$. For such $\acute{x}$ we apply the inverse function theorem to ϕ,

$$H(h)\phi(0) = \phi(\acute{x}, \gamma(\acute{x})) = \phi(0) + \beta\gamma(0) + O\left(h^2\right). \qquad (19)$$

Putting (19) and (18) into (17) we get

$$\gamma(0) = hG\left(\frac{1}{\beta}\Delta'\phi(0)\right) + o(h). \qquad (20)$$

Furthermore, differentiating both sides of $H(h)\phi(0) = \phi(\acute{x}, \gamma(\acute{x}))$ gives

$$\begin{aligned}\phi_{x_i} + \phi_{x_n}\gamma_{x_i} &= 0 \\ \phi_{x_ix_j} + \phi_{x_ix_n}\gamma_{x_j} + \phi_{x_nx_j}\gamma_{x_i} + \phi_{x_nx_n}\gamma_{x_j}\gamma_{x_i} + \phi_{x_n}\gamma_{x_ix_j} &= 0\end{aligned}$$

for $j, i = 1, \ldots, n-1$. We deduce $\gamma_{x_i}(0) = 0$ from the first equality and rewrite the second one for $i = j$

$$\phi_{x_jx_j}(0) + \phi_{x_n}(0)\,\gamma_{x_jx_j}(0) = 0.$$

After a summation over j this becomes

$$\frac{1}{\beta}\Delta' u(0) = -\Delta\gamma(0).$$

It remains to put this relation into (20) to get the desired equality (16). □

3.3 The convergence result for a wider class of G

In this subsection we construct the thresholding function $F(M_1, M_2)$ so, that the corresponding convolution thresholding scheme (5), (6) would give the convergence to the viscosity solution $u(x,t)$ of (3)

$$H^m_{\frac{t}{m}} g(x) \to u(x,t) \text{ as } m \mapsto \infty.$$

We start with $F(M_1 C(x,h), M_2 C(x,h))$, where

$$M_i C(x,h) = \int_C \rho_i (x-y)\, dy.$$

For each ρ_i we write (26), i.e.

$$M_i [\phi \le H(h)\phi(0)](0,h) = A_i + \sqrt{h} v C_i + \sqrt{h}\Delta\gamma(0) B_i + O\left(h^{3/2}\right), \quad (21)$$

where $i = 1, 2$. This is a system of linear algebraic equation for $\Delta\gamma(0)$ and v. We choose the kernels so, that the determinant of this system is positive

$$D = C_1 B_2 - C_2 B_1 > 0,$$

denote $N_i = M_i [\phi \le H(h)\phi(0)](0,h) - A_i$ and write the solution

$$\begin{aligned} v = \frac{\gamma(0)}{h} &= \frac{1}{\sqrt{h}} \frac{N_1 B_2 - N_2 B_1}{C_1 B_2 - C_2 B_1} + O(h), \\ \Delta\gamma(0) &= \frac{1}{\sqrt{h}} \frac{N_2 C_1 - N_1 C_2}{C_1 B_2 - C_2 B_1} + O(h). \end{aligned}$$

Looking back at the Lemma 2, we see, that the operator H will be consistent with the PDE in (3) if we take

$$\begin{aligned} F(N_1, N_2) &= v - G(-\Delta\gamma(0)) = \\ &= \frac{1}{\sqrt{h}} \frac{N_1 B_2 - N_2 B_1}{D} - G\left(\frac{1}{\sqrt{h}} \frac{N_1 C_2 - N_2 C_1}{D}\right). \end{aligned} \quad (22)$$

Let us now take a look at the inclusion principle, which implies the monotonicity of the scheme. In the case of thresholding function of one variable, the inclusion principle holds when F is nondecreasing. Analogously, in the case of two variables we require

$$\frac{\partial F}{\partial N_1} = \frac{B_2}{D} - \frac{C_2}{D} G' \ge 0, \quad (23)$$

$$\frac{\partial F}{\partial N_2} = -\frac{B_1}{D} + \frac{C_1}{D} G' \ge 0. \quad (24)$$

This implies

$$\frac{B_1}{C_1} \leq G' \leq \frac{B_2}{C_2}. \tag{25}$$

therefore, at least for a while, we consider G with bounded and positive derivative.

Next, we state some auxiliary results.

Lemma 3. *Suppose (23) and (24) hold and $\mathcal{H}$ is defined by (5), then $\forall h \in \mathbb{R}_+$,*

1. $\mathcal{H}(h)(\mathbb{R}^n) = \mathbb{R}^n$,
   $\mathcal{H}(h)(\emptyset) = \emptyset$,
2. $\forall a, b \in \mathbb{X} : a \subseteq b \Rightarrow \mathcal{H}(h)\, a \subseteq \mathcal{H}(h)\, b$.

Proof.

1. It is enough to show that $F(M_1(\mathbb{R}^n)(x,h), M_2(\mathbb{R}^n)(x,h)) \geq 0$ and $F(M_1(\emptyset)(x,h), M_2(\emptyset)(x,h)) < 0$. First we observe, that $F(A_1, A_2) = 0$, $M_i(\mathbb{R}^n)(x,h) \geq A_i$ and $M_i(\emptyset)(x,h) = 0 < A_i$. This, together with $\frac{\partial F}{\partial N_i} > 0$ gives the desired inequalities.
2. Since $M_i(b) \geq M_i(a)$, $F(M_1(b), M_2(b)) \geq F(M_1(a), M_2(a))$, therefore $[F(M_1(a), M_2(a)) \geq 0] \subseteq [F(M_1(b), M_2(b)) \geq 0]$, which is equivalent to $\mathcal{H}(h)\, a \subseteq \mathcal{H}(h)\, b$.

