How fast are the two-dimensional Gaussian waves?

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ABSTRACT

For a stationary two-dimensional random field evolving in time, we derive the intensity distributions of appropriately defined velocities of crossing contours. The results are based on a generalization of the Rice formula. The theory can be applied to practical problems where evolving random fields are considered to be adequate models. We study dynamical aspects of deep sea waves by applying the derived results to Gaussian fields modeling irregular sea surfaces. In doing so, we obtain distributions of velocities for the sea surface as well as for the envelope field based on this surface. Examples of wave and wave group velocities are computed numerically and illustrated graphically.

KEY WORDS: directional spectrum, Gaussian sea, Rice formula, velocities, level crossing contours, wave groups.

INTRODUCTION

In this paper, we are interested in analyzing the dynamics of the sea by studying the distributions of different notions of velocity. In order to accomplish this goal, we proceed in two steps. Firstly, we identify different motions of the surface through appropriately defined velocities. Secondly, we derive the distributions of the defined

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velocities and we compute their densities given the spectrum of the underlying field. Distributions of the velocities can be studied at various regions of the surface such as points of local extremes, level crossing contours or regions with large curvature and so on. These distributions are different even if the same notion of velocity is considered. This leads to studies of the distribution of a random field given that another field (describing for example, level crossing contours) takes a fixed value. For computation of such a distribution we utilize a generalized Rice formula.

Notation and Assumptions

The sea surface elevation at a position $\mathbf{p} = (x, y)$ and time t is represented by a homogeneous real Gaussian field $W(\mathbf{p}, t)$. Statistical properties of the field $W(\mathbf{p}, t)$ are uniquely identified by its spectrum $S(\boldsymbol{\lambda})$. For example in the case of discrete spectrum, the Gaussian field becomes a sum of cosine functions with random amplitudes and phases

$$W(\mathbf{p}, t) = \sum_{\boldsymbol{\lambda}\in\Lambda} \sqrt{2S(\boldsymbol{\lambda})\Delta(\boldsymbol{\lambda})} R_j \cos(\lambda_1 x + \lambda_2 y + \lambda_3 t + \epsilon_j), \tag{1}$$

where Λ is any subset of $\mathbb{R}^3$ such that $-\Lambda \cap \Lambda = \emptyset$ and $-\Lambda \cup \Lambda = \mathbb{R}^3 - \{\mathbf{0}\}$ for example, $\Lambda = \big\{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 : \lambda_3 \geq 0\big\}$, $\Delta(\boldsymbol{\lambda})$ are infinitesimally small increments, $\big\{R_j\big\}$ is a sequence of independent Rayleigh random variables having density $f_R(r) = re^{-\frac{r^2}{2}}, r \geq 0$ and $\big\{\epsilon_j\big\}$ is a sequence of independent uniformly distributed random variables in $[0, 2\pi]$ also independent of $\big\{R_j\big\}$.

In this case the covariance function can be written as the Fourier integral

$$R(\mathbf{p}, t) = \int_{\mathbb{R}^3} \exp(i \cdot (\lambda_1 x + \lambda_2 y + \lambda_3 t)) S(\boldsymbol{\lambda}) \, \mathrm{d}\boldsymbol{\lambda}. \tag{2}$$

The Hilbert transform of the process W is defined as

$$\hat{W}(\mathbf{p}, t) = \sum_{\boldsymbol{\lambda}\in\Lambda} \sqrt{2S(\boldsymbol{\lambda})\Delta(\boldsymbol{\lambda})} R_j \sin(\lambda_1 x + \lambda_2 y + \lambda_3 t + \epsilon_j) \tag{3}$$

and the real envelope process is given by

$$E(\mathbf{p}, t) = \sqrt{W(\mathbf{p}, t)^2 + \hat{W}(\mathbf{p}, t)^2}. \tag{4}$$

We shall write the first and second order derivatives of W with respect to x,y and t as

$$W_u = \frac{\partial W}{\partial u}, \qquad W_{uv} = \frac{\partial^2 W}{\partial u \partial v}, \qquad u, v = x, y, t.$$

The spectral moments λ_{ijk}, if they are finite, are defined as $\lambda_{ijk} = 2\int_\Lambda \lambda_1^i \lambda_2^j \lambda_3^k S(\boldsymbol{\lambda}) \, \mathrm{d}\boldsymbol{\lambda}$. Some of the most often used covariances of the field, its Hilbert transform and their derivatives are given by the spectral moments of W in the following fashion

$$\begin{aligned} \mathrm{Var}(W) &= \lambda_{000}, & \mathrm{Var}(\hat{W}) &= \lambda_{000}, \\ \mathrm{Cov}(W_x, W_y) &= \lambda_{110}, & \mathrm{Cov}(W_x, W_t) &= \lambda_{101}, \\ \mathrm{Cov}(W_{xx}, W_{yy}) &= \lambda_{220}, & \mathrm{Cov}(W_{xx}, W_{xt}) &= \lambda_{301}, \\ \mathrm{Cov}(\hat{W}_x, W) &= \lambda_{100}, & \mathrm{Cov}(\hat{W}, W_x) &= -\lambda_{100}. \end{aligned}$$

Next we introduce a convenient coordinate system by rotating the plane in such a way that the spatial partial derivatives of the field are independent. We can choose the rotation so that the x-direction has larger variance than the y-direction. That is $\mathrm{Var}(W_x) = \lambda_{200} \geq \lambda_{020} = \mathrm{Var}(W_y)$. This is equivalent to the fact that the intensity of zero-crossings along any line passing through the origin attains its maximum on the x-axis and its minimum on the y-axis.

In addition to the above notation, we use $\mathbf{n}_\alpha$ to denote the vector $(\cos\alpha, \sin\alpha)$ and $\mathbf{n}^*_\alpha$ for the perpendicular vector $(-\sin\alpha, \cos\alpha)$. We also use $\mathbf{n}^T_\alpha$ for the transpose of $\mathbf{n}_\alpha$. For a rigorous treatment of Gaussian random fields see e.g.(Cramér & Leadbetter, 1967).

