Operator synthesis. I.
Synthetic sets, bilattices and tensor algebras.

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1 Introduction

The classical notion of spectral synthesis is related to the Galois correspondence between ideals J of a commutative regular Banach algebra $\mathcal{A}$ and closed subsets E of its character space $X(\mathcal{A})$: $\ker J = \{t \in X(\mathcal{A}) : t(a) = 0, \text{ for any } a \in J\}$, $\text{hull } E = \{a \in \mathcal{A} : t(a) = 0, \text{ for any } t \in E\}$. Namely, a set E is called synthetic (or a set of spectral synthesis) if $\ker J = E$ implies $J = \text{hull } E$. Note, that the converse implication holds for any closed $E \subseteq X(\mathcal{A})$.

In the invariant subspace theory the central object is a Galois correspondence between operator algebras $\mathcal{M}$ and strongly closed subspace lattices $\mathcal{L}$: $\text{lat } \mathcal{M} = \{L : TL \subseteq L, \text{for any } T \in \mathcal{M}\}$, $\text{alg } \mathcal{L} = \{T : TL \subseteq L, \text{for any } L \in \mathcal{L}\}$. A lattice $\mathcal{L}$ can be called *operator synthetic* if $\text{lat } \mathcal{M} = \mathcal{L}$ implies $\mathcal{M} = \text{alg } \mathcal{L}$.

W.Arveson [A] proved that if one restricts the map lat to the variety of algebras, containing a fixed maximal abelian selfadjoint algebra (masa), then the above formal analogy becomes very rich and fruitful. In particular, answering a question of H.Radjavi and P.Rosenthal, he proved the failure of operator synthesis in the class of σ-weakly closed algebras, containing masa (Arveson algebras, in terminology of [ErKS]), by using the famous L.Schwartz's example of a non-synthetic set for the group algebra $L^1(\mathbb{R}^3)$. Note, that among other brilliant results, [A] contains the implication $\mathcal{M} = \text{alg } \mathcal{L} \Rightarrow \mathcal{L} = \text{lat } \mathcal{M}$, for an Arveson algebra $\mathcal{M}$ (in full analogy with the classical situation).

The results in [A] indicate, in fact, that the problematics of the operator synthesis obtains a more natural setting if instead of algebras and lattices one considers bimodules over masas and their bilattices (see the definitions below). We choose this point of view aiming at the investigation of various faces of operator synthesis, that reflect its connections with measure theory, approximation theory, linear operator equations and spectral theory of multiplication operators, synthesis in modules, Haagerup tensor products and Varopoulos tensor algebras.

Let us list some results, proved in this first part of our work. We show the equivalence of several different definitions of operator synthesis. Answering a question of W.Arveson [A][Problem, p.469] we prove the existence of a minimal Arveson algebra (bimodule) with a given invariant subspace lattice (bilattice), without the assumption of separability of the underlying Hilbert space. For separable case our coordinate approach does not need a choice of a topology, replacing it by the pseudo-topology, naturally related to the measure spaces. This allows to consider simultaneously the synthesis for a more wide class of subsets and to avoid the use of pseudo-integral operators and the complicated theory of integral decompositions of measures (see [A] and [Da]). This approach admits also the use of measurable sections which leads to an "inverse image theorem" (Theorem 4.7) for operator synthesis, implying in particular Arveson's theorem on synthesis for finite width lattices. We prove that a closed subset in a product of two compact sets is a set of spectral synthesis for the Varopoulos algebra if it is operator synthetic for any choice of measures (Theorem 6.1)(Proposition 6.1 shows that the converse implication fails). This, together with the above mentioned inverse image theorem, gives some sufficient conditions for spectral synthesis, implying, for example, the well known Drury's theorem on non-triangular sets (Corollary 6.1).

In the second part of the work we are going to consider the individual operator synthesis and its connections with linear operator equations.

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2 Synthetic sets (measure-theoretic approach)

Let (X, μ), (Y, ν) denote σ-finite separable spaces with standard measures. We use standard measure-theoretic terminology. A subset of the Cartesian product $X \times Y$ is said to be a measurable rectangle if it has the form $A \times B$ with measurable $A \subseteq X$, $B \subseteq Y$. A set $E \subseteq X \times Y$ is called *marginally null set* if $E \subseteq (X_1 \times Y) \cup (X \times Y_1)$, where $\mu(X_1) = \nu(Y_1) = 0$. If subsets α, β of $X \times Y$ are marginally equivalent (i.e. their symmetric difference is marginally null) we write $\alpha \cong \beta$. Following [ErKS] we define w-topology on $X \times Y$ such that the w-open (pseudo-open) sets are, modulo marginally null sets, countable union of measurable rectangles. The complements of w-open sets are called w-closed (pseudo-closed). The complement to a set A will be denoted by A^c.

Let $\Gamma(X,Y) = L_2(X,\mu)\hat{\otimes} L_2(Y,\nu)$ be the projective tensor product, i.e. the space of all functions $f : X \times Y \to \mathbb{C}$ which admit a representation

$$F(x, y) = \sum_{n=1}^{\infty} f_n(x) g_n(y) \tag{1}$$

where $f_n \in L_2(X, \mu)$, $g_n \in L_2(Y, \nu)$ and $\sum_{n=1}^{\infty} ||f_n||_{L_2} \cdot ||g_n||_{L_2} < \infty$. Such a function F is defined marginally almost everywhere (m.a.e.) in that, if f_n, g_n are changed on null sets then F will change on a marginally null set. Then $L_2(X,\mu)\hat{\otimes} L_2(Y,\nu)$-norm of such a function F is

$$||F||_\Gamma = \inf \sum_{n=1}^{\infty} ||f_n||_{L_2} \cdot ||g_n||_{L_2},$$

where the infinum is taken over all sequences f_n, g_n for which (1) holds m.a.e. In what follows we identify two functions in $\Gamma(X, Y)$ which coincides m.a.e.

By [ErKS][Theorem 6.5], any function $F \in \Gamma(X,Y)$ is pseudo-continuous (continuous with respect to the ω-topology defined above). We say that $F \in \Gamma(X,Y)$ vanishes on a set $K \subseteq X \times Y$ if $F\chi_K = 0$ (m.a.e), where χ_K is the characteristic function of K. For arbitrary $K \subseteq X \times Y$ denote by $\Phi(K)$ the set of all functions $F \in \Gamma(X,Y)$ vanishing on K. Clearly $\Phi(K)$ is a subspace of $\Gamma(X,Y)$.

Lemma 2.1. *Any convergent in norm sequence $\{F_n\} \in \Gamma(X,Y)$ has a subsequence which converges marginally almost everywhere.*

Proof. We may assume that $\{F_n\}$ converges to zero in norm. Then there exist functions $f_k^{(n)} \in L_2(X,\mu)$, $g_k^{(n)} \in L_2(Y,\nu)$ such that

$$F_n(x,y) = \sum_{k=1}^{\infty} f_k^{(n)}(x) g_k^{(n)}(y), \quad \sum_{k=1}^{\infty} ||f_k^{(n)}||_{L_2}^2 \to 0, \text{ and } \sum_{k=1}^{\infty} ||g_k^{(n)}||_{L_2}^2 \to 0.$$

By the Riesz theorem applied to the functions $f^{(n)}(x) = \sum_{k=1}^{\infty} |f_k^{(n)}(x)|^2$ and $g^{(n)}(y) = \sum_{k=1}^{\infty} |g_k^{(n)}(y)|^2$ there exists a subsequence $\{F_{n_j}\}$ such that $f^{(n_j)}(x)$ and $g^{(n_j)}(y)$ converge to zero almost everywhere. Therefore, there exist $M \subset X$, $N \subset Y$, $\mu(M) = 0$, $\nu(N) = 0$, such that $f^{(n_j)}(x) \to 0$ and $g^{(n_j)}(y) \to 0$ for any $x \in X \backslash M$, $y \in Y \backslash N$, and since $|F_{n_j}(x,y)| \leq f^{(n_j)}(x) g^{(n_j)}(y)$, this implies $F_{n_j}(x,y) \to 0$ for any $(x,y) \in (X \setminus M) \times (Y \setminus N)$. □

Proposition 2.1. $\Phi(K)$ *is closed.*

Proof. Let $F \in \overline{\Phi(K)}$. By Lemma 2.1 there exists a sequence $F_n \in \Phi(K)$ which converges to F marginally almost everywhere. Taking away a countable union of marginally null sets we can assume that all F_n vanish on the rest of the set K and therefore $F\chi_K = 0$ m.a.e. □

If $F \in \Gamma(X,Y)$ vanishes on K then by pseudo-continuity it vanishes on the pseudo-closure of K so that without loss of generality we can restrict ourselves to pseudo-closed sets K.

Given arbitrary subset $\mathcal{F} \subseteq \Gamma(X,Y)$, we define the null set of $\mathcal{F}$, null $\mathcal{F}$, to be the largest, up to marginally null sets, pseudo-closed set such that each function $F \in \mathcal{F}$ vanishes on it. To see the existence of such a set take a countable dense subset $\mathcal{A} \subseteq \mathcal{F}$ and consider $K = \cap_{F \in \mathcal{A}} F^{-1}(0)$. Clearly, K is pseudo-closed, $\mathcal{A} \subseteq \Phi(K)$ and, by Proposition 2.1, $\mathcal{F} = \overline{\mathcal{A}} \subseteq \Phi(K)$. The maximality of K is obvious.

Let $\Phi_0(K)$ be the closure in $\Gamma(X,Y)$ of the set of all functions which vanish on neighbourhoods of K (pseudo-open sets containing K). $\Phi_0(K)$ is a closed subspace of $\Phi(K)$.

Proposition 2.2. null $\Phi_0(K) = K =$ null $\Phi(K)$.

Proof. We work modulo marginally null sets. Let $\alpha \subseteq X$, $\beta \subseteq Y$ be measurable sets such that $(\alpha \times \beta) \cap K = \emptyset$. Then the function $\chi_\alpha(x)\chi_\beta(y)$ belongs to $\Phi_0(K)$ and therefore null $\Phi_0(K) \subset (\alpha \times \beta)^c$. Since K is pseudo-closed, $K = (\cup_{k=1}^{\infty} \alpha_k \times \beta_k)^c$ for some measurable α_k, β_k so that $(\alpha_k \times \beta_k) \cap K = \emptyset$ and thus null $\Phi_0(K) \subset K$. We have also that null $\Phi_0(K) \supset$ null $\Phi(K) \supset K$ which implies our result. □

Clearly, the subspaces $\Phi_0(K)$ and $\Phi(K)$ are invariant with respect to the multiplication by functions $f \in L_\infty(X,\mu)$ and $g \in L_\infty(Y,\nu)$ (we simply say invariant).

Theorem 2.1. *If $A \subseteq \Gamma(X,Y)$ is an invariant closed subspace then*

$$\Phi_0(\text{null } A) \subseteq A \subseteq \Phi(\text{null A}). \tag{2}$$

The second inclusion is obvious. The proof of the first one is postponed till Section 4. This theorem justifies the following definition.

Definition 2.1. *We say that a pseudo-closed set $K \subseteq X \times Y$ is synthetic (or $\mu \times \nu$-synthetic) if*

$$\Phi_0(K) = \Phi(K).$$

We shall also refer to synthetic sets as sets of operator synthesis or sets of $\mu \times \nu$-synthesis when the measures need to be specified.

We shall see that the sets of operator synthesis can be defined in several different ways. The relation to operator theory is based on the fact that elements of $\Gamma(X, Y)$ are the kernels of the nuclear (trace class) operators from $H_2 = L_2(Y, \nu)$ to $H_1 = L_2(X, \mu)$ and the space $\mathfrak{S}^1(H_2, H_1)$ of all such operators is isometrically isomorphic to $\Gamma(X, Y)$ (see [A]). The space of bounded operators, $B(H_1, H_2)$, from H_1 to H_2 is dual to $\mathfrak{S}^1(H_2, H_1)$ and therefore to $\Gamma(X, Y)$. The duality between $\Gamma(X, Y)$ and $B(H_1, H_2)$ is given by

$$\langle T, F \rangle = \sum_{n=1}^{\infty} (T f_n, \bar{g}_n),$$

with $T \in B(H_1, H_2)$ and $F \in \Gamma(X, Y)$ having a representation $F(x, y) = \sum_{n=1}^{\infty} f_n(x) g_n(y)$, where $f_n \in L_2(X, \mu)$, $g_n \in L_2(Y, \nu)$ and $\sum_{n=1}^{\infty} ||f_n||_{L_2} \cdot ||g_n||_{L_2} < \infty$. This will allow us to introduce the notion of "operator" synthesis for some sets of pairs of projections - bilattices - which (for separable H_i) bijectively correspond to ω-closed subsets in the product of measure spaces.

3 Bilattices, bimodules and operator synthesis

First, we introduce the concept of a bilattice and give some notations. Let $\mathcal{R}_1$, $\mathcal{R}_2$ be von Neumann algebras on Hilbert spaces H_1 and H_2. We write $\mathcal{P}_{\mathcal{R}_i}$ for the set of all selfadjoint projections in $\mathcal{R}_i$. By a *bilattice*, S, in $\mathcal{R}_1 \times \mathcal{R}_2$, we mean a set of pairs, (P, Q) with $P \in \mathcal{P}_{\mathcal{R}_1}$, $Q \in \mathcal{P}_{\mathcal{R}_2}$, which contains $(0, 1)$, $(1, 0)$, is closed under the operations $(\bigvee, \bigwedge)$ and $(\bigwedge, \bigvee)$ (i.e. if $(P_1, Q_1) \in S$ and $(P_2, Q_2) \in S$ then $(P_1 \bigvee P_2, Q_1 \bigwedge Q_2) \in S$ and $(P_1 \bigwedge P_2, Q_1 \bigvee Q_2) \in S$), is decreasing (i.e. $(P, Q) \in S$ implies $(P', Q') \in S$ for any $P' \in \mathcal{P}_{\mathcal{R}_1}$, $Q' \in \mathcal{P}_{\mathcal{R}_2}$ such that $P' \leq P$, $Q' \leq Q$), and is closed in the strong operator topology.

