On risk aversion and maximal expected utility from terminal wealth in the Black-Scholes model

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Abstract

Consider an investor in the Black-Scholes model during the period $[0, T]$, where $T > 0$ and where the mean rate of return on the stock is assumed to be strictly greater than the rate of return on the bond. The investor strives to maximize the expected utility from terminal wealth. Suppose the investor has the initial wealth $x > 0$ and let $\hat{\alpha}_x$ be the amount invested in the stock at time zero. Set $\hat{\pi}_x = \hat{\alpha}_x / x$. It will be proved that the function $x \to \hat{\pi}_x$ decreases if the investor possesses an increasing relative risk aversion and that the function $x \to \hat{\alpha}_x$ increases if the investor possesses a decreasing absolute risk aversion.

Similar results are obtained by Arrow in a so called one period market model with time consisting of two points [1].

1. Introduction

This paper studies connections between an investor's wealth, portfolio, and risk aversion in the Black-Scholes model. The investor is assumed to act so that the expected utility from terminal wealth is maximal at each point of time. Below two types of risk aversion will be studied, viz. increasing relative risk aversion and decreasing absolute risk aversion. Before going into details it is natural to give a few definitions.

The Black-Scholes capital market model has two assets, one risky asset called a stock and another, secure asset called a bond, both given in the period $[0,T]$, where $T>0$. The stock price $S(t)$ at time t is given by the equation

$$S(t)=S(0)e^{(\mu-\frac{\sigma^2}{2})t+\sigma W(t)}$$

where $(W(t))_{0\leq t\leq T}$ is a normalized real-valued Wiener process and the mean rate of return μ and the volatility σ are real and positive constants, respectively. Furthermore, the bond price $B(t)$ at time t satisfies the equation

$$B(t)=B(0)e^{rt}$$

where the rate of interest r is a positive constant. Here the initial stock and bond prices $S(0)$ and $B(0)$ are positive constants. The Black-Scholes model is free from transaction costs.

In this paper a so called utility function $U:\,]0,\infty[\,\rightarrow\mathbf{R}$ is supposed to satisfy the following assumptions

$$U\in\mathcal{C}^{(2)} \tag{1.1}$$

$$U \text{ is strictly increasing} \tag{1.2}$$

and

$$U \text{ is strictly concave.} \tag{1.3}$$

Note that these assumptions imply that $U'(x)>0$.

If U is a utility function, the Arrow-Pratt measures of relative risk aversion and absolute risk aversion corresponding to the wealth $x>0$ are given by

$$R^U_{rel}(x)=-\frac{xU''(x)}{U'(x)}$$

and

$$R^U_{abs}(x)=-\frac{U''(x)}{U'(x)}$$

respectively (see Arrow [1] and Pratt [8]).

In continuous time portfolio theory it is common to assume that a utility function U satisfies the additional condition

$$U'(+\infty)=0. \tag{1.4}$$

The inverse function U'^{-1} then maps the open interval $]0, U'(0+)[$ onto the open interval $]0, +\infty[$. Set

$$I(y) = I^U(y) = \begin{cases} U'^{-1}(y), & \text{if } 0 < y < U'(0+) \\ 0, & \text{if } U'(0+) \leq y < +\infty. \end{cases}$$

In continuous time portfolio theory, some important results also rely on the assumption that

$$I(e^x) \leq Ae^{B|x|}, \; x \in \mathbf{R} \tag{1.5}$$

for appropriate positive constants A and B (see e.g. Karatzas et alias [5], [6], and Korn [7]). Below, for short, $U(0+)$ is written $U(0)$ and $\mathcal{UF}$ denotes the class of all utility functions U satisfying (1.1) – (1.5).

Let us go back to the Black-Scholes model and insert the additional assumption that the mean rate of return on the stock is strictly greater than the rate of return on the bond, that is $\mu > r$. An investor has the initial wealth $x > 0$ and will invest in the stock and the bond according to a self-financing strategy with a non-negative wealth process $(X(t))_{0 \leq t \leq T}$ so that the expected utility $E[U(X(T))]$ from terminal wealth is maximal. Here $U \in \mathcal{UF}$ is the investor's utility function. Let $(\hat{X}(t))_{0 \leq t \leq T}$ be the optimal wealth process and $V(x) = E\left[U(\hat{X}(T))\right]$ the so called value function (at time zero). Then $\hat{X}([0, T[) \subseteq \;]0, \infty[$ with probability one and $\hat{X}(T)$ vanishes with positive probability, if $U'(0+) < +\infty$. Let for any $t \in [0, T[$, $\hat{\pi}_x(t)$ be the fraction of wealth $\hat{X}(t)$ invested in the stock time at time t and let $\hat{\alpha}_x(t) = \hat{\pi}_x(t)\hat{X}(t)$ be the amount invested in the stock at time t.