Contours

For the sea surface elevation $W(\mathbf{p}, t)$ let us define the *level crossing contour*

$$\mathcal{C}(w) = \{\mathbf{p} \in \mathbb{R}^2 : W(\mathbf{p}, t) = w\}, \tag{5}$$

where $w \in \mathbb{R}$ is a fixed level. Obviously the set $\mathcal{C}(w)$ depends on the value of t although this is not explicit in the notation. Assuming that the second order derivatives of W exist and are continuous, the contour $\mathcal{C}(w)$ is a curve. This curve represents for example, the boundary of the excursion of a wave above the level w. Obviously, it is evolving in time, and we are interested in the statistical properties of this evolution. Often, instead of contour lines one is interested in isolated points such as local extremes. This leads to the case of contours in which the field $W(\mathbf{p}, t)$ is replaced by the two-dimensional random field $\dot{\mathbf{W}}(\mathbf{p}, t) = (W_x(\mathbf{p}, t), W_y(\mathbf{p}, t))$ and the intersection of the crossing contours of the spatial derivatives of the field W constitutes a set of isolated points that are positions of the local extremes of W at time t. For random seas the motion of these contours illustrates the motion of the troughs and crests of the waves.

To summarize, in this work we deal with two types of contours: $\mathcal{C}(w)$ if we consider w-crossings of the one-dimensional random field W, and $\mathcal{C}(\mathbf{w})$ if we consider $\mathbf{w}$-crossings of the two-dimensional random field $\mathbf{W}$. Under appropriate regularity conditions imposed on the random field (see for details in Adler (1981)), $\mathcal{C}(w)$ is a curve while $\mathcal{C}(\mathbf{w})$ consists of isolated points. In the following we do not distinguish in notation between w and $\mathbf{w}$.

Rice formula

To each point of the crossing contour as defined above, we can attach a vector velocity $\mathbf{V}$ creating in this way a velocity field. We are interested in the distribution of this field. Such distributions can be obtained by measuring the part of the contour $\mathcal{C}(w)$ on which $\mathbf{V}$ has some property, say A. More specifically, let $B \subseteq \mathbb{R}^2$ be the region in the sea surface, where we study the distribution of the velocity $\mathbf{V}$, then $\mathcal{C}_B(w) = \mathcal{C}(w) \cap B$ represents the part of the level crossing contour which is in the region B. Typically, the set B has the form of a rectangle, that is $B = [0, x] \times [0, y]$. Following, by $\mathcal{C}_1(w)$ we denote the special case of $\mathcal{C}_B(w)$ for B being the unit rectangle $[0, 1] \times [0, 1]$. Let $\mathcal{H}(\mathcal{C}_B(w))$ be the Hausdorff measure of the set $\mathcal{C}_B(w)$, which is the number of the points in the set $\mathcal{C}_B(w)$ if the crossing contour $\mathcal{C}(w)$ consists of isolated points, or the length of the set $\mathcal{C}_B(w)$ if $\mathcal{C}(w)$ is a curve. Further let $\mathcal{H}(\mathcal{C}_B(w), A)$ be the Hausdorff measure of the part of $\mathcal{C}_B(w)$ on which the field $\mathbf{V}$ satisfies the property A. Then the empirical measure of the field $\mathbf{V}$ on the contour $\mathcal{C}_B(w)$ equals to the following ratio

$$P_{emp}(A) = \frac{\mathcal{H}(\mathcal{C}_B(w), A)}{\mathcal{H}(\mathcal{C}_B(w))}. \tag{6}$$

When B expands without bound and the involved random fields follow the ergodic theorem, the empirical measure defined in Eq. 6 converges almost surely to the limit

$$P(A) = \frac{E\big[\mathcal{H}(\mathcal{C}_1(w), A)\big]}{E\big[\mathcal{H}(\mathcal{C}_1(w))\big]}. \tag{7}$$

The above distribution of $\mathbf{V}$ can be computed by utilizing a generalized Rice formula. More exactly, let $\mathbf{V}(\mathbf{p}) \in \mathbb{R}^m$ and $\mathbf{W}(\mathbf{p}) \in \mathbb{R}^n$ with $\mathbf{p} \in \mathbb{R}^2$ and $n \leq 2$, be a pair of jointly homogeneous random fields. Denote the matrix of the partial derivatives of $\mathbf{W}(\mathbf{p})$ by $\dot{\mathbf{W}}(\mathbf{p})$ and the Jacobian of this matrix by $|\dot{\mathbf{W}}(\mathbf{p})|$. The field $\mathbf{V}(\mathbf{p}) = \mathbf{V}(\mathbf{p}, t)$, shall represent a velocity measured at point $\mathbf{p}$ and fixed time t. The velocity can be either vector valued or scalar. Now, an appropriate version of the generalized Rice formula (given for example in Zähle (1984)) in a form that suits our applications, allows us to rewrite Eq. 7 in the form

$$P(A) = \frac{E\Big[\{\mathbf{V}(\mathbf{0}) \in A\} \cdot |\dot{\mathbf{W}}(\mathbf{0})| \Big| \mathbf{W}(\mathbf{0}) = \mathbf{w}\Big]}{E\Big[|\dot{\mathbf{W}}(\mathbf{0})| \Big| \mathbf{W}(\mathbf{0}) = \mathbf{w}\Big]}, \tag{8}$$

where $\mathbf{w}$ is the level of the crossing contour and $\{\mathbf{V}(\mathbf{0}) \in A\}$ is the indicator function equal to one if $\mathbf{V}(\mathbf{0}) \in A$ and zero otherwise. We are interested in velocity fields that can be measured on level crossing contours and extremal points. Statistically this means sampling from different types of sets. In this paper we encounter the following three cases of sampling

- Sampling on a level crossing contour of the scalar $\mathbf{W}(\mathbf{p}) := W(\mathbf{p}, 0) = w$, where w is for example the still water mean level. In this case $|\dot{\mathbf{W}}(\mathbf{0})| = \sqrt{W_x^2 + W_y^2}$ and $\mathcal{H}$ measures the length of a contour on $\mathbb{R}^2$.

- Sampling at level crossing points of $\mathbf{W}(\mathbf{p}) := W(x, 0, t)$ (along the line $y = 0$). In this case $|\dot{\mathbf{W}}(\mathbf{0})| = W_x$ and $\mathcal{H}$ counts points along the x-axis.
- Sampling at specular points, i.e. points at which the vector $\mathbf{W}(\mathbf{p}) := (W_x(\mathbf{p}, t), W_y(\mathbf{p}, t))$ takes a specified value. Then $|\dot{\mathbf{W}}(\mathbf{0})| = |W_{xx}W_{yy} - W_{xy}^2|$ and $\mathcal{H}$ counts points on the plane.