The space $B(H_1, H_2)$ is a left $B(H_2)$-module and a right $B(H_1)$-module and so we can consider it as $\mathfrak{A}_1 \times \mathfrak{A}_2$-bimodule for any subalgebras $\mathfrak{A}_i \subset B(H_i)$. We shall also refer to $\mathfrak{A}_1 \times \mathfrak{A}_2$-subbimodule of $B(H_1, H_2)$ as $\mathfrak{A}_1 \times \mathfrak{A}_2$-bimodule or just bimodule when no confusion can arise.

Given a subset $U \subseteq B(H_1, H_2)$, we can define a bilattice

$$\mathrm{bil}_{\mathcal{R}_1, \mathcal{R}_2} U = \{(P, Q) \in \mathcal{P}_{\mathcal{R}_1} \times \mathcal{P}_{\mathcal{R}_2} \mid QTP = 0, \text{ for any } T \in U\}.$$

Conversely, each bilattice $S \subset \mathcal{R}_1 \times \mathcal{R}_2$ determines an $\mathcal{R}_1' \times \mathcal{R}_2'$-bimodule by

$$\mathfrak{M}(S) = \{T \in B(H_1, H_2) \mid \mathrm{bil}_{\mathcal{R}_1, \mathcal{R}_2} T \supset S\},$$

$\mathcal{R}_i'$ being the commutant of $\mathcal{R}_i$. Then $\mathrm{bil}_{\mathcal{R}_1, \mathcal{R}_2}(\mathfrak{M}(S)) \supset S$ and $\mathfrak{M}(\mathrm{bil}_{\mathcal{R}_1, \mathcal{R}_2} U) \supset U$.

A bilattice $S \subset \mathcal{R}_1 \times \mathcal{R}_2$ is called *reflexive* if

$$S = \mathrm{bil}_{\,\mathcal{R}_1, \mathcal{R}_2} \mathfrak{M}(S),$$

or, equivalently, if $S = \text{bil}\,_{\mathcal{R}_1,\mathcal{R}_2}U$ for some $U \subset B(H_1, H_2)$. Strictly speaking, we have to use the term $\mathcal{R}_1 \times \mathcal{R}_2$-reflexive. Dependence of the notion on $\mathcal{R}_1 \times \mathcal{R}_2$ is not very strong. Indeed, let Bil U denote bil $_{B(H_1),B(H_2)}U$. Then it is easy to see that

$$\text{bil}\,_{\mathcal{R}_1,\mathcal{R}_2}\mathfrak{M}(S) = \mathcal{R}_1 \times \mathcal{R}_2 \cap \text{Bil}\,\mathfrak{M}(S)$$

for any bilattice $S \subset \mathcal{R}_1 \times \mathcal{R}_2$, and, conversely, Bil $\mathfrak{M}(S)$ consists of all pairs majorized by elements of S.

In the remainder of the section S stands for commutative bilattice, i.e. $S \subset \mathcal{D}_1 \times \mathcal{D}_2$, where $\mathcal{D}_1$ and $\mathcal{D}_2$ are maximal commutative selfadjoint algebras on H_1 and H_2 respectively. In this case we shall write simply bil U instead of $\text{bil}_{\mathcal{D}_1,\mathcal{D}_2}U$ for $U \subseteq B(H_1, H_2)$.

Theorem 3.1. *Let S be a commutative bilattice in $\mathcal{D}_1 \times \mathcal{D}_2$. Then* bil $\mathfrak{M}(S) = S$*. Moreover, if $\mathfrak{M}$ is an ultraweakly closed $\mathcal{D}_1 \times \mathcal{D}_2$-bimodule such that* bil $\mathfrak{M} = S$ *then $\mathfrak{M} \subseteq \mathfrak{M}(S)$.*

Proof. The second statement follows directly from the definition of $\mathfrak{M}(S)$. The first one can be reduced, by a 2×2-matrix trick, to Arveson's theorem on reflexivity of commutative subspace lattices, [A] (for a coordinate-free proof see [Da] or [Sh1]). Indeed, consider the set, $\mathcal{L}$, of all projections $\begin{pmatrix} p & 0 \\ 0 & 1-q \end{pmatrix} \in \mathcal{P}_{B(H_1 \oplus H_2)}$, where $(p, q) \in S$. We see at once that $\mathcal{L}$ is a commutative lattice and is closed in the strong operator topology. Therefore, $\mathcal{L}$ is reflexive, i.e., lat alg $\mathcal{L} = \mathcal{L}$. Since S contains all pairs $(p, 0)$, $(0, q)$, $p \in \mathcal{P}_{\mathcal{D}_1}$, $q \in \mathcal{P}_{\mathcal{D}_2}$ and $\mathcal{D}_1$, $\mathcal{D}_2$ are maximal commutative selfadjoint algebras, one can easily check that if $T = (T_{ij})_{i,j=1}^2 \in B(H_1 \oplus H_2)$ belongs to alg $\mathcal{L}$ then $T_{12} = 0$, $T_{11} \in \mathcal{D}_1$ and $T_{22} \in \mathcal{D}_2$. Simple arguments give us also $T_{21} \in \mathfrak{M}(S)$ and

$$\text{alg}\,\mathcal{L} = \{T = (T_{ij})_{i,j=1}^2 \in B(H_1 \oplus H_2) \mid T_{11} \in \mathcal{D}_1, T_{22} \in \mathcal{D}_2, T_{21} \in \mathfrak{M}(S), T_{12} = 0\}.$$

Therefore, if $q\mathfrak{M}(S)p = 0$ for some $p \in \mathcal{D}_1$, $q \in \mathcal{D}_2$ then $p \oplus (1 - q) \in$ lat alg $\mathcal{L} = \mathcal{L}$, i.e. $(p, q) \in S$. This yields bil $\mathfrak{M}(S) \subseteq S$. The reverse inclusion is obvious. □

The above theorem shows that $\mathfrak{M}(S)$ is the largest ultraweakly closed bimodule such that bil $\mathfrak{M}(S) = S$. Now we are going to define the smallest one.

Given a state φ on $B(l_2)$, consider a slice operator $L_\varphi : B(l_2 \otimes H_1, l_2 \otimes H_2) \to B(H_1, H_2)\}$ defined by $L_\varphi(A \otimes B) = \varphi(A)B$. Let Conv S denote the weakly closed (or uniformly closed, see Lemma 3.1) convex hull of S and let $\mathcal{R}_1 = B(l_2)\bar{\otimes}\mathcal{D}_1$, $\mathcal{R}_2 = B(l_2)\bar{\otimes}\mathcal{D}_2$, the von Neumann algebras generated by elementary tensors $A \otimes B$, $A \in B(l_2)$, $B \in \mathcal{D}_i$. Set

$$F_S = \{(A, B) \in \mathcal{R}_1 \times \mathcal{R}_2 \mid (L_\varphi(A), L_\varphi(B)) \in \text{Conv}\,S\}$$

$$\tilde{S} = \{(P, Q) \in \mathcal{P}_{\mathcal{R}_1} \times \mathcal{P}_{\mathcal{R}_2} \mid (P, Q) \in F_S\}$$

and define

$$\mathfrak{M}_0(S) = \{X \in B(H_1, H_2) \mid 1 \otimes X \in \mathfrak{M}(\tilde{S})\},$$

where 1 is the identity operator on l_2. Here and subsequently we mean $\text{bil}_{\mathcal{R}_1,\mathcal{R}_2} 1 \otimes U$ for the chosen above von Neumann algebras $\mathcal{R}_1$ and $\mathcal{R}_2$ when we write bil $1 \otimes U$ for $U \subseteq B(H_1, H_2)$. Clearly, like $\mathfrak{M}(S)$, the space $\mathfrak{M}_0(S)$ is a $\mathcal{D}_1 \times \mathcal{D}_2$-bimodule. Moreover, both bimodules are ultraweakly closed.

Lemma 3.1. *Let S be a commutative bilattice in $\mathcal{D}_1 \times \mathcal{D}_2$. Then*

$$\overline{\text{conv } S}^u = \overline{\text{conv } S}^w = \{(A, B) \in \mathcal{D}_1 \times \mathcal{D}_2 \mid 0 \le A \le 1, 0 \le B \le 1, \\ (E_A([\alpha, 1]), E_B([\beta, 1])) \in S, \alpha + \beta > 1\}.$$

where "u" and "w" indicate the "uniform" and the "weak operator topology" closure of the convex hull, conv S*, of S and $E_X(\cdot)$ is the spectral projection measure of selfadjoint operator X.*

Proof. Let D denote the set to the right. To see that $D \subset \overline{\text{conv } S}^u$, set $A_n = \sum_{i=1}^{n} \frac{1}{n} E_A([\frac{i}{n}, 1])$, $B_n = \sum_{i=1}^{n} \frac{1}{n} E_B([\frac{i}{n}, 1])$ for $(A, B) \in D$. Clearly, $A_n \to A$ and $B_n \to B$ uniformly as $n \to \infty$. Then, since $(E_A([\frac{i}{n}, 1]), E_B([\frac{n-i+1}{n}, 1]) \in S$ and

$$(A_n, B_n) = \frac{1}{n} \sum_{i=1}^{n} ((E_A([\frac{i}{n}, 1]), E_B([\frac{n-i+1}{n}, 1])),$$

we have $(A_n, B_n) \in \text{conv } S$ and therefore $(A, B) \in \overline{\text{conv } S}^u$.

Next claim is that D is convex. In fact, for (A_1, B_1), $(A_2, B_2) \in D$, we have

$$E_{(A_1+A_2)/2}([\alpha, 1]) = \bigvee_n E_{A_1}([\varepsilon_n, 1]) E_{A_2}([2\alpha - \varepsilon_n, 1]),$$
$$E_{(B_1+B_2)/2}([\beta, 1]) = \bigvee_m E_{B_1}([\varepsilon_m, 1]) E_{B_2}([2\beta - \varepsilon_m, 1]),$$

where $\alpha, \beta \in [0, 1)$, $\{\varepsilon_n\}$ is a countable dense subset of $[0, 1]$. Fix α, β such that $\alpha + \beta > 1$. Then for n, $m \in \mathbb{Z}^+$, we have either $\varepsilon_n + \varepsilon_m > 1$ which gives $(E_{A_1}([\varepsilon_n, 1]), E_{B_1}([\varepsilon_m, 1])) \in S$ and therefore $(E_{A_1}([\varepsilon_n, 1]) E_{A_2}([2\alpha - \varepsilon_n, 1]), E_{B_1}[\varepsilon_m, 1]) E_{B_2}([2\beta - \varepsilon_m, 1, 1])) \in S$, or $(2\alpha - \varepsilon_n) + (2\beta - \varepsilon_m) > 1$ which implies $(E_{A_1}([\varepsilon_n, 1]) E_{A_2}([2\alpha - \varepsilon_n, 1])), E_{B_1}([\varepsilon_m, 1]) E_{B_2}([2\beta - \varepsilon_m, 1])) \in S$. Since S is a bilattice,

$$(E_{(A_1+A_2)/2}([\alpha, 1]), E_{(B_1+B_2)/2}([\beta, 1])) \in S.$$

Next step is to prove that D is weakly closed. Since it is convex it is enough to prove that it is strongly closed. Let $\{(A_n, B_n)\} \subset D$ be a sequence strongly converging to $(A, B) \in \mathcal{D}_1 \times \mathcal{D}_2$. Then, for any $\varepsilon > 0$ and α, $\beta < 1$, we have

$$E_A([\alpha, 1]) \le s.\lim_{n\to\infty} E_{A_n}([\alpha + \varepsilon, 1]) \text{ and } E_B([\beta, 1)) \le s.\lim_{n\to\infty} E_{B_n}([\beta + \varepsilon, 1])$$

(the strong limit). Since $(E_{A_n}([\alpha+\varepsilon, 1]), E_{B_n}([\beta+\varepsilon, 1]) \in S$ if $\alpha+\beta > 1$ and S is decreasing and closed in the strong operator topology, we obtain $(E_A([\alpha, 1]), E_B([\beta, 1]) \in S$. If one of α, β equals 1, then that $(E_A([\alpha, 1]), E_B([\beta, 1]) \in S$ follows from $E_A(\{1\}) = s.\lim_{\varepsilon\to 0} E_A([1-\varepsilon, 1])$, $E_B(\{1\}) = s.\lim_{\varepsilon\to 0} E_B([1 - \varepsilon, 1])$. So we can conclude that $(A, B) \in D$.

We have therefore

$$S \subset D \subset \overline{\text{conv } S}^u \subset \overline{\text{conv } S}^w,$$

and, since D is convex and weakly closed, $D = \overline{\text{conv } S}^u = \overline{\text{conv } S}^w$. □

Definition 3.1. *We say that a bilattice, S, is synthetic if there exists only one ultraweakly closed bimodule whose bilattice is S.*

Theorem 3.2. *Let $S \subset \mathcal{D}_1 \times \mathcal{D}_2$ be a commutative bilattice and let $\mathfrak{M}$ be an ultraweakly closed $\mathcal{D}_1 \times \mathcal{D}_2$-bimodule such that* bil $\mathfrak{M} \subset S$. *Then* bil $1 \otimes \mathfrak{M} \subset \tilde{S}$.