Suppose $0 < x_0 < x_1$. If the function R^U_{rel} increases, Theorem 5.1, a) shows that $\hat{\pi}_{x_0}(t) \geq \hat{\pi}_{x_1}(t)$, $0 \leq t < T$. Here $\hat{\pi}_{x_0}(t) > \hat{\pi}_{x_1}(t)$, $0 \leq t < T$, if R^U_{rel} is not constant. The proof is based on the Brunn-Minkowski inequality in the functional form put forward by Andras Prêkopa (see Prêkopa's book [9]). Furthermore, by Theorem 5.1, b), if the function R^U_{abs} decreases, then $\hat{\alpha}_{x_0}(t) < \hat{\alpha}_{x_1}(t)$, $0 \leq t < T$. These results are quite similar to those obtained by Arrow under slightly different assumptions [1] . Arrow´s investigations are based on a so called one period market model, where time consists of two points only.

If the function R^U_{rel} increases, Theorem 3.1, b) shows that the value function $V \in \mathcal{UF}$ and the function R^V_{rel} increases. Moreover, if R^U_{abs} decreases, then, by Theorem 4.1, b), $V \in \mathcal{UF}$ and R^V_{abs} decreases in the strict sense.

The investigations below are based on the fundamental and elegant martingale approach to optimal portfolio theory ([5], [6]). This approach is based on a variety of auxiliary functions and inserting some additional lines will make the paper easy of access to readers with no previous knowledge of optimal portfolio theory. Another reason for completeness is the need of slightly different conditions on the utility function U than what seems to be standard. For example, if the function R^U_{rel} is increasing, the function I need not be convex, which is a common assumption in literature.

2. A review of the martingale approach to maximal expected utility from terminal wealth in the Black-Scholes model

In this section some important parts of the martingale approach to optimal portfolio theory in the Black-Scholes capital market model will be recalled. There is no consumption at all in the development below, which simplifies the presentation very much. Throughout the section it is assumed that $\mu \neq r$ and $U \in \mathcal{UF}$.

Let

$$\theta = \frac{\mu - r}{\sigma}$$

be the market price of risk and introduce the P-martingale

$$Z(t) = e^{-\frac{1}{2}\theta^2 t - \theta W(t)},\ 0 \leq t \leq T.$$

By the Cameron-Martin theorem, the process

$$W^Q(t) = W(t) + \theta t,\ 0 \leq t \leq T$$

is a normalized Wiener process relative to the so called martingale measure Q defined by the equation

$$dQ = Z(T)dP.$$

Note that

$$\begin{aligned} dS(t) &= S(t)(\mu dt + \sigma dW(t)) \\ &= S(t)(r dt + \sigma dW^Q(t)) \end{aligned}$$

so that the process $(S(t)e^{-rt})_{0 \leq t \leq T}$ is a Q-martingale.

Let $\{\mathcal{F}_t\}_{0\leq t\leq T}$ denote the P-augmentation of the natural filtration generated by $(W(t))_{0\leq t\leq T}$. Set

$$H(t) = Z(t)e^{-rt}, \ 0 \leq t \leq T$$

and recall from option pricing that a contingent claim which pays the amount $Y \in L^2(\Omega, \mathcal{F}_T, Q)$ to its owner at maturity T has the value

$$E\left[H(T)Y\right] = E^Q\left[e^{-rT}Y\right]$$

at time 0.

A portfolio process is a real-valued process $\alpha(t)$, $0 \leq t \leq T$, which is progressively measurable with respect to the filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$ and satisfies

$$\int_0^T \alpha^2(t)dt < \infty.$$

Here the amount $\alpha(t)$ is invested in the stock at time t and the corresponding wealth $X(t) = X^\alpha(t)$ is supposed to satisfy the equation

$$\begin{aligned} dX(t) &= \alpha(t)\frac{dS(t)}{S(t)} + (X(t) - \alpha(t))\frac{dB(t)}{B(t)} \\ &= rX(t)dt + \sigma\alpha(t)dW^Q(t) \end{aligned}$$

or, stated otherwise,

$$X(t)e^{-rt} = X(0) + \int_0^t \sigma\alpha(s)e^{-rs}dW^Q(s).$$

It is assumed that the process $(X(t))_{0\leq t\leq T}$ is given in a version possessing continuous sample functions with probability one. The process $(X(t)e^{-rt})_{0\leq t\leq T}$ is a local Q-martingale. Let $x > 0$ be given and write $\alpha \in \mathcal{A}(x)$ if the corresponding wealth process satisfies $X(0) = x$ and $X(t) \geq 0$ for each $t \in [0,T]$. Note that in this case the process $(X(t)e^{-rt})_{0\leq t\leq T}$ is a non-negative Q-supermartingale and

$$E\left[H(T)X(T)\right] = E^Q\left[e^{-rT}X(T)\right] \leq x.$$

Moreover, if

$$t_0 = \min(T, \inf\{t \in [0,T]; \ X(t) = 0\})$$

then $X(t) = 0$, $t_0 < t \leq T$ with probability one (for details, see [5]). Below it will be seen that the so called optimal wealth process is strictly positive in the interval $[0, T[$ with probability one.