□

Proposition 1. *Define H by (6) and $\mathcal{H}$ by (5), then for each $h > 0$ and $u \in \mathbb{B}(\mathbb{R}^n)$ one has $H(h)\, u \in \mathbb{B}(\mathbb{R}^n)$.*

Proof. Without loss of generality we assume, that $S_1 \leq u(x) \leq S_2$ for some $S_1, S_2 \in \mathbb{R}$. From $\forall h \in \mathbb{R}_+ \quad \mathcal{H}(h)(\mathbb{R}^n) = \mathbb{R}^n$ and $\mathcal{H}(h)(\emptyset) = \emptyset$ follows $x \in \mathcal{H}(h)[u \leq S_2]$ and $x \notin \mathcal{H}(h)[u \leq S_1]$. Therefore, we see that:

$$S_1 \leq H(h)\, u(x) = inf\, \{\lambda \in \mathbb{R} : x \in \mathcal{H}(h)[u \leq \lambda]\} \leq S_2.$$

□

With the results above, we are ready to state the convergence of the approximations $H(t/m)^m g$ to the unique viscosity solution of (3).

Theorem 4. *Let* $H(h)$ *be defined by*

$$[H(h)u](x) = \sup\{\lambda \in \mathbb{R} : x \in \mathcal{H}_h[u \geq \lambda]\}$$

with

$$\mathcal{H}_h C = \{x \in \mathbb{R}^n : F(M_1(C)(x,h), M_2(C)(x,h)) \leq 0\},$$

where

$$F(N_1, N_2) = \frac{1}{\sqrt{h}} \frac{N_1 B_2 - N_2 B_1}{D} - G\left(\frac{1}{\sqrt{h}} \frac{N_2 C_1 - N_1 C_2}{D}\right),$$

where $\tilde{\rho}_1, \tilde{\rho}_2$ *have compact support, and* G *is continuous nondecreasing satisfying (25).*

Then

$$H^m_{\frac{t}{m}} g(x) \to u(x,t)$$

locally uniformly when $m \to \infty$*. Here* $u(x,t)$ *is the unique viscosity solution of*

$$\begin{cases} u_t = |Du|\, G\left(div\left(\frac{Du}{|Du|}\right)\right) & in\ \mathbb{R}^n \times (0,T) \\ u = g(x) \in BUC(\mathbb{R}^n) & on\ \mathbb{R}^n \times \{0\}. \end{cases}$$

with G *satisfying (25).*

Proof. Our aim is to show here that the operator $H(h)$ satisfies the conditions of the Theorem 3.

1. The monotonicity of H_h is ensured by Lemma 1 and Lemma 3.
2. The stability of H is exactly the result of the Proposition 1: $H(h)u \in \mathbb{B}(\bar{\Omega})$
3. Another property of $H(h)$ that is required is that it must commute with addition of constant, i.e:

   $$\forall a \in \mathbb{R} \quad H(h)(u(x) + a) = H(h)u(x) + a$$

   This equality can be easily obtained from the very definition of $H(h)$:

$$H(h)(u(x) + a) = inf\{\lambda \in \mathbb{R} : x \in \mathcal{H}(h)[u(x) + a \leq \lambda]\} =$$

$$= inf\{\beta + a \in \mathbb{R} : x \in \mathcal{H}(h)[u(x) \leq \beta]\} = H(h)u(x) + a$$

4. The limit we are interested in is:

$$\lim_{h\to 0} \frac{H(h)\,u(x_0)}{h} = 0$$

for u of the form $u(x) = f(|x - x_0|)$, where $f \in C^2([0,\infty))$ with $f(0) = f'(0) = f''(0) = 0$ and $f''(r) > 0$ for $r > 0$.

It is enough to show, that this is true for $x_0 = 0$. First, we observe, that $\mathcal{H}_h^{-1}[\{0\}] = \{u \leq \lambda_1\}$, where $\lambda_1 = H(h)\,u(0)$. Since both ρ_1 and ρ_2 have compact support, we can be sure, that there exists R s. t. $\left\{|x| \leq R\sqrt{h}\right\} \supseteq \mathcal{H}_h^{-1}[\{0\}]$. Now we observe, that $\left\{|x| \leq R\sqrt{h}\right\} = \{u \leq \lambda_2\}$ for some $\lambda_2 > \lambda_1$. From the latter equality we deduce $\lambda_2 = O\left(h^{\frac{3}{2}}\right)$ and conclude by

$$\lim_{h\to 0} \frac{H(h)\,u(x_0)}{h} \leq \lim_{h\to 0} \frac{O\left(h^{\frac{3}{2}}\right)}{h} = 0.$$