Proof. Let $(P, Q) \in \text{bil } 1 \otimes \mathfrak{M}$. Fix $\xi \in l_2$, $||\xi|| = 1$. Consider the corresponding state $\varphi_\xi(A) = (A\xi, \xi)$ and denote the corresponding operator L_{φ_ξ} simply by L_ξ. It is sufficient to show that $(L_\xi(P), L_\xi(Q)) \in \text{Conv } S$. By definition of L_ξ, we have $(L_\xi(K)x, x) = (K(\xi \otimes x), \xi \otimes x)$ for any operator K on $l_2 \otimes H$ and, in particular, if $K = P$ (a selfadjoint projection) then $(L_\xi(P)x, x) = ||P(\xi \otimes x)||^2$. Therefore, for $A \in \mathfrak{M}$ the following holds

$$(AL_\xi(P)A^*x, x) = (L_\xi(P)A^*x, A^*x) = ||P(\xi \otimes A^*x)||^2 =$$
$$||P(1 \otimes A^*)Q^\perp(\xi \otimes x)||^2 \leq ||A||^2||Q^\perp(\xi \otimes x)||^2 = ||A||^2(L_\xi(Q^\perp)x, x).$$

We obtain now the inequality $AL_\xi(P)A^* \leq ||A||^2 L_\xi(Q^\perp)$. Let $L_\xi(P) = K^2$, $L_\xi(Q^\perp) = L^2$, where K, $L \geq 0$. Then $||KA^*x|| \leq ||A||||Lx||$ for any $A \in \mathfrak{M}$ and $x \in H$ or, equivalently, $||KA^*L^{-1}|| \leq ||A||$. Since $\mathfrak{M}$ is a bimodule, $KA^*L^{-1} \in \mathfrak{M}^*$. Writing now KA^*L^{-1} instead of A^* we get $||K^2A^*L^{-2}|| \leq ||KA^*L^{-1}|| \leq ||A||$. Proceeding in this fashion we obtain $||K^nA^*L^{-n}|| \leq ||A||$ and hence

$$||K^nA^*x|| \leq ||A||||L^nx||, \quad x \in H. \tag{3}$$

Fix $x \in E_L([0, \varepsilon])$, where $E_L(\cdot)$ is the spectral projection measure of L. Then $||L^nx|| \leq C\varepsilon^n$ and, by (3), we obtain $A^*x \in E_K([0, \varepsilon])$. Thus $\mathfrak{M}^*E_L([0, \varepsilon]) \subset E_K([0, \varepsilon])$ or, equivalently, $E_K([\varepsilon', 1])\mathfrak{M}^*E_L([0, \varepsilon]) = 0$ if $\varepsilon' > \varepsilon$. This implies $E_{K^2}([\varepsilon', 1])\mathfrak{M}^*E_{1-L^2}([1-\varepsilon, 1]) = 0$, as $\varepsilon' > \varepsilon$, i.e.

$$E_{L_\xi(Q)}([\alpha, 1])\mathfrak{M}E_{L_\xi(P)}([\beta, 1]) = 0, \quad \alpha + \beta > 1.$$

Since $(E_{L_\xi(Q)}([\alpha, 1]), E_{L_\xi(P)}([\beta, 1])) \in \text{bil } \mathfrak{M} \subset S$ as $\alpha + \beta > 1$, by Theorem 3.1, we obtain $(L_\xi(Q), L_\xi(P)) \in \text{Conv } S$ for any $\xi \in H$. □

Corollary 3.1. *Let $S \subset \mathcal{D}_1 \times \mathcal{D}_2$ be a commutative bilattice and let $\mathfrak{M}$ be an ultraweakly closed $\mathcal{D}_1 \times \mathcal{D}_2$-bimodule such that* bil $\mathfrak{M} \subset S$. *Then $\mathfrak{M}_0(S) \subset \mathfrak{M}$.*

Proof. Let $T \in \mathfrak{M}_0(S)$. To see that $T \in \mathfrak{M}$ we choose an ultraweakly continuous linear functional φ such that $\varphi(\mathfrak{M}) = 0$. Then there exist $F \in l_2 \otimes H_1$, $G \in l_2 \otimes H_2$ such that $\varphi(A) = ((1 \otimes A)F, G)$, $A \in B(H_1, H_2)$, moreover, $(1 \otimes \mathfrak{M})F \perp G$. Denoting by P_F and P_G the projections on $\overline{[(1 \otimes \mathcal{D}_1)F]}$ and $\overline{[(1 \otimes \mathcal{D}_2)G]}$ we have $P_G(1 \otimes \mathfrak{M})P_F = 0$, i.e. $(P_F, P_G) \in \text{bil } 1 \otimes \mathfrak{M}$. It follows now from the definition of $\mathfrak{M}_0(S)$ and Theorem 3.2 that $(P_F, P_G) \in \text{bil } 1 \otimes \mathfrak{M} \subset \tilde{S} \subset \text{bil } 1 \otimes T$ and therefore $P_G(1 \otimes T)P_F = 0$, i.e. $\varphi(T) = 0$. From the arbitrariness of φ we obtain $T \in \mathfrak{M}$. □

Summarising we have the following statement.

Theorem 3.3. *Let $S \subset \mathcal{D}_1 \times \mathcal{D}_2$ be a commutative bilattice. If $\mathfrak{M}$ is an ultraweakly closed $\mathcal{D}_1 \times \mathcal{D}_2$-bimodule such that* bil $\mathfrak{M} = S$ *then $\mathfrak{M}_0(S) \subset \mathfrak{M} \subset \mathfrak{M}(S)$.*

Theorem 3.4. *Given a bilattice $S \subset \mathcal{D}_1 \times \mathcal{D}_2$, bil $\mathfrak{M}_0(S) = S$.*

Theorem 3.3 and 3.4 state that $\mathfrak{M}_0(S)$ is the smallest ultraweakly closed $\mathcal{D}_1 \times \mathcal{D}_2$-bimodule whose bilattice is S and that a commutative bilattice S is synthetic if and only if $\mathfrak{M}(S) = \mathfrak{M}_0(S)$.

We shall prove Theorem 3.4 in Section 5 after treating the case of bilattices on separable Hilbert spaces.

4 Separably acting bilattices

If Hilbert spaces H_1 and H_2 are separable then there exist finite separable measure spaces (X, μ) and (Y, ν) with standard measures μ, ν, such that $H_1 = L_2(X, \mu)$, $H_2 = L_2(Y, \nu)$ and the multiplication algebras $\mathcal{D}_1$ and $\mathcal{D}_2$ are $L_\infty(X, \mu)$ and $L_\infty(Y, \nu)$ respectively. Denote by P_U and Q_V the multiplication operators by the characteristic functions of $V \subset X$ and $U \subset Y$. Given $E \subset X \times Y$, we define S_E to be the set of all pairs of projections (P_U, Q_V), where $V \subseteq X$, $U \subseteq Y$ and $(V \times U) \cap E \cong \emptyset$.

Theorem 4.1. *S_E is a bilattice.*

Proof. We shall prove only the closedness of S_E, because other conditions trivially hold. Let $(P_n, Q_n) \in S_E$, $P_n \to P$, $Q_n \to Q$ in the strong operator topology. Then there exist $A \subset X$, $B \subset Y$ such that $P = P_A$, $Q = Q_B$. Changing, if necessarily, P_n to $P_n P$, Q_n to $Q_n Q$, we may assume that $P_n \leq P$ and $Q_n \leq Q$. We have therefore $P_n = P_{A_n}$, $Q_n = Q_{B_n}$, for some $A_n \subseteq X$, $B_n \subseteq Y$ such that $(A_n \times B_n) \cap E \cong \emptyset$ and $\mu(A \setminus A_n) \to 0$, $\nu(B \setminus B_n) \to 0$. Given $\varepsilon > 0$, $k \in \mathbb{N}$, choose n_k such that $\mu(A \setminus A_{n_k}) < \dfrac{\varepsilon}{2^k}$ and $\nu(B \setminus B_{n_k}) < \dfrac{\varepsilon}{2^k}$. Set

$$A_\varepsilon = \cap_{k=1}^\infty A_{n_k}, \quad B_\varepsilon = \cup_{k=1}^\infty B_{n_k}.$$

Then $\mu(A \setminus A_\varepsilon) \leq \varepsilon$, $\nu(B \setminus B_\varepsilon) = 0$ and $(A_\varepsilon \times B_\varepsilon) \cap E \cong \emptyset$. Taking now $A_0 = \cup_{n=1}^\infty A_{1/n}$ and $B_0 = \cap_{n=1}^\infty B_{1/n}$, we obtain $\mu(A \setminus A_0) = 0$, $\nu(B \setminus B_0) = 0$, $(A_0 \times B_0) \cap E \cong \emptyset$ so that $(P, Q) = (P_{A_0}, Q_{B_0}) \in S_E$.

□

Theorem 4.2. *Let $S \subset \mathcal{D}_1 \times \mathcal{D}_2$ be a bilattice. Then there exists a unique, up to a marginally null set, pseudo-closed set $E \subseteq X \times Y$ such that $S = S_E$.*

Proof. Let $\{(P_n, Q_n)\}$ be a strongly dense sequence in the bilattice S, and let $A_n \subseteq X$, $B_n \subseteq Y$ be such that $P_n = P_{A_n}$ and $Q_n = Q_{B_n}$. The set $E = (X \times Y) \setminus (\cup_{n=1}^\infty A_n \times B_n)$ is clearly pseudo-closed. We will show that $S = S_E$.

Since S_E is closed in the strong operator topology, we have the inclusion $S \subset S_E$. For the reverse inclusion, we first show that if a rectangular, $A \times B$, lies in the union of a finite number of rectangulars, say $C_k \times D_k$ $(1 \leq k \leq n)$, such that $(P_{C_k}, Q_{D_k}) \in S$, then $(P_A, Q_B) \in S$. We use the induction by n (the case $n = 1$ being obvious from the decreasing condition on S). If $A \times B \subseteq \cup_{k=1}^n C_k \times D_k$, then $(A \setminus C_1) \times B \subseteq \cup_{k=2}^n (C_k \times D_k)$ and so, by the induction hypothesis, we have that $(P_{A \setminus C_1}, Q_B) \in S$. Similarly, $(P_A, Q_{B \setminus D_1}) \in S$. Therefore, $(P_{A \cap C_1}, Q_{B \setminus D_1}) \in S$, which together with $(P_{C_1}, P_{D_1}) \in S$ gives us $(P_{A \cap C_1}, P_B) \in S$, S being closed under the operation $(\bigvee, \bigwedge)$. Using now closeness under the operation $(\bigwedge, \bigvee)$, we obtain $(P_A, Q_B) \in S$.

Let now $(P, Q) = (P_A, Q_B) \in S_E$. Deleting null sets from A, B we may assume that $A \times B \subset \cup_{n=1}^\infty A_n \times B_n$. Then, by [ErKS][Lemma 3.4,d], given $\varepsilon > 0$, there exist $A_\varepsilon \subseteq A$, $B_\varepsilon \subseteq B$ with $\mu(A \setminus A_\varepsilon) < \varepsilon$, $\nu(B \setminus B_\varepsilon) < \varepsilon$ such that $A_\varepsilon \times B_\varepsilon$ is contained in the union of a finite number of sets $\{A_n \times B_n\}$. By the statement we have just proved, $(P_{A_\varepsilon}, Q_{B_\varepsilon}) \in S$, and, since $P_{A_\varepsilon} \to P$, $Q_{B_\varepsilon} \to Q$ strongly, as $\varepsilon \to 0$, we have $(P, Q) \in S$. This proves $S = S_E$.

To see the uniqueness, let E_1 be a pseudo-closed set such that $S_{E_1} = S_E$. Then $(P_A, Q_B) \in S_E$ for any $A \times B \in E_1^c$ and therefore $A \times B \subset E^c$ up to a marginally null set. As E_1^c is pseudo-open, we have $E_1^c \subset E^c$ up to a marginally null set. Similarly, we have the reverse inclusion and therefore $E_1^c \cong E^c$ and $E_1 \cong E$. □

We say that $T \in B(H_1, H_2)$ is *supported* in $E \subseteq X \times Y$ if bil $T \supset S_E$, i.e., if $P_U T Q_V = 0$ for each sets $U \subseteq Y$, $V \subseteq X$ such that $(U \times V) \cap F \cong \emptyset$. Clearly,

$$\mathfrak{M}(S_E) = \{T \in B(H_1, H_2) \mid T \text{ is supported in } E\}.$$

For any subset $U \subseteq B(H_1, H_2)$ there exists the smallest (up to a marginally null set) pseudo-closed set, supp U, which supports any operator $T \in U$, namely, supp U is the pseudo-closed set E such that bil $U = S_E$. The support of an operator $T \in B(H_1, H_2)$ will be denoted by supp T. We will use also the notations $\mathfrak{M}_{max}(E)$ and $\mathfrak{M}_{min}(E)$ for the bimodules $\mathfrak{M}(S_E)$ and $\mathfrak{M}_0(S_E)$. Theorem 3.3 says now that

$$\mathfrak{M}_{min}(E) \subset \mathfrak{M} \subset \mathfrak{M}_{max}(E)$$

if supp $\mathfrak{M} = E$. Clearly, supp $\mathfrak{M}_{max}(E) = E$ and therefore $\mathfrak{M}_{max}(E)$ is the largest ultraweakly closed bimodules whose support is E. By proving now that supp $\mathfrak{M}_{min}(E) = E$ we would also have that $\mathfrak{M}_{min}(E)$ is the smallest ultraweakly closed bimodules whose support is E, justifying the notations.

Let Ψ be a subspace of $\Gamma(X, Y)$. Using the duality of $B(H_1, H_2)$ and $\Gamma(X, Y)$ we denote by $\Psi^\perp$ the subspace of all operators $T \in B(H_1, H_2)$ such that $\langle T, F\rangle = 0$ for any $F \in \Psi$. Clearly, if Ψ is invariant then $\Psi^\perp$ is an $(\mathcal{D}_1, \mathcal{D}_2)$-bimodule.

