ON BOUND STATES FOR SYSTEMS OF WEAKLY COUPLED SCHRÖDINGER EQUATIONS IN ONE SPACE DIMENSION

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ABSTRACT. We establish the Birman-Schwinger relation for a class of Schrödinger operators $-d^2/dx^2 \otimes 1_{\mathcal{H}} + V$ on $L^2(\mathbb{R}, \mathcal{H})$, where $\mathcal{H}$ is an auxiliary Hilbert space and V is an operator-valued potential. As an application we give an asymptotic formula for the bound states which may arise for a weakly coupled Schrödinger operator with a matrix potential (having one or more thresholds). In addition, for a two-channel system with eigenvalues embedded in the continuous spectrum we show that, under a small perturbation, such eigenvalues turn into resonances.

1. INTRODUCTION

In a recent paper [22] (see also [21]) we studied spectral and scattering theory for the two-channel Schrödinger operator

$$\mathbf{H} = \tilde{\mathbf{H}}_0 + \mathbf{V} = \begin{pmatrix} -\frac{d^2}{dx^2} & 0 \\ 0 & -\frac{d^2}{dx^2} + 1 \end{pmatrix} + \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \tag{1.1}$$

on the Hilbert space $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. In the low-energy limit, where the spectral parameter tends to the boundary point of the continuous spectrum of $\mathbf{H}$, viz. the point zero, we deduced asymptotic expansions for the resolvent of $\mathbf{H}$ and, as an application, we obtained asymptotic expansions for the scattering matrix associated with the pair $(\mathbf{H}, \tilde{\mathbf{H}}_0)$ as the energy parameter tends to zero. Besides being interesting from the mathematical point of view, the study of spectral and scattering theory for $\mathbf{H}$, having thresholds at 0 and 1, also works as a useful exercise towards analogous investigations for various multichannel quantum system with more than one threshold (see, e.g., [23]) because it describes many actual physical phenomena to a good approximation.

If we replace $\tilde{\mathbf{H}}_0$ in (1.1) by $\mathbf{H}_0 = -d^2/dx^2 \otimes 1_{\mathbb{C}^N}$ and $\mathbf{V}$ by an $N \times N$ matrix potential, we obtain the (usual) matrix Schrödinger operator on $L^2(\mathbb{R}, \mathbb{C}^N)$ having a single threshold at 0. The latter, of course, has attracted a lot of attention during the years. Among recent results we mention low-energy asymptotics for the corresponding scattering

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matrix [2, 3], Levinson's theorem [14], Lieb-Thirring inequalities [20, 4] and quantum design [7].

A natural question, which seems not to have been addressed in the literature, concerns how negative energy levels may arise in a system of weakly coupled Schrödinger equations. In the scalar-valued setting, weakly coupled bound states for Schrödinger operators have been investigated in various dimensions (see [19, Chapter VI] and [31, 15, 16]).

In this work we generalize the scalar-valued result obtained by Simon in dimension one [31] to the analogous matrix-valued setting.

We begin in the more abstract framework of Schrödinger operators with operator-valued potentials given formally by $H = -d^2/dx^2 \otimes 1_{\mathcal{H}} + V$ on $L^2(\mathbb{R}, \mathcal{H})$, where $\mathcal{H}$ is an auxiliary Hilbert space and the potential V is a $\mathcal{B}(\mathcal{H})$-valued, measurable function on $\mathbb{R}$ such that $V(x)$ is symmetric for almost all x. In Section 3 we define the Hamiltonian H by means of quadratic forms (Proposition 3.1) and in Section 4 we establish the celebrated Birman-Schwinger relation (Proposition 4.2), which transforms the eigenvalue problem for H into an eigenvalue problem for a compact operator; the so-called Birman-Schwinger operator.

Equipped with the Birman-Schwinger relation we study weakly coupled bound states in Section 5. We restrict our attention to Schrödinger operators with matrix-valued potentials. In Section 5.1 we consider two-channel Hamiltonians with one and two thresholds, resp. First we consider $\mathbf{H}(g) = -d^2/dx^2 \otimes 1_{\mathbb{C}^2} + g\mathbf{V}(x)$, where $\mathbf{V}$ is a 2×2 matrix potential. Theorem 5.2 reveals how non-positive eigenvalues of an auxiliary matrix $\mathbf{S}$, defined in (5.2), give rise to negative eigenvalues E_{ij} of $\mathbf{H}(g)$ provided g is small enough. The eigenvalues E_{ij} satisfy an asymptotic perturbation formula in which we derive the first few coefficients explicitly (see (5.3)). Second, we consider the above-mentioned Hamiltonian (1.1), henceforth denoted $\tilde{\mathbf{H}}(g)$, having thresholds at 0 and 1. In Theorem 5.6 we show how a negative eigenvalue of an auxiliary matrix $\tilde{\mathbf{S}}$, defined in (5.9), generates a negative eigenvalue of $\tilde{\mathbf{H}}(g)$. However, if one compares the proofs of Theorems 5.2 and 5.6 (in particular, the expressions for the matrices $\mathbf{T}_0$ and $\tilde{\mathbf{T}}_0$), it seems that the argument used in the proof of Theorem 5.2(ii), cannot be modified in order to treat the situation where zero is an eigenvalue of $\tilde{\mathbf{S}}$. Thus, it remains an attractive open problem to show that the zero eigenvalue of $\tilde{\mathbf{S}}$ (may) gives rise to a negative eigenvalue of $\tilde{\mathbf{H}}(g)$. In Section 5.2 we state the generalization of Theorem 5.2 to the N-channel Hamiltonian $-d^2/dx^2 \otimes 1_{\mathbb{C}^N} + \mathbf{V}(x)$, where $\mathbf{V}$ is an $N \times N$ matrix potential.

Having studied how negative eigenvalues arise for multichannel Hamiltonians under weak coupling, it is natural to address the problem of perturbation of embedded eigenvalues for a multichannel Schrödinger operator with a matrix-valued potential. In Section 6 we consider a two-channel Hamiltonian having eigenvalues embedded in its continuous spectrum. When perturbed by a "short range" potential, we show that such eigenvalues move into the complex plane and become resonances. In particular, we verify Fermi's golden rule (see, e.g., [27, 32]).

There is a vast literature on 2×2 operator-valued matrices, e.g. in system theory (see e.g. [8]) and in semigroup theory (see e.g. [11]). Most notably in this context is the substantial number of questions of a general nature which have been answered on spectral theory recently, see e.g. the survey [33]. However, the methods therein are not related to ours although some of the questions addressed clearly are, e.g. the appearance of resonances discussed by Mennicken and Motovilov [24].

2. Preliminaries

1) *Vector-valued functions.* Let $\mathcal{H}$ be a separable Hilbert space with scalar product and norm denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\| \cdot \|_{\mathcal{H}}$. Then a function $\psi(x)$ from $\mathbb{R}$ to $\mathcal{H}$ is measurable if the scalar-valued functions $\langle \psi(x), \phi \rangle_{\mathcal{H}}$ are measurable, where ϕ denotes an arbitrary vector of $\mathcal{H}$. If $\psi(x)$ is such a measurable function, then $\|\psi(x)\|_{\mathcal{H}}$ is also measurable (as a function with non-negative values). Thus $L^p(\mathbb{R}, \mathcal{H})$ is defined as the set of equivalence classes of measurable functions $\psi(x)$ from $\mathbb{R}$ to $\mathcal{H}$, which satisfy that $\int_{\mathbb{R}} \|\psi(x)\|_{\mathcal{H}}^p dx$ is finite if $p < \infty$ and $\|\psi\|_\infty = \text{ess sup } \|\psi(x)\|_{\mathcal{H}} < \infty$ if $p = \infty$. The measure dx is the Lebesgue measure. For any p the $L^p(\mathbb{R}, \mathcal{H})$ space is a Banach space with norm $\| \cdot \|_p = (\int_{\mathbb{R}} \| \cdot \|_{\mathcal{H}}^p dx)^{1/p}$. In the case $p = 2$, $L^2(\mathbb{R}, \mathcal{H})$ is a complex and separable Hilbert space with scalar product $\langle \phi, \psi \rangle_2 = \int_{\mathbb{R}} \langle \phi, \psi \rangle_{\mathcal{H}} dx$ and corresponding norm $\|\psi\|_2 = \langle \psi, \psi \rangle_2^{1/2}$. For $n \in \mathbb{N}$, $1 \le p < \infty$, the Sobolev space $W^{n,p}(\mathbb{R}, \mathcal{H})$ is defined as the space of those $\psi \in L^p(\mathbb{R}, \mathcal{H})$, for which all derivatives (weak sense) up to order n are in $L^p(\mathbb{R}, \mathcal{H})$. If $p = 2$, $W^{n,2}(\mathbb{R}, \mathcal{H})$ is a separable Hilbert space denoted by $H^n(\mathbb{R}, \mathcal{H})$ with scalar product $\langle \phi, \psi \rangle_{H^n(\mathbb{R},\mathcal{H})} = \int_{\mathbb{R}} \sum_{\alpha=0}^n \langle (d/dx)^\alpha \phi, (d/dx)^\alpha \psi \rangle_{\mathcal{H}}$ and norm denoted by $\|\psi\|_{H^n(\mathbb{R},\mathcal{H})}$.

2) *Operators.* Below $\mathcal{H}$, $\mathcal{H}_1$, $\mathcal{H}_2$ are separable Hilbert spaces. For a linear operator T, the notations $\mathcal{D}(T)$, $\text{Ran}\,(T)$, $\text{Ker}\,(T)$, T^*, $\overline{T}$, $\sigma(T)$, $\rho(T)$ are standard, see for example [25]. By I we denote the identity operator. The resolvent of a self-adjoint operator T is denoted by $R(T, z) = (T - zI)^{-1}$. By $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathbf{S}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ we denote respectively the sets of bounded and compact operators acting from $\mathcal{H}_1$ into $\mathcal{H}_2$. With the usual operator norm $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is a Banach space. We set $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$ and $\mathbf{S}_\infty(\mathcal{H}) := \mathbf{S}_\infty(\mathcal{H}, \mathcal{H})$.