VELOCITIES DEFINED ON CONTOURS

In this section we present various notions of velocities that can be defined for the random sea surface. All these velocities reflect different aspects of the kinematic features of the sea and therefore can be used accordingly to the problem at hand. Let $W(\mathbf{p}, t)$ represent an evolving surface observed at time t and point $\mathbf{p}$.

Velocity in fixed direction $\mathbf{V}_\alpha$

Let α denote the angle between the x-axis and some specified horizontal direction. This angle can be fixed, like in the case of the azimuth of a ship traveling on the sea when we are interested in the sea waves along its route. However, in general, we allow for variable α as in the case of the gradient direction. We use W_α to denote the directional derivative of the field W in the direction α, i.e $W_\alpha = \dot{\mathbf{W}} \cdot \mathbf{n}_\alpha^T$. For a fixed reference time t_0 and point $\mathbf{p}_0$, let $\mathcal{C}(s)$ be the contour defined by

$$\mathcal{C}(s) = \{\mathbf{p} : W(\mathbf{p}, t_0 + s) = W(\mathbf{p}_0, t_0), \quad (\mathbf{p}(s) - \mathbf{p}_0) \cdot \mathbf{n}_\alpha^* = 0\} \tag{9}$$

The points on this contour move with time. Let $\mathbf{p}(s)$ represent the motion of the point which at time $s = 0$ is at $\mathbf{p}_0$. By differentiating $\mathbf{p}(s)$ with respect to s we obtain that the velocity $\mathbf{V}_\alpha = \mathrm{d}\mathbf{p}(s)/\,\mathrm{d}s = V_\alpha \cdot \mathbf{n}_\alpha^T$, is given by $V_\alpha = -W_t/W_\alpha$. Indeed the points $\mathbf{p}(s)$ satisfy the system

$$W(\mathbf{p}(s), s) = W(\mathbf{p}_0, 0), \tag{10}$$

$$(\mathbf{p}(s) - \mathbf{p}_0) \cdot \mathbf{n}(\alpha)^* = 0. \tag{11}$$

Eq. 10 guarantees that all points $\mathbf{p}(s)$ stay on the same level contour, while Eq. 11 makes sure that these points always move to the direction α. Use of the implicit function theorem yields for the velocity $\mathbf{V}_\alpha$, the matrix equivalent of Eq. 10 and 11

$$\begin{bmatrix} W_x & W_y \\ -\sin\alpha & \cos\alpha \end{bmatrix} \mathbf{V}_\alpha = -\begin{bmatrix} W_t \\ 0 \end{bmatrix}. \tag{12}$$

Velocity in the direction of gradient $\mathbf{V}_{gr}$

Probably the most interesting example of velocity $\mathbf{V}_\alpha$ is for α being the angle between the x-axis and the gradient $\dot{\mathbf{W}}$. This angle will be denoted by β. By the previous section, $\mathbf{V}_{gr} = V_{gr} \cdot \mathbf{n}_\beta = -\frac{W_t}{W_\beta} \cdot \mathbf{n}_\beta$. If we substitute in Eq.12, $\cos\beta = W_x/\dot{W}$ and $\sin\beta = W_y/\dot{W}$ we obtain

$$\begin{bmatrix} W_x & W_y \\ -W_y & W_x \end{bmatrix} \mathbf{V}_{gr} = -\begin{bmatrix} W_t \\ 0 \end{bmatrix}. \tag{13}$$

This velocity describes the motion of points that stay on the same level contour while they move in the steepest direction (the gradient direction). It is interesting to note that $V_{gr} \le V_\alpha$ for any choice of α. This follows from the representation of speeds

$$V_\alpha = V_{gr} / \cos(\alpha - \beta).$$

Velocity of specular points $\mathbf{V}_{sp}$

We could also define contours using any other field than W. Consider for example the fields W_x and W_y and the corresponding level crossing contours $\mathcal{C}^1(s)$ and $\mathcal{C}^2(s)$:

$$\mathcal{C}^1(s) = \Big\{\mathbf{p}(s) : W_x(\mathbf{p}(s), t+s) = W_x(\mathbf{p}_0, t)\Big\},$$

$$\mathcal{C}^2(s) = \Big\{\mathbf{p}(s) : W_y(\mathbf{p}(s), t+s) = W_y(\mathbf{p}_0, t)\Big\}.$$

Let $\mathbf{p}(s)$ be a point in the intersection $\mathcal{C}^1(s) \cap \mathcal{C}^2(s)$, moving with time s. Then the motion of this point can be obtained by differentiating the following system

$$\begin{aligned} W_x(\mathbf{p}(t+s), t+s) &= W_x(\mathbf{p}(t), t), \\ W_y(\mathbf{p}(t+s), t+s) &= W_y(\mathbf{p}(t), t). \end{aligned}$$

The resulting system is used to define the velocity $\mathbf{V}_{sp}$

$$\begin{bmatrix} W_{xx} & W_{xy} \\ W_{yx} & W_{yy} \end{bmatrix} \mathbf{V}_{sp} = -\begin{bmatrix} W_{xt} \\ W_{yt} \end{bmatrix}. \tag{14}$$

The velocity $\mathbf{V}_{sp}$ describes the motion of points with a fixed value of gradient. In particular it describes the motion of local extremes on the surface. This type of velocity was considered by Longuet-Higgins (1957).

Although it is very easy to recognize the specular points as the points where light is reflected, this velocity has the disadvantage of involving second order derivatives. These derivatives are difficult to model for irregular surfaces. The large spreading of the specular velocity is partly due to the variability of the sea surface. In practice, the high frequencies can be excluded by putting the spectrum equal to zero above a certain frequency and study the velocity $\mathbf{V}_{sp}$ on the smoothed surface instead. In opposite to $\mathbf{V}_{gr}$, $\mathbf{V}_{sp}$ recognizes properly a drift of the surface, however it cannot reflect the motion due to the rising and lowering of the surface. In the extreme case of such a motion, the specular points may not move, while the contour lines do move.

These difficulties inspired us to the following notion of velocity, which describes the drift motion better than $\mathbf{V}_{gr}$ and at the same time it is easier to analyze statistically than the $\mathbf{V}_{sp}$.