Recall from the definition of the class $\mathcal{UF}$ that the function $I(y)$ satisfies (1.5) and introduce

$$\mathcal{X}(y) = \mathcal{X}^U(y) = E\left[H(T)I(yH(T))\right] = e^{-rT}E^Q\left[I(yH(T))\right], \ y > 0.$$

The function $\mathcal{X}$ is strictly decreasing with $\mathcal{X}(0+) = +\infty$ and $\mathcal{X}(+\infty) = 0$. Moreover, by the Morera Theorem, $\mathcal{X}$ is real analytic since

$$\mathcal{X}(e^\eta) = e^{-rT}\int_{-\infty}^{\infty} I(e^{-\lambda})e^{-\frac{(\lambda+\eta-(r-\frac{\theta^2}{2})T)^2}{2\theta^2 T}}\frac{d\lambda}{\sqrt{2\pi\theta^2 T}}, \ \eta \in \mathbf{R}$$

(for more details, see John's book [3]). The inverse function $x = \mathcal{X}^{-1}(y)$, $y > 0$, is denoted by $y = \mathcal{Y}(x)$, $x > 0$.

Below it is useful to know that the function $\mathcal{X}$ determines I uniquely. To prove this claim suppose $U_0 \in \mathcal{UF}$ and $\mathcal{X}^{U_0} = \mathcal{X}$. Set $J = I - I^{U_0}$ so that

$$E\left[H(T)J(e^\eta H(T))\right] = 0, \text{ all } \eta \in \mathbf{R}.$$

But then the convolution

$$J(e^{\theta\sqrt{T}\eta}) * e^{-\frac{\eta^2}{2}} = 0$$

and Fourier transformation shows that $J = 0$, that is $I = I_0$.

The main objective in this paper is the optimization problem

$$\max_{\alpha \in \mathcal{A}^U(x)} E\left[U(X^\alpha(T))\right]$$

where

$$\mathcal{A}^U(x) = \left\{\alpha \in \mathcal{A}(x);\ E\left[U^-(X^\alpha(T))\right] < \infty\right\}$$

and $U^- = \max(-U, 0)$. Note that the inequality

$$U(I(y)) \geq U(x) + y(I(y) - x), \ y > 0 \tag{2.1}$$

yields

$$U^-(I(\mathcal{Y}(x)H(T))) \leq \mid U(1) \mid + \mathcal{Y}(x)H(T)(I(\mathcal{Y}(x)H(T)) + 1)$$

and, accordingly from this,

$$E\left[U^{-}(I(\mathcal{Y}(x)H(T)))\right] \leq\mid U(1)\mid +\mathcal{Y}(x)(x+e^{-rT}) < \infty.$$

Given $x > 0$, let

$$\hat{X}(t) = \hat{X}_x(t) = e^{-r\tau} E^Q\left[I(\mathcal{Y}(x)H(T)) \mid \mathcal{F}_t\right],\ 0 \leq t \leq T \tag{2.2}$$

where it is assumed that the process $(\hat{X}(t))_{0\leq t\leq T}$ is given in a version possessing continuous sample functions with probability one. Note that for each fixed $t \in [0, T[$, $\hat{X}(t) > 0$ with probability one since the random variable $I(\mathcal{Y}(x)H(T))$ is positive with positive probability. If $U'(0+) < +\infty$, clearly $\hat{X}(T) = 0$ with positive probability. By martingale representation the process $(\hat{X}(t))_{0\leq t\leq T}$ is the wealth process of a unique portfolio process $\hat{\alpha}_x \in \mathcal{A}^U(x)$ and, in fact, $\hat{\alpha}_x$ is an optimal solution to the maximization problem above. Indeed, if $\alpha \in \mathcal{A}^U(x)$ and $(X(t))_{0\leq t\leq T}$ is the corresponding wealth process, by (2.1)

$$U(I(\mathcal{Y}(x)H(T))) \geq U(X(T)) + \mathcal{Y}(x)(H(T)I(\mathcal{Y}(x)H(T)) - H(T)X(T))$$

and it follows that

$$E\left[U(I(\mathcal{Y}(x)H(T))\right] \geq E\left[U(X(T))\right].$$

Since U is strictly concave $\hat{\alpha}_x$ is the unique optimal solution to the above maximization problem.

Throughout this section it will be assumed that

$$\mathcal{X}'(y) < 0,\ \text{all } y > 0 \tag{2.3}$$

so that, in particular, the function $\mathcal{Y}$ is real analytic. Then

$$\hat{\alpha}_x(0) - \frac{\mu - r}{\sigma^2}\frac{\mathcal{Y}(x)}{\mathcal{Y}'(x)} \tag{2.4}$$

and setting

$$\hat{\pi}_x(0) = \frac{\hat{\alpha}_x(0)}{\hat{X}(0)}$$

it follows that

$$\hat{\pi}_x(0) = -\frac{\mu - r}{\sigma^2}\frac{\mathcal{Y}(x)}{x\mathcal{Y}'(x)}.$$

In literature the formula (2.4) is often proved under the additional assumption that the function $x = I(y)$ is convex (which is not necessarily the case for a utility function U with increasing relative risk aversion). A short direct proof of (2.4) reads as follows. Set

$$\hat{X}(t) = f(t, H(t)).$$

To be more explicit, if $\tau = T - t > 0$ and $h > 0$,

$$f(t,h) = e^{-r\tau} E^Q \left[I(\mathcal{Y}(x) h e^{-(r+\frac{\theta^2}{2})\tau - \theta(W(T)-W(t))}) \right]$$

$$= e^{-r\tau} \int_{-\infty}^{\infty} I(\mathcal{Y}(x)e^{-\lambda}) e^{-\frac{(\lambda + \ln h - (r - \frac{\theta^2}{2})\tau)^2}{2\theta^2\tau}} \frac{d\lambda}{\sqrt{2\pi\theta^2\tau}}.$$