3) *Trace classes of compact operators.* If $T \in \mathbf{S}_\infty(\mathcal{H})$ then the non-zero eigenvalues of $|T| = \sqrt{T^*T}$ are called the singular numbers or s-numbers of T. Let $\{s_j(T)\}$ denote the (possibly finite) non-increasing sequence of the singular numbers of T; every number counted according to its multiplicity as an eigenvalue of $|T|$. For $0 < p < \infty$ the von Neumann-Schatten class $\mathbf{S}_p(\mathcal{H}_1, \mathcal{H}_2)$ is the set of $T \in \mathbf{S}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ for

which the functional

$$\|T\|^p_{\mathbf{S}_p(\mathcal{H}_1,\mathcal{H}_2)} := \sum_j [s_j(T)]^p$$

is finite. The functional $\|\cdot\|_{\mathbf{S}_p(\mathcal{H}_1,\mathcal{H}_2)}$ is a norm for $p \geq 1$ and the normed space $\mathbf{S}_p(\mathcal{H}_1,\mathcal{H}_2)$ is a Banach space. For $p < 1$ the functional is a quasinorm. For additional properties of the spaces $\mathbf{S}_p$ of compact operators we refer [5, Chapter 11]. The sets $\mathbf{S}_1(\mathcal{H}_1,\mathcal{H}_2)$ and $\mathbf{S}_2(\mathcal{H}_1,\mathcal{H}_2)$ are called the trace class and Hilbert-Schmidt class, respectively.

4) *Operator-valued functions.* Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two separable Hilbert spaces. From above, a function $\mathbb{R} \ni x \to \psi(x) \in \mathcal{H}$ is measurable if and only if all the functions $\mathbb{R} \ni x \to \langle \psi(x), \phi\rangle_{\mathcal{H}} \in \mathbb{C}$ are measurable. As a result of Pettis Measurability Theorem (see, e.g.,[10, Theorem II.1.2]) the following properties are equivalent for a $\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$-valued function $\mathbb{R} \ni x \mapsto T(x)$:
(i) $\forall \phi \in \mathcal{H}_2$, $\forall \psi \in \mathcal{H}_1$, $\mathbb{R} \ni x \to \langle \phi, T(x)\psi\rangle_{\mathcal{H}_2} \in \mathbb{C}$ is measurable,
(ii) $\forall \psi \in \mathcal{H}_1$, $\mathbb{R} \ni x \to T(x)\psi \in \mathcal{H}_2$ is measurable.
We say that a function $\mathbb{R} \ni x \mapsto T(x) \in \mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$ is measurable if it satisfies any one of the above properties (i)-(ii). In the affirmative case, $\|T(x)\|_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)}$ is also measurable because

$$\|T(x)\|_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)} = \sup_{\psi\in\mathcal{D}_1} \left(\|T(x)\psi\|_{\mathcal{H}_2}/\|\psi\|_{\mathcal{H}_1}\right),$$

where $\mathcal{D}_1$ is a countable dense subset of $\mathcal{H}_1$. Moreover, we can define $L^p(\mathbb{R}, \mathcal{B}(\mathcal{H}_1,\mathcal{H}_2))$ as the linear space of (equivalence classes of) measurable functions $T : \mathbb{R} \to \mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$ such that $\|T(\cdot)\|_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)} \in L^p(\mathbb{R})$.

For the functional calculus for self-adjoint operators we recall the following result which can be found in, e.g., [6, Proposition V.1.2].

Proposition 2.1. *If for each $x \in \mathbb{R}$, $T(x)$ is a self-adjoint operator on $\mathcal{H}$ and $\{E_{T(x)}(A); A$ Borel set of $\mathbb{R}\}$ denotes its resolution of the identity, the following three properties are equivalent:*
(i) $\mathbb{R} \ni x \to E_{T(x)}(A) \in \mathcal{B}(\mathcal{H})$ is measurable for all Borel sets A,
(ii) $\mathbb{R} \ni x \to e^{-itT(x)} \in \mathcal{B}(\mathcal{H})$ is measurable for all $t \in \mathbb{R}$,
(iii) $\mathbb{R} \ni x \to (T(x)-\zeta)^{-1} \in \mathcal{B}(\mathcal{H})$ is measurable for all $\zeta \in \mathbb{C}\backslash\mathbb{R}$.

5) *Fourier transform.* Suppose $\psi \in L^1(\mathbb{R}^d, \mathcal{H})$. Then we define its Fourier transform $(\mathcal{F}\psi)(\xi) = \widehat{\psi}(\xi) := (2\pi)^{-1/2}\int_{\mathbb{R}} e^{ix\xi}\psi(x)\,dx$ which is an element of $L^\infty(\mathbb{R}, \mathcal{H})$. If $\psi \in L^1(\mathbb{R},\mathcal{H}) \cap L^2(\mathbb{R},\mathcal{H})$, then $\widehat{\psi} \in L^2(\mathbb{R},\mathcal{H})$ with $\|\widehat{\psi}\|_{L^2} = \|\psi\|_{L^2}$. The Fourier transform can then be extended by continuity to a unitary mapping of the Hilbert space $L^2(\mathbb{R},\mathcal{H})$ into itself.

We have the following criterion.

Lemma 2.2. *Let T be an operator on $L^2(\mathbb{R},\mathcal{H})$ defined by*

$$(T\phi)(x) = \int_{\mathbb{R}} t(x,\xi)\phi(\xi)\,d\xi, \tag{2.1}$$

where $t(x,\xi)\in\mathcal{B}(\mathcal{H})$ *for each* (x,ξ)*. Then* T *is a Hilbert-Schmidt operator on* $L^2(\mathbb{R},\mathcal{H})$ *if and only if*

$$\int_{\mathbb{R}_x}\int_{\mathbb{R}_\xi}\mathrm{tr}_{\mathcal{H}}[t(x,\xi)^*t(x,\xi)]\,d\xi dx<\infty.$$

In this case,

$$\|T\|^2_{\mathbf{S}_2(L^2(\mathbb{R},\mathcal{H}))}=\int_{\mathbb{R}_x}\int_{\mathbb{R}_\xi}\mathrm{tr}_{\mathcal{H}}[t(x,\xi)^*t(x,\xi)]\,d\xi dx.$$

Proof. The Hilbert space $\mathcal{H}$ is isomorphic to some $L^2(Y)$ space and therefore it suffices to establish the statement for an operator T on $L^2(\mathbb{R},L^2(Y))$ defined by (2.1) for some $t(x,\xi)\in\mathcal{B}(L^2(Y))$. Since $L^2(\mathbb{R},L^2(Y))$ is isomorphic to $L^2(\mathbb{R}\times Y)$, the rephrased assertion follows immediately from [25, Theorem VI.23]. □

3. The Hamiltonian $H=H_0+V$

As in the scalar-valued case the quadratic form

$$h_0[\psi,\psi]:=\int_{\mathbb{R}}\|(d/dx)\psi(x)\|^2_{\mathcal{H}}\,dx \tag{3.1}$$

is closed in $L^2(\mathbb{R},\mathcal{H})$ on the domain $H^1(\mathbb{R},\mathcal{H})$. Thus, this form generates a self-adjoint operator H_0 on $L^2(\mathbb{R},\mathcal{H})$. The free Hamiltonian H_0 corresponds to the "Laplacian" $-d^2/dx^2\otimes 1_{\mathcal{H}}$ on $L^2(\mathbb{R},\mathcal{H})$.

A potential V is a $\mathcal{B}(\mathcal{H})$-valued, measurable function on $\mathbb{R}$. Assume that $V(x)$ is symmetric for almost all x, i.e. $V(x)^*=V(x)$ for almost all x. The operator $V(x)\in\mathcal{B}(\mathcal{H})$ has a unique representation[1] in the form $V(x)=U(x)|V(x)|$, where $|V(x)|$ is the modulus of $V(x)$ defined by $|V(x)|=(V(x)^*V(x))^{1/2}=(V(x)V(x))^{1/2}$. We have that $|V(x)|$ is a non-negative, selfadjoint operator belonging to $\mathcal{B}(\mathcal{H})$ and, moreover, $\||V(x)|\|_{\mathcal{B}(\mathcal{H})}=\|V(x)\|_{\mathcal{B}(\mathcal{H})}$. The operator $U(x)$ is a partial isometry with initial domain $\overline{\mathrm{Ran}\,|V(x)|}$, final domain $\overline{\mathrm{Ran}\,V(x)}$ and $\mathrm{Ker}\,U(x)=\mathrm{Ker}\,V(x)$. Observe that $U(x)^*U(x)=P_{\overline{\mathrm{Ran}\,|V(x)|}}$ and $U(x)U(x)^*=P_{\overline{\mathrm{Ran}\,V(x)}}$, where P_M denotes the orthogonal projection onto a closed subspace M. The modulus $|V(x)|$ possesses exactly one non-negative, self-adjoint square-root $|V(x)|^{1/2}\in\mathcal{B}(\mathcal{H})$. The square-root $|V(x)|^{1/2}$ commutes with every bounded operator which commutes with $|V(x)|$. We may define $V(x)^{1/2}=U(x)|V(x)|^{1/2}$ such that $V(x)=V(x)^{1/2}|V(x)|^{1/2}$. Moreover, $V(x)^{1/2}\in\mathcal{B}(\mathcal{H})$ with $\|V(x)^{1/2}\|_{\mathcal{B}(\mathcal{H})}=[\|V(x)\|_{\mathcal{B}(\mathcal{H})}]^{1/2}$ and adjoint $(V(x)^{1/2})^*=|V(x)|^{1/2}U(x)^*$. From Proposition 2.1 it follows that $|V|$, $|V|^{1/2}$ and $V^{1/2}$ are $\mathcal{B}(\mathcal{H})$-valued measurable functions on $\mathbb{R}$.