5. To see that our approximation operator is consistent with the PDE, we use Lemma 2. It is enough to prove the following

$$\gamma(0) = hG(-\Delta\gamma(0)) + o(h),$$

where $x_n = \gamma(\acute{x})$ is a parametrisation of the surface

$$\{x \in \mathbb{R}^n : u(x) = H(h)\,u(0)\}$$

near $\acute{x} = 0$. To show this, we use the fact that:

$$F\left(M_1[u \leq \mu], M_2[u \leq \mu]\right)\Big|_{x=0} = 0$$

We begin by writing the expressions for M_i in detail:

$$M_i = \left(\chi_{[u\leq\mu]} \star \frac{1}{h^{\frac{n}{2}}}\rho_i\left(\frac{|\cdot|}{\sqrt{h}}\right)\right)(0) = \int_{\mathbb{R}^n} \chi_{[u\leq\mu]}(y)\, \frac{1}{h^{\frac{n}{2}}}\rho_i\left(\frac{|y|}{\sqrt{h}}\right) dy =$$

$$= \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\gamma(\acute{y})} \frac{1}{h^{\frac{n}{2}}}\rho_i\left(\frac{|y|}{\sqrt{h}}\right) dy_n d\acute{y} = A_i + \int_{\mathbb{R}^{n-1}} \int_0^{\frac{1}{\sqrt{h}}\gamma\left(\sqrt{h}\acute{y}\right)} \rho_i(|y|)\, dy_n d\acute{y}.$$

Here

$$A_i = \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{0} \rho_i(|y|)\, dy_n d\acute{y}.$$

Expanding $\gamma(h\acute{y})$ in a Taylor series we get:

$$\frac{1}{\sqrt{h}}\gamma\left(\sqrt{h}\acute{y}\right) = \sqrt{h}\frac{\gamma(0)}{h} + \frac{\sqrt{h}}{2}\sum_{i,j=1}^{n-1}\gamma_{y_i y_j}(0)\, y_i y_j + \\ +\frac{h}{6}\sum_{i,j,l=1}^{n-1}\gamma_{y_i y_j y_l}(0)\, y_i y_j y_l + O\left(h^{3/2}\acute{y}^4\right).$$

Observing that $\gamma(0) = O\left(\sqrt{h}\right)$, we denote $\frac{\gamma(0)}{h} = v$. The expression for M_i becomes:

$$\begin{aligned} M_i &= A_i + \\ &+ \int_{\mathbb{R}^{n-1}} \rho_i(\acute{y}, 0)\left[\sqrt{h}v + \frac{\sqrt{h}}{2}\sum_{i,j=1}^{n-1}\gamma_{y_i y_j}(0)\, y_i y_j + O\left(h^{3/2}\acute{y}^4\right)\right] dy_n d\acute{y} \\ &= A_i + \sqrt{h}vC_i + \sqrt{h}\Delta\gamma(0)\, B_i + O\left(h^{3/2}\right). \end{aligned} \tag{26}$$

Here

$$C_i = \int_{\mathbb{R}^{n-1}} \rho_i(\acute{y}, 0)\, d\acute{y}, \tag{27}$$

$$B_i = \frac{1}{2}\int_{\mathbb{R}^{n-1}} y_k^2 \rho_i(\acute{y}, 0)\, d\acute{y}. \tag{28}$$

Remark 1. *At this point it is easy to see, that a scheme with a thresholding depending only on one variable can be consistent with the PDE (2) only in the case of a linear G. The thresholding condition on the front becomes*

$$F\left(A + \sqrt{h}vC + \sqrt{h}\Delta\gamma(0)\, B + O\left(h^{3/2}\right)\right) = 0.$$

As was required by the inclusion principle, the function F is non decreasing. This implies, that F' exists almost everywhere, and we can write:

$$F(A) + F'(A)\left(\sqrt{h}Cv + \sqrt{h}\Delta\gamma(0)\, B\right) + o(h) = 0.$$

Thus

$$v = \frac{\gamma(0)}{h} = -\frac{B}{C}\Delta\gamma(0) - \frac{F(A)}{\sqrt{h}CF'(A)} + o\left(\sqrt{h}\right).$$

Comparing this relationship with the one in Lemma 2, we see that the only G's we can resolve by thresholding dependent on one variable are the linear ones: $G(k) = const \cdot k + const$.

Let us denote $k = \Delta\gamma(0)$.

Now we can express v and k in terms of M_i and constants A_i, B_i and C_i :

$$v = \frac{1}{\sqrt{h}} \frac{N_1 B_2 - N_2 B_1}{C_1 B_2 - C_2 B_1} + O(h),$$

$$k = \frac{1}{\sqrt{h}} \frac{N_2 C_1 - N_1 C_2}{C_1 B_2 - C_2 B_1} + O(h).$$

Since $F(M_1, M_2) = v - G(-k) = 0$, we have:

$$\frac{\gamma(0)}{h} = v = G(-\Delta\gamma(0)) + o(1),$$

or

$$\gamma(0) = hG(-\Delta\gamma(0)) + o(h).$$

□

Remark 2. *As was already mentioned above, convolution kernels $\tilde{\rho}_i$ can also be taken with unbounded support. For example, the exponential decay for large arguments is sufficient in order for Theorem 4 to hold.*

The requirement (25) is quite restrictive. Our next result shows, that it is enough to take G_ϵ satisfying (25) and uniformly close to G in order to approximate the solutions of (3).

Proposition 2. *Suppose G_ϵ, G are continuous and $G_\epsilon \to G$ uniformly on $\mathbb{R}$ as $\epsilon \to 0$. Then $\mathcal{F}(G) = \mathcal{F}(G_\epsilon)$.*

Proof. Suppose $f \in \mathcal{F}(G)$. It means, that $f(0) = f'(0) = f''(0)$, $f(r) > 0$ for $r > 0$ and

$$\lim_{p \mapsto 0} f'(p)\, G\left(\frac{1}{p}\right) = \lim_{p \mapsto 0} f'(p)\, G\left(\frac{-1}{p}\right) = 0$$

Since $G_\epsilon \mapsto G$ uniformly, $G(k) = G_\epsilon(k) + o_\epsilon(1)\,\alpha(k)$, where $\alpha \in \mathbb{B}(\mathbb{R})$. We write

$$0 = \lim_{p \mapsto 0} f'(p)\, G\left(\frac{1}{p}\right) = \lim_{p \mapsto 0} f'(p)\left(G_\epsilon\left(\frac{1}{p}\right) + o_\epsilon(1)\,\alpha\left(\frac{1}{p}\right)\right) = \lim_{p \mapsto 0} f'(p)\, G_\epsilon\left(\frac{1}{p}\right)$$

to see, that $f \in \mathcal{F}(G_\epsilon)$.