Now

$$d\hat{X}(t) = f'_h(t, H(t)) dH(t) + (...)dt$$
$$= -\theta H(t) f'_h(t, H(t)) dW^Q(t) + (...)dt$$

since

$$dH(t) = -H(t)(r dt + \theta W(t)).$$

Thus

$$\sigma \hat{\alpha}_x(t) = -\theta H(t) f'_h(t, H(t))$$

and, in particular,

$$\hat{\alpha}_x(0) = -\frac{\mu - r}{\sigma^2} f'_h(0,1).$$

Since

$$f(0,h) = \mathcal{X}(\mathcal{Y}(x)h), \;\; h > 0$$

the equation (2.3) yields

$$f'_h(0,1) = \frac{\mathcal{Y}(x)}{\mathcal{Y}'(x)}$$

and the relation (2.4) follows at once.

The value function $V(x)$ is defined by the equation

$$V(x) = E\left[U(\hat{X}(T))\right]$$

that is

$$V(x) = E\left[U(I(\mathcal{Y}(x)H(T)))\right].$$

The inequality (2.1) combined with the inequality

$$U(I(y)) \leq U(1) + U'(1)(I(y) - 1),\ y > 0$$

yields

$$\mid U(I(y)) \mid \leq 3 \mid U(1) \mid + 2y + U'(1)I(y),\ y > 0$$

and it follows that V is real analytic (by Morera's Theorem for example).

In the next step it will be proved that $V' = \mathcal{Y}$ (which is well known at least if I is convex). To this end, let $x_0 > 0$ be fixed and replace (x, y) by $(I(\mathcal{Y}(x)H(T)), \mathcal{Y}(x_0)H(T))$ in the inequality (2.1) to obtain

$$U(I(\mathcal{Y}(x_0)H(T)))$$

$$\geq U(I(\mathcal{Y}(x)H(T))) + \mathcal{Y}(x_0)H(T)(I(\mathcal{Y}(x_0)H(T)) - I(\mathcal{Y}(x)H(T))).$$

By integrating this inequality with respect to P,

$$V(x_0) \geq V(x) + \mathcal{Y}(x_0)(x_0 - x).$$

Thus V is concave and $V'(x_0) = \mathcal{Y}(x_0)$. From this it is readily seen that $V \in \mathcal{UF}$. Moreover,

$$\hat{\pi}_x(0) = -\frac{\mu - r}{\sigma^2}\frac{V'(x)}{xV''(x)} = \frac{\mu - r}{\sigma^2}\frac{1}{R^V_{rel}(x)} \tag{2.5}$$

and

$$\hat{\alpha}_x(0) = -\frac{\mu - r}{\sigma^2}\frac{V'(x)}{V''(x)} = \frac{\mu - r}{\sigma^2}\frac{1}{R^V_{abs}(x)}. \tag{2.6}$$

3. Increasing relative risk aversion

Let

$$\varepsilon = (\varepsilon_0, \varepsilon_1)$$

be a vector such that $\varepsilon_0, \varepsilon_1 > 0$ and $\varepsilon_0 + \varepsilon_1 = 1$. By abuse of language, ε will be referred to as a probability vector. If x_j, $j = 0, 1$, are vectors in the same vector space, let $x_\varepsilon = \varepsilon_0 x_0 + \varepsilon_1 x_1$.

Lemma 3.1. *Suppose the utility function U satisfies* (1.1) – (1.4). *The following assertions are equivalent:*

a) the function R^U_{rel} is increasing
b) $U'(x)$ is a log-concave function of $\ln x$
c) $I(y)$ is a log-concave function of $\ln y$.

Proof. Since

$$\frac{d}{d\xi} \ln U'(e^{\xi}) = -R^U_{rel}(e^{\xi})$$

a) and b) are equivalent.

To prove that b) $\Rightarrow$ c), first choose $y_0, y_1 \in]0, U'(0+)[$ arbitrarily and then $\xi_0, \xi_1 \in \mathbf{R}$ such that

$$y_0 = U'(e^{\xi_0}) \text{ and } y_1 = U'(e^{\xi_1}).$$

Then, if $\varepsilon = (\varepsilon_0, \varepsilon_1)$ is a probability vector,

$$U'(e^{\xi_\varepsilon}) \geq (U'(e^{\xi_0}))^{\varepsilon_0}(U'(e^{\xi_1}))^{\varepsilon_1}.$$

Hence

$$e^{\xi_\varepsilon} \leq I(y_0^{\varepsilon_0} y_1^{\varepsilon_1})$$

or

$$I^{\varepsilon_0}(y_0) I^{\varepsilon_1}(y_1) \leq I(y_0^{\varepsilon_0} y_1^{\varepsilon_1}). \tag{3.1}$$

If $I(y_0)$ or $I(y_1)$ vanishes, the inequality (3.1) is obviously true. This proves c).

The implication c) $\Rightarrow$ b) is proved in a similar way. This completes the proof of Lemma 3.1.

Before proceeding it is instructive to give an example, which will be used later on.