We want to establish the following result.

[1]The representation is not unique if the potential vanishes on a set of positive measure

Proposition 3.1.
(i) If $V \in L^1(\mathbb{R}, \mathcal{B}(\mathcal{H}))$ *then the real-valued quadratic form*

$$v[\psi,\psi] := \int_{\mathbb{R}} \langle V(x)^{1/2}\psi(x), |V(x)|^{1/2}\psi(x)\rangle_{\mathcal{H}}\, dx$$

is H_0 *form-bounded with relative bound zero.*
(ii) If $V \in L^1(\mathbb{R}, \mathbf{S}_2(\mathcal{H}))$ *then* v *is* H_0 *form-compact.*

It follows from Proposition 3.1(i) and the KLMN theorem [26, Theorem X.17] that the form sum

$$h[\psi,\psi] := h_0[\psi,\psi] + v[\psi,\psi]$$

is closed and semi-bounded from below on $H^1(\mathbb{R}^d, \mathcal{H})$ and thus generates a self-adjoint operator $H = H_0 + V$ on $L^2(\mathbb{R}, \mathcal{H})$. From Proposition 3.1(ii) and Weyl's essential spectrum theorem it follows that $\sigma_{ess}(H) = \sigma_{ess}(H_0) = [0, \infty)$.

Proof of Proposition 3.1. The "kernel" of the resolvent of H_0 is given by (see, e.g., [28, Theorem 9.5.2])

$$Q(x-y; \sqrt{|E|}) = \frac{e^{-\sqrt{|E|}|x-y|}}{2\sqrt{|E|}}, \quad E < 0. \tag{3.2}$$

(i) To show that the form v is infinitesimally H_0 form-bounded, it suffices to show that the form

$$w[\phi] = \langle |V|^{1/2}(H_0 - E)^{-1/2}\phi, V^{1/2}(H_0 - E)^{-1/2}\phi\rangle_{L^2(\mathbb{R},\mathcal{H})}$$

is bounded on $L^2(\mathbb{R}, \mathcal{H})$ and that its norm

$$\|w\| := \inf_{\phi \in L^2(\mathbb{R},\mathcal{H})} \frac{|\langle |V|^{1/2}(H_0 - E)^{-1/2}\phi, V^{1/2}(H_0 - E)^{-1/2}\phi\rangle|}{\|\phi\|^2}$$

tends to zero as $E \to -\infty$. By the definition of $\|w\|$, and since U in $V^{1/2} = U|V|^{1/2}$ is a partial isometry, we have that

$$\|w\| \le \left\| |V|^{1/2}(H_0 - E)^{-1/2} \right\|^2_{\mathcal{B}(L^2(\mathbb{R},\mathcal{H}))}. \tag{3.3}$$

Therefore, it is enough to show that the right-hand side of the latter tends to zero as $E \to -\infty$.

We consider first $V \in L^\infty(\mathbb{R}, \mathcal{B}(\mathcal{H}))$. For such V we have that

$$\left\| |V|^{1/2}(H_0 - E)^{-1/2} \right\|^2_{\mathcal{B}(L^2)} = \left\| |V|^{1/2}(H_0 - E)^{-1}|V|^{1/2} \right\|_{\mathcal{B}(L^2)} \tag{3.4}$$

Let $\alpha = \sqrt{|E|}$ and $\phi \in L^2(\mathbb{R}, \mathcal{H})$. Then Hölder's inequality yields that

$$\begin{aligned} &\left\| [|V|^{1/2}(H_0 + \alpha^2)^{-1}|V|^{1/2}\phi](x) \right\|_{\mathcal{H}} \\ &\quad \le \frac{1}{2\alpha} \|V(x)\|^{1/2}_{\mathcal{B}(\mathcal{H})} \|V\|_{L^1(\mathbb{R},\mathcal{B}(\mathcal{H}))} \|\phi\|_{L^2(\mathbb{R},\mathcal{H})}. \end{aligned}$$

The latter implies that

$$\begin{aligned}&\big\||V|^{1/2}(H_0+\alpha^2)^{-1}|V|^{1/2}\phi\big\|^2_{L^2(\mathbb{R},\mathcal{H})}\\ &\quad\le \int_{\mathbb{R}}\frac{1}{4\alpha^2}\,\|V(x)\|_{\mathcal{B}(\mathcal{H})}\,\|V\|^2_{L^1(\mathbb{R},\mathcal{B}(\mathcal{H}))}\,\|\phi\|^2_{L^2(\mathbb{R},\mathcal{H})}\,dx\\ &\quad\le \frac{1}{4\alpha^2}\|V\|^3_{L^1(\mathbb{R},\mathcal{B}(\mathcal{H}))}\|\phi\|^2_{L^2(\mathbb{R},\mathcal{H})}.\end{aligned}$$

In conjunction with (3.4), the latter shows that the right-hand side of (3.3) tends to zero as $E\to-\infty$, which establishes assertion (i) for $V\in L^\infty(\mathbb{R},\mathcal{H})$. A standard approximation argument yields the assertion for general V.

(ii). It suffices to show that the form

$$w[\phi]=\langle|V|^{1/2}(H_0-E)^{-1/2}\phi,V^{1/2}(H_0-E)^{-1/2}\phi\rangle$$

defines a compact operator in $L^2(\mathbb{R},\mathcal{H})$. Under the assumption in (i) we already know that w generates a bounded, self-adjoint operator W in $L^2(\mathbb{R},\mathcal{H})$. Let us show that W is a Hilbert-Schmidt operator. From

$$\operatorname{tr}_{L^2}(W^*W)=\operatorname{tr}\left(V^{1/2}(H_0-E)^{-1}(V^{1/2})^*|V|^{1/2}(H_0-E)^{-1}(|V|^{1/2})^*\right),$$

we see that it is enough to show that $W_1=|V|^{1/2}(H_0-E)^{-1}(|V|^{1/2})^*$ and $W_2=V^{1/2}(H_0-E)^{-1}(V^{1/2})^*$ are Hilbert-Schmidt operators on $L^2(\mathbb{R},\mathcal{H})$. It is enough to show it for W_2; the proof for W_1 is similar. The operator W_2 has integral "kernel"

$$K_{W_2}(x-y;\alpha)=V(x)^{1/2}(2\alpha)^{-1}e^{-\alpha|x-y|}(V(y)^{1/2})^*,\quad \alpha=\sqrt{-E}>0.$$

Using the criterion in Lemma 2.2 and the assumption in (ii), we estimate as follows:

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}}\operatorname{tr}_{\mathcal{H}}[K_{W_2}(x-y;\alpha)^*K_{W_2}(x-y;\alpha)]\,dxdy=\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\frac{e^{-\alpha|x-y|}}{2\alpha}\right)^2\\ &\qquad\times\operatorname{tr}_{\mathcal{H}}[(V(y)^{1/2})^{**}(V(x)^{1/2})^*V(x)^{1/2}(V(y)^{1/2})^*]\,dxdy\\ &\quad=\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\frac{e^{-\alpha|x-y|}}{2\alpha}\right)^2\\ &\quad\times\operatorname{tr}_{\mathcal{H}}[V(y)^{1/2}|V(x)|^{1/2}U(x)^*U(x)|V(x)|^{1/2}(V(y)^{1/2})^*]\,dxdy\\ &\quad=\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\frac{e^{-\alpha|x-y|}}{2\alpha}\right)^2\operatorname{tr}_{\mathcal{H}}[|V(x)||V(y)|]\,dxdy\\ &\quad\le\frac{1}{4\alpha^2}\int_{\mathbb{R}}\|V(x)\|_{\mathbf{S}_2(\mathcal{H})}\,dx\int_{\mathbb{R}}\|V(y)\|_{\mathbf{S}_2(\mathcal{H})}\,dy.\end{aligned}$$

This shows that W_2 is a Hilbert-Schmidt operator in $L^2(\mathbb{R},\mathcal{H})$. □

We note that $V\in L^1(\mathbb{R},\mathcal{B}(\mathcal{H}))$ implies that $|V|\in L^1(\mathbb{R},\mathcal{B}(\mathcal{H}))$ and, in view of Proposition 3.1(i), $|V|$ is infinitesimally H_0 form-bounded.

Consequently, the following mappings are bounded:

$$\begin{aligned} V,\ |V| &: H^1(\mathbb{R},\mathcal{H}) \to H^{-1}(\mathbb{R},\mathcal{H}) && (3.5)\\ |V|^{1/2}, V^{1/2} &: H^1(\mathbb{R},\mathcal{H}) \to L^2(\mathbb{R},\mathcal{H}) && (3.6)\\ |V|^{1/2}, V^{1/2} &: L^2(\mathbb{R},\mathcal{H}) \to H^{-1}(\mathbb{R},\mathcal{H}). && (3.7) \end{aligned}$$

The qualitative behaviour of any possible negative eigenvalues of $H_0 + gV$ as $g \to 0$ is described by the following simple result.

Proposition 3.2. *If $V \in L^1(\mathbb{R}, \mathcal{B}(\mathcal{H}))$ then any negative eigenvalues of $H_0 + gV$ approach zero as g tends to zero.*

Proof. Following [31] it suffices to show that there are positive constants g_0 and C such that $H_0 + gV \geq -Cg$ for all $g_0 > g > 0$.