The proof of the reverse inclusion is analogous. □

Lemma 4. *Suppose G_ϵ, G are nondecreasing continuous and $G_\epsilon \to G$ uniformly on $\mathbb{R}$ as $\epsilon \to 0$. Suppose also, that for each $\epsilon > 0$ the operator H_ϵ is monotone, stable, commuting with additions of constants and consistent with*

$$\frac{\partial u_\epsilon}{\partial t} = |Du_\epsilon|\, G_\epsilon\left(div \frac{Du_\epsilon}{|Du_\epsilon|}\right). \tag{29}$$

Additionally, let the following limit hold

$$\lim_{h \mapsto 0} \frac{H_h(h)\left(f\left(|x - x_0|\right)\right)(x_0)}{h} = 0 \tag{30}$$

for each $f \in \mathcal{F}(G)$. Then

$$H^m_{\frac{t}{m}}\left(\frac{t}{m}\right) u_0(x) \to u(x,t)$$

locally uniformly as $m \to \infty$, where $u(x,t)$ is the unique viscosity solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} = |Du|\, G\left(div \dfrac{Du}{|Du|}\right) & on \quad \mathbb{R}^n \times (0,T) \\ u(x,0) = u_0(x) & on \quad \mathbb{R}^n \times \{0\}. \end{cases} \tag{31}$$

Proof. We show here, that the operator $H_h(h)$ satisfies the conditions of the Theorem 3. This operator commutes with additions of constants and satisfies limit (30) by the assumption, therefore it is enough to check the stability, monotonicity and consistency.

1. Stability
Since the operator H_ϵ is stable for all $\epsilon > 0$, it is in particular stable for $\epsilon = h$ for each $h > 0$.

2. Monotonicity
Since the operator H_ϵ is monotonous for all $\epsilon > 0$, it is in particular monotonous

for $\epsilon = h$ for each $h > 0$.

3. Consistency
We have to show, that for each $\phi \in C^{\infty}(\mathbb{R}^n)$ at each point where $|D\phi| \neq 0$

$$H_h(h)\phi(x) - \phi(x) = h|D\phi(x)|G\left(div\frac{D\phi(x)}{|D\phi(x)|}\right) + o(h). \tag{32}$$

Since the operator H_ϵ is consistent with the equation (29) and $G_h(k) = G(y) + o_h(1)\alpha(k)$ for some $\alpha \in \mathbb{B}(\mathbb{R})$, we write

$$\begin{aligned} H_h(h)\phi(x) - \phi(x) &= h|D\phi(x)|G_h\left(div\frac{D\phi(x)}{|D\phi(x)|}\right) + o(h) = \\ = h|D\phi(x)|\left(G\left(div\frac{D\phi(x)}{|D\phi(x)|}\right)\right. &+ \left. o_h(1)\alpha\left(div\frac{D\phi(x)}{|D\phi(x)|}\right)\right) + o(h) = \\ &= h|D\phi(x)|G\left(div\frac{D\phi(x)}{|D\phi(x)|}\right) + o(h). \end{aligned}$$

□

Theorem 5. *Consider a convolution-thresholding scheme*

$$\begin{aligned} H_\epsilon(h)u(x) &= \inf\{\lambda \in \mathbb{R} : x \in \mathcal{H}_\epsilon(h)[u \leq \lambda]\} \\ \mathcal{H}_\epsilon(h)C &= \{x \in \mathbb{R}^n : F_\epsilon(M_1(C)(x,h), M_2(C)(x,h)) \geq 0\}, \end{aligned}$$

where the thresholding function $F_\epsilon(M_1, M_2)$ is choosen so, that the scheme monotone and is consistent with the equation (29) and the convolution kernels have compact support. If $G_\epsilon \mapsto G$ uniformly then

$$H^m_{\frac{t}{m}}\left(\frac{t}{m}\right)u_0(x) \to u(x,t)$$

locally uniformly as $m \to \infty$, where $u(x,t)$ is the unique viscosity solution of (31).

Proof. In order to establish the convergence by means of the Lemma (4) we have to show that the limit (30) holds. Let us set $x_0 = 0$, then the set $[f(|x|) \leq \lambda]$ is a ball centred at the origin with radius $O\left(\lambda^{1/3}\right)$. We denote $H_h(h)f(0) = \lambda_1$. Observe, that λ_1 can be characterised as a number for which $\mathcal{H}_h(h)[f \leq \lambda_1] = \{0\}$. Since we know, that $F_h(A_1, A_2) > 0$, the radius of $[f \leq \lambda_1]$ must be less or equal to the radius of the greatest support of the kernel: $O\left(\lambda_1^{1/3}\right) \leq R\sqrt{h}$. From this inequality we deduce $H_h(h)f(0) = \lambda_1 \leq O\left(h^{3/2}\right)$. This establishes the desired limit (30). □