Theorem 4.3. *Let $E \subset X \times Y$ be a pseudo-closed set. Then*

$$\Phi_0(E)^\perp = \mathfrak{M}_{max}(E).$$

Proof. We begin by showing the inclusion $\mathfrak{M}_{max}(E) \subseteq \Phi_0(E)^\perp$. Let $A \in \mathfrak{M}_{max}(E)$, $F \in \Phi_0(K)$. By [ErKS][Lemma 3.4], E is ε-compact, so that, for any $\varepsilon > 0$, there exist $X_\varepsilon \subseteq X$, $Y_\varepsilon \subseteq Y$ with $\mu(X_\varepsilon) < \varepsilon$, $\nu(Y_\varepsilon) < \varepsilon$ such that

$$F_\varepsilon(x, y) = F(x, y)\chi_{X_\varepsilon^c}(x)\chi_{Y_\varepsilon^c}(y)$$

vanishes on an open-closed neighbourhood of E ($\cong$ the union of a finite number of rectangulars). Clearly, $F_\varepsilon \to F$ as $\varepsilon \to 0$. It remains to show that $\langle A, F_\varepsilon\rangle = 0$. Choose measurable sets $\{X_j\}_{j=1}^N$, $\{Y_i\}_{i=1}^M$ in a way that

$$X = \cup_{j=1}^N X_j, \quad Y = \cup_{i=1}^M Y_i \quad \text{and} \quad \text{null } F_\varepsilon \supseteq \cup_{(i,j)\in J} X_j \times Y_i \supseteq E$$

for some index set J. If $(i, j) \in J$ then $\langle P_{Y_i} A Q_{X_j}, F_\varepsilon\rangle = \langle A, F_\varepsilon \chi_{X_j}\chi_{Y_i}\rangle = 0$. If $(i, j) \notin J$ then $P_{Y_i} A Q_{X_j} = 0$ since supp $A \subseteq E$. Therefore, $\langle P_{Y_i} A Q_{X_j}, F_\varepsilon\rangle = 0$ for any pair (i, j) and hence $\langle A, F_\varepsilon\rangle = 0$.

Let A be an operator in $B(H_1, H_2)$ such that $\langle A, F\rangle = 0$ for any $F \in \Phi_0(E)$. Consider $V \subseteq X$, $U \subseteq Y$ such that $(V \times U) \cap E = \emptyset$ (m.a.e.). Then $F(x, y)\chi_V(x)\chi_U(y) \in \Phi_0(E)$ for any $F \in \Gamma(X, Y)$ and $\langle P_U A Q_V, F\rangle = \langle A, F \cdot \chi_V \chi_U\rangle = 0$, which implies $P_U A Q_V = 0$. □

Let $\mathcal{D}_i^+$ denote the set of positive functions in $\mathcal{D}_i$. Operators $A \in B(l_2)\bar{\otimes}\mathcal{D}_1$ and $B \in B(l_2)\bar{\otimes}\mathcal{D}_2$ can be identified with operator-valued functions $A(x) : X \to B(l_2)$ and $B(y) : Y \to B(l_2)$. If A, B are projections then $A(x)$, $B(y)$ are projection-valued functions. We say that a pair of projections $(P, Q) \in (B(l_2)\bar{\otimes}\mathcal{D}_1) \times (B(l_2)\bar{\otimes}\mathcal{D}_2)$ is an *E-pair* if $P(x)Q(y)$ vanishes on E. If, additionally, P and Q take only finitely many values then the pair (P, Q) is said to be a *simple E-pair*.

Lemma 4.1. *Let E be a pseudo-closed subset of $X \times Y$. Then*

$$\text{conv } S_E = \{(a(x), b(y)) \in \mathcal{D}_1^+ \times \mathcal{D}_2^+ : a(x) + b(y) \leq 1, \ \textit{m.a.e on } E\},$$

$$F_{S_E} = \{(A, B) \in (B(l_2)\bar{\otimes}\mathcal{D}_1)^+ \times (B(l_2)\bar{\otimes}\mathcal{D}_2)^+ \mid A(x) + B(y) \leq 1, \ \textit{m.a.e on } E\},$$

and

$$\tilde{S}_E = \{(P, Q) \mid (P, Q) \textit{ is an } E\textit{-pair}\}.$$

Proof. The first statement follows easily from Lemma 3.1. To see the second equality take $\xi \in l_2$ and $(A, B) \in F_{S_E}$, identifying the operators with the corresponding operator-valued functions. Set now $a(x) = (A(x)\xi, \xi)$ and $b(y) = (B(y)\xi, \xi)$. It is easy to see that $(L_\xi(A)f)(x) = a(x)f(x)$ and $(L_\xi(B)g)(y) = b(y)g(y)$. By the definition of F_{S_E} and the first statement, we have $(A(x)+B(y)\xi, \xi) = (A(x)\xi, \xi)+(B(y)\xi, \xi) = a(x)+b(y) \leq 1$ (m.a.e.) on E and therefore $A(x) + B(y) \leq 1$ (m.a.e.) on E. If, additionally, A and B are projections, the inequality gives $A(x)B(y) = 0$ (m.a.e.) on E, completing the proof. □

Theorem 4.4. *Let $E \subset X \times Y$ be a pseudo-closed set. Then*

$$\Phi(E)^\perp = \mathfrak{M}_{min}(E).$$

Proof. Let $(P, Q) \in \tilde{S}_E$ and let $\vec{x}(x) = P(x)\xi$ and $\vec{y}(y) = Q(y)\eta$ for some ξ, $\eta \in l_2$. By Lemma 4.1, $(P(x), Q(y))$ is an E-pair which implies $(\vec{x}(x), \vec{y}(y)) = 0$ m.a.e. on E. Clearly, the function $F : (x, y) \mapsto (\vec{x}(x), \vec{y}(y))$ belongs to $\Gamma(X, Y)$ and therefore $F \in \Phi(E)$. For any $T \in B(H_1, H_2)$ we have $\langle T, F\rangle = ((1 \otimes T)\vec{x}, \vec{y})$ and if $T \in \Phi(E)^\perp$ we obtain $((1 \otimes T)\vec{x}, \vec{y}) = 0$ and $Q(1 \otimes T)P = 0$, i.e. $T \in \mathfrak{M}_{min}(E)$.

To see the converse we observe that any function $F \in \Phi(E)$ can be written as $(\vec{x}(x), \vec{y}(y))$, where $\vec{x}(x)$, $\vec{y}(y) \in l_2$ and $\vec{x}(x) \perp \vec{y}(y)$ if $(x, y) \in E$ m.a.e. Denoting by $P(x)$ and $Q(y)$ the projections onto one-dimensional spaces generated by $\vec{x}(x)$ and $\vec{y}(y)$ yields $P(x)Q(y) = 0$ m.a.e. on E and $(P, Q) \in \tilde{S}_E$. For any $T \in \mathfrak{M}_{min}(E)$ we have

$$\langle T, F\rangle = ((1 \otimes T)\vec{x}(x), \vec{y}(y)) = (Q(1 \otimes T)P\vec{x}(x), \vec{y}(y)) = 0.$$

This implies $T \in \Phi(E)$. □

Corollary 4.1.

$$\text{bil } \mathfrak{M}_{min}(E) = S_E.$$

Proof. It suffices to show that $Q_U\mathfrak{M}_{min}(E)P_V = 0$ with measurable $U \subseteq Y$, $V \subseteq X$ implies that $(V \times U) \cap E$ is marginally null. In fact, this would imply $S_E \supset \text{bil } \mathfrak{M}_{min}(E)$ which together with $S_E = \text{bil } \mathfrak{M}_{max}(E) \subset \text{bil } \mathfrak{M}_{min}(E)$ gives us the statement, the last inclusion being true since $\mathfrak{M}_{min}(E) \subset \mathfrak{M}_{max}(E)$.

Assume that $E_0 = (V \times U) \cap E$ is not marginally null. Then $\Phi(E_0)$ does not contain $\chi_{V\times U}$ and therefore does not equal to $\Gamma(V, U)$. Since $\Phi(E_0)$ is closed in $\Gamma(V, U)$, there exists an operator $A_0 \in B(P_V H_1, Q_U H_2)$ such that $0 \neq A_0 \perp \Phi(E_0)$. Extend A_0 to an operator $A \in B(H_1, H_2)$ so that $Q_U AP_V|_{L_2(V)} = A_0$ and $A = Q_U AP_V$. Then $A \perp \Phi(E)$ and, by Theorem 4.4, $A \in \mathfrak{M}_{min}(E)$. Since $Q_U AP_V \neq 0$, we obtain a contradiction. □

Corollary 4.2. *Let $\mathfrak{M} \subset B(H_1, H_2)$ be an ultraweakly closed bimodule, E be a pseudo-closed set. Then supp $\mathfrak{M} = E$ iff*

$$\mathfrak{M}_{min}(E) \subseteq \mathfrak{M} \subseteq \mathfrak{M}_{max}(E).$$

Proof. It follows from Theorem 3.1, Corollory 3.1,4.1 and the fact that bil $\mathfrak{M} = S_E$ if and only if supp $\mathfrak{M} = E$. □

Proof of Theorem 2.1. Let $E = \text{supp } A^\perp$. By Corollary 4.2

$$\mathfrak{M}_{min}(E) \subseteq A^\perp \subseteq \mathfrak{M}_{max}(E)$$

and therefore, by Theorem 4.3, 4.4,

$$\Phi_0(E) \subseteq A \subseteq \Phi(E)$$

which also implies null $A = E$. □

The next corollary is an analogue of Wiener's Tauberian Theorem.

Corollary 4.3. *If* $\Psi \subset \Gamma(X,Y)$ *and* null $\Psi \cong \emptyset$ *then* Ψ *is dense in* $\Gamma(X,Y)$.

Proof. Follows from Theorem 2.1, since $\Phi_0(\emptyset) = \Gamma(X,Y)$. □

Corollary 4.4.

$$\text{bil } 1 \otimes \mathfrak{M}_{min}(E) = \tilde{S}_E = \{(P,Q) : (P,Q) \textit{ is an } E\textit{-pair } \},$$

Proof. By Corollary 4.1, bil $\mathfrak{M}_{min}(E) = S_E$ which together with Theorem 3.2 imply bil $1 \otimes \mathfrak{M}_{min}(E) \subset \tilde{S}_E$. On the other hand, bil $1 \otimes \mathfrak{M}_{min}(E) \supset \tilde{S}_E$ by the definition of $\mathfrak{M}_{min}(E)$. The second equality is proved in Lemma 4.1. □

Remark 4.1. For sets that are graphs of preoders (that is for lattices) the result was, in fact, proved in [A][Cor.1 of Theorem 2.1.5].

Theorem 4.5. *Let E be a pseudo-closed set. Then*

$$\text{bil } 1 \otimes \mathfrak{M}_{max}(E) = \overline{\{(P,Q) : (P,Q) \textit{ is a simple } E \textit{ pair } \}}^s,$$

where "s" indicates the strong operator topology closure.

Proof. Consider the commutative lattice, $\mathcal{L}$, of all projections $\begin{pmatrix} p & 0 \\ 0 & 1-q \end{pmatrix} \in \mathcal{P}_{B(H_1 \oplus H_2)}$, where $(p,q) \in S_E$. By [Sh1],

$$\mathcal{P}_{B(l_2)} \otimes \mathcal{L} = \text{lat } (1 \otimes \text{alg } \mathcal{L}), \tag{4}$$

where the tensor product on the left hand side denotes the smallest (strongly closed) lattice containing the elementary tensors $A \otimes B$, $A \in \mathcal{P}_{B(l_2)}$, $B \in \mathcal{L}$. Moreover, it is shown in [Sh1] that

$$\text{lat } (1 \otimes \text{alg } \mathcal{L}) = \lim_n \mathcal{P}_{B(l_2)} \otimes \mathcal{L}_n,$$

where $\{\mathcal{L}_n\}$ is a sequence of finite sublattices of $\mathcal{L}$. It is easy to check that for a finite sublattice $\mathcal{L}_n \subset \mathcal{L}$, $\mathcal{P}_{B(l_2)} \otimes \mathcal{L}_n \subset \{P \oplus (1-Q) : (P,Q) \text{ is a simple } E\text{-pair}\}$, whence

$$\mathcal{P}_{B(l_2)} \otimes \mathcal{L} = \overline{\{P \oplus (1-Q) : (P,Q) \text{ is a simple } E\text{-pair}\}}^s.$$

Since

$$\text{alg } \mathcal{L} = \{T = (T_{ij})_{i,j=1}^2 \in B(H_1 \oplus H_2) \mid T_{11} \in \mathcal{D}_1, T_{22} \in \mathcal{D}_2, T_{21} \in \mathfrak{M}_{max}(E), T_{12} = 0\}$$

(see the proof of Theorem 3.1), one can easily check that $\begin{pmatrix} P & 0 \\ 0 & 1-Q \end{pmatrix} \in \mathcal{P}_{B(l_2\otimes H_1 \oplus l_2 \otimes H_2)}$, where $(P,Q) \in \text{bil}\ (1\otimes \mathfrak{M}_{max}(E))$, belongs to lat $(1 \otimes \text{alg}\ \mathcal{L})$. By (4) we have bil $1 \otimes \mathfrak{M}_{max}(E) \subset \overline{\{(P,Q) : (P,Q) \text{ is a simple } E \text{ pair }\}}^s$. The reverse inclusion is obvious. □

In the following theorem we list several possible definitions of a set of operator synthesis.

Theorem 4.6. *Let $E \subseteq X \times Y$ be a pseudo-closed set. Then the following are equivalent:*
(i) E is a set of synthesis;
(ii) $\mathfrak{M}_{min}(E) = \mathfrak{M}_{max}(E)$;
(iii) $\langle T, F\rangle = 0$ for any $T \in B(H_1,H_2)$ and $F \in \Gamma(X,Y)$, supp $T \subset E \subset$ null F;
(iv) any E-pair can be s-approximated in the strong operator topology of $B(l_2 \otimes H_1) \times B(l_2 \otimes H_2)$ by simple E-pairs;
(v) any E-pair can be approximated by simple E-pairs almost everywhere in the strong operator topology of $B(l_2)$.

Proof. (i) ⇔ (ii): obviously follows from the definition and Theorems 4.3,4.4.

(ii) ⇒ (iii): if $T \in \mathfrak{M}_{min}(E)$ then, by Theorem 4.4, $\langle T, F\rangle = 0$ for any $F \in \Gamma(X,Y)$, such that $E \subset$ null F, which shows the implication.

(iii) ⇒ (ii): Let $T \in \mathfrak{M}_{max}(E)$. Then supp $T \subset E$ and, therefore, $\langle T, F\rangle = 0$ for any $F \in \Phi(E)$. By Theorem 4.4, $T \in \mathfrak{M}_{min}(E)$, which gives us the necessary inclusion $\mathfrak{M}_{max}(E) \subset \mathfrak{M}_{min}(E)$.