Let $\mathcal{F}$ denote the Fourier transform of vector-valued functions in $L^2(\mathbb{R}, \mathcal{H})$ (see Part 5 in Section 2). We observe that, as for scalar-valued functions, a function ϕ whose Fourier transform is integrable is bounded and continuous with the usual estimate

$$\|\phi(x)\|_{\mathcal{H}} \leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \|\mathcal{F}\phi(\xi)\|_{\mathcal{H}}\, d\xi. \tag{3.8}$$

For an arbitrary $\gamma > 0$, Hölder's inequality yields that

$$\begin{aligned} &\left(\int_{\mathbb{R}} \|\mathcal{F}\phi(\xi)\|_{\mathcal{H}}\, d\xi\right)^2 \\ &\quad\leq \left(\int_{\mathbb{R}} (\xi^2+\gamma^2)^{-1}\, d\xi\right)\left(\int_{\mathbb{R}} (\xi^2+\gamma^2)\|\mathcal{F}\phi(\xi)\|^2_{\mathcal{H}}\, d\xi\right) \\ &\quad= \frac{\pi}{\gamma}\|(-i\xi+\gamma)\mathcal{F}\phi\|^2_{L^2(\mathbb{R},\mathcal{H})} = \frac{\pi}{\gamma}\|(H_0^{1/2}+\gamma)\phi\|^2_{L^2(\mathbb{R},\mathcal{H})} \\ &\quad\leq \frac{2\pi}{\gamma}\left\{\|H_0^{1/2}\phi\|^2_{L^2(\mathbb{R},\mathcal{H})} + \|\gamma\phi\|^2_{L^2(\mathbb{R},\mathcal{H})}\right\}. \end{aligned} \tag{3.9}$$

Let $d = \max\{1/\gamma, 1\}$. Then (3.8) and (3.9) imply that

$$\|\phi(x)\|_{\mathcal{H}} \leq d\left\{\|H_0^{1/2}\phi\|^2_{L^2(\mathbb{R},\mathcal{H})} + \|\phi\|^2_{L^2(\mathbb{R},\mathcal{H})}\right\} \tag{3.10}$$

for any $\phi \in \mathcal{D}(H_0^{1/2}) = H^1(\mathbb{R}, \mathcal{H})$. The Sobolev type inequality (3.10) implies that

$$\begin{aligned} h_g[\phi] &\geq \|H_0^{1/2}\|^2_{L^2(\mathbb{R},\mathcal{H})} - g\int_{\mathbb{R}} \|V(x)\|_{\mathcal{B}(\mathcal{H})}\|\phi(x)\|^2_{\mathcal{H}}\, dx \\ &\geq (1-gd_1)\|H_0^{1/2}\|^2_{L^2(\mathbb{R},\mathcal{H})} - gd_1\|\phi\|^2_{L^2(\mathbb{R},\mathcal{H})} \end{aligned}$$

where $d_1 = d\|V\|^2_{L^1(\mathbb{R},\mathcal{B}(\mathcal{H}))}$. When we take $C = d_1$, $g_0 = d_1^{-1}$ and $0 < g < g_0$, we arrive at $h_g[\phi] \geq -Cg$ as desired. □

4. The Birman-Schwinger relation

The Birman-Schwinger relation has been established rigorously for various classes of operators in the scalar-valued setting (see, e.g., [30, 31, 17]. It asserts that E is a negative eigenvalue of $H = -d^2/dx^2 + V$ if and only if -1 is an eigenvalue of the operator $V^{1/2}(-d^2/dx^2 - E)^{-1}|V|^{1/2}$. Formally this is obvious since $\phi = V^{1/2}\psi$ is a solution to $V^{1/2}(-d^2/dx^2 - E)^{-1}|V|^{1/2}\phi = -\phi$.

Here we provide a simple proof of the Birman-Schwinger relation in our concrete operator-valued setting. For this purpose we introduce the Birman-Schwinger operator $K_E(V) = V^{1/2}(H_0 - E)^{-1}|V|^{1/2}$, $E < 0$, where H_0 is the nonnegative, self-adjoint operator associated with the quadratic form h_0 in (3.1). Setting $\alpha^2 = -E$, its integral "kernel" is given by

$$K_\alpha(x,y) = V(x)^{1/2}(2\alpha)^{-1}e^{-\alpha|x-y|}|V(y)|^{1/2}, \quad \alpha > 0.$$

We have the following result.

Lemma 4.1. *If $V \in L^1(\mathbb{R}, \mathbf{S}_2(\mathcal{H}))$ then the Birman-Schwinger operator $K_E(V)$, $E < 0$, is a Hilbert-Schmidt operator on $L^2(\mathbb{R}, \mathcal{H})$; in particular $K_E(V)$ is a compact operator. Moreover, $\|K_E(V)\|_{\mathcal{B}(L^2(\mathbb{R},\mathcal{H}))} \to 0$ as $E \to -\infty$.*

Proof. We argue as for the operator W_2 in the proof of Proposition 3.1(ii). We omit the details. □

Having introduced the compact Birman-Schwinger operator we may formulate the Birman-Schwinger relation.

Proposition 4.2. *Let $V \in L^1(\mathbb{R}, \mathbf{S}_2(\mathcal{H}))$. Then $E < 0$ is an eigenvalue of $H = H_0 + V$ (defined by a quadratic form) having multiplicity $\varkappa$ if and only if -1 is an eigenvalue of $K_E(V)$ having geometric multiplicity $\varkappa$.*

To establish Proposition 4.2 we need the following two results.

Lemma 4.3. *If $V \in L^1(\mathbb{R}, \mathcal{B}(\mathcal{H}))$ then for $E < 0$ the operators $|V|^{1/2} \times (H_0 - E)^{-1/2}$ and $V^{1/2}(H_0 - E)^{-1/2}$ are bounded on $L^2(\mathbb{R}, \mathcal{H})$.*

Proof. Since V is H_0 form-bounded, it follows immediately from (3.6) and (3.7) in conjunction with the fact that the operator $(H_0 - E)^{-1/2}$ is a bounded map from the domain $L^2(\mathbb{R}, \mathcal{H})$ to the range $H^1(\mathbb{R}, \mathcal{H})$. □

Lemma 4.4. *Let S and T be bounded operators on the Hilbert space $\mathcal{K}$. Then $\sigma(ST)\backslash\{0\} = \sigma(TS)\backslash\{0\}$. Moreover, $\lambda \neq 0$ is an eigenvalue of ST having geometric multiplicity m if and only if λ is an eigenvalue of TS having geometric multiplicity m.*

Proof. This is a simplified version of Theorem 2(i) in [9]. □

Proof of Proposition 4.2. Let h_0 be the form of H_0, let v be the form of V and let $h = h_0 + v$ be their form sum. According to Lemma 4.3

the operators $|V|^{1/2}(H_0-E)^{-1/2}$ and $V^{1/2}(H_0-E)^{-1/2}$ are bounded on $L^2(\mathbb{R},\mathcal{H})$ and, consequently, the operator $I+[|V|^{1/2}(H_0-E)^{-1/2}]^*V^{1/2}$ $(H_0-E)^{-1/2}$ is bounded on $L^2(\mathbb{R},\mathcal{H})$. Moreover, the operator $A^{-1}=(H_0-E)^{1/2}$ has domain $H^1(\mathbb{R},\mathcal{H})$ and range $L^2(\mathbb{R},\mathcal{H})$. Thus, we may introduce an auxiliary sesquilinear form a defined on the form domain $H^1(\mathbb{R},\mathcal{H})\times H^1(\mathbb{R},\mathcal{H})$ by

$$a[\phi,\psi]=\left\langle\left[I+(|V|^{1/2}A)^*V^{1/2}A\right]A^{-1}\phi,A^{-1}\psi\right\rangle. \tag{4.1}$$

We re-write a and find that

$$a[\phi,\psi]=\underbrace{\langle A^{-1}\phi,A^{-1}\psi\rangle}_{a_1[\phi,\psi]}+\underbrace{\langle(|V|^{1/2}A)^*V^{1/2}AA^{-1}\phi,A^{-1}\psi\rangle}_{a_2[\phi,\psi]}. \tag{4.2}$$

Clearly,

$$a_1[\phi,\psi]=h_0[\phi,\psi]-E\langle\phi,\psi\rangle \tag{4.3}$$

and, since $|V|^{1/2}A$ is bounded on $L^2(\mathbb{R},\mathcal{H})$,

$$\begin{aligned} a_2[\phi,\psi] &= \langle V^{1/2}AA^{-1}\phi,[|V|^{1/2}A]^{**}A^{-1}\psi\rangle \\ &= \langle V^{1/2}\phi,|V|^{1/2}AA^{-1}\psi\rangle=v[\phi,\psi]. \end{aligned} \tag{4.4}$$

Hence, (4.2)-(4.4) shows that the forms a and $h-E$ are identical.

Suppose that $E<0$ is an eigenvalue of $H=H_0+V$, i.e. there exists an eigenfunction $\psi\in\mathcal{D}(H)$, $\psi\neq 0$, such that $(H-E)\psi=0$. This is equivalent to $(h-E)[\psi,\phi]=0$ for all $\phi\in H^1(\mathbb{R},\mathcal{H})$. Since the forms h and a are identical, we may introduce $u=A^{-1}\phi$ and deduce that

$$\begin{aligned} 0=a[\psi,\phi] &= \left\langle\left[I+(|V|^{1/2}A)^*V^{1/2}A\right]A^{-1}\psi,A^{-1}\phi\right\rangle,\ \forall\phi\in H^1(\mathbb{R},\mathcal{H}), \\ &= \left\langle\left[I+(|V|^{1/2}A)^*V^{1/2}A\right]A^{-1}\psi,u\right\rangle,\ \forall\phi\in L^2(\mathbb{R},\mathcal{H}), \end{aligned}$$

because u runs through $L^2(\mathbb{R},\mathcal{H})$ as ϕ runs through $H^1(\mathbb{R},\mathcal{H})$. Consequently, $\left(I+[|V|^{1/2}A]^*V^{1/2}A\right)v=0$, where $v=A^{-1}\psi=(H_0-E)^{1/2}\psi$, so $-1\in\sigma_p([|V|^{1/2}A]^*V^{1/2}A)$. By reversing the arguments leading to the latter conclusion, we infer that

$$E\in\sigma_p(H_0+V)\text{ if and only if }-1\in\sigma_p([|V|^{1/2}A]^*V^{1/2}A) \tag{4.5}$$

Since $(H_0-E)^{1/2}$ is injective from the domain $L^2(\mathbb{R},\mathcal{H})$ to the range $H^1(\mathbb{R},\mathcal{H})$, the arguments above also show that the multiplicities of the eigenvalues E and -1 must be equal. In view of Lemma 4.4 and the definition of A, (4.5) implies that

$$\begin{aligned} &E\in\sigma_p(H_0+V)\text{ if and only if} \\ &\qquad -1\in\sigma_p(V^{1/2}(H_0-E)^{-1/2}[|V|^{1/2}(H_0-E)^{-1/2}]^*) \end{aligned} \tag{4.6}$$

and the multiplicities of E and -1 are equal. But $[|V|^{1/2}(H_0-E)^{-1/2}]^*=(H_0-E)^{-1/2}|V|^{1/2})$ and therefore, in view of the definition of $K_E(V)$, (4.6) yields that $E\in\sigma_p(H_0+V)$ if and only if $-1\in\sigma_p(K_E(V))$. $\square$

If g is fixed and $\lambda(\alpha)$ is an eigenvalue of $K_\alpha(V)$ then the Birman-Schwinger relation asserts that any solution $\alpha_g > 0$ of

$$g\lambda(\alpha_g) = -1 \tag{4.7}$$

is associated to the eigenvalue $E(g) = -\alpha_g^2$ of $H(g)$. The latter equation plays a crucial role in Section 5.