Let us now consider $G(k) = k|k|^{\alpha-1}$ with $\alpha > 1$. We set

$$G_m(k) = \begin{cases} (1-\alpha) m^{\alpha} + \alpha m^{\alpha-1} k & \text{for } k < -n \\ m^{1-\alpha} k & \text{for } |k| < 1/n \\ -(1-\alpha) m^{\alpha} + \alpha m^{\alpha-1} k & \text{for } k > n \\ k|k|^{\alpha-1} & \text{elsewhere.} \end{cases}$$

G_m is continuous, increasing and its derivative is bounded below and above: $m^{1-\alpha} \leq G_m^{'} \leq \alpha m^{\alpha-1}$. Moreover, $G_m \mapsto k|k|^{\alpha-1}$ locally uniformly as $m \mapsto \infty$. Using Theorem 2 it is easy to show the following

Theorem 6. *Let u_m be the viscosity solution of*

$$\frac{\partial u_m}{\partial t} = |Du_m| G_m \left(div \frac{Du_m}{|Du_m|} \right) \quad in\ \mathcal{O},$$

where G_m is defined above. Then $u_m \mapsto u$ locally uniformly as $m \mapsto \infty$, where u is the viscosity solution of

$$\frac{\partial u}{\partial t} = |Du| G \left(div \frac{Du}{|Du|} \right) \quad in\ \mathcal{O},$$

with $G(k) = k|k|^{\alpha-1}$, alpha > 1.

Proof. First we establish the inclusion $\mathcal{F}(G) \subset \mathcal{F}(G_m)$. Take $f \in \mathcal{F}(G)$. By the definition of $\mathcal{F}(G)$, $f'(x) = o(x^{\alpha})$. This immediately gives

$$\lim_{p \mapsto 0} f'(p) G_m(1/p) = \lim_{p \mapsto 0} f'(p)/p = 0,$$

since $\alpha > 1$. We observe also, that the remaining conditions of Theorem 2 are satisfied. Hence a subsolution and a supersolution u^+ and u_+ can be constructed by means of (11) and (12). Since the equaton has the strong comparison property (see [14]), $u^+ = u_+$ and the result follows. ☐

Remark 3. *In a more general case when $G(k) = O(k^{\alpha})$, $\alpha > 1$, one can pick a sequence of increasing functions with derivative bounded below and above and apply the Theorem 2 to get a result similar to Theorem 6.*

4 Numerical implementation

This section is devoted to a description of our numerical implementations of the convolution-thresholding scheme developed in Section 3.

We will always consider the evolution of just one surface in $\mathbb{R}^n$, which is a level-set of a function $u : \mathbb{R}^n \mapsto \mathbb{R}$ satisfying the Cauchy problem (3).

Given a compact set $C \subset \mathbb{R}^n$, we fix convolution kernels ρ_1, ρ_2 and the time step h and approximate C_t at a time moment $t = mh$ by $(\mathcal{H}(h))^m C$. The algorithm of computations consists of the following steps:

1. Compute convolutions and the thresholding function

$$M_i C(x,h) = \int_{\mathbb{R}^n} \chi_C(y)\, \rho_i(x-y)\, dy \quad i = 1,2 \tag{33}$$

$$F(x,h) = F(M_1 C(x,h), M_2 C(x,h)). \tag{34}$$

2. Find the evolved set $\mathcal{H}(h)\, C = \{x \in \mathbb{R}^n : F(x,h) \geq 0\}$.
3. Repeat the procedure with the evolved set to get $\mathcal{H}^2(h)C$ and so on.

Due to the calculation of a convolution in (33) the main computational effort falls into the first step of the algorithm. We have implemented two different algorithms for the calculation of the convolution step.

4.1 Spatial discretization

We observe, that the surface under consideration is always an isosurface of some function. From the very beginning, it is some isosurface of the initial data of (2), but after each application of $\mathcal{H}(h)$ it is the zero isosurface of the thresholding function F: $\{x \in \mathbb{R}^n : F(M_1(x,h), M_2(x,h)) = 0\}$. We assume that initially our surface is closed and is contained in a unit cube.

In our implementation we use a modification of so called Marching Cubes algorithm for extracting an isosurface. It was originally proposed by Lorensen and Cline in [27]. This procedure was first applied for the mean curvature flow calculations by Ruuth in [32]. The purpose of the algorithm is to create an adaptive spatial discretization of the unit cube. For the sake of simplicity, we describe this procedure in the case $n = 2$ (see Fig. 4.1):

1. Choose a rough initial grid, i.e divide the unit square into $N_0 \times N_0$ equals squares and assign 1 to the points that are inside the curve and 0 to ones outside.

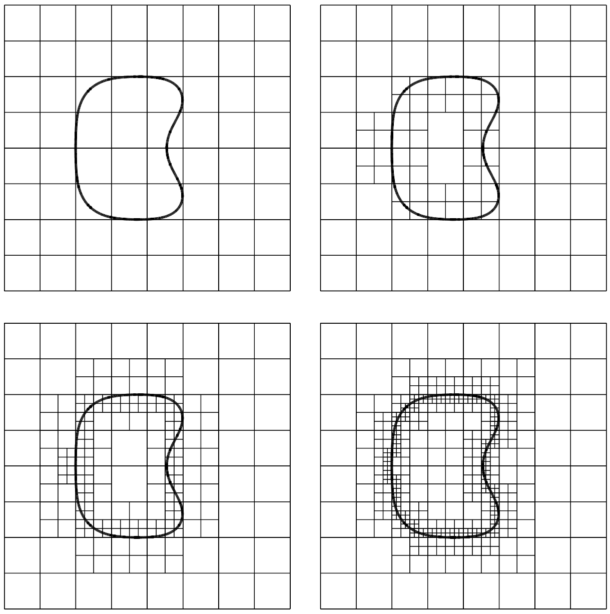

Figure 2: On the spatial discretization.