Velocity of constant gradient direction V

Consider again the contour $\mathcal{C}(t)$ and identify all points in this contour with fixed gradient direction β. Then these points should satisfy

$$\begin{aligned} W(\mathbf{p}(s), t_0 + s) &= W(\mathbf{p}_0, t_0), \\ \frac{W_x(\mathbf{p}_0, t_0)}{W_x(\mathbf{p}(s), t_0 + s)} &= \frac{W_y(\mathbf{p}_0, t_0)}{W_y(\mathbf{p}(s), t_0 + s)}. \end{aligned} \tag{15}$$

If we differentiate in Eq. 15 with respect to s, we obtain the following system of differential equations which is used to define $\mathbf{V}$

$$\begin{bmatrix} W_x & W_y \\ W_{xx}W_y - W_{yx}W_x & W_{xy}W_y - W_{yy}W_x \end{bmatrix} \mathbf{V} = \begin{bmatrix} -W_t \\ W_{yt}W_x - W_{xt}W_y \end{bmatrix}. \tag{16}$$

From the first relation in Eq. 16 is obvious that the projection of the velocity $\mathbf{V}$ on the vector $\mathbf{n}_\beta$ is equal to the velocity in the direction of gradient V_{gr}. The second relation in Eq. 16 defines the vector difference between $\mathbf{V}$ and $\mathbf{V}_{gr}$. Let us denote by V_{gr}^* the projection of $\mathbf{V}$ onto $\mathbf{n}_\beta^*$.

Consequently (V_{gr}, V_{gr}^*) is the vector $\mathbf{V}$ in the coordinate system rotated counter-clockwise by an angle β, that is $\mathbf{V} = V_{gr}\mathbf{n}_\beta + V_{gr}^*\mathbf{n}_\beta^*$. Note that this coordinate system is variable, i.e. $\mathbf{n}_\beta$ and thus $\mathbf{n}_\beta^*$ are varying from point to point and from instant to instant.

Although $\mathbf{V}$ and $\mathbf{V}_{gr}$ formally are two different velocities, both describe motions which stay on the same vertical level, so both describe some aspects of the motion of crossing contours. But, $\mathbf{V}_{gr}$ represents the motion of a point on the contour towards the direction of the gradient, while $\mathbf{V}$ represents the motion of a point on the contour towards a point on the next instant contour having the same gradient direction as the original point.

Group velocity. Wave groups are rather hard to define in a formal way but roughly speaking, they are collections of waves with large ones in the center together with small vanishing waves at the ends. Interest in wave grouping arises in the safety of marine structures where wave groups rather than individual waves are responsible for the damage.

There are more than one ways to define velocities for wave groups. The method we follow in this work, is the one proposed in Longuet-Higgins (1957). This method suggests the use of envelope-crossing wave groups.

The envelope is a positive process that always stays higher than the sea elevation process. In the case of narrow-band process, the envelope is passing close to the local maxima of the process and hence can be used to describe the evolution of wave groups. This allows us to use the definitions of velocities given in the previous section, but this time defined on the envelope field. In this work we just consider the case of the velocity in direction α. The speed of this velocity for the wave group, applied on the envelope field given by Eq. 4, is given by

$$V_E = -\frac{W_t \cdot W + \hat{W}_t \cdot \hat{W}}{W_x \cdot W + \hat{W}_x \cdot \hat{W}}. \tag{17}$$

Distributions of velocities

Let $W(\mathbf{p}, t)$ be a homogeneous real valued Gaussian field having continuous second order derivatives. For a convenience, we also assume that the field W is ergodic. In what follows we utilize the generalized Rice formula to derive distributions of velocities on crossing contours.

We use three models in order to illustrate the results. The first model is a sea with a directional spectrum given in Fig. 1(*Top*). The spectrum (created using WAFO toolbox see (Brodtkorb, 2000)) is a Torsethaugen[1] frequency spectrum (see (Torsethaugen, 1996)) representing a pure wind sea, i.e., a sea with no swell, multiplied by a frequency dependent spreading function. The spreading function is given by $D(\theta) = G_0 \cos^{2s}(\frac{\theta}{2})$ with $\theta \in \big[-\pi, \pi\big]$ and $s = 10$. The spectrum has peak frequency $w_p = 0.57\text{rad/s}$. Frequencies exceeding $3w_p$ were cut off.

For the second and third models we consider a superposition of swell and wind spectra. The wind component is the same as in the first model. The swell component, representing swell coming from a direction perpendicular to the main direction, can be seen in Fig. 1(*Bottom*). Frequencies exceeding $2.15w_p$ were cut off. This spectrum is also a Torsethaugen frequency spectrum, multiplied by the frequency dependent spreading function given above, with parameter $s = 25$. In the second model, the energy of the swell is approximately 11% of the wind energy, while in the third this percentage is increased to 33%.

Comparison of velocities for individual waves and wave groups along a line

We start with the velocity $\mathbf{V}_\alpha$. We can focus on the case $\alpha = 0$ and study this velocity along the x-axis. The distribution of the speed $V_0 = -W_t/W_x$ when we sample from points that satisfy $W(x, 0, 0) = u$, is well known and given by

$$V_0 \stackrel{d}{=} -\frac{1}{\lambda_{200}}\Big(\lambda_{101} - \sqrt{\lambda_{200}\lambda_{002} - \lambda_{101}^2}\,\frac{T_2}{\sqrt{2}}\Big),$$

where T_2 is distributed as a t random variable with 2 degrees of freedom, (see Podgórski et al. (2000b))

Similarly the distribution of V_E along the line $y = 0$ when we sample from points satisfying $E(x, 0, 0) = u$ is given by

[1] A Torsethaugen spectrum is given by $S_w(w) + S_s(w)$, where S_w and S_s are modified JONSWAP spectra for wind and swell peak respectively. The parameters in these examples, have been chosen so that we obtain a spectrum that represents a pure wind sea and another one that represents only swell.