Define the operators L_α and M_α by their "kernels":

$$L_\alpha(x,y) = \frac{1}{2\alpha} V(x)^{1/2} |V(y)|^{1/2}, \tag{4.8}$$

$$M_\alpha(x,y) = \frac{1}{2\alpha} V(x)^{1/2} [[e^{-\alpha|x-y|} - 1]] |V(y)|^{1/2} \tag{4.9}$$

Moreover, we introduce the operator M_0 with "kernel"

$$M_0(x,y) = -\frac{1}{2} V(x)^{1/2} |x-y| |V(y)|^{1/2}. \tag{4.10}$$

Imitating [31] we obtain the following result.

Lemma 4.5. *If $\int_{\mathbb{R}} (1+|x|^2) \, \|V(x)\|_{\mathbf{S}_2(\mathcal{H})} \, dx < \infty$ then the following assertions are valid:*
(i) The operator M_0 is Hilbert-Schmidt on $L^2(\mathbb{R}, \mathcal{H})$.
(ii) As $\alpha \downarrow 0$, the operator M_α converges to M_0 in the Hilbert-Schmidt norm on $L^2(\mathbb{R}, \mathcal{H})$.
(iii) The Birman-Schwinger operator $gK_\alpha(V)$ has eigenvalue -1 if and only if the same is true for $g(1+gM_\alpha)^{-1}L_\alpha$.

Proof.
(i) It follows from the estimate

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{tr}_{\mathcal{H}} [M_0(x,y)^* M_0(x,y)] \, dxdy \\ \leq \frac{1}{2} \int \int (|x|^2 + |y|^2) \, \||V(x)|\|_{\mathbf{S}_2(\mathcal{H})} \, \||V(y)|\|_{\mathbf{S}_2(\mathcal{H})} \, dxdy < \infty.$$

(ii) We want to show that

$$\int \int \mathrm{tr}_{\mathcal{H}} \, [(M_\alpha - M_0)(x,y)^* (M_\alpha - M_0)(x,y)] \, dxdy \longrightarrow 0 \tag{4.11}$$

as $\alpha \downarrow 0$. Now,

$$\mathrm{tr}_{\mathcal{H}} \, [(M_\alpha - M_0)(x,y)^* (M_\alpha - M_0)(x,y)] \\ = \left| \frac{1}{2\alpha} (e^{-\alpha|x-y|} - 1) + \frac{1}{2} |x-y| \right|^2 \mathrm{tr}_{\mathcal{H}} \, [|V(x)||V(y)|],$$

and since $\left| \frac{1}{2\alpha}(e^{-\alpha|x-y|} - 1) + \frac{1}{2}|x-y| \right| \to 0$ as $\alpha \downarrow 0$, we have the pointwise convergence

$$\mathrm{tr}_{\mathcal{H}} \, [(M_\alpha - M_0)(x,y)^* (M_\alpha - M_0)(x,y)] \longrightarrow 0 \text{ as } \alpha \downarrow 0. \tag{4.12}$$

Moreover,

$$\operatorname{tr}_{\mathcal{H}}[M_\alpha(x,y)^* M_\alpha(x,y)] = \left|\frac{1}{2\alpha}(e^{-\alpha|x-y|}-1)\right|^2 \operatorname{tr}_{\mathcal{H}}[|V(x)||V(y)|]$$
$$\leq \left|\frac{1}{2}|x-y|\right|^2 \operatorname{tr}_{\mathcal{H}}[|V(x)||V(y)|] = \operatorname{tr}_{\mathcal{H}}[M_0(x,y)^* M_0(x,y)]. \quad (4.13)$$

It follows from (i) and (4.12)-(4.13) in conjunction with Lebesgue's dominated convergence theorem that (4.11) holds.
(iii) It follows from (4.13) that $\|M_\alpha\|_{\mathcal{B}(L^2(\mathbb{R},\mathcal{H}))} \leq \|M_\alpha\|_{HS} \leq \|M_0\|_{HS}$. Hence, $\|M_\alpha\|_{\mathcal{B}(L^2(\mathbb{R},\mathcal{H}))}$ is bounded *independently* of $\alpha \in (0,\alpha_0]$ for some $\alpha_0 > 0$. Therefore, for g small enough, $\|gM_\alpha\|_{\mathcal{B}(L^2(\mathbb{R},\mathcal{H}))} < 1$ and, consequently, $(1+gM_\alpha)^{-1}$ exists and is bounded for these g and α. In particular, we may write $1+gK_\alpha(V) = (1+gM_\alpha)[1+g(1+gM_\alpha)^{-1}L_\alpha]$ from which the assertion follows. □

5. Weakly coupled bound states

Throughout this section operators (resp. vectors) are denoted by boldface capital (resp. small) letters to emphasize their matrix (resp. vector) structure.

5.1. Two-channel Hamiltonians with matrix-valued potentials.

We consider the case where the potential is a 2×2 matrix-valued potential $\mathbf{V}(x)$ with measurable functions V_{ij} on $\mathbb{R}$ as entries. The Euclidean inner product and norm in $\mathbb{C}^2$ are denoted by $\langle\cdot,\cdot\rangle_{\mathbb{C}^2}$ and $\|\cdot\|_{\mathbb{C}^2}$, respectively.

Assumption 5.1.
(a) $\mathbf{V}(x)$ is symmetric, i.e. $\overline{V_{ji}} = V_{ij}$.
(b)
$$\int_{\mathbb{R}} (1+|x|^2)\,\|\mathbf{V}(x)\|_{\mathcal{B}(\mathbb{C}^2)}\,dx < \infty.$$
(c) The functions V_{ij} are real-valued.

5.1.1. *Two-channel Hamiltonian with a single threshold.* First we consider the Hamiltonian $\mathbf{H}(g) = \mathbf{H}_0 + g\mathbf{V}(x)$ in $L^2(\mathbb{R},\mathbb{C}^2)$, defined in Proposition 3.1 by means of forms. Formally, we may write the Hamiltonian as

$$\mathbf{H}(g) = \mathbf{H}_0 + g\mathbf{V} = \begin{pmatrix} -\frac{d^2}{dx^2} & 0 \\ 0 & -\frac{d^2}{dx^2} \end{pmatrix} + g\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \quad (5.1)$$

in $L^2(\mathbb{R},\mathbb{C}^2) = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. Under Assumption 5.1 we know that its essential spectrum equals the half-axis starting at the (threshold) point zero.

Define the matrices $\mathbf{S}$ and $\mathbf{T}$ by

$$\mathbf{S} = \int_{\mathbb{R}} \mathbf{V}(x)\,dx, \quad \mathbf{T}_0 = -\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{V}(x)|x-y|\mathbf{V}(y)\,dydx. \quad (5.2)$$

We establish the following result.

Theorem 5.2. *Let* $\mathbf{V}$ *obey Assumption 5.1(a)-(b) and let* $\mathbf{H}(g) = \mathbf{H}_0 + g\mathbf{V}(x)$ *be the self-adjoint Hamiltonian on* $L^2(\mathbb{R}, \mathbb{C}^2)$ *defined in Proposition 3.1 by means of forms.*

(i) Assume that the matrix $\mathbf{S}$*, defined in (5.2), has* n (≤ 2) *negative eigenvalues, denoted by* s_i*, with multiplicities* $\varkappa_i$*. Then, for a small enough* g*, the two-channel Hamiltonian* $\mathbf{H}(g)$ *has precisely* $\sum_{i=1}^{n} \varkappa_i$ *negative eigenvalues (taking into account multiplicity)* E_{ij} *satisfying the formulas*

$$(-E_{ij}(g))^{1/2} = -\frac{g}{2}s_i + \frac{g^2}{2}\langle \mathbf{v}_{ij}, \mathbf{T}_0\mathbf{v}_{ij}\rangle_{\mathbb{C}^2} + O(g^3), \qquad i = 1, \ldots, n, \quad j = 1, \ldots, \varkappa_i, \tag{5.3}$$

where $\mathbf{T}_0$ *is defined in (5.2) and* $\mathbf{v}_{ij}$ *are the eigenvectors corresponding to the eigenvalue* s_i *of* $\mathbf{S}$*.*

(ii) Suppose that $\mathbf{V}$ *obey Assumption 5.1(c) and that the matrix* $\mathbf{S}$ *has* n *non-positive eigenvalues, denoted by* s_i*, with multiplicities* $\varkappa_i$*. If the eigenvectors* $\mathbf{v}_{0j}$*,* $j = 1, \ldots, \varkappa_i$*, associated with the eigenvalue zero of* $\mathbf{S}$ *satisfy* $\langle \mathbf{v}_{0j}, \mathbf{T}_0\mathbf{v}_{0j}\rangle_{\mathbb{C}^2} \neq 0$ *then the conclusion of part (i) remains valid.*