(ii) ⇒ (iv): if $\mathfrak{M}_{min}(E) = \mathfrak{M}_{max}(E)$ then bil $1 \otimes \mathfrak{M}_{min}(E) =$ bil $1 \otimes \mathfrak{M}_{max}(E)$ and by Corollary 4.4 and Theorem 4.5 we obtain that any E-pair can be s-approximated by simple E-pairs.

(iv) ⇔ (v). We prove that the approximation of operator-valued functions in the strong operator topology in $B(l_2 \otimes L_2(X,\mu))$ is equivalent to the approximation almost everywhere in the strong operator topology in $B(l_2)$. In fact, let $P_n(x)$, $P(x) \in B(l_2 \otimes L_2(X,\mu))$, $P_n(x) \to P(x)$ almost everywhere on (X,μ) in the strong operator topology in $B(l_2)$ and take $\varphi = \sum_{k=1}^N \varepsilon_k(x)\vec{\xi_k}$, where $\varepsilon_k(\cdot)$ is the characteristic function of a set of finite measure and $\xi_k \in l_2$. It easily follows from the Lebesgue theorem that $||P_n\varphi - P\varphi|| \to 0$ as $n \to \infty$. Since the measure μ is sigma-finite, the set of all such φ is dense in $l_2 \otimes L_2(X,\mu)$. Therefore $||P_n\varphi - P\varphi|| \to 0$, $n \to \infty$, for any $\varphi \in l_2 \otimes L_2(X,\mu)$.

If now a sequence, $\{P_n\}$, of projection-valued functions converges to P in the strong operator topology in $B(l_2 \otimes L_2(X,\mu))$, then there exists a subsequence converging almost everywhere on (X,μ) in the strong operator topology in $B(l_2)$. To see this choose a dense set of vectors, $\{\vec{\xi_n}\}$, in l_2. Then

$$\int_A ||P_n(x)\vec{\xi_k} - P(x)\vec{\xi_k}||d\mu(x) \to 0,\ n \to \infty$$

for each k and each measurable set A of finite measure. Let $A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$ be a sequence of sets of finite measure such that $X = \cup_{j=1}^\infty A_j$. By the Riesz theorem there exists a subsequence $\{P_{k1}\}_{k=1}^\infty$ such that $\lim_{k\to\infty} P_{k1}(x)\vec{\xi_1} = P(x)\vec{\xi_1}$ a.e. on A_1. Then choose a subsequence $\{P_{k2}\}_{k=1}^\infty$ of $\{P_{k1}\}_{k=1}^\infty$ such that $\lim_{k\to\infty} P_{k2}(x)\vec{\xi_1} = P(x)\vec{\xi_1}$ a.e. on A_2. Proceeding in this fashion we obtain a series of sequences

$$\{P_n\}_{n=1}^\infty \supset \{P_{k1}\}_{k=1}^\infty \supset \{P_{k2}\}_{k=1}^\infty \supset \ldots \supset \{P_{kj}\}_{k=1}^\infty \supset \ldots,$$

such that $\lim_{k\to\infty} P_{kj}(x)\vec{\xi_1} = P(x)\vec{\xi_1}$ almost everywhere on A_j.

Consider now the diagonal sequence $\{P_{kk}\}_{k=1}^{\infty}$. Clearly $\lim_{k\to\infty} P_{kk}(x)\vec{\xi_1} = P(x)\vec{\xi_1}$ a.e. on each A_j and therefore on X. Set $P^{l1} = P_{ll}$, $l = 1, 2, \ldots$. Using the same arguments we can find a subsequence, $\{P^{l2}\}_{l=1}^{\infty}$, of $\{P^{l1}\}_{l=1}^{\infty}$ such that $\lim_{l\to\infty} P^{l2}(x)\vec{\xi_2} = P(x)\vec{\xi_2}$ a.e. on X and then $\{P^{lk}\}_{l=1}^{\infty}$, of $\{P^{l1}\}_{l=1}^{\infty}$ such that $\lim_{l\to\infty} P^{lk}(x)\vec{\xi_m} = P(x)\vec{\xi_m}$ a.e. on X for any $m \le k$ so that $\lim_{l\to\infty} P^{ll}(x)\vec{\xi_k} = P(x)\vec{\xi_k}$ a.e. on X for any k. Since $\{\vec{\xi_k}\}$ is dense in l_2 and the sequence $\{P^{ll}\}_{l=1}^{\infty}$ is bounded,

$$\lim_{l\to\infty} P^{ll}(x)\vec{\xi} = P(x)\vec{\xi} \text{ a.e. on } X \text{ for any } \vec{\xi} \in l_2.$$

$(iv) \Rightarrow (ii)$: if $T \in \mathfrak{M}_{max}(E)$, we have bil $1 \otimes T \supset \overline{\{(P,Q) : (P,Q) \text{ is a simple } E \text{ pair }\}}^{s}$, due to Theorem 4.5; (iv) implies now bil $1 \otimes T \supseteq \tilde{S}_E$ and hence $T \in \mathfrak{M}_{min}(E)$. □

Remark 4.2. The equivalence $(i) \Leftrightarrow (iii)$ was essentially proved in [A] and $(i) \Leftrightarrow (ii)$ in [Da] but using some other methods.

We use the equivalence $(i) \Leftrightarrow (v)$ to obtain the following result.

Theorem 4.7 (Inverse Image Theorem). *Let (X,μ), (Y,ν), (X_1,μ_1) and (Y_1,ν_1) be standard Borel spaces with measures, $\varphi : X \mapsto X_1$, $\psi : Y \mapsto Y_1$ Borel mappings. Suppose that the measures $\varphi_*\mu$, $\psi_*\nu$ are absolutely continuous with respect to the measures μ_1 and ν_1 respectively. If a Borel set $E_1 \subset X_1 \times Y_1$ is a set of $\mu_1 \times \nu_1$-synthesis then $(\varphi \times \psi)^{-1}(E_1)$ is a set of $\mu \times \nu$ synthesis.*

Proof. To prove the theorem we will need to prove first an auxiliary lemma.

Lemma 4.2. *Let (X,μ), (Y,ν) be standard Borel spaces with measures and $f : X \to Y$ be a Borel map. Then there exists a ν-measurable set $N \subset f(X)$, $\nu(N) = 0$, such that $f(X) \setminus N$ is Borel and if $u : X \to \mathbb{R}$ is a bounded Borel function then for any $\varepsilon > 0$ there exists a Borel map $g : f(X) \setminus N \to X$ such that $f(g(y)) = y$ for every $y \in f(X) \setminus N$ and $u(g(f(x))) > u(x) - \varepsilon$ a.e. on X.*

Proof. Assume first that the map $f : X \to Y$ is surjective. For any such map there exists a Borel section, i.e., a map $g : Y \to X$ which satisfies $f(g(y)) = y$, $y \in Y$ (see, for example, [Ta]). Since $u : X \to \mathbb{R}$ is bounded, $u(X) \subset [a,b]$. Let $a = a_0 < a_1 < \ldots < a_n = b$ be a partition of $[a,b]$ such that $a_{i+1} - a_i < \varepsilon$. Set

$$X_j = u^{-1}([a_j, a_{j+1})),\ Y_j = f(X_j),\ Y_j' = Y_j \setminus (\cup_{k>j} Y_k).$$

Then each Y_j' is the image of $X_j' = X_j \setminus (\cup_{k>j} f^{-1}(Y_k))$. We have also that $\cup_j Y_j' = Y$, $Y_i' \cap Y_j' = \emptyset$, $i \ne j$, and since every Y_i' is an analytic space, we obtain that Y_i' must be Borel (see, for example, [Ta, Theorem A.3]). Let $g_j : Y_j' \to X_j'$ be a Borel section for $f|_{X_j'}$. Then the functions g_j determine a Borel section, g, for f. Clearly, $g(Y_j) \subset \cup_{i\ge j} X_i$ so that $u(g(y)) \ge a_j$ for each $y \in Y_j$ and therefore $u(g(f(x))) \ge a_j$ for any $x \in X_j$. As $u(x) \in [a_j, a_{j+1})$ for $x \in X_j$, we obtain $u(g(f(x))) > u(x) - \varepsilon$ for each $x_j \in X_j$ and therefore for each $x \in X$.

For the general case consider the image $f(X)$ which is an analytic subset of Y. By [Ta, Theorem A.13] there exists a ν-measurable set $N \subset f(X)$ of zero measure such that $f(X) \setminus N$ is Borel. Set $\tilde{X} = f^{-1}(f(X) \setminus N)$. Then f is a Borel map from the Borel set $\tilde{X}$ onto $f(X) \setminus N$. Thus, given $\varepsilon > 0$, there exists a Borel map $g : f(X) \setminus N \to X$ such that $f(g(y)) = y$ for every $y \in f(X) \setminus N$ and $u(g(f(x))) > u(x) - \varepsilon$ on $\tilde{X}$. Since $X \setminus \tilde{X} \subset f^{-1}(N)$, we have that $\mu(X \setminus \tilde{X}) = 0$ and the inequality holds almost everywhere on X. □

Set $E = (\varphi \times \psi)^{-1}(E_1)$. By Theorem 4.6, we shall have established the theorem if we prove that any E-pair can be approximated a.e. in the strong operator topology of $B(l_2)$ by simple E-pairs. Since, by Theorem 4.5, the approximated pairs form a bilattice it would be enough to prove that any E-pair is majorized by an approximated pair.

Let (P, Q) be an E-pair. Choose a dense sequence ξ_n in l_2 and a sequence $\varepsilon_n > 0$, $\varepsilon_n \to 0$. Set $u_n(x) = (P(x)\xi_n, \xi_n)$. By Lemma 4.2, there are null sets $N_n \subset X_1$, $M_n \subset X$ and a Borel map $g_n : \varphi(X) \setminus N_n \to X$, such that $\varphi(g_n(x_1)) = x_1$, for $x_1 \in \varphi(X) \setminus N_n$, and $u_n(g_n(\varphi(x))) > u_n(x) - \varepsilon_n$, for $x \in X \setminus M_n$.

For $x_1 \in \varphi(X) \setminus N$, where $N = \cup_{n=1}^{\infty} N_n$, set

$$\hat{P}(x_1) = \bigvee_n P(g_n(x_1)).$$

Then for any $x \in X \setminus M$, where $M = \cup_{n=1}^{\infty} M_n$, one has

$$\begin{aligned}(P(x)\xi_n, \xi_n) = u_n(x) < u_n(g_n(\varphi(x))) + \varepsilon_n = \\ = (P(g_n(\varphi(x)))\xi_n, \xi_n)) + \varepsilon_n \le (\hat{P}(\varphi(x))\xi_n, \xi_n) + \varepsilon_n.\end{aligned}$$

It follows easily that

$$P(x) \le \hat{P}(\varphi(x)), \quad x \in X \setminus M. \tag{5}$$

Similarly, we construct null sets $M' \subset Y$, $N' \subset Y_1$, functions $g'_n : \psi(Y) \setminus N' \to Y$ and set $\hat{Q}(y_1) = \bigvee_n Q(g'_n(y_1))$ with

$$Q(y) \le \hat{Q}(\psi(y)), \quad y \in Y \setminus M'. \tag{6}$$

Thus (P, Q) is majorized by $(\hat{P} \circ \varphi, \hat{Q} \circ \psi)$.

It follows easily that $(\hat{P}, \hat{Q})$ is an E_1-pair. Indeed, let $(x_1, y_1) \in E_1$, $x_1 \notin N$, $y_1 \notin N'$, then

$$P(g_n(x_1)) \perp Q(g'_n(y_1))$$

for any n, m. Hence

$$\hat{P}(x_1) \perp \hat{Q}(y_1).$$

It follows that there are simple E_1-pairs $(\hat{P}_n, \hat{Q}_n)$ with $\hat{P}_n(x_1) \to \hat{P}(x_1)$ a.e. $(x \notin S)$, $\hat{Q}_n(y_1) \to \hat{Q}(y_1)$ a.e. $(y_1 \notin S')$. Let

$$P_n(x) = \hat{P}_n(\varphi(x)), \quad Q_n(y) = \hat{Q}_n(\psi(y)).$$

Then $P_n(x) \to \hat{P}(\varphi(x))$ a.e., $Q_n(y) \to \hat{Q}(\psi(y))$ a.e. Indeed, let $\tau = \{x : \varphi(x) \in S\}$, then

$$\mu(\tau) = \mu(\{x : \varphi(x) \in S\}) = \varphi_*\mu(S) = 0,$$

because $\varphi_*\mu$ is absolutely continuous with respect to μ_1. Similarly, $\nu(\tau') = 0$, where $\tau' = \{y : \psi(y) \in S'\}$. This shows that the pair $(\hat{P} \circ \varphi, \hat{Q} \circ \psi)$ is approximable by simple pairs. The proof is complete. □

Corollary 4.5. *Let $E \subseteq X \times Y$ be a set of synthesis with respect to a pair of measures (μ_1, ν_1), $\mu_1 \in M(X)$, $\nu_1 \in M(Y)$. Then E is a set of (μ, ν)-synthesis for any $\mu \in M(X)$, $\nu \in M(Y)$ such that $\mu \le \mu_1$, $\nu \le \nu_1$.*

Proof. Follows from Theorem 4.7 applied to the identity mappings φ and ψ. □

Suppose that φ_i and ψ_i, $i = 1, \ldots, n$, are Borel maps of standard Borel spaces (X, μ) and (Y, ν) into an ordered standard Borel space $(Z, \leq)$. Then the set $E = \{(x, y) \mid \varphi_i(x) \leq \psi_i(y), i = 1, \ldots, n\}$ is called a set of width n.