Example 3.1. Suppose the relative risk aversion R^U_{rel} is constant. Then either

$$U(x) = a \ln x + b$$

or

$$U(x) = \frac{a}{\gamma} x^{\gamma} + b$$

where $a > 0$, $b \in \mathbf{R}$, and $0 \neq \gamma < 1$. In the first case

$$I(y) = \frac{a}{y} \text{ and } \mathcal{X}(y) = \frac{c}{y}$$

for a suitable constant $c > 0$, and, in the second case

$$I(y) = \frac{y^{\frac{1}{\gamma-1}}}{a^{\frac{1}{\gamma-1}}} \text{ and } \mathcal{X}(y) = d y^{\frac{1}{\gamma-1}}$$

for a suitable positive constant $d > 0$. Thus, in both cases, $\ln I(y)$ is an affine function of $\ln y$.

Observe that, for any $\alpha > 0$,

$$R_{rel}^{\alpha U} = R_{rel}^{U}$$

and

$$I^{\alpha U}(y) = I^{U}(\frac{y}{\alpha}).$$

Accordingly from these equations, for any $\delta, \kappa > 0$, there exists a $U_0 \in \mathcal{UF}$ with constant relative risk aversion such that

$$\mathcal{X}^{U_0}(y) = \delta y^{-\kappa}.$$

Lemma 3.2. *Suppose $\mu > r$, $U \in \mathcal{UF}$, and that the function R_{rel}^U is increasing. Then the function $x = \mathcal{X}(y)$ is a log-concave function of $\ln y$ and the inverse function $y = \mathcal{Y}(x)$ is a log-concave function of $\ln x$. In particular,* (2.3) *holds.*

If R_{rel}^U is increasing and non-constant, then the function $x = \mathcal{X}(y)$ is a strictly log-concave function of $\ln y$ and the inverse function $y = \mathcal{Y}(x)$ is a strictly log-concave function of $\ln x$.

The proof of Lemma 3.2 depends on Prêkopa's inequality. Suppose $\varepsilon = (\varepsilon_0, \varepsilon_1)$ is a probability vector and $f, g, h : \mathbf{R}^n \to [0, \infty[$ Borel functions such that

$$h(x_\varepsilon) \geq f^{\varepsilon_0}(x_0) g^{\varepsilon_1}(x_1)$$

for all $x_i \in \mathbf{R}^n$, $i = 0, 1$. Then, Prêkopa's inequality states that

$$\int_{\mathbf{R}^n} h(x)dx \geq (\int_{\mathbf{R}^n} f(x)dx)^{\varepsilon_0}(\int_{\mathbf{R}^n} g(x)dx)^{\varepsilon_1}$$

([9]). A proof of this inequality based on the Itô lemma is given in my paper [2] (thereby supporting H. P. McKean's claim that "People in financial mathematics make their living (in part) by Itô´s lemma" [4]).

In what follows, P denotes Wiener measure on the Banach space Ω of all real-valued continuous functions defined on $[0, T]$ and W is the identity map on Ω. If $\varepsilon = (\varepsilon_0, \varepsilon_1)$ is a probability vector and $f, g, h : \Omega \to [0, \infty[$, are cylindrical Borel functions such that

$$h(\omega_\varepsilon) \geq f^{\varepsilon_0}(\omega_0)g^{\varepsilon_1}(\omega_1)$$

for all $\omega_i \in \Omega$, $i = 0, 1$, Prêkopa's inequality gives

$$E[h] \geq (E[f])^{\varepsilon_0}(E[g])^{\varepsilon_1}.$$

Proof of lemma 3.2. Let $\varepsilon = (\varepsilon_0, \varepsilon_1)$ be a probability vector, let y_0, y_1 be positive numbers, and suppose $\omega_0, \omega_1 \in \Omega$. Now writing

$$H(T) = H(T, \omega)$$

it follows that

$$H(T, \omega_\varepsilon) = (H(T, \omega_0))^{\varepsilon_0}(H(T, \omega_1))^{\varepsilon_1}.$$

Since, by Lemma 3.1, the function $I(y)$ is a log-concave function of $\ln y$,

$$I(y_0^{\varepsilon_0}y_1^{\varepsilon_1}H(T, \omega_\varepsilon)) = I((y_0 H(T, \omega_0))^{\varepsilon_0}(y_1 H(T, \omega_1))^{\varepsilon_1})$$

$$\geq I^{\varepsilon_0}(y_0 H(T, \omega_0))I^{\varepsilon_1}(y_1 H(T, \omega_1))$$

and, accordingly from this,

$$H(T, \omega_\varepsilon)I(y_0^{\varepsilon_0}y_1^{\varepsilon_1}H(T, \omega_\varepsilon))$$

$$\geq (H(T, \omega_0)I(y_0 H(T, \omega_0)))^{\varepsilon_0}(H(T, \omega_1)I(y_1 H(T, \omega_1)))^{\varepsilon_1}.$$

Thus, by Prêkopa's inequality,

$$\mathcal{X}(y_0^{\varepsilon_0}y_1^{\varepsilon_1}) \geq \mathcal{X}^{\varepsilon_0}(y_0)\mathcal{X}^{\varepsilon_1}(y_1).$$

Proceeding in a similar way as in the proof of Lemma 3.1 it follows that the function $y = \mathcal{Y}(x)$ is a log-concave function of $\ln x$.