Proof. According to the Birman-Schwinger relation formulated in Proposition 4.2, $E(g) < 0$ is an eigenvalue of $\mathbf{H}(g)$ if and only if -1 is an eigenvalue of $g\mathbf{K}_\alpha(\mathbf{V})$ with $\alpha^2 = -E(g)$. Furthermore, in view of Lemma 4.5(iii), the operator $g\mathbf{K}_\alpha(\mathbf{V})$ has eigenvalue -1 if and only if the same is true for $g(1 + g\mathbf{M}_\alpha)^{-1}\mathbf{L}_\alpha$. Now let us denote the (unknown) eigenvalues and eigenfunctions of $(1 + g\mathbf{M}_\alpha)^{-1}\mathbf{L}_\alpha$ by $\mu_k(g, \alpha)$ and $\mathbf{\Psi}_k(x; g, \alpha)$, respectively, viz.

$$(1 + g\mathbf{M}_\alpha)^{-1}\mathbf{L}_\alpha\mathbf{\Psi}_k(x; g, \alpha) = \mu_k(g, \alpha)\mathbf{\Psi}_k(x; g, \alpha). \tag{5.4}$$

Let $\mathbf{u}_k \in \mathbb{C}^2$ be a constant vector. We insert

$$\mathbf{\Psi}_k(x; g, \alpha) = \frac{1}{2\alpha}(1 + g\mathbf{M}_\alpha)^{-1}|\mathbf{V}(x)|^{1/2}\mathbf{u}_k$$

into (5.4) and obtain

$$\mathbf{R}_g\mathbf{u}_k = \mu_k(g, \alpha)\mathbf{u}_k, \tag{5.5}$$

where $\mathbf{R}_g$ is the matrix

$$\mathbf{R}_g = \frac{1}{2\alpha}\int_{\mathbb{R}} \mathbf{V}^{1/2}(x)[(1 + g\mathbf{M}_\alpha)^{-1}|\mathbf{V}|^{1/2}](x)\, dx.$$

Define $\mathbf{S}$ as in (5.2) and, moreover, define

$$\mathbf{T}(\alpha) = \int_{\mathbb{R}} \mathbf{V}^{1/2}(x)[\mathbf{M}_\alpha|\mathbf{V}|^{1/2}](x)\, dx.$$

Then we have that

$$\mathbf{R}_g = \frac{1}{2\alpha}\mathbf{S} - \frac{g}{2\alpha}\mathbf{T}(\alpha) + \mathrm{O}(g^2)$$

for small g.
(i) By assumption the matrix $\mathbf{S}$ has n negative eigenvalues, denoted by s_k. For simplicity we assume that the eigenvalues s_k are simple. The corresponding eigenvectors are denoted by $\mathbf{v}_k$. We apply the regular perturbation theory to the eigenvalue problem (5.5) and we find that

$$\mu_k(g,\alpha) = \frac{1}{2\alpha} s_k - \frac{g}{2\alpha}\langle \mathbf{v}_k, \mathbf{T}(\alpha)\mathbf{v}_k\rangle_{\mathbb{C}^2} + \mathrm{O}(g^2).$$

Define the matrix $\mathbf{T}_0$ as in (5.2) and

$$\mathbf{T}_1 = \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{V}(x)|x-y|^2\mathbf{V}(y)\,dxdy.$$

Then we have that $\mathbf{T}(\alpha) = \mathbf{T}_0 + \alpha\mathbf{T}_1 + \mathrm{O}(\alpha^2)$. In this way we find that the eigenvalues associated with the eigenvalue problem (5.5) are

$$\mu_k(g,\alpha) = \frac{1}{2\alpha} s_k - \frac{g}{2\alpha}\langle \mathbf{v}_k, \mathbf{T}_0\mathbf{v}_k\rangle_{\mathbb{C}^2} + \mathrm{O}(g^2).$$

Together with the comments following the proof of Proposition 4.2, the latter implies that the solution to (4.7) is

$$\alpha_g = -\frac{g}{2} s_k + \frac{g^2}{2}\langle \mathbf{v}_k, \mathbf{T}_0\mathbf{v}_k\rangle_{\mathbb{C}^2} + \mathrm{O}(g^2). \tag{5.6}$$

Clearly, (5.6) implies that each negative eigenvalue s_k of $\mathbf{S}$ gives rise to precisely one negative eigenvalue $E_k(g)$ of $\mathbf{H}(g)$ obeying the asymptotic formula

$$(-E_k(g))^{1/2} = -\frac{g}{2} s_k + \frac{g^2}{2}\langle \mathbf{v}_k, \mathbf{T}_0\mathbf{v}_k\rangle_{\mathbb{C}^2} + \mathrm{O}(g^3).$$

(ii) We investigate the situation where zero is an eigenvalue of $\mathbf{S}$ (as above we restrict ourselves to the case where zero is simple). Let $\mathbf{S}\mathbf{v}_0 = 0$ for some $\mathbf{v}_0 \neq 0$. Taylor's formula yields

$$\begin{aligned}
&\left\langle \mathbf{v}_0, \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{V}(x)\frac{e^{-\alpha|x-y|}}{2\alpha}\mathbf{V}(y)\,dx\,dy\mathbf{v}_0\right\rangle_{\mathbb{C}^2} \\
&\quad = \frac{1}{2\alpha}\langle \mathbf{v}_0, \mathbf{S}^2\mathbf{v}_0\rangle_{\mathbb{C}^2} - \frac{1}{2}\left\langle \mathbf{v}_0, \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{V}(x)|x-y|\mathbf{V}(y)\,dx\,dy\mathbf{v}_0\right\rangle_{\mathbb{C}^2} \\
&\quad +\alpha\left\langle \mathbf{v}_0, \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{V}(x)O(|x-y|^2)\mathbf{V}(y)\,dx\,dy\mathbf{v}_0\right\rangle_{\mathbb{C}^2}.
\end{aligned}$$

Since $\mathbf{S}\mathbf{v}_0 = 0$ by assumption, the first term equals zero. As $\alpha \downarrow 0$, we obtain that

$$\begin{aligned}
\langle \mathbf{v}_0, \mathbf{T}_0\mathbf{v}_0\rangle_{\mathbb{C}^2} &= \lim_{\alpha\downarrow 0}\left\langle \mathbf{v}_0, \int\int \mathbf{V}(x)\frac{e^{-\alpha|x-y|}}{2\alpha}\mathbf{V}(y)\,dxdy\,\mathbf{v}_0\right\rangle_{\mathbb{C}^2} \\
&= \lim_{\alpha\downarrow 0}\sum_{i,j,k}\int\int \frac{e^{-\alpha|x-y|}}{2\alpha} V_{ik}(x)V_{kj}(y)(\mathbf{v}_0)_i\overline{(\mathbf{v}_0)_j}\,dxdy. \qquad (5.7)
\end{aligned}$$

Let $\mathcal{F}$ denote the one-dimensional Fourier transform and let $(\mathcal{F}\mathbf{V})(\xi)$ denote the matrix with elements $(\mathcal{F}\mathbf{V})_{ij}(\xi) = (\mathcal{F}V_{ij})(\xi)$ satisfying $\mathcal{F}V_{ij} = \mathcal{F}V_{ji}$ because V is symmetric and V_{ij} are real-valued. Using the latter in conjunction with the Fourier transform of $(1/2\alpha)e^{-\alpha|x|}$, which equals $1/(\xi^2+\alpha^2)$, we find that

$$\begin{aligned} \text{r.h.s. of (5.7)} &= \lim_{\alpha\downarrow 0}\sum_{i,j,k}\int \frac{1}{\xi^2+\alpha^2}(\mathcal{F}V_{ik})(\xi)\overline{(\mathcal{F}V_{kj})(\xi)}(\mathbf{v}_0)_i\overline{(\mathbf{v}_0)_j}\,d\xi \\ &= \int \frac{1}{\xi^2}\langle \mathbf{v}_0, (\mathcal{F}\mathbf{V})^*(\xi)(\mathcal{F}\mathbf{V})(\xi)\mathbf{v}_0\rangle_{\mathbb{C}^2}\,d\xi \\ &= \int \frac{1}{\xi^2}\,\|(\mathcal{F}\mathbf{V})(\xi)\mathbf{v}_0\|^2_{\mathbb{C}^2}\,d\xi \geq 0, \end{aligned} \tag{5.8}$$

By assumption, $\langle \mathbf{v}_0, \mathbf{T}_0\mathbf{v}_0\rangle_{\mathbb{C}^2} \neq 0$ and therefore (5.8) implies that there is also a negative eigenvalue of $\mathbf{H}(g)$ associated with the eigenvalue zero of $\mathbf{S}$. □

Remark 5.3. The reasoning in the proof of Theorem 5.2(ii) requires that the entries V_{ij} in the potential $\mathbf{V}$ are real-valued. A substantial improvement would be to establish the result for complex-valued entries.

Example 5.4 (Square-well potentials). Let $\chi_{[0,1]}$ denote the characteristic function associated with the interval $[0,1]$. Choose the following entries of $\mathbf{V}$:

$$\begin{aligned} V_{11}(x) &= -5\chi_{[0,1]}(x), \quad V_{22}(x) = -3\chi_{[0,1]}(x), \\ V_{12}(x) &= V_{21}(x) = -3a\chi_{[0,1]}(x), \;\; a>0. \end{aligned}$$

Then the matrix $\mathbf{S}$ equals

$$\mathbf{S} = \begin{pmatrix} -5 & -3a \\ -3a & -3 \end{pmatrix}$$

and it has two real eigenvalues given by $-4\pm\sqrt{1+9a^2}$. Thus the following cases are possible: 1) If $a>\sqrt{5/3}$ there is exactly one negative eigenvalue of $\mathbf{S}$, namely $-4-\sqrt{1+9a^2}$. 2) If $a<\sqrt{5/3}$ there are two negative eigenvalues of $\mathbf{S}$, namely $-4\pm\sqrt{1+9a^2}$. 3) If $a=\sqrt{5/3}$ there are two nonpositive eigenvalues of $\mathbf{S}$, namely $-4-\sqrt{1+9a^2}$ and 0.