2. Divide each square that has different values assigned to its corners into 4 equal squares and assign the corresponding values to the new grid points that appear.

3. Refine also all the squares that have more than one grid point between their corners.

4. Repeat steps 2 and 3 while the side of the smallest square is longer than a desired spatial accuracy.

In doing so, we significantly reduce the number of grid points. Besides that, the accurate piecewise linear approximation of the curve (surface) can be arranged.

4.2 Spectral method

One can use Fourier series to calculate the convolutions (33). The numerical aspects of this has been presented by Ruuth in [32].

We expand both kernels ρ_i and the characteristic function of a set C into the

Fourier series to get the Fourier coefficients of convolutions $M_i(x,h)$. The surface position after time period h is an isosurface

$\{x \in \mathbb{R}^n : F(M_1(x,h), M_2(x,h)) = 0\}$.

The adaptive refinement procedure introduces one technical complication, namely: the standard discrete fast Fourier transform no longer applies, because the grid is not structured and not equally spaced. The way to proceed with the spectral method in this situation is in using unequally spaced approximate fast Fourier transform algorithm developed by Beylkin in [6]. The method involves projecting the functions on a subspace of multi-resolution analysis, and applies scaling functions from the spline family. The way to generalise the procedure to the functions on $\mathbb{R}^n$ is also given in [6].

As it was calculated in [32], the numerical cost of the transform algorithm combined with the Marching Cubes procedure is

$$O\left(m^n N_p + N_f^n \log\left(N_f\right)\right), \tag{35}$$

where m is a constant depending on a desired accuracy in the calculation of the Fourier coefficients (in case $m = 23$ the accuracy is comparable with the machine truncation error) and N_f is a number of the Fourier modes along each axis.

In the case of homogeneous grid the number of operations is

$$O\left(N_{ph} \log N_{ph}^{1/n}\right), \tag{36}$$

where N_{ph} is the number of points in the homogeneous grid. Let us choose a spatial step $\Delta x = 2^{-M}$. In order to have such discretisation, the number of grid points in the case of homogeneous grid is $N_{ph} = 2^{nM}$ while in the case of adaptive grid-$N_p = C' L 2^{(n-1)M}$, where L is $n-1$ dimensional Hausdorff measure of the boundary ∂C and C' is a constant of order one. Comparing (36) and (35), we see that in the case of adaptive discretisation one needs to perform $O\left(\frac{m^4 C'' L}{M 2^M} + \zeta\right)$ times less operations. Here $\zeta = \frac{N_f^n \log N_f}{2^{nM} M \log 2}$ is small. In our numerical experiments with different curves on the plane this acceleration factor turned out to be around $1:50$ on the first stages of the evolution. Since L decreases to zero as time progresses, the advantage of the adaptive grid becomes even bigger at large time values.

4.3 Direct method

If ρ_1 and ρ_2 are simple enough and have compact support, their convolutions with χ_C can be calculated explicitly. Let us choose

$$\begin{aligned}\tilde{\rho}_1(x) &= \begin{cases} \frac{1}{|\mathcal{B}_1|} & \text{if } x < 1 \\ 0 & \text{otherwise} \end{cases} \\ \tilde{\rho}_2(x) &= \frac{1}{\alpha^n}\tilde{\rho}_1\left(\frac{x}{\alpha}\right)\end{aligned}$$

where $|\mathcal{B}_1|$ is a Lebesgue measure of an unit ball in $\mathbb{R}^n$ and $\alpha \in \mathbb{R}_+,\ \alpha < 1$. In this case, the convolutions in (33) are proportional the measure of the intersection of C with a ball of radius proportional to $\sqrt{h}$ placed in the point x.

We present expressions for the thresholding function $F(M_1, M_2))$ in the case $n = 2$:

$$F(M_1, M_2) = v - G(k),$$

where

$$\begin{aligned} v &= \frac{\pi\alpha(2\alpha M_1 - 2M_2 - \alpha + 1)}{4\sqrt{h}(\alpha^2 - 1)} \\ k &= \frac{-3\pi(2M_1 - 2\alpha M_2 + \alpha - 1)}{2\sqrt{h}(\alpha^2 - 1).} \end{aligned}$$

In this case convolutions M_1 and M_2 can be calculated as follows. We represent C as a disjoint union of squares and triangles (or cubes in tetrahedron in case $n = 3$) using the adaptive grid procedure described in subsection 4.2 and calculate the area (volume) of intersection of the ball (suppρ) with each square and triangle. The numerical cost of each step of the evolution can be estimated by $O(N_p * N_i + N_p)$, where N_i is the number of points inside the ball of radius dt with the center at some grid point. When dt is large, the accuracy of the method is low, therefore one can take less grid points. Thus, N_i is entirely determined by the desired accuracy.

4.4 Computed examples

We begin by presenting some non-trivial examples of applications of the above algorithms in case of curve evolution.