To prove that (2.3) holds first note that $\mathcal{X}'(y) \leq 0$ for all $y > 0$. By the first part of Lemma 3.2 as already proved, the function

$$\eta \to e^{\eta}\frac{\mathcal{X}'(e^{\eta})}{\mathcal{X}(e^{\eta})}$$

decreases. Thus if $\mathcal{X}'(y_0) = 0$, it follows that $\mathcal{X}'(y) = 0$ for all $0 < y \leq y_0$, which is absurd since the function $\mathcal{X}$ is strictly decreasing.

To prove the very last part of Lemma 3.1, first note that the function $\ln \mathcal{X}(e^{\eta})$, $\eta \in \mathbf{R}$, is affine on an appropriate non-empty open subinterval of $\mathbf{R}$ if and only if the function $\ln \mathcal{Y}(e^{\xi})$, $\xi \in \mathbf{R}$, is affine on an appropriate non-empty open subinterval of $\mathbf{R}$.

Now suppose there exist $a, b \in \mathbf{R}$ and a non-empty open subinterval J of $\mathbf{R}$ such that

$$\ln \mathcal{X}(e^{\eta}) = -a\eta + b, \ \ \eta \in J.$$

Here $a > 0$ since $\mathcal{X}$ is strictly decreasing and, moreover

$$\mathcal{X}(e^{\eta}) = e^b e^{-a\eta}, \ \ \eta \in J.$$

By using the real analyticity of $\mathcal{X}$ it follows that

$$\mathcal{X}(y) = e^b y^{-a}, \text{ all } y > 0.$$

Recall from Section 2 that $\mathcal{X}$ determines I uniquely. Therefore, in view of Example 3.1, the function R_{rel}^U must be constant, which, by assumption, is not the case. The function $\ln \mathcal{X}(e^{\eta})$, $\eta \in \mathbf{R}$, is therefore not affine on any non-empty open subinterval of $\mathbf{R}$. This completes the proof of Lemma 3.1.

Theorem 3.1. *Suppose $U \in \mathcal{UF}$ and that the function R_{rel}^U increases. Furthermore, assume $\mu > r$.*

a) The quantity

$$\hat{\pi}_x(0) = -\frac{\mu - r}{\sigma^2}\frac{\mathcal{Y}(x)}{x\mathcal{Y}'(x)}$$

is a decreasing function of initial wealth x. If R^U_{rel} is not constant, $\hat{\pi}_x(0)$ is a strictly decreasing function of x.

b*) The function R^V_{rel} increases. Moreover, if R^U_{rel} is not constant, R^V_{rel} is strictly increasing.*

Proof. By Lemma 3.2, the strictly positive function

$$-\frac{x\mathcal{Y}'(x)}{\mathcal{Y}(x)}, \ x > 0$$

is increasing. Moreover, this function is strictly increasing if the function R^U_{rel} is not constant. This proves Part a) in Theorem 3.1. Part b) now follows at once from (2.5), which concludes the proof of Theorem 3.1.

4. Decreasing absolute risk aversion

In connection with the problems faced in this paper, decreasing absolute risk aversion is slightly simpler to handle than increasing relative risk aversion. For example, in contrast to the previous section, the results in this section do not lean on any special inequalities.

Lemma 4.1. *Suppose the utility function U satisfies* (1.1) $-$ (1.4). *The following assertions are equivalent:*

a) the function R^U_{abs} is decreasing

b) $U'(x)$ is a log-convex function of x

c) $I(y)$ is a convex function of $\ln y$.

If the function R^U_{abs} is decreasing, Lemma 4.1 implies that the function I is convex.

Proof of Lemma 4.1. Since

$$\frac{d}{dx}\ln U'(x) = -R^U_{abs}(x)$$

a) and b) are equivalent.

To prove that b) $\Rightarrow$ c), let $\eta_0, \eta_1 < \ln U'(0+)$ and choose $x_0, x_1 > 0$ such that

$$e^{\eta_0} = U'(x_0) \text{ and } e^{\eta_1} = U'(x_1).$$

Then, if $\varepsilon = (\varepsilon_0, \varepsilon_1)$ is a probability vector,

$$U'(x_\varepsilon) \leq (U'(x_0))^{\varepsilon_0}(U'(x_1))^{\varepsilon_1}.$$

Hence

$$\varepsilon_0 I(e^{\eta_0}) + \varepsilon_1 I(e^{\eta_1}) \geq I(e^{\eta_\varepsilon}).$$

Thus the non-negative continuous function $I(e^\eta)$, $\eta \in \mathbf{R}$, is convex in the interval $]-\infty, \ln U'(0+)[$ and, since $I(e^{\ln U'(0+)}) = 0$, it must be convex everywhere, which proves c).

The implication c) $\Rightarrow$ b) is proved in a similar way. This completes the proof of Lemma 4.1.