Theorem 4.8. *Any set of finite width is synthetic with respect to measures μ, ν.*

Proof. Let E be a set of width n, i.e. $E = \{(x, y) \in X \times Y \mid f_i(x) \leq g_i(x), i = 1, \ldots, n\}$, where $f_i : X \to Z$, $g_i : Y \to Z$ are Borel functions. We define mappings $F : X \to Z^n$ and $G : Y \to Z^n$ by setting $F(x) = (f_1(x), \ldots, f_n(x))$, $G(y) = (g_1(y), \ldots, g_n(y))$. Put $\mu_1 = F_*\mu$, $\nu_1 = G_*\nu$. Let $E_1 = \{(x, y) \in Z^n \times Z^n \mid x_i \leq y_i, i = 1, \ldots, n\}$. By [A], E_1 is a set of $\mu_1 \times \nu_1$-synthesis if the measures μ_1 and ν_1 are equal. In general, consider the measure $\lambda = \mu_1 + \nu_1$, then we can conclude that E_1 is a set of $\lambda \times \lambda$-synthesis and applying now Corollary 11 we obtain that E_1 is a set of synthesis with respect to μ_1, ν_1. It follows now from Theorem 4.7 that $(F \times G)^{-1}(E_1) = E$ is a set of $\mu \times \nu$-synthesis. □

Remark 4.3. Arveson, [A], introduced the class of finite width lattices as those which are generated by a finite set of nests (linearly ordered lattices). He proved that all finite width lattices are synthetic. Todorov, [T], defined a subspace map (see [Er]) of finite width and proved that such subspace maps are synthetic. This result is in fact equivalent to our, actually a subspace map is a counterpart of a bilattice. Synthesizeability of special sets of width two ("nontriangular" sets) was proved in [KT, Sh2].

5 General bilattices

Let h_0 be the function on $[0, 1]$ such that $h_0(0) = 0$ and $h_0(t) = 1$ for $t \neq 0$, and let $h_1(t) = 1 - h_0(1 - t)$. It is clear that for any positive contraction A, $h_0(A)$ is the projection onto the range of A, $h_1(A)$ is the projection onto the subspace of invariant vectors. It is easy to see (for example, approximating $h_1(t)$ by t^α, $\alpha \to 0$) that h_i are operator monotone, i.e., if A, $B \in B(H)$, $0 \leq A \leq B \leq 1$, then $h_i(A) \leq h_i(B)$.

Lemma 5.1. *Let $\mathcal{D}_1$, $\mathcal{D}_2$ be commutative von Neumann algebras in Hilbert spaces H_1, H_2 and let S be a bilattice in $\mathcal{D}_1 \times \mathcal{D}_2$ Then, for any $(A, B) \in F_S$, $(h_0(A), h_1(B)) \in F_S$ and $(h_1(A), h_0(B)) \in F_S$.*

Proof. If $\mathcal{D}_1$, $\mathcal{D}_2$ are masas in separable spaces H_1, H_2, then the assertion follows from Lemma 4.1. Indeed, if $A(x) + B(y) \leq 1$, then

$$h_0(A(x)) \leq h_0(1 - B(y)) = 1 - h_1(B(y))$$

and $(h_0(A), h_1(B)) \in F_S$. Similarly, $(h_1(A), h_0(A)) \in F_S$.

Assume now that $\mathcal{D}_1$, $\mathcal{D}_2$ are arbitrary commutative von Neumann algebras acting on separable Hilbert spaces. Let x_1 and x_2 be separating vectors for $\mathcal{D}_1$ and $\mathcal{D}_2$, and let $K_i = \overline{[\mathcal{D}_i x_i]}$, $i = 1, 2$. Then the restriction of $B(l_2)\bar{\otimes}\mathcal{D}_i$ to $l_2 \otimes K_i$ is injective. Now, since the restriction of $\mathcal{D}_i$ to K_i is a masa and the restriction of $(A, B) \in F_S$ to $(l_2 \otimes K_1) \times (l_2 \otimes K_2)$ is in the restriction of F_S, the problem is reduced to the above.

Furthermore, the statement is true when $\mathcal{D}_1$, $\mathcal{D}_2$ are countably generated. To see this it is enough to prove that if $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are vectors in $l_2 \otimes H_1$ and $l_2 \otimes H_2$, then there exist a pair $(C, D) \in F_S$ such that $h_0(A)x_i = Cx_i$ and $h_1(B)y_i = Dy_i$, $i = 1, \ldots, n$. If $x_k = (x_{kj})$, $y_k = (y_{kj})$, $x_{kj} \in H_1$, $y_{kj} \in H_2$, we define K_1 and K_2 to be the closed linear spans of vector Xx_{kj}, $X \in \mathcal{D}_1$, and Yy_{kj}, $Y \in \mathcal{D}_2$, respectively. Then K_1 and K_2 are separable and we come to the previous.

Now, to prove the assertion in general situation, it is sufficient to show that each $\mathcal{D}_i$ contains countably generated von Neumann algebras, $\hat{\mathcal{D}}_i$, such that $(A, B) \in F_{\hat{S}}$, where $\hat{S}$ is the intersection of S with $\hat{\mathcal{D}}_1 \times \hat{\mathcal{D}}_2$. For this take a dense sequence of unit vectors, $\{\xi_n\}$, in l_2. For each pair $(L_{\xi_n}(A), L_{\xi_n}(B))$ there exists a sequence, (A_k^n, B_k^n), from the convex linear span, conv S, of S, which converges to the pair uniformly. Let S' be the set of all pairs of projections $(p, q) \in S$ which participate in the linear combinations for (A_k^n, B_k^n). Then $\hat{\mathcal{D}}_1$ and $\hat{\mathcal{D}}_2$ can be defined as von Neumann algebras generated by $\pi_1(S')$ and $\pi_2(S')$, π_i being the projection onto the i-th coordinate. □

Lemma 5.2. *$\tilde{S}$ is a billatice.*

Proof. Let $(P, Q) \in \tilde{S}$ and $P_1 \in \mathcal{P}_{B(l_2)\bar{\otimes}\mathcal{D}_1}$, $Q_1 \in \mathcal{P}_{B(l_2)\bar{\otimes}\mathcal{D}_2}$, $P_1 \leq P$, $Q_1 \leq Q$. Then $L_\varphi(P_1) \leq L_\varphi(P)$, $L_\varphi(Q_1) \leq L_\varphi(Q)$ for each state φ on $B(l_2)$ so that

$$E_{L_\varphi(P_1)}([a,1]) \leq E_{L_\varphi(P)}([a,1]) \text{ and } E_{L_\varphi(Q_1)}([b,1]) \leq E_{L_\varphi(Q)}([b,1])$$

for any $0 \leq a, b \leq 1$. Applying now Lemma 3.1 we obtain $(P_1, Q_1) \in \tilde{S}$.

That $\tilde{S}$ is closed under the operations $(\bigvee, \bigwedge)$, $(\bigwedge, \bigvee)$ follows from

$$(P_1 \bigvee P_2, Q_1 \bigwedge Q_2) = (h_0((P_1 + P_2)/2), h_1((Q_1 + Q_2)/2),$$
$$(P_1 \bigwedge P_2, Q_1 \bigvee Q_2) = (h_1((P_1 + P_2)/2), h_0((Q_1 + Q_2)/2)$$

and the previous lemma. □

Our next goal is to show that $\tilde{S}$ is reflexive. We will deduce this from a general criteria of reflexivity. To formulate it we need some definitions and notations.

Let S be a bilattice in $\mathcal{R} \times \mathcal{R}$, where $\mathcal{R}$ is a von Neumann algebra on a Hilbert space H, and let $\mathcal{M}$ be a von Neumann algebra on H. Denote by $I(\mathcal{M})$ the semigroup of all isometries in $\mathcal{M}$. We say that S is *$\mathcal{M}$-invariant* if

- S contains all pairs $(P, 1-P)$, $P \in \mathcal{P}_\mathcal{M}$.
- If $U \in I(\mathcal{M})$ then a pair $(P, Q) \in \mathcal{R} \times \mathcal{R}$ belongs to S if and only if (UPU^*, UQU^*) belongs to S.

For any bilattice S we set

$$\Omega_S = \{(x, y) \in H \times H \mid \exists (P, Q) \in S \text{ with } Px = x, Qy = y\}.$$

If S is clear we write Ω instead of Ω_S. A bilattice S is called *stable* if Ω_S is norm-closed in $H \oplus H$.

Theorem 5.1. *Any bilattice in $\mathcal{R}\times\mathcal{R}$ which is stable and invariant with respect to a properly infinite von Neumann algebra is reflexive.*

Proof. Suppose that S is stable and $\mathcal{M}$-invariant, where $\mathcal{M}$ is properly infinite. Note first that $\mathfrak{M}(S) \subset \mathcal{M}'$. Indeed, if $T \in \mathfrak{M}_S$ then $(1-P)TP = 0$ for any $P \in \mathcal{P}_\mathcal{M}$, and similarly $PT(1-P)$, hence $TP = PT$ and $T \in \mathcal{M}'$, because $\mathcal{P}_\mathcal{M}$ generates $\mathcal{M}$. For $(x, y) \in H \times H$, we denote by $v_{x,y}$ the restriction of the vector state $w_{x,y}$ to $\mathcal{M}'$.

Claim 1. Let $U \in I(\mathcal{M})$. If $(U^*x, y) \in \Omega$ then $(x, Uy) \in \Omega$.

Indeed, let $(P, Q) \in S$ such that $PU^*x = U^*x$, $Qy = y$. Consider $P_1 = UPU^*$, $Q_1 = UQU^*$. Then $P_1x = UU^*x$, $Q_1Uy = Uy$ and thus $Uy \in Q_1H \cap UU^*H$. Set

$$P_2 = P_1 \bigvee (1 - UU^*), \quad Q_2 = Q_1 \bigwedge UU^*.$$

Then P_2H contains UU^*x and $(1 - UU^*)x$, hence P_2H contains x, i.e. $P_2x = x$. On the other hand Q_2H contains Uy. So $Q_2Uy = Uy$. Clearly, $(P_2, Q_2) \in S$ and we get $(x, Uy) \in \Omega$.

Now we prove the converse statement.

Claim 2. If $(x, Uy) \in \Omega$, $U \in I(\mathcal{M})$ then $(U^*x, y) \in \Omega$.

Indeed, let $(P, Q) \in S$, $Px = x$, $QUy = Uy$. Set

$$P_1 = P \bigvee (1 - UU^*), Q_1 = Q \bigwedge UU^*.$$

Then $P_1x = x$, $Q_1Uy = Uy$, $P_1 \geq 1 - UU^*$, $Q_1 \leq UU^*$. It follows that P_1, Q_1 commute with UU^*. Hence $P_2 = U^*P_1U$ and $Q_2 = U^*Q_1U$ are projections. To see that $(P_2, Q_2) \in S$ note that $(UP_2U^*, UQ_2U^*) = (UU^*P_1, UU^*Q_1) \in S$, since $UU^*P_1 \leq P_1$, $UU^*Q_1 \leq Q_1$. It remains to show that $P_2U^*x = U^*x$ and $Q_2y = y$. Indeed,

$$P_2U^*x = U^*P_1UU^*x = U^*UU^*P_1x = U^*UU^*x = U^*x,$$
$$Q_2y = U^*Q_1Uy = U^*Uy = y.$$

Our claim is proved.

Claim 3. If $(x, y) \in \Omega$, $v_{x,y} = v_{x,z}$ then $(x, z) \in \Omega$.

To show this set $t = y - z$. Then $v_{x,t} = 0$, $\mathcal{M}'x \perp \mathcal{M}'t$. Defining R to be the projection onto $\overline{\mathcal{M}'x}$ we have $R \in \mathcal{M}$, $Rx = x$ and $(1 - R)t = t$.

Let now $(P, Q) \in S$, $Px = x$, $Qy = y$. Set $P_1 = P \bigwedge R$, $Q_1 = Q \bigvee (1 - R)$. Then $(P_1, Q_1) \in S$, $P_1x = x$, $Q_1z = Q_1(y - t) = y - t = z$. We proved that $(x, z) \in \Omega$.

Since $\mathcal{M}$ is properly infinite there are U_1, $U_2 \in I(\mathcal{M})$ with $U_1H \perp U_2H$. We fix such a pair of isometries.

Claim 4. If $(x_1, y_1) \in \Omega$ and $v_{x_1,y_1} = v_{x_2,y_2}$ then $(x_2, y_2) \in \Omega$.

Indeed, set $x = U_1x_1 + U_2x_2$. Then $x_1 = U_1^*x$. Hence $(U_1^*x, y_1) \in \Omega$. By Claim 1, $(x, U_1y_1) \in \Omega$. Since

$$v_{x,U_1y_1} = v_{U_1^*x,y_1} = v_{x_1,y_1} = v_{x_2,y_2} = v_{x,U_2y_2},$$

we obtain from Claim 3 that $(x, U_2y_2) \in \Omega$. Now by Claim 2, $(U_2^*x, y_2) \in \Omega$, that is $(x_2, y_2) \in \Omega$. The claim is proved.

Set now

$$W = \{v_{x,y} \mid (x, y) \in \Omega\}.$$

Claim 5. W is a linear subspace in the space $(\mathcal{M}')_*$ of all σ-weakly continuous functionals on $\mathcal{M}'$.

Indeed,

$$v_{x_1,y_1} + v_{x_2,y_2} = v_{x,U_1^*y_1} + v_{x,U_2^*y_2} = v_{x,y},$$

where $x = U_1x_1 + U_2x_2$. We know from the preceding claim that $(x, U_1^*y_1)$ and $(x, U_2^*y_2)$ belong to Ω. Let $(P_1, Q_1) \in S$, $(P_2, Q_2) \in S$ such that

$$P_1x = x,\ Q_1U_1^*y_1 = U_1^*y_1,\ P_2x = x,\ Q_2U_2^*y_2 = U_2^*y_2.$$

Then setting $P = P_1 \bigwedge P_2$, $Q = Q_1 \bigvee Q_2$ we have $Px = x$, $Qy = y$. Thus $(x, y) \in \Omega$ and $W + W \subset W$.

Claim 6. W is norm-closed.

Let $\varphi_n \to \varphi$, $\varphi_n \in W$. Since φ is σ-weakly continuous and $\mathcal{M}'$ has a separating vector, $\varphi = v_{x,y}$ for some $x \in H$, $y \in H$. Since $\mathcal{M}'$ has the properly infinite commutant, there are x_n, $y_n \in H$ such that $\varphi_n = v_{x_n,y_n}$, $||x_n - x|| \to 0$, $||y_n - y|| \to 0$ ([Sh1]). By Claim 4, $(x_n, y_n) \in \Omega$. Since S is stable, $(x, y) \in \Omega$ and $\varphi \in W$. We proved that W is norm-closed.