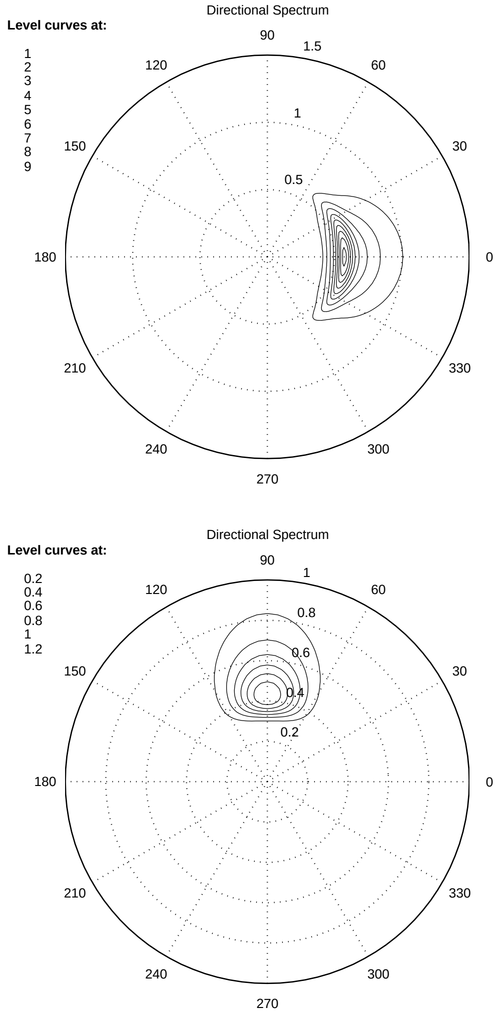

FIG. 1 *Top*: Directional wind spectrum S_w, defining the sea $W(x, y, t)$ in the first model. *Bottom*: Directional swell spectrum S_s, used together with S_w for defining the sea in the second and third model.

$$V_E \stackrel{d}{=} -\frac{1}{\lambda_{200} - \frac{\lambda_{100}^2}{\lambda_{000}}} \times \Bigg(\lambda_{101} - \frac{\lambda_{100}\lambda_{001}}{\lambda_{000}} + \\ + \sqrt{\left(\lambda_{200} - \frac{\lambda_{100}^2}{\lambda_{000}}\right)\left(\lambda_{002} - \frac{\lambda_{001}^2}{\lambda_{000}}\right) - \left(\lambda_{101} - \frac{\lambda_{100}\lambda_{001}}{\lambda_{000}}\right)^2} \, \frac{T_2}{\sqrt{2}} \Bigg),$$

with T_2 defined as above. It is interesting to notice that both these distributions do not depend on the level u. The derivation of the distributions is a result of a rather standard decomposition of the Gaussian random field into a sum of two independent components and an application of the Rice formula. (For more details see Baxevani et al. (2002)).

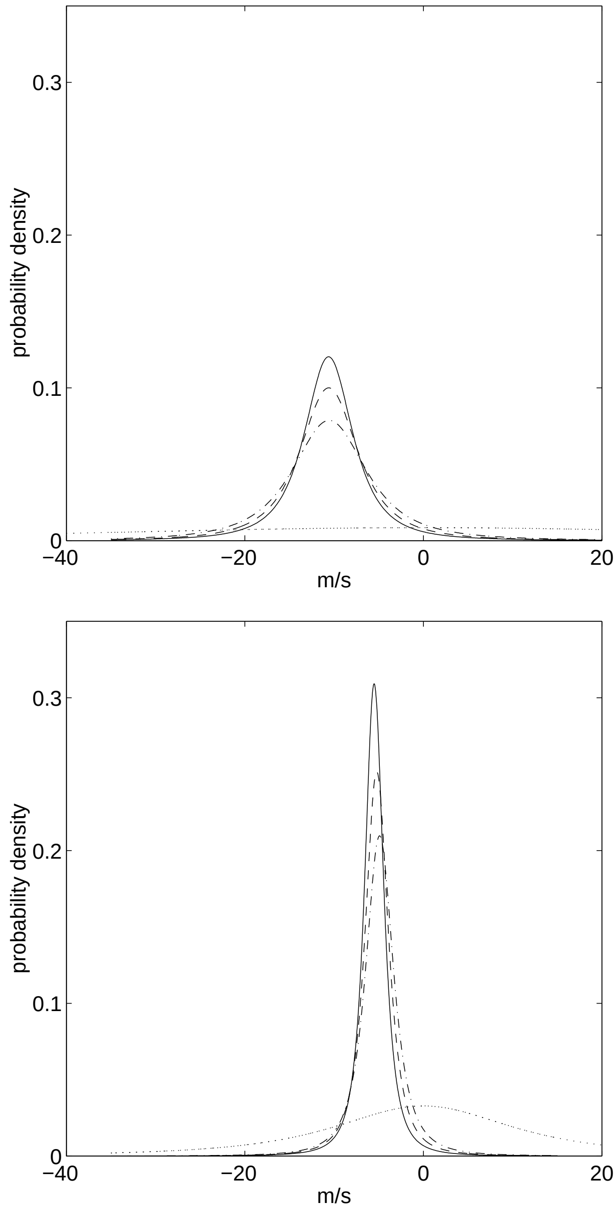

FIG. 2 *Top*: Density of the speed V_0. *Bottom*: Density of the speed V_E. Densities for the wind spectrum are with solid line, for the second model with dash-dotted line and for the third model with dashed line. Dotted lines have been used for the swell spectrum moving along the y-axis.

Numerically computed densities of the speed V_0 for the different sea models are given in Fig. 2(*Top*). The speed has mean values equal to -10.6251m/s, -10.6020m/s and -10.5561m/s and scale factors 4.1538m/s, 4.9985m/s and 6.3518m/s for the three sea models. It should be noted that smaller scale factor implies that the speed is more concentrated around its mean value. Hence, we can conclude that the introduction of the swell waves moving along a line perpendicular to the main direction of propagation, does not influence the speed significantly, it just increases the variability. The dotted line is the density of the speed V_0 for pure swell waves moving along the y-axis. This speed is in average equal to zero, something to be expected since V_0 describes the motion of the waves along the line $y = 0$, and is very noisy.

Next, we consider the velocity of the envelope field given in Eq. 17. As we have already mentioned, the speed of this velocity is used to describe the motion of wave groups. As it can be seen in Fig. 2(*Bottom*), the wave groups move almost half as fast as the individual waves. Their average speed equals -5.5201m/s, -5.2303m/s and -4.9110m/s for the three different sea models. The scale factors equal 1.6168m/s, 1.9874m/s and 2.3848m/s, which means that the densities for the wave groups are more concentrated around the mean than the densities for the individual waves. The influence of the swell waves is again restricted to the increase of variability although in a more profound way. The speed of the pure swell waves is again in average equal to zero with very large variability.