5.1.2. *Two-channel Hamiltonian with two thresholds.* As an example of a Hamiltonian with more than one threshold, we consider the one in (1.1), having thresholds at 0 and 1. Henceforth its free Hamiltonian is denoted by $\tilde{\mathbf{H}}_0$. The essential spectrum of $\tilde{\mathbf{H}}_0$ is the union of the half-axes starting at the thresholds, i.e. $\sigma_{ess}(\tilde{\mathbf{H}}_0) = [0,\infty)$. The resolvent of $\tilde{\mathbf{H}}_0$ is given by

$$(\tilde{\mathbf{H}}_0+\alpha^2)^{-1} = \begin{pmatrix} (-d^2/dx^2+\alpha^2)^{-1} & 0 \\ 0 & (-d^2/dx^2+\alpha^2+1)^{-1} \end{pmatrix}, \quad \alpha>0,$$

where the entries have the integral kernels

$$\frac{1}{2\alpha}e^{-\alpha|x-y|} \quad \text{and} \quad \frac{1}{2\sqrt{\alpha^2+1}}e^{-\sqrt{\alpha^2+1}|x-y|}.$$

It is easy to show that the assertions of Proposition 3.1 are valid if one replaces $\mathbf{H}_0$ by $\tilde{\mathbf{H}}_0$. In this way we obtain a self-adjoint realization of the formal Hamiltonian $\tilde{\mathbf{H}}_0 + g\mathbf{V}$ in $L^2(\mathbb{R}, \mathbb{C}^2)$. Moreover, the Birman-Schwinger relation in Proposition 4.2 holds for $\tilde{\mathbf{H}}(g) = \tilde{\mathbf{H}}_0 + g\mathbf{V}$.

Define the operators $\tilde{\mathbf{L}}_\alpha$ and $\tilde{\mathbf{M}}_\alpha$ by their "kernels"

$$\begin{aligned}
\tilde{\mathbf{L}}_\alpha(x,y) &= \frac{1}{2\alpha}\mathbf{V}(x)^{1/2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}|\mathbf{V}(y)|^{1/2},\\
\tilde{\mathbf{M}}_\alpha(x,y) &= \mathbf{V}(x)^{1/2}\begin{pmatrix}\frac{1}{2\alpha}[e^{-\alpha|x-y|}-1] & 0\\ 0 & \frac{e^{-\sqrt{\alpha^2+1}|x-y|}}{2\sqrt{\alpha^2+1}}\end{pmatrix}|\mathbf{V}(y)|^{1/2}.
\end{aligned}$$

Moreover, we introduce the operator $\tilde{\mathbf{M}}_0$ by its kernel

$$\tilde{\mathbf{M}}_0(x,y) = \mathbf{V}(x)^{1/2}\begin{pmatrix}-\frac{1}{2}|x-y| & 0\\ 0 & \frac{1}{2}e^{-|x-y|}\end{pmatrix}|\mathbf{V}(y)|^{1/2}.$$

By making a few obvious changes to the proof of Lemma 4.5 we obtain the following result.

Lemma 5.5. *Assume that* $\int_{\mathbb{R}}(1+|x|^2)\,\|\mathbf{V}(x)\|_{\mathcal{B}(\mathbb{C}^2)}\,dx < \infty$. *If* K_α, L_α, M_α *and* M_0 *in Lemma 4.5 are replaced by* $\tilde{\mathbf{K}}_\alpha$, $\tilde{\mathbf{L}}_\alpha$, $\tilde{\mathbf{M}}_\alpha$ *and* $\tilde{\mathbf{M}}_0$ *then the assertions (i)-(iii) of Lemma 4.5 are still valid.*

Define the matrices

$$\tilde{\mathbf{S}} = \begin{pmatrix}\int_{\mathbb{R}} V_{11}(x)\,dx & \int_{\mathbb{R}} V_{12}(x)\,dx\\ 0 & 0\end{pmatrix}, \tag{5.9}$$

$$\tilde{\mathbf{T}}_0 = \int_{\mathbb{R}}\int_{\mathbb{R}}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\mathbf{V}(x)\begin{pmatrix}-\frac{1}{2}|x-y| & 0\\ 0 & \frac{e^{-|x-y|}}{2}\end{pmatrix}\mathbf{V}(y)\,dxdy. \tag{5.10}$$

For the Hamiltonian $\tilde{\mathbf{H}}(g)$ we are able to derive an analogue of part (i) in Theorem 5.2.

Theorem 5.6. *Let* $\mathbf{V}$ *obey Assumption 5.1(a)-(c) and let* $\tilde{\mathbf{H}}(g) = \tilde{\mathbf{H}}_0 + g\mathbf{V}(x)$ *be the self-adjoint Hamiltonian on* $L^2(\mathbb{R}, \mathbb{C}^2)$ *defined in Proposition 3.1 by means of forms.*
Assume that the matrix $\tilde{\mathbf{S}}$*, defined in (5.9), has a negative eigenvalue* $\tilde{s}$ *(such an eigenvalue is simple if it exists). Then, for a small enough coupling constant* g*, the eigenvalue* $\tilde{s}$ *of* $\tilde{\mathbf{S}}$ *gives rise to exactly one negative eigenvalue* $\tilde{E}$ *of the two-channel Hamiltonian* $\tilde{\mathbf{H}}(g)$*. The negative eigenvalue* $\tilde{E}$ *satisfies the formula*

$$(-\tilde{E}(g))^{1/2} = -\frac{g}{2}\tilde{s} + \frac{g^2}{2}\langle\tilde{\mathbf{v}}, \tilde{\mathbf{T}}_0\tilde{\mathbf{v}}\rangle_{\mathbb{C}^2} + O(g^3), \tag{5.11}$$

where $\tilde{\mathbf{T}}_0$ *is defined in (5.10) and* $\tilde{\mathbf{v}}$ *is the eigenvector corresponding to the eigenvalue* $\tilde{s}$ *of* $\tilde{\mathbf{S}}$.

Proof. Imitating the proof of Theorem 5.2 we arrive at the eigenvalue problem

$$\tilde{\mathbf{R}}_g \mathbf{u}_k = \mu_k(g,\alpha)\mathbf{u}_k, \tag{5.12}$$

where $\tilde{\mathbf{R}}_g$ is the matrix

$$\tilde{\mathbf{R}}_g = \frac{1}{2\alpha}\int_{\mathbb{R}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{V}^{1/2}(x)[(1+g\tilde{\mathbf{M}}_\alpha)^{-1}|\mathbf{V}|^{1/2}](x)\,dx.$$

Define the matrix $\tilde{\mathbf{S}}$ as in (5.9) and, moreover, define

$$\tilde{\mathbf{T}}(\alpha) = \int_{\mathbb{R}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{V}^{1/2}(x)[\tilde{\mathbf{M}}_\alpha|\mathbf{V}|^{1/2}](x)\,dx.$$

Then we may write

$$\tilde{\mathbf{R}}_g = \frac{1}{2\alpha}\tilde{\mathbf{S}} - \frac{g}{2\alpha}\tilde{\mathbf{T}}(\alpha) + \mathrm{O}(g^2)$$

for small g. From here on everything depends on the possible eigenvalues of $\tilde{\mathbf{S}}$. Let

$$a = \int_{\mathbb{R}} V_{11}(x)\,dx.$$

The following cases may occur: I. If $a \neq 0$ then there are two subcases. I.1. If $a > 0$ then $\tilde{\mathbf{S}}$ has the eigenvalue zero and the positive eigenvalue a, each of multiplicity one. I.2. If $a < 0$ then $\tilde{\mathbf{S}}$ has the eigenvalue zero and the negative eigenvalue a, each having multiplicity one. II. If $a = 0$ then $\tilde{\mathbf{S}}$ has the eigenvalue zero with multiplicity one.

Repeating the reasoning in the first part of the proof of Theorem 5.2 we show that a negative eigenvalue of $\tilde{\mathbf{S}}$ (from I.2 it has multiplicity one) generates exactly one negative eigenvalue of $\tilde{\mathbf{H}}(g)$ provided g is small enough. □

Remark 5.7. The matrix $\tilde{\mathbf{S}}$ always has the eigenvalue zero. It remains an open problem to settle whether or not the latter gives rise to a negative eigenvalue of $\tilde{\mathbf{H}}(g)$ for a sufficiently small g.

5.2. **N-channel Hamiltonian with matrix-valued potentials.** In this section we consider the case where the potential is a $N \times N$ matrix-valued potential $\mathbf{V}(x)$ with measurable functions V_{ij} on $\mathbb{R}$ as entries.

Assumption 5.8.
(a) $\mathbf{V}(x)$ is symmetric, i.e. $\overline{V_{ji}} = V_{ij}$.
(b)

$$\int_{\mathbb{R}} (1+|x|^2)\,\|V(x)\|_{\mathcal{B}(\mathbb{C}^N)}\,dx < \infty.$$

(c) The functions V_{ij} are real-valued.

Define the matrices $\mathbf{S}$ and $\mathbf{T}_0$ by

$$\mathbf{S} = \int_{\mathbb{R}} \mathbf{V}(x)\,dx, \quad \mathbf{T}_0 = -\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{V}(x)|x-y|\mathbf{V}(y)\,dydx. \tag{5.13}$$

We have the following result.