Recall that $\mathcal{M}'$ is the dual of $(\mathcal{M}')_*$. So for $\mathcal{A} \subset \mathcal{M}'$, $\mathcal{B} \subset \mathcal{M}$ we write

$$\mathcal{A}_\perp = \{\varphi \in (\mathcal{M}')_* \mid \mathcal{A} \subset \ker\varphi\},\quad \mathcal{B}^\perp = \{T \in \mathcal{M} \mid \varphi(T) = 0, \forall\varphi \in \mathcal{B}\}.$$

By the usual duality argument, $(\mathcal{B}^\perp)_\perp$ coincides with the norm closure of $\mathcal{B}$, for any linear subspace $\mathcal{B} \subset (\mathcal{M}')_*$.

Claim 7. $W = (\mathfrak{M}(S))_\perp$.

Indeed, suppose that $T \in W^\perp$. Then for any $(P, Q) \in S$, $QTP = 0$, because

$$(QTPx, y) = w_{Px,Qy}(T) = 0.$$

Thus $W^\perp = \mathfrak{M}_S$ and, by duality, $W = \mathfrak{M}(S)_\perp$, since W is closed.

Now we can finish the proof of the theorem.

If $(P_0, Q_0) \in \operatorname{bil}\mathfrak{M}(S)$ then $w_{P_0x,Q_0y}(T) = 0$ for any $T \in \mathfrak{M}(S)$. Hence $w_{P_0x,Q_0y} \in \mathfrak{M}(S)_\perp = W$. On the other hand, for any $x \in P_0H$, $y \in Q_0H$ there are $(P_{x,y}, Q_{x,y}) \in S$ with $x \in P_{x,y}H$, $y \in Q_{x,y}H$. Set

$$P_x = \bigwedge_{y\in Q_0H} P_{x,y},\quad Q_x = \bigvee_{y\in Q_0H} Q_{x,y}.$$

Then $x \in P_xH$, $Q_0H \subset Q_xH$. Let

$$P = \bigvee_{x\in P_0H} P_x,\ Q = \bigwedge_{x\in P_0H} Q_x,$$

then $(P, Q) \in S$, $P_0 \le P$, $Q_0 \le Q$. We proved that $(P_0, Q_0) \in S$. □

Let S be a bilattice in $\mathcal{R}_1 \times \mathcal{R}_2$ and let $B(S)$ denote the bilattice in $(\mathcal{R}_1 \oplus \mathcal{R}_2) \times (\mathcal{R}_1 \oplus \mathcal{R}_2)$ generated by all pairs $(p \oplus (1 - q), (1 - p) \oplus q) \in S$. It is easy to see that $B(S)$ consists of all pairs $(p_1 \oplus q_1, p_2 \oplus q_2)$, where $(p_1, q_2) \in S$ and $p_2 \le 1 - p_1$, $q_1 \le 1 - q_2$.

Proposition 5.1. *A bilattice S in $\mathcal{R}_1 \times \mathcal{R}_2$ is reflexive if and only if the bilattice $B(S)$ in $(\mathcal{R}_1 \oplus \mathcal{R}_2) \times (\mathcal{R}_1 \oplus \mathcal{R}_2)$ is reflexive.*

Proof. Since S is a bilattice, $(p, 0)$, $(0, q) \in S$ for any $p \in \mathcal{R}_1$, $q \in \mathcal{R}_2$. This implies

$$\begin{aligned} \mathfrak{M}(B(S)) &= \{(T_{ij})_{i,j=1}^2 \mid (1-p)T_{11}p = qT_{22}(1-q) = qT_{21}p = 0, \\ &\quad (1-p)T_{12}(1-q) = 0, \forall(p,q) \in S\} = \\ &= \{(T_{ij})_{i,j=1}^2 \mid T_{ii} \in \mathcal{R}_i', i = 1, 2, T_{21} \in \mathfrak{M}(S), T_{12} = 0\} \end{aligned}$$

and

$$\text{bil } \mathfrak{M}(B(S)) = \{(p_1 \oplus p_2, q_1 \oplus q_2) \mid q_i T_{ii} p_i = q_2 T_{12} p_1 = 0, \forall T = (T_{ij})_{i,j=1}^2 \in \mathfrak{M}(B(S))\}$$
$$= \{(p_1 \oplus p_2, q_1 \oplus q_2) \mid q_1 p_1 = q_2 p_2 = 0, (p_1, q_2) \in \text{bil } \mathfrak{M}(S)\}$$

giving the statement. □

Let now S be again a commutative bilattice in $\mathcal{D}_1 \times \mathcal{D}_2$ and let $\tilde{S}$ be the bilattice defined above.

Theorem 5.2. *The bilattice $\tilde{S}$ is reflexive.*

Proof. By Proposition 5.1 and Theorem 5.1 it is sufficient to prove that the bilattice $B(\tilde{S})$ is stable and $B(l_2) \otimes 1$-invariant.

Let $(x_n^1 \oplus x_n^2, y_n^1 \oplus y_n^2) \in \Omega_{B(\tilde{S})}$, x_n^i, $y_n^i \in l_2 \otimes H_i$, $i = 1, 2$, and $x_n^i \to x_i$, $y_n^i \to y_i$ as $n \to \infty$. Then $p_n^i x_n^i = x_n^i$, $q_n^i = y_n^i$ for some $(p_n^1 \oplus p_n^2, q_n^1 \oplus q_n^2) \in B(\tilde{S})$. We have $(p_n^1, q_n^2) \in \tilde{S}$ and $p_n^2 \le 1 - p_n^1$, $q_n^1 \le 1 - q_n^2$. We can also assume that the sequences $\{p_n^i\}$, $\{q_n^i\}$ are weakly convergent:

$$p_n^i \to a_i, \quad q_n^i \to b_i.$$

Clearly, $a_i x_i = x_i$, $b_i y_i = y_i$ and $a_2 \le 1 - a_1$, $b_1 \le 1 - b_2$. Let $P_i = h_1(a_i)$ and $Q_i = h_1(b_i)$ be the projections onto invariant vectors of a_i and b_i, $i = 1, 2$. It is easy to check that $(a_1, b_2) \in F_{\tilde{S}}$. By Lemma 3.1, $(P_1, Q_2) \in \tilde{S}$. Moreover, $P_2 \le 1 - P_1$, $Q_1 \le 1 - Q_2$. Thus $(P_1 \oplus P_2, Q_1 \oplus Q_2) \in B(\tilde{S})$, $(x_1 \oplus x_2, y_1 \oplus y_2) \in \Omega_{B(\tilde{S})}$ and $B(\tilde{S})$ is stable.

In order to prove $B(l_2) \otimes 1$-invariance we note first that for any unit vector $\xi \in l_2$, $u \in I(B(l_2))$ and $P \in B(l_2) \bar{\otimes} \mathcal{D}_i$, $i = 1, 2$,

$$L_\xi(P) = L_{u\xi}((u \otimes 1)P(u \otimes 1)^*) \text{ and } L_\xi((u \otimes 1)P(u \otimes 1)^*) = L_{u^*\xi}(P)$$

implying that

$$(p, q) \in \tilde{S} \text{ iff } ((u \otimes 1)p(u \otimes 1)^*, (u \otimes 1)q(u \otimes 1)^*) \in \tilde{S}. \tag{7}$$

Since u is an isometry, we have also that for any $p \in B(l_2) \bar{\otimes} \mathcal{D}_i$

$$p \le 1 - q \Leftrightarrow (u \otimes 1)p(u \otimes 1)^* \le 1 - (u \otimes 1)q(u \otimes 1)^*$$

From this and (7) it follows that $(p_1 \oplus p_2, q_1 \oplus q_2) \in B(\tilde{S})$ if and only if $((u \otimes 1)(p_1 \otimes p_2)(u \otimes 1)^*, (u \otimes 1)(q_1 \otimes q_2)(u \otimes 1)^*) \in B(\tilde{S})$.

Since for a state φ on $B(l_2)$ and $p \in \mathcal{P}_{B(l_2)}$,

$$(L_\varphi(p \otimes 1), L_\varphi((1 - p) \otimes 1) = (\varphi(p), 1 - \varphi(p)) = \varphi(p)(1, 0) + (1 - \varphi(p))(0, 1) \in \text{Conv } S,$$

we have also

$$(p \otimes 1 \oplus p \otimes 1, (1 - p) \otimes 1 \oplus (1 - p) \otimes 1) \in B(\tilde{S}).$$

We proved therefore that $B(\tilde{S})$ is $B(l_2) \otimes 1$-invariant. □

Proof of Theorem 3.4. Since bil $\mathfrak{M}_0(S) \supset S$, we have only to prove the reverse inclusion. Let $(p, q) \in$ bil $\mathfrak{M}_0(S)$. Then $(1 \otimes p, 1 \otimes q) \in$ bil $\mathfrak{M}(\tilde{S})$. By Theorem 5.2, $(1 \otimes p, 1 \otimes q) \in \tilde{S}$ and therefore $(p, q) \in S$. □

6 Operator synthesis and spectral synthesis

We recall first a definition of a set of spectral synthesis. Let $\mathcal{A}$ be a unital semi-simple regular commutative Banach algebra with spectrum X, which is thus a compact Hausdorff space. We will identify $\mathcal{A}$ with a subalgebra of the algebra $C(X)$ of continuous complex-valued functions on X in our notation. If $E \subset X$ is closed, let

$$\begin{aligned} I_{\mathcal{A}}(E) &= \{a \in \mathcal{A} : a(x) = 0 \text{ for } x \in E\} \\ I^0_{\mathcal{A}}(E) &= \{a \in \mathcal{A} : a(x) = 0 \text{ in a nbhd of } E\} \\ \text{and } J_{\mathcal{A}}(E) &= \overline{I^0_{\mathcal{A}}(E)}. \end{aligned}$$

We say that E is a set of *spectral synthesis* for $\mathcal{A}$ if $I_{\mathcal{A}}(E) = J_{\mathcal{A}}(E)$ (this definition is equivalent to the one given in the introduction).

The Banach algebra we will be mainly deal with is the projective tensor product $V(X,Y) = C(X)\hat{\otimes}C(Y)$, where X and Y are compact Hausdorff spaces. Recall that $V(X,Y)$ (the Varopoulos algebra) consists of all functions $\Phi \in C(X \times Y)$ which admit a representation

$$\Phi(x,y) = \sum_{i=1}^{\infty} f_i(x)g_i(y), \tag{8}$$

where $f_i \in C(X)$, $g_i \in C(Y)$ and

$$\sum_{i=1}^{\infty} ||f_i||_{C(X)}||g_i||_{C(Y)} < \infty.$$

$V(X,Y)$ is a Banach algebra with the norm

$$||\Phi||_V = \inf \sum_{i=1}^{\infty} ||f_i||_{C(X)}||g_i||_{C(Y)},$$

where inf is taken over all representations of Φ in the form $\sum f_i(x)g_i(y)$ (shortly, $\sum f_i \otimes g_i$) satisfying the above conditions (see [V1]). We note that $V(X,Y)$ is a semi-simple regular Banach algebra with spectra $X \times Y$.

For $B \in V(X,Y)'$ and $F \in V(X,Y)$, define FB in $V(X,Y)'$ by $\langle FB, \Psi\rangle = \langle B, F\Psi\rangle$. Define the support of B by

$$\text{supp } (B) = \{(x,y) \in X \times Y \mid FB \neq 0 \text{ whenever } F(x,y) \neq 0\}.$$

Then it is known that for a closed set $E \subseteq X \times Y$,

$$J_{V(X,Y)}(E)^{\perp} = \{B \in V(X,Y)' \mid \text{supp } (B) \subset E\}$$

and hence E is a set of spectral synthesis for $V(X,Y)$ if $I_{V(X,Y)}(E)^{\perp} = \{B \in V(X,Y)' \mid \text{supp } (B) \subset E\}$, i.e., if

$$\langle B, F\rangle = 0$$

for any $B \in V(X,Y)'$, supp $(B) \subseteq E$, and any $F \in V(X,Y)$ vanishing on E. Any element of $V(X,Y)'$ can be identified with a bounded bilinear form $\langle B, f\otimes g\rangle = B(f,g)$ on $C(X)\times C(Y)$ which we also call a bimeasure.

We will need also to consider the class of all functions Φ on $X \times Y$ representable in the form (8) (i.e. $\Phi(X,Y) = \sum_{i=1}^{\infty} f_i(x)g_i(y)$, where $f_i \in C(X)$, $g_i \in C(Y)$) with

$$\sup_x \sum |f_i(x)|^2 < \infty, \quad \sup_y \sum |g_i(x)|^2 < \infty$$

(with the pointwise convergence of the series). It is called the extended Haagerup tensor product and denoted by $C(X)\hat{\otimes}_{eh}C(Y)$. Clearly $V(X,Y) \subset C(X)\hat{\otimes}_{eh}C(Y)$. The inclusion is strict, moreover $C(X)\hat{\otimes}_{eh}C(Y)$ contains some discontinuous functions. Indeed let $f(x) \in C(\mathbb{R})$ such that $|f(x)| \leq 1$, $f(x) = 0$ for any $x \in (-\infty, 1] \cup [3/2, +\infty)$ and $f(x) = 1$ on the interval $[1+\varepsilon, 3/2-\varepsilon]$, ε being small enough. Setting $f_k(x) = f(2^k x)$ and $u(x,y) = \sum f_k(x)\bar{f}_k(y)$, we obtain $\sup \sum |f_k(x)|^2 = 1$ and therefore $u(x,y) \in C(X)\hat{\otimes}_{eh}C(Y)$. However, $u(x,x) = \sum |f_k(x)|^2$ does not converge to zero as $x \to 0$ while $u(x,0) = u(0,y) = 0$, i.e. $u(x,y)$ is not continuous in $(0,0)$.

The following theorem connects operator synthesis and synthesis with respect to the Varopoulos algebra $V(X,Y)$. Let $M(X)$, $M(Y)$ be the spaces of finite Borel measures on X and Y respectively.