Velocities in the direction of gradient and in constant gradient direction

We turn now to the study of the distribution of $\mathbf{V}_{gr}$. Instead of studying the joint distribution of its coordinates it is easier to study the joint distribution of (V_{gr}, β), where V_{gr} is the seed of the velocity $\mathbf{V}_{gr}$ and β is the angle between the gradient and the x-axis. We suppose that we sample from points on the fixed level contour $\mathcal{C}(w)$. First we give the distribution of β and the conditional distribution of $\dot{\mathbf{W}}$ given the value of the direction β. Direct application of the generalized Rice formula as given in Eq. 8 leads to

$$F_{\beta,\dot{\mathbf{W}}}(\phi, u) = \frac{E\Big[|\dot{\mathbf{W}}| \cdot \Big\{\beta \le \phi, \dot{\mathbf{W}} \le u\Big\} \Big| W = w\Big]}{E\Big[|\dot{\mathbf{W}}| \Big| W = w\Big]}. \tag{18}$$

Here, the indicator function $\Big\{\beta \le \phi, \dot{\mathbf{W}} \le u\Big\}$ equals one if the condition inside the braces is satisfied and zero otherwise. The generalized Jacobian $|\dot{\mathbf{W}}|$ as well as the angle β and the gradient $\dot{\mathbf{W}}$ are independent of W, which simplifies Eq. 18 to

$$F_{\beta,\dot{\mathbf{W}}}(\phi, u) = \frac{E\Big[|\dot{\mathbf{W}}| \cdot \Big\{\beta \le \phi, \dot{\mathbf{W}} \le u\Big\}\Big]}{E\Big[|\dot{\mathbf{W}}|\Big]}. \tag{19}$$

Omitting the mathematical details of the derivation that can be found in Longuet-Higgins (1957) and Baxevani et al. (2002), we obtain the marginal density of β and the conditional density of $\dot{\mathbf{W}}$ given the value of β. First,

$$f(\beta) = \frac{\gamma^2}{4\mathcal{E}(\sqrt{1-\gamma^2})} \frac{1}{(\gamma^2 \cos^2\beta + \sin^2\beta)^{\frac{3}{2}}}, \tag{20}$$

where $\beta \in [-\pi, \pi]$ and $\mathcal{E}(\kappa) = \int_0^{\frac{\pi}{2}} \sqrt{1-\kappa^2 \sin^2\beta}\, \mathrm{d}\beta$ is the Legendre elliptic integral. The parameter $\gamma = \sqrt{\frac{\lambda_{020}}{\lambda_{200}}}$ which is referred as short-crestedness, equals $\gamma = 0$ for the long-crested sea and $\gamma = 1$ for the most irregular or short-crested sea. Next,

$$\dot{\mathbf{W}} \stackrel{d}{=} R \cdot s(\beta) \cdot (\cos\beta, \sin\beta), \tag{21}$$

where R has density $f_R(r) = \sqrt{\frac{2}{\pi}} r^2 e^{-\frac{r^2}{2}}$ and is independent of β, and

$$s(\beta) = \frac{\sqrt{\lambda_{020}}}{\sqrt{\gamma^2 \cos^2\beta + \sin^2\beta}}.$$

The letter d above the equality sign means equality of distributions.

The density of the speed in the direction of gradient V_{gr} follows immediately from the relation $V_{gr} = -W_t/||\dot{\mathbf{W}}||$ and the form of the above densities. The conditional distribution of V_{gr} given the direction β, is given by

$$V_{gr} \stackrel{d}{=} \mathbf{v}_{max}^T \mathbf{n}_\beta + \frac{s_E}{s(\beta)} \frac{T_3}{\sqrt{3}}, \tag{22}$$

where T_3 is a t-distribution with 3 degrees of freedom. The constant s_E^2 is the conditional variance of W_t given the gradient $\dot{\mathbf{W}}$, i.e

$$s_E^2 = \lambda_{002} - \frac{\lambda_{101}^2}{\lambda_{200}} - \frac{\lambda_{011}^2}{\lambda_{020}},$$

and

$$\mathbf{v}_{max} = (-\frac{\lambda_{101}}{\lambda_{200}}, -\frac{\lambda_{011}}{\lambda_{020}})$$

is referred to as the *principal velocity.*

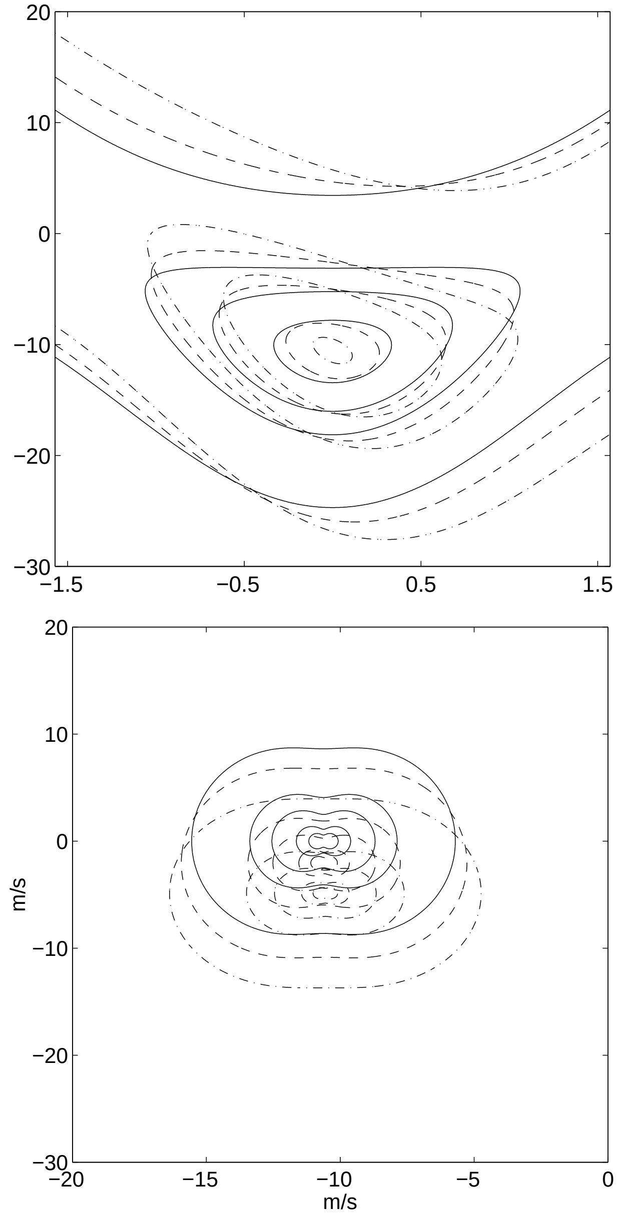

FIG. 3 *Top*: Joint density of V_{gr} and β. *Bottom*: Asymptotic biased density of $\mathbf{V}$. The isolines in all figures are drawn at the levels: 0.025, 0.01, 0.005, 0.001, 0.0005.