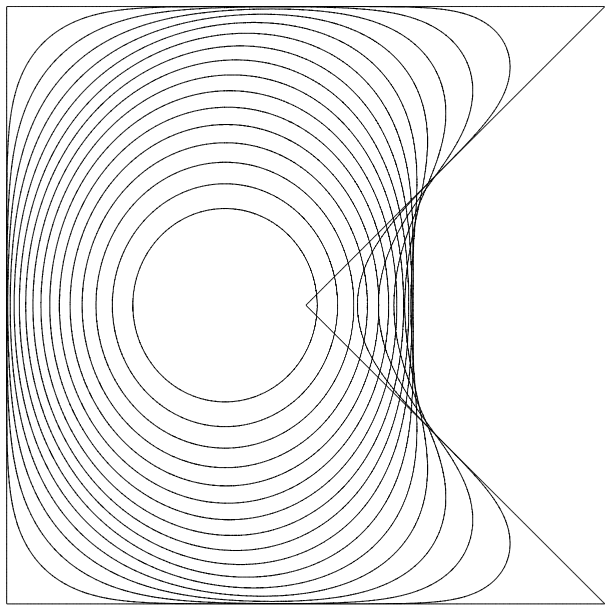

Figure 3: The mean curvature evolution of a non-smooth, non-convex curve

Let us first look at the mean curvature evolution. In this case, according to the Von Neumann-Mullins parabolic law, the area enclosed by a simple curve which moves by mean curvature decays linearly in time i.e.

$$\frac{dS}{dt} = -2\pi.$$

Consider a non-convex, non-smooth initial curve depicted on Fig 3. The mean curvature evolution of this curve was calculated using the direct method with timestep values $dt = 1/600$ and $1/6000$. The shape of the curve is plotted on the Fig. 3 for times $t = 1/600, 2/600, \cdots$, when calculated with the fine timestep. The comparison between local relative errors

$$e_i = \frac{|S_i - S_{i+1} - 2\pi dt|}{2\pi dt} \tag{37}$$

for calculations with different timesteps is seen on the Fig. 4. One can observe, that the error indeed depends linearly on the timestep: taking ten times smaller timestep we achieve ten times better accuracy.

The local area error dependence on time (37) depicted on the fig. 4 is in a sense typical. Since the curve is not smooth initially, we obtain quite high

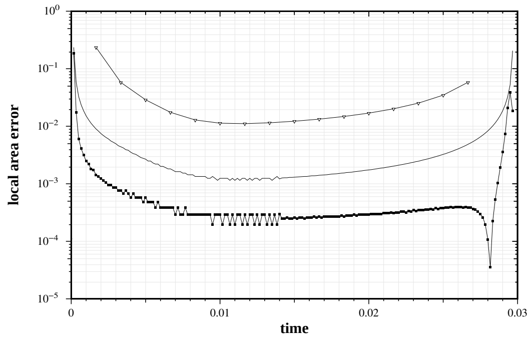

Figure 4: Local area error dependence on time. The first order method with timestep 1/600 – the line with triangle markers; the first order method with timestep 1/6000 – the thin line; the second order method with timestep 1/6000 – the line with square markers.

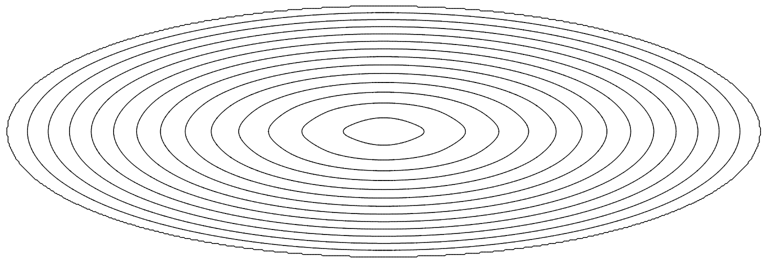

Figure 5: The evolution $v = k^{1/3}$ of an ellipse.

e_i values for the first several time steps. Calculating further, the error stays small until the curve turns into a small circle and begins to collapse rapidly.

Let us illustrate the evolution with the velocity $v = k^{1/3}$. In this case the flow is affine invariant [1], hence the eccentricity e of the evolving ellipse remains constant. The evolving ellipse is depicted on Figure 5.

Remark 4. *Function $G(k) = k^{1/3}$ has unbounded derivative at 0 and also $G' \mapsto 0$ as $|k| \mapsto \infty$. Thus, the convergence theorems 4 and 5 do not apply. However, the curvature of an ellipse is bounded above and below by positive constants. Furthermore, it is bounded by positive constants during the* *evolution for some time period.*

$$\epsilon \leq k(x,t) \leq 1/\epsilon \;\; \forall x \in \mathcal{E}, t \in (0,T) \tag{38}$$

Therefore, in order to track the evolution of an ellipse with $v = k^{1/3}$*, it is enough to construct a convolution thresholding scheme for a function* G_ϵ *which is equal to* $k^{1/3}$ *only on the interval* $(\epsilon, 1/\epsilon)$ *and has an arbitrary behaviour near the origin and at the infinity.*

In figures 1, 6 - 8 computed 3D mean curvature evolution of non-convex surfaces is represented.

4.5 On the higher order schemes for the mean curvature motion

Let us now look at approximations to the mean curvature evolution. It is easy to see, that if the surface is smooth, the BMO method gives the first order approximation in time. A higher order scheme in time was proposed by Ruuth in [32]. The author uses an extrapolation argument to obtain the results. We show here, how to construct higher order approximations to the mean curvature evolution using some properties of the convolution kernels.