Theorem 6.1. *If a closed set $E \subseteq X \times Y$ is a set of synthesis with respect to any pair of measures (μ,ν), $\mu \in M(X)$, $\nu \in M(Y)$, then E is synthetic with respect to $V(X,Y)$.*

Proof. Assume that E is not a set of spectral synthesis for the algebra $V(X,Y)$. Then there exists a bimeasure B, supp $(B) \subseteq E$ and $F \in V(X,Y)$, $F\chi_E = 0$, such that $\langle B, F\rangle \neq 0$. By the Grothendick theorem, there exist measures $\mu \in M(X)$ and $\nu \in M(Y)$ and a constant C such that

$$|\langle B, f \otimes g\rangle| = |B(f,g)| \leq C||f||_{L_2(X,\mu)}||g||_{L_2(Y,\nu)} \tag{9}$$

Since $V(X,Y)$ can be densely embedded into $L_2(X,\mu)\hat{\otimes}L_2(Y,\nu)$, it follows from (9) that the linear functional $\Phi \mapsto \langle B, \Phi\rangle$ defined on $V(X,Y)$ can be extended to a continuous linear functional on $L_2(X,\mu)\hat{\otimes}L_2(Y,\nu)$. Therefore, there exists an operator $T \in B(L_2(X,\mu), L_2(Y,\nu))$ such that

$$\langle B, \Phi\rangle = \langle T, \Phi\rangle,$$

the left hand side being the pairing in the sense of duality between $V(X,Y)$ and $V(X,Y)'$ and the right hand side is the pairing in the sense of duality between $L_2(X,\mu)\hat{\otimes}L_2(Y,\nu)$ and $B(L_2(X,\mu), L_2(Y,\nu))$.

We shall have established the theorem if we prove that T is supported in E. Since E is closed, for every closed sets α, β such that $(\alpha \times \beta) \cap E = \emptyset$, there exist open sets $\alpha_0 \supset \alpha$, $\beta_0 \supset \beta$ such that $\overline{\alpha_0} \times \overline{\beta_0}$ does not intersect E. For every functions $f \in C(X)$, $g \in C(Y)$ which are equal to zero outside the set α_0 and β_0 respectively, we have $(Tf,g) = \langle T, f \otimes g\rangle = \langle B, f \otimes g\rangle = 0$. Since $C(\alpha_0)$ and $C(\beta_0)$ are dense in $L_2(\alpha_0,\mu)$ and $L_2(\beta_0,\nu)$, we obtain $P_{\beta_0}TQ_{\alpha_0} = 0$ and $P_\beta TQ_\alpha = 0$. By the regularity of measures μ and ν it follows that this is true for any Borel sets α, β. □

Corollary 6.1. *Suppose that $\varphi_i : X \mapsto Z$ and $\psi_i : Y \mapsto Z$, $i = 1,\ldots,n$, are continuous functions from compact metric spaces X and Y to an ordered compact metric space Z. Then the set $E = \{(x,y) \mid \varphi_i(x) \leq \psi_i(y), i = 1,\ldots,n\}$ is a set of synthesis with respect to the algebra $V(X,Y)$.*

Proof. It follows from Theorems 4.8,6.1. □

This corollary yields the theorem of Drury on synthesizeability of "nontriangular" sets, which are sets of width two (see [D]).

We will see that converse of Theorem 6.1 is false in general.

Lemma 6.1. *If $E \subset X \times Y$ is a set of synthesis with respect to a pair of finite measures then so is its intersection with any measurable rectangular.*

Proof. Let $\mu \in M(X)$, $\nu \in M(Y)$, let $K \times S$ be a measurable rectangular in $X \times Y$, let $T \in B(L_2(X,\mu), L_2(Y,\nu))$ and $F \in \Gamma(X,Y)$ be such that $\text{supp } T \subset E \cap (K \times S) \subset \text{null } F$. Then $T = P_S T Q_K$ and $\text{supp } T \subset E$. Moreover, the function $F'(x,y) = \chi_K(x)\chi_S(y)F(x,y)$ belongs to $\Gamma(X,Y)$ and vanishes on E. Since E is a set of synthesis, we obtain

$$\langle T, F\rangle = \langle P_S T Q_K, F\rangle = \langle T, F'\rangle = 0,$$

finishing the proof. □

Proposition 6.1. *There exists a closed set $E \subset X \times Y$ and a pair (μ,ν) of finite measures on X and Y such that E is set of synthesis in $V(X,Y)$, but not of operator synthesis with respect to the pair (μ,ν).*

Proof. It will be sufficient to find a closed set $E \subset X \times Y$ and a closed rectangular $K \times S$ in $X \times Y$ such that E is synthetic with respect to $V(X,Y)$ but not $E \cap (K \times S)$. In fact, if E were a set of synthesis with respect to any pair of finite measures we would obtain, by Lemma 6.1, that so would be its intersection with any measurable rectangular and, by Theorem 6.1, the intersection $E \cap (K \times S)$ would be synthetic for $V(X,Y)$.

Let X, Y be compact metric spaces and let $G \subset X \times Y$ be a non-synthetic set with respect to $V(X,Y)$. Let I denote the unit interval $[0,1]$ and $d((x,y),G)$ be the distance between (x,y) and G. In $(X \times I) \times Y$ consider the set

$$E = \{((x,t),y) \in (X \times I) \times Y \mid d((x,y),G) \le t\}.$$

Then E is a set of synthesis with respect to $V(X \times I, Y)$. To see this take a function $F((x,t),y) = \sum_{k=1}^{\infty} f_k(x,t)g_k(y)$ in $V(X \times I, Y)$ such that

$$\sum_{k=1}^{\infty} \sup |f_k(x,t)|^2 \sum_{k=1}^{\infty} \sup |g_k(y))|^2 < \infty \tag{10}$$

and null $F \subset E$, and consider $F_n((x,t),y)) = F((x,t+1/n),y)$, $n \in \mathbb{N}$. Clearly, F_n vanishes on

$$E_n = \{((x,t),y) \in (X \times I) \times Y \mid d((x,y),G) < t + 1/n\},$$

an open set containing the set E. Now

$$F_n((x,t),y) - F((x,t),y) = \sum_{k=1}^{\infty} (f_k(x,t+1/n) - f_k(x,t))g_k(y)$$

and

$$||F_n((x,t),y) - F((x,t),y)||_V \le \sum_{k=1}^{\infty} \sup |(f_k(x,t+1/n) - f_k(x,t)|^2 \sum_{k=1}^{\infty} \sup |g_k(y)|^2.$$

Fix $\varepsilon > 0$. By (10) one can find $K > 0$ such that $\sum_{k=K+1}^{\infty} \sup |(f_k(x,t+1/n) - f_k(x,t)|^2 < \varepsilon$. Since all f_k, $k = 1, \ldots, K$, are continuous on the compact $X \times I$, they are uniformly continuous. Therefore there exists $N > 0$ such that, for any $n \geq N$, we have $\sup |f_k(x,t + 1/n) - f_k(x,t))| < \sqrt{\varepsilon/K}$ $k = 1, \ldots K$. This yields $\sum_{k=1}^{K} \sup |(f_k(x,t+1/n) - f_k(x,t)|^2 < \varepsilon$ and

$$\sum_{k=1}^{\infty} \sup |(f_k(x,t+1/n) - f_k(x,t)|^2 < 2\varepsilon,$$

showing $F_n \to F$ as $n \to \infty$ in $V(X \times I, Y)$.

Consider now

$$E^* = E \cap ((X \times \{0\}) \times Y) = \{(x,0), y) \in (X \times I) \times Y \mid (x,y) \in G\}.$$

Our goal is to show that E^* is not synthetic in $V(X \times I, Y)$. Given a function $\Phi(x,y) = \sum_{k=1}^{\infty} f_k(x) g_k(y) \in V(X,Y)$, null $\Phi \subset G$, consider $F((x,t),y) = \Phi(x,y)$ in $V(X \times I, Y)$. Assume that E^* is synthetic. Then F can be approximated in $V(X \times I, Y)$ by functions $F_n((x,t),y)$ which vanish in neighbourhood of E^*. This implies that Φ can be approximated by $F_n((x,0),y)$ in $V(X,Y)$. Clearly, each $F_n((x,0),y)$ vanishes on nbhd of G. By arbitriness of Φ, we obtain that G is a set of synthesis, contradicting our assumption. □

Remark 6.1. *The construction of the set E uses an idea of N. Varopoulos [V2].*

Thus the sets of universal (independent on the choice of measures) operator synthesis form a more narrow class than the sets of spectral synthesis. It is of interest to clarify which known classes it includes.

A closed set $E \subseteq X \times Y$ such that any bimeasure concentrated on E is a measure (*a set without true bimeasure*) is a set of spectral synthesis in $V(X,Y)$.

Proposition 6.2. *A closed set without true bimeasures is a set of operator synthesis with respect to any pair (μ, ν) of finite measures.*

Proof. Let $\mu \in M(X)$, $\nu \in M(Y)$ and let E be a closed set without true bimeasure. Consider $T \in B(L_2(X,\mu), L_2(Y,\nu))$ such that T is supported in E. It defines a bimeasure B_T by $(Tu, \bar{v}) = B_T(u,v)$, where $u \in C(X)$ and $v \in C(Y)$. Moreover, supp $(B_T) \subseteq E$. By the condition of the theorem, there exists a measure $m \in M(X \times Y)$ such that supp $(m) \subseteq E$ and

$$(Tu, \bar{v}) = \int u(x) v(y) dm(x,y), \tag{11}$$

for every $u \in C(X)$, $v \in C(Y)$.

Let $F(x,y) = \sum_{n=1}^{\infty} u_n(x) v_n(y) \in C(X) \hat{\otimes}_{eh} C(Y)$ and let $F_k(x,y) = \sum_{n=1}^{k} u_n(x) v_n(y)$, $E_k(x,y) = \sum_{n=k+1}^{\infty} |u_n(x)|^2 + |v_n(y)|^2$. Then $E_k(x,y) \to 0$, $k \to \infty$, for every $(x,y) \in X \times Y$,

$$|F(x,y) - F_k(x,y)| \leq E_k(x,y)$$

and therefore $F_k(x,y) \to F(x,y)$, $k \to \infty$, everywhere on $X \times Y$. Moreover, $|F_k(x,y)| \leq E_0(x,y)$ and $E_0(x,y)$ is integrable over m, as m is finite. Thus, by the theorem on majorized convergence,

$$\int F_k(x,y) dm(x,y) \to \int F(x,y) dm(x,y).$$

On the other hand, $||F - F_k||_\Gamma \leq \int E_k(x,y)d\mu(x)d\nu(y)$ and $\int E_k(x,y)d\mu(x)d\nu(y) \to 0$, which imply $||F - F_k||_\Gamma \to 0$ and $\langle T, F_k\rangle \to \langle T, F\rangle$ as $k \to \infty$.

We now obtain the equality

$$\langle T, F\rangle = \int F(x,y)dm(x,y), \quad F \in C(X)\hat{\otimes}_{eh}C(Y).$$

Since m is supported in E, this gives $\langle T, F\rangle = 0$ with F vanishing on E.

Consider now $F \in \Gamma(X,Y)$, null $F \supset E$. Then there exist $f_i \in L_2(X,\mu)$, $g_i \in L_2(Y,\nu)$ such that $F(x,y) = \sum_{i=1}^{\infty} f_i(x)g_i(y)$ (m.a.e.) and $\sum_{i=1}^{\infty}||f_i||_{L_2}^2 \sum_{i=1}^{\infty}||g_i||_{L_2}^2 < \infty$. Given $\varepsilon > 0$, we can find compact sets $X_\varepsilon \subset X$, $Y_\varepsilon \subset Y$ such that $\mu(X \setminus X_\varepsilon) < \varepsilon$, $\nu(Y \setminus Y_\varepsilon) < \varepsilon$ and

$$\sum_{i=1}^{\infty}|f_i(x)|^2 < C_\varepsilon, \quad x \in X_\varepsilon, \quad \sum_{i=1}^{\infty}|g_i(y)|^2 < C_\varepsilon, \quad y \in Y_\varepsilon,$$

Moreover, we can assume that f_i, g_i are continuous by the Lusin theorem so that the restriction F_ε of F to $X_\varepsilon \times Y_\varepsilon$ belongs to $C(X_\varepsilon)\hat{\otimes}_{eh}C(Y_\varepsilon)$. Clearly, if E is a set without true bimeasure, so is $E \cap (X_\varepsilon \times Y_\varepsilon)$. If now $T \in B(L_2(X,\mu), L_2(Y,\nu))$ is supported in E then supp $P_{Y_\varepsilon}TQ_{X_\varepsilon} \subset E \cap (X_\varepsilon \times Y_\varepsilon)$ and

$$\langle P_{Y_\varepsilon}TQ_{X_\varepsilon}, F_\varepsilon\rangle = 0.$$

Letting $\varepsilon \to 0$, we obtain $\langle T, F\rangle = 0$. □

Remark 6.2. In [V1], Varopoulos established a deep connection between the algebra $V(G) = C(G)\hat{\otimes}C(G)$ and the Fourier algebra $A(G)$ of compact Abelian groups G. Using the relationships he showed that a closed set $E \subseteq G$ is a set of spectral synthesis for $A(G)$ if and only if the diagonal set $E^* = \{(x,y) \in G \times G \mid x + y \in E\}$ is a set of spectral synthesis for $V(G)$. Recently the same result was proved for non-Abelian compact groups in [ST] using the established there connection between $A(G)$ and the Haagerup tensor product $C(G)\hat{\otimes}_h C(G)$ which is the Varopoulos algebra, renormed. An analogous result for sets of operator synthesis in $G \times G$ was obtained in [F] for locally compact Abelian groups G and in [ST] for compact non-Abelian groups G. Namely, a closed set $E \subset G$ is a set of spectral synthesis for $A(G)$ if and only if E^* is a set of operator synthesis with respect to the Haar measure (for the reverse, synthesizeability with respect to all pairs of finite measures is not required, as in Theorem 6.1).

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