Numerical illustrations of the joint density of V_{gr} and β are given in Fig. 3 (*Top*). The gradient may point in any direction but due to the symmetry of the densities, we have restricted the domain interval to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Note that the negative values of V_{gr} indicate that the waves move in the opposite direction to the one pointed by the gradient, which could be interpreted as the velocity of the back of the wave. The wave fronts move with speed approximately equal to $\mathbf{v}_{max} = -10.6$m/s. In models two and three, we have introduced swell of different energies that is moving in a direction perpendicular to the wind. The energy of the swell waves apparently is too small to create any obvious changes except the slight change in direction, which becomes more obvious as the swell energy increases. We can also see that the waves with gradient in the direction of the x-axis are moving only with negative speeds.

We turn now to the velocity $\mathbf{V}$ sampled at points of the fixed level contour $\mathcal{C}(w)$ defined in Eq. 5, as w tends to infinity. The distribution is given by

$$\mathbf{V} \stackrel{d}{=} \mathbf{v}_{max} + \frac{1}{\gamma^2 \cos^2 \beta + \sin^2 \beta} \frac{s_E}{s(\beta)} \frac{X}{R} \begin{bmatrix} \gamma^2 \cos^2 \beta \\ \sin^2 \beta \end{bmatrix}, \tag{23}$$

where X is a standard normal variable, R has density $f_R(r) = \sqrt{\frac{2}{\pi}} r^2 e^{-\frac{r^2}{2}}$ and X and R are independent. For a derivation of the distribution of $\mathbf{V}$ for the non-asymptotic case as well as for a proof of Eq. 23, see Baxevani et al. (2002). The asymptotic density of $\mathbf{V}$, which is useful for applications, can be seen in Fig. 3 (*Bottom*). A comparison of the two densities given in Fig. 3 reveals that the velocity in constant gradient direction $\mathbf{V}$ is more sensitive than the velocity in the direction of gradient $\mathbf{V}_{gr}$. Indeed the introduction of swell causes the isolines to move a lot faster than before. Moreover the increase of the energy, the swell waves are caring, is also very influential.

Velocities of specular points

Finally we study the density of $\mathbf{V}_{sp}$ sampled from specular points as well as from points of a fixed level contour. Let $\mathbf{Y} = (Y_1, Y_2, Y_3, Y_4)$ be a zero mean Gaussian vector with covariance matrix:

$$\mathbf{\Sigma}(v_1, v_2) = \Big(\mathbf{A}^T(v_1, v_2) \ddot{\mathbf{\Lambda}}^{-1} \mathbf{A}(v_1, v_2)\Big)^{-1},$$

where

$$\ddot{\mathbf{\Lambda}} = \begin{bmatrix} \lambda_{400} & \lambda_{220} & \lambda_{310} & \lambda_{301} & \lambda_{211} \\ \lambda_{220} & \lambda_{040} & \lambda_{130} & \lambda_{121} & \lambda_{031} \\ \lambda_{310} & \lambda_{130} & \lambda_{220} & \lambda_{211} & \lambda_{121} \\ \lambda_{301} & \lambda_{121} & \lambda_{211} & \lambda_{202} & \lambda_{112} \\ \lambda_{211} & \lambda_{031} & \lambda_{121} & \lambda_{112} & \lambda_{022} \end{bmatrix} \tag{24}$$

and

$$\mathbf{A}(v_1, v_2) = \begin{bmatrix} 1 & 0 & 0 & v_1 & 0 \\ 0 & 1 & 0 & 0 & v_2 \\ 0 & 0 & 1 & v_2 & v_1 \end{bmatrix}. \tag{25}$$

First let us sample $\mathbf{V}_{sp}$ from points satisfying the condition $\dot{\mathbf{W}} = (W_x, W_y) = (u_1, u_2)$, that is from specular points. Longuet-Higgins (1957) has derived the density of $\mathbf{V}_{sp}$ for this case. This density is given by

$$f_{\mathbf{V}_{sp}}(v_1, v_2) = c \cdot \frac{E(Y_1 Y_2 - Y_3^2)}{E|\det \ddot{\mathbf{W}}|}, \tag{26}$$

being the Hessian of the field W. The normalization constant c equals

$$c = \frac{1}{2\pi}\sqrt{\frac{\det \mathbf{\Sigma}(v_1, v_2)}{\det \ddot{\mathbf{\Lambda}}}}. \tag{27}$$

The density of $\mathbf{V}_{sp}$ can be seen in Fig. 4 (*Top*). It is interesting that this density, defined by Eq. 26, does not depend on the fixed gradient level (u_1, u_2). The velocity $\mathbf{V}_{sp}$ when sampled as in the previous case unfortunately describes the motion of all local maxima. Most of these points are of small amplitude and hence are of limited interest for applications. What is of interest is the behavior of high waves, that is of local maxima above a certain level.