For the sake of simplicity, we present the construction in $\mathbb{R}^2$.We begin by writing the system of linear algebraic equations (21), where each equation is a relationship between the convolution value, mean curvature and the velocity on the evolved front. However, now we will keep an additional term in each equation with the fourth order derivatives to have the error $O\left(t^{5/2}\right)$:

$$\begin{aligned} M_1 &= A_1 + \sqrt{h} v C_1 + \sqrt{h}\gamma''(0) B_1 + h\sqrt{h}\gamma''''(0) E_1 + O\left(h^{5/2}\right) \\ M_2 &= A_2 + \sqrt{h} v C_2 + \sqrt{h}\gamma''(0) B_2 + h\sqrt{h}\gamma''''(0) E_2 + O\left(h^{5/2}\right) \end{aligned}$$

We multiply the first equation by E_2, the second by E_1 and subtract one from another to obtain

$$E_2 N_1 - E_1 N_2 = \sqrt{h}[(E_2 C_1 - E_1 C_2)v + (E_2 B_1 - E_1 B_2)\gamma''(0)] + O\left(h^{5/2}\right).$$

This relationship motivates to take the thresholding function $F(N_1, N_2) = E_2 N_1 - E_1 N_2$ to have the mean curvature evolution with the second order accuracy for smooth curves. However this thresholding function does not simultaneously satisfy (23) and (24) and, therefore, the stability of the numerical scheme is not guaranteed by the previous argumentation.

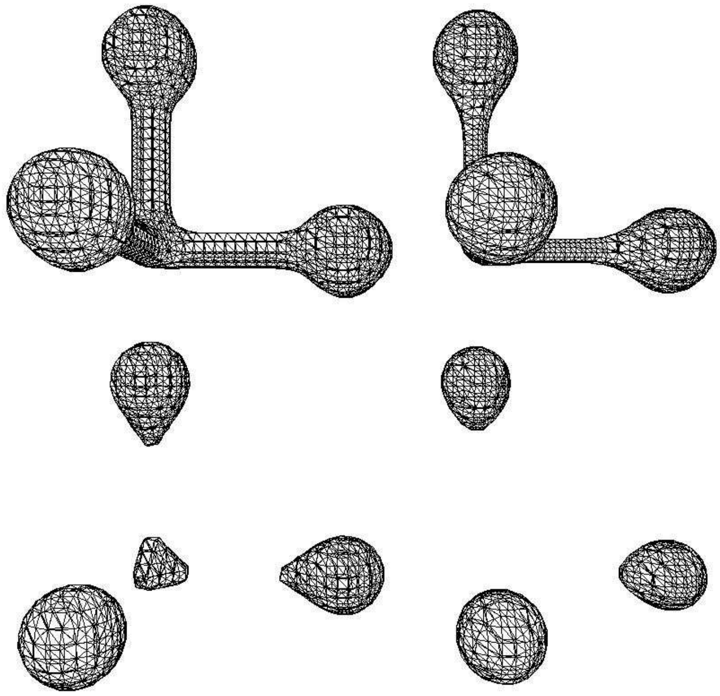

Figure 6: Computed mean curvature evolution

The calculations with the above thresholding function were performed. No sign of instability was observed in the numerical experiments and, as one can see on the Fig. 4, the accuracy was increased.

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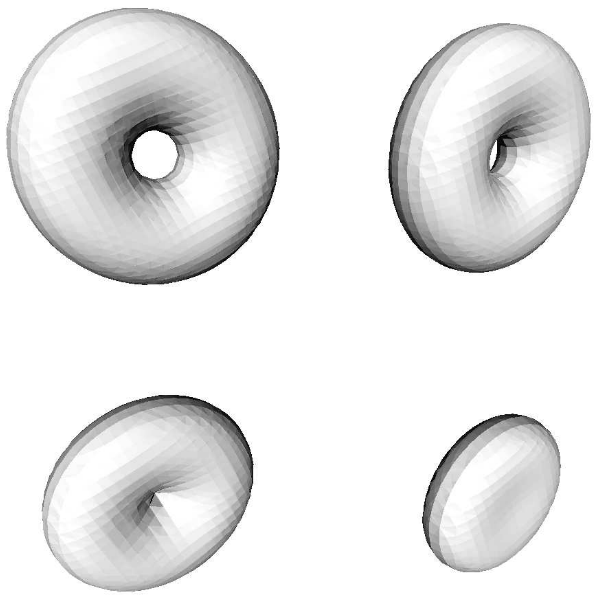

Figure 7: Computed mean curvature evolution

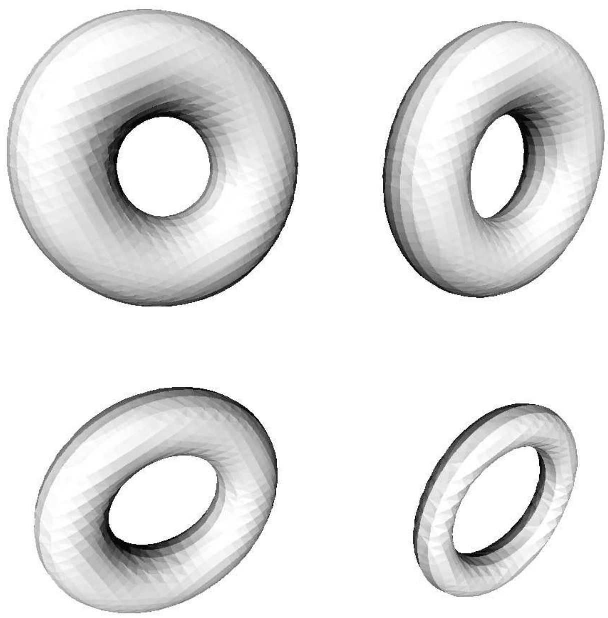

Figure 8: Computed mean curvature evolution

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