If the function R^U_{abs} is constant $U(x) = a - be^{-cx}$ for appropriate constants $a \in \mathbf{R}$ and $b, c > 0$. Moreover, $I(y) = \frac{1}{c}\ln^+ \frac{bc}{y}$ and $\mathcal{X}(y)$ equals the Black-Scholes price at time 0 of a simple European option which pays the amount $\frac{1}{c}\ln^+ \frac{bc}{yH(T)}$ to its owner at maturity T. The function $\mathcal{X}(y)$ is possible to compute explicitly in terms of the distribution function of a $N(0,1)$-distributed random variable and $\mathcal{X}(y)$ turns out to be a strictly convex function of $\ln y$. In fact the following is true.

Lemma 4.2. *Suppose $\mu > r$, $U \in \mathcal{UF}$, and that the function R^U_{abs} decreases. Then the function $x = \mathcal{X}(y)$ is a strictly convex function of* $\ln y$ *and the function $y = \mathcal{Y}(x)$ is a strictly log-convex function of x. In particular,* (2.3) *holds.*

Proof of lemma 4.2. Let $\varepsilon = (\varepsilon_0, \varepsilon_1)$ be a probability vector and let y_0, y_1 be positive numbers. Since, by Lemma 4.1, the function $I(y)$ is a convex function of $\ln y$,

$$I(y_0^{\varepsilon_0} y_1^{\varepsilon_1} H(T)) = I((y_0 H(T))^{\varepsilon_0}(y_1 H(T))^{\varepsilon_1})$$

$$\leq \varepsilon_0 I(y_0 H(T)) + \varepsilon_1 I(y_1 H(T)).$$

By taking Q-expectations it follows that

$$\mathcal{X}(y_0^{\varepsilon_0} y_1^{\varepsilon_1}) \leq \varepsilon_0 \mathcal{X}(y_0) + \varepsilon_1 \mathcal{X}(y_1)$$

and, hence, the function $x = \mathcal{X}(y)$ is a convex function of $\ln y$. From this it is simple to deduce that the function $y = \mathcal{Y}(x)$ is a log-convex function of x. The function $\mathcal{X}(y)$ is an affine function of $\ln y$ on an appropriate open non-empty subinterval of $\mathbf{R}$ if and only if the function $\ln \mathcal{Y}(x)$ is an affine function of x on an appropriate non-empty open subset of $\mathbf{R}$. On the other hand, if $a, b \in \mathbf{R}$, and $\mathcal{X}(y) = a \ln y + b$ for all y belonging to an open non-empty subset of $\mathbf{R}$, the same relation holds for all $y > 0$ since $\mathcal{X}$ is real analytic. This contradicts the positivity of $\mathcal{X}$ and the first part in Lemma 4.2 is proved.

To prove (2.3) first note that $\mathcal{X}'(y) \leq 0$ for all $y > 0$. By the first part of Lemma 4.2 as already proved, the function

$$\eta \to e^{\eta} \mathcal{X}'(e^{\eta})$$

is strictly increasing. This proves (2.3) and completes the proof of Lemma 4.2.

Lemmas 4.1 and 4.2 and the relation (2.6) now give

Theorem 4.1. *Suppose $U \in \mathcal{UF}$ and that the function R^U_{abs} decreases. Furthermore, assume $\mu > r$.*

a) The quantity

$$\hat{\alpha}_x(0) = -\frac{\mu - r}{\sigma^2} \frac{\mathcal{Y}(x)}{\mathcal{Y}'(x)}$$

is a strictly increasing function of initial wealth x.

b) The function R^V_{abs} is strictly decreasing.

5. Monotonicity properties of the optimal portfolio processes

As above let $\hat{\alpha}_x$ be the optimal portfolio process and define the corresponding relative portfolio process by

$$\hat{\pi}_x(t) = \frac{\hat{\alpha}_x(t)}{\hat{X}(t)},\ 0 \le t < T.$$

In this section monotonicity properties of the processes $(\hat{\pi}_x(t))_{0\le t<T}$ and $(\hat{\alpha}_x(t))_{0\le t<T}$ as functions of initial wealth x will be investigated under similar conditions on the utility function as in Sections 3 and 4.

To begin with, write $\mathcal{X}(y) = \mathcal{X}(T,y)$ and $\mathcal{Y}(y) = \mathcal{Y}(T,y)$. Using this notation, for any $t \in [0,T[$,

$$\begin{aligned}\hat{X}(t) &= e^{-r(T-t)}E^Q\left[I(\mathcal{Y}(T,x)H(T)) \mid \mathcal{F}_t\right]\\ &= e^{-r(T-t)}E\left[Z(T)I(\mathcal{Y}(T,x)H(T)) \mid \mathcal{F}_t\right]\\ &= e^{-r(T-t)}\frac{1}{Z(t)}E\left[Z(T)I(\mathcal{Y}(T,x)H(T)) \mid \mathcal{F}_t\right]\end{aligned}$$

by the "Bayes rule". Hence

$$\hat{X}(t) = E\left[\frac{H(T)}{H(t)}I(\mathcal{Y}(T,x)H(t)\frac{H(T)}{H(t)}) \mid \mathcal{F}_t\right] = \mathcal{X}(T-t,\mathcal{Y}(T,x)H(t)). \quad (5.1)$$