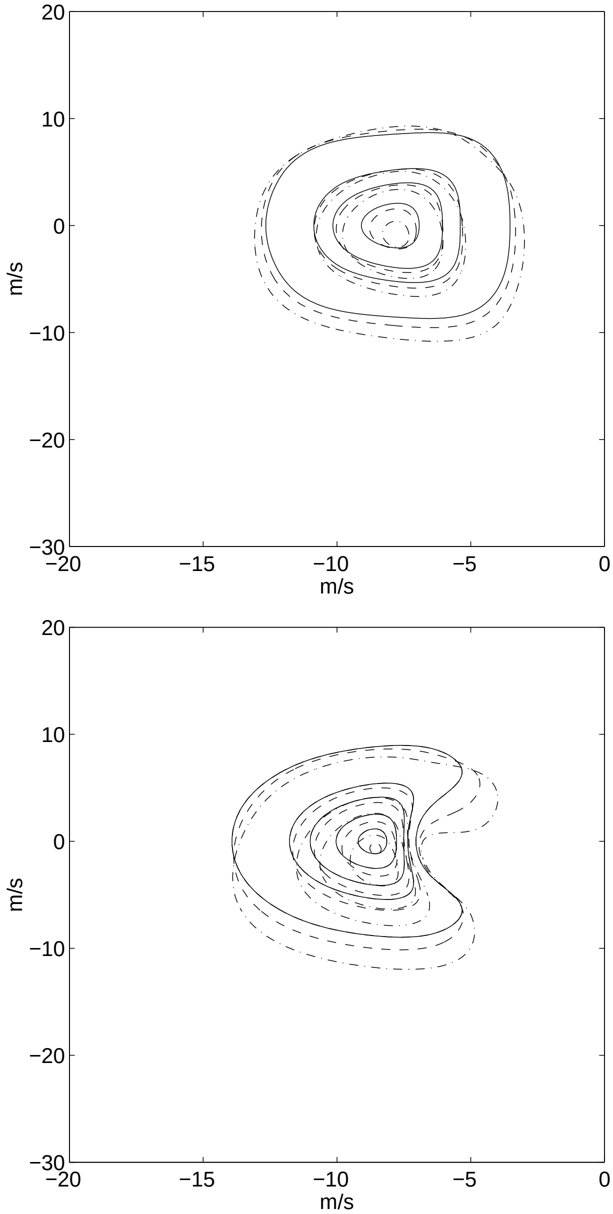

FIG. 4 *Top*: Density of the vector velocity $\mathbf{V}_{sp}$ given by Eq. 26. *Bottom*: Conditional density of $\mathbf{V}_{sp}$ given that $W \geq 2\sqrt{\lambda_{000}}$.

This is the reason of turning now to the joint density of $\mathbf{V}_{sp}$ and W. The generalized Rice formula gives us the joint distribution of $\mathbf{V}_{sp}$ and W

$$
\begin{aligned}
F_{\mathbf{V}_{sp},W}(\mathbf{v}, w) & \\
&= \frac{E\left(|\det \ddot{\mathbf{W}}|\{\mathbf{V}_{sp} \le \mathbf{v}\}\{W \le w\}\big| W_x = u_1, W_y = u_2\right)}{E\left(|\det \ddot{\mathbf{W}}|\big| W_x = u_1, W_y = u_2\right)} \\
&= \frac{E\left(|\det \ddot{\mathbf{W}}|\{\mathbf{V}_{sp} \le \mathbf{v}\}\{W \le w\}\right)}{E|\det \ddot{\mathbf{W}}|}.
\end{aligned}
$$

Both $\ddot{\mathbf{W}}$ and $\mathbf{V}_{sp}$ depend only on the second order derivatives of W and therefore are independent of $\dot{\mathbf{W}}$, which allows the simplification in conditioning. Again omitting the mathematical details of the derivation we obtain the following joint density $(\mathbf{V}_{gr}, W)$, for sampling from specular points

$$
\begin{aligned}
f_{\mathbf{V}_{sp},W}(v_1, v_2, w) & \\
&= c \cdot \frac{E\left[(Y_1 Y_2 - Y_3^2)^2 \big| Y_4 = w\right]}{E|\det \ddot{\mathbf{W}}|} \cdot \frac{1}{\sqrt{2\pi\sigma_{44}}} e^{-\frac{w^2}{2\sigma_{44}}}.
\end{aligned}
$$

In Fig. 4 (*Bottom*) we can see the density of $\mathbf{V}_{sp}$ at a subset of specular points. To be more specific, it is the marginal density $\int f_{\mathbf{V}_{sp},W}(v_1, v_2, w)\, dw$, for values of W above the critical level $2\sqrt{\lambda_{000}}$. In the special case of gradient equal to zero (extremal points), this velocity corresponds to the velocity of crests of high waves. Both these densities, shown in Fig. 4, appear to be quite similar. The velocity of all local maxima has more spreading caused obviously from the presence of all this noise, small waves and is not really affected by the swell waves. An explanation for this is that the wind is more irregular than the swell, hence points like local maxima are still defined mainly by the wind. On the other hand, the introduction of the swell is quite influential for the velocity of high waves. The most probable values of the tops of the high waves are moving faster after the swell was introduced.

Conclusions

The statistical properties of a sea surface changing in time and modeled by a Gaussian field are essentially different when observed both in time and space than their analogs observed only in time and in one of the two spatial dimensions. The dynamics of the motion for one dimensional records of some of the studied velocities can also be found elsewhere. (See, for example, Longuet-Higgins (1957) or Podgórski at al. (2000b)). In this work we study spatio-temporal waves, that is we deal with random variability of the sea elevation in both space and time.

The main purpose of this paper was to analyze the dynamical aspects of deep sea by studying the statistical properties of the wave motion. Several notions of velocity were introduced and their densities were derived. For this two dimensional Gaussian sea evolving in time, waves were identified with the help of contours of spatial mean level-crossings, contours of spatial local maxima and points of the extremal spatial crests.

One of the most important notions of velocity, that of a group, was defined for the spatio-temporal sea using the envelope field. Also a new velocity that describes the motion of a point towards the point residing on the next instant contour and having the gradient pointing in the same direction as the original point was defined and studied.

The densities of the above mentioned velocities, among others, were computed and compared for three different sea models. One of the main assumptions of this work is that the sea is well described by a real homogeneous Gaussian random field. In the first model the sea had a directional wind spectrum (no swell), while in the other two models we have added swell of different energies coming from a direction perpendicular to the main direction of propagation.

As was expected, the wave groups appear to move almost twice as slowly as the individual waves while their densities are more concentrated around the mean value. Furthermore, the group velocity is not really affected by the swell waves coming from different direction. Finally, we have derived the asymptotic distribution of the new proposed velocity of constant gradient direction. This velocity appeared to be the most sensitive to the drift motion, a very important property that was also entertained by the velocity of the high local maxima of the sea surface. Here it should also be noted that the velocity of constant gradient direction showed more variability than the velocity of the high local maxima since the first velocity describes the drift motion as well as the motion of the contour due to the changes of the overall level of the sea surface, while the latter velocity describes solely the drift motion.

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