Thus

$$\begin{aligned}d\hat{X}(t) &= \mathcal{X}'_y(T-t,\mathcal{Y}(T,x)H(t))\mathcal{Y}(T,x)dH(t) + (...)dt\\ &= -\theta H(t)\mathcal{Y}(T,x)\mathcal{X}'_y(T-t,\mathcal{Y}(T,x)H(t))dW^Q(t) + (...)dt\end{aligned}$$

and, accordingly from this,

$$\hat{\alpha}_x(t) = -\frac{\theta}{\sigma}H(t)\mathcal{Y}(T,x)\mathcal{X}'_y(T-t,\mathcal{Y}(T,x)H(t)).$$

Here if either R^U_{rel} increases or R^U_{abs} decreases, Lemmas 3.2 and 4.2 yield $\mathcal{X}'_y(T-t,\mathcal{Y}(T,x)H(t)) < 0$ and it follows that

$$\hat{\alpha}_x(t) = -\frac{\mu-r}{\sigma^2}\frac{H(t)\mathcal{Y}(T,x)}{\mathcal{Y}'_x(T-t,\mathcal{X}(T-t,\mathcal{Y}(T,x)H(t)))}.$$

Now using (5.1), $H(t)\mathcal{Y}(T,x) = \mathcal{Y}(T-t,\hat{X}(t))$ and thus

$$\hat{\alpha}_x(t) = -\frac{\mu-r}{\sigma^2}\frac{\mathcal{Y}(T-t,\hat{X}(t))}{\mathcal{Y}'_x(T-t,\hat{X}(t))} \quad (5.2)$$

and

$$\hat{\pi}_x(t) = -\frac{\mu - r}{\sigma^2} \frac{\mathcal{Y}(T-t, \hat{X}(t))}{\hat{X}(t)\mathcal{Y}'_x(T-t, \hat{X}(t))}. \tag{5.3}$$

If the relative risk aversion is constant, then either

$$U(x) = a \ln x + b$$

or

$$U(x) = \frac{a}{\gamma} x^\gamma + b$$

where $a > 0$, $b \in \mathbf{R}$, and $0 \neq \gamma < 1$. In the first case $\mathcal{Y}(T, x) = ax^{-1}$ and in the second case $\mathcal{Y}(T, x) = c(T)x^{\gamma-1}$ for a suitable positive constant $c(T)$. Thus in both cases the quantity

$$-\frac{\mathcal{Y}(T, x)}{x\mathcal{Y}'(T, x)}$$

is independent of (T, x). Therefore, if the relative risk aversion is constant, $\hat{\pi}_x(t)$ is independent of time t and initial wealth x.

Theorem 5.1. *Suppose $U \in \mathcal{UF}$ and $\mu > r$ and let $0 < x_0 < x_1$.*

a) If the function R^U_{rel} is increasing and not constant, then $\hat{\pi}_{x_0}(t) > \hat{\pi}_{x_1}(t)$ for all $0 \leq t < T$ with probability one.

b) If the function R^U_{abs} is decreasing, then $\hat{\alpha}_{x_0}(t) < \hat{\alpha}_{x_1}(t)$ for all $0 \leq t < T$ with probability one.

Proof. The equation (2.2) implies that $\hat{X}_{x_0}(t) < \hat{X}_{x_1}(t)$ for each t. Theorem 5.1, a) now follows from Lemma 3.2 and (5.3). In a similar way, Theorem 5.1, b) follows from Lemma 4.2 and (5.2). This concludes the proof of Theorem 5.1.

Theorem 5.1 applies to a so called HARA utility function

$$U(x) = \frac{1-\gamma}{\gamma}(\frac{\beta x}{1-\gamma} + \delta)^\gamma$$

if the parameters satisfy the conditions $0 \neq \gamma < 1$ and $\beta, \delta > 0$. In fact,

$$R^U_{rel}(x) = \frac{\beta x}{\frac{\beta x}{1-\gamma} + \delta}$$

is a non-constant and increasing function of x and

$$R^U_{abs}(x) = \frac{\beta}{\frac{\beta x}{1-\gamma} + \delta}$$

is a decreasing function of x.

References

1. Arrow, K. J. (1971) Essays in the Theory of Risk-Bearing. North-Holland Publishing Company, Amsterdam.

2. Borell, Ch. (2002) Isoperimetry, log-concavity and elasticity of option prices. In New Directions in Mathematical Finance, edited by Wilmott, P. and Rasmussen, H.. John Wiley&Sons, Chichester.

3. John, F. (1982) Partial Differential Equations. Springer-Verlag, New York.

4. McKean, H.P. (2002) Brownian Motion and the General Diffusion: Scale & Clock. In Mathematical Finance-Bachelier Congress 2000, edited by Geman, H., Madan, D., Pliska, S.R., and Vorst, T. Springer-Verlag, Berlin, Heidelberg, New York.

5. Karatzas, I., Lehoczky, J. P., Shreve, S. E. (1987) Optimal portfolio and consumption decisions. SIAM Journal on Control and Optimization **25**, 1557-1586.

6. Karatzas, I. (1989) Optimization problems in continuous trading. SIAM Journal on Control and Optimization **27**, 1221-1259.

7. Korn, R. (1997) Optimal Portfolios. World Scientific, Singapore.

8. Pratt, J. W. (1964) Risk aversion in the small and in the large. Econometrica **32**, 122-136.

9. Prékopa, A. (1995) Stochastic Programming. Kluwer Academic, Dordrecht.

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