Theorem 5.9. *Let* $\mathbf{V}$ *obey Assumption 5.8(a)-(b) and let* $\mathbf{H}(g) = \mathbf{H}_0 + g\mathbf{V}(x)$ *be the self-adjoint Hamiltonian on* $L^2(\mathbb{R}, \mathbb{C}^2)$ *defined in Proposition 3.1 by means of forms.*
(i) Assume that the matrix $\mathbf{S}$*, defined in (5.13), has n negative eigenvalues, denoted by* s_i*, with multiplicities* $\varkappa_i$*. Then, for a small enough* g*, the* N*-channel Hamiltonian* $\mathbf{H}(g)$ *has precisely* $\sum_{i=1}^{n} \varkappa_i$ *negative eigenvalues (taking into account multiplicity)* E_{ij} *satisfying the formulas*

$$(-E_{ij}(g))^{1/2} = -\frac{g}{2}s_i + \frac{g^2}{2}\langle \mathbf{v}_{ij}, \mathbf{T}_0\mathbf{v}_{ij}\rangle_{\mathbb{C}^N} + O(g^3), \qquad (5.14)$$
$$i = 1, \ldots, n, \quad j = 1, \ldots, \varkappa_i,$$

where $\mathbf{T}_0$ *is defined in (5.13) and* $\mathbf{v}_{ij}$ *are the eigenvectors corresponding to the eigenvalue* s_i *of* $\mathbf{S}$.
(ii) Suppose that $\mathbf{V}$ *obey Assumption 5.8(c) and that the matrix* $\mathbf{S}$ *has n non-positive eigenvalues, denoted by* s_i*, with multiplicities* $\varkappa_i$*. If the eigenvectors* $\mathbf{v}_{0j}$*,* $j = 1, \ldots, \varkappa_i$*, associated with the eigenvalue zero of* $\mathbf{S}$ *satisfy* $\langle \mathbf{v}_{0j}, \mathbf{T}_0\mathbf{v}_{0j}\rangle_{\mathbb{C}^N} \neq 0$ *then the conclusion of part (i) remains valid.*

Proof. The proof is a straightforward generalization of the proof of Theorem 5.2. □

Remark 5.10. One of the referees pointed out that Theorem 5.9 was proven in Šeba [29]. Therein, however, Theorem 3 is incorrect because the quantity $\int_{\mathbb{R}}(1/p^2)(\mathbf{a}_0, \mathcal{F}(V)^2(p)\mathbf{a}_0)_n\, dp$ (in Šeba's notation) is not necessarily different from zero. Moreover, the Birman-Schwinger relation (in the matrix-valued setting) is stated without proof.

6. Perturbation of embedded eigenvalues

For the sake of completeness we consider perturbation of two-channel diagonal Hamiltonians with one-dimensional Schrödinger operators as component Hamiltonians, having eigenvalues embedded in its continuous spectrum.

6.1. **Two-channel Hamiltonians.** Consider the formal expression

$$\mathbf{H}(g) = \mathbf{H}(0) + g\mathbf{V} = \begin{pmatrix} H_{11} & 0 \\ 0 & H_{22} \end{pmatrix} + g\begin{pmatrix} V_{11}(x) & V_{12}(x) \\ V_{21}(x) & V_{22}(x) \end{pmatrix} \qquad (6.1)$$

in $\mathcal{H} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, where

$$H_{11} = -\frac{d^2}{dx^2} + W_{11}(x) \text{ and } H_{22} = -\frac{d^2}{dx^2} + 1 + W_{22}(x). \qquad (6.2)$$

We impose the following assumptions on the potentials W_{jj}, $j = 1, 2$.

Assumption 6.1. Suppose that the real-valued, measurable functions W_{jj}, $j = 1, 2$, satisfy:
(a) $W_{jj} \neq 0$.
(b) The bound

$$|W_{jj}(x)| \leq C\,(1 + |x|^2)^{-1-\delta} \tag{6.3}$$

holds for some $C, \delta > 0$ and all x.
(c) $\int_{\mathbb{R}} W_{jj}(x)\, dx \leq 0$.
(d) W_{jj} extends to a function analytic in the sector

$$\mathcal{A}_{\alpha_0} = \{\, z \in \mathbb{C} \;:\; |\arg z| \leq \alpha_0 \,\}$$

for some $\alpha_0 > 0$. Moreover, the bound (6.3) holds in this sector.

Under Assumption 6.1(a)-(c) the operator $H_{11} = -\frac{d^2}{dx^2} + W_{11}(x)$ is self-adjoint in $L^2(\mathbb{R})$ and $\sigma(H_{11}) = \sigma_d(H_{11}) \cup \sigma_{ess}(H_{11}) = \sigma_d(H_{11}) \cup [0, \infty)$ with a non-empty discrete spectrum $\mu_1 < \mu_2 < \cdots < \mu_N < 0$, which is simple and finite [31]. The corresponding normalized eigenfunctions ϕ_n , $n = 1, 2, \ldots, N$, are exponentially decaying. The analyticity requirement in Assumption 6.1(d) is convenient to adopt for analyzing the resonance behaviour. Similarly, the operator $H_{22} = -\frac{d^2}{dx^2} + 1 + W_{22}(x)$ is self-adjoint in $L^2(\mathbb{R})$ and $\sigma(H_{22}) = \sigma_d(H_{22}) \cup \sigma_{ess}(H_{22}) = \sigma_d(H_{22}) \cup [1, \infty)$ with a non-empty discrete spectrum $\nu_1 < \nu_2 < \cdots < \nu_M < 1$ which is simple and finite. The corresponding normalized eigenfunctions χ_m , $m = 1, 2, \ldots, M$, are exponentially decaying.

Consider the unperturbed Hamiltonian $\mathbf{H}(0) = \operatorname{diag}(H_{11}, H_{22})$. Assumption 6.1 ensures that

$$\begin{aligned}\sigma_c(\mathbf{H}(0)) &= \sigma_{ess}(\mathbf{H}(0)) = \sigma_{ess}(H_{11}) \cup \sigma_{ess}(H_{22}) \\ &= [0, \infty) \cup [1, \infty) = [0, \infty).\end{aligned}$$

Thus, the continuous spectrum of $\mathbf{H}(0)$ is the union of the two half-lines starting at 0 and 1. This motivates the definition of the threshold set $\mathsf{T} = \{\, 0, 1 \,\}$. Furthermore, $\sigma_p(\mathbf{H}(0)) = \sigma_p(H_{11}) \cup \sigma_p(H_{22})$. Among this (finite) set of eigenvalues, a (finite) subset is isolated or situated at the threshold 0, while the rest satisfying the condition $0 < \nu_m < 1$ is embedded in the continuous spectrum of $\mathbf{H}(0)$. For the sake of simplicity we make the following assumption.

Assumption 6.2. Suppose that none of the embedded eigenvalues ν_m of $\mathbf{H}(0)$ coincide with the threshold 0.

We impose the following conditions on the components of the perturbation $\mathbf{V}$.

Assumption 6.3. Suppose that the real-valued, measurable functions V_{ij}, $i, j = 1, 2$, satisfy:
(a) The bound

$$|V_{ij}(x)| \leq C\,(1 + |x|^2)^{-1-\delta} \tag{6.4}$$

holds for some $C, \delta > 0$ and all x.
(b) V_{ij} extends to a function analytic in the sector $\mathcal{A}_{\alpha_0}$ (see Assumption 6.1(d)) for some $\alpha_0 > 0$. Moreover, the bound (6.4) holds in this sector.

6.2. **Complex dilation.** We use a complex deformation. For θ real define S_θ on $L^2(\mathbb{R})$ by the unitary operator

$$(S_\theta \psi) = e^{\theta/2}\psi(e^\theta x), \ \ \psi \in L^2(\mathbb{R}). \tag{6.5}$$

S_θ is a one-parameter unitary group on $L^2(\mathbb{R})$. It is easy to see that S_θ leave $\mathcal{D}(-d^2/dx^2) = H^2(\mathbb{R})$ invariant and that

$$H_{11,\theta} := S_\theta H_{11} S_\theta^{-1} = -e^{-2\theta}\frac{d^2}{dx^2} + W_{11,\theta}(x) = -e^{-2\theta}\frac{d^2}{dx^2} + W_{11}(e^\theta x),$$

Let $\mathcal{A}_0 = \{\, \theta \,:\, |\mathrm{Im}\,\theta| \le \min\{\alpha_0, \pi/4\,\}$ (cf. Assumption 6.1(d)). Under Assumption 6.1, $H_{11,\theta}$ obviously has a continuation to a type (A) family of m-sectorial operators analytic in the sense of Kato [13] for $\theta \in \mathcal{A}_0$. Likewise,

$$H_{22,\theta} := S_\theta H_{22} S_\theta^{-1} = -e^{-2\theta}\frac{d^2}{dx^2} + 1 + W_{22,\theta}(x) = -e^{-2\theta}\frac{d^2}{dx^2} + 1 + W_{22}(e^\theta x)$$

has a continuation to a type (A) analytic family of operators on $\mathcal{A}_0$. From standard Aguilar-Combes theory [1] we determine the spectra of $H_{11,\theta}$ and $H_{22,\theta}$:

$$\sigma(H_{11,\theta}) = \{\mu_1, \mu_2, \dots, \mu_N\} \cup \{e^{-2\theta}\lambda \,:\, \lambda \in [0,\infty)\,\},$$
$$\sigma(H_{22,\theta}) = \{\nu_1, \nu_2, \dots, \nu_M\} \cup \{e^{-2\theta}\lambda + 1 \,:\, \lambda \in [0,\infty)\,\}.$$

Having extended S_θ in (6.5) analytically to $\mathcal{A}_0$ we may define

$$\mathbf{S}_\theta \Psi = \begin{pmatrix} S_\theta & 0 \\ 0 & S_\theta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \Psi \in \mathcal{H}.$$

Due to its diagonal structure, the Hamiltonian

$$\mathbf{H}_\theta(0) := \mathbf{S}_\theta \mathbf{H}(0) \mathbf{S}_\theta^{-1} = \begin{pmatrix} H_{11,\theta} & 0 \\ 0 & H_{22,\theta} \end{pmatrix}$$

has a continuation to a type (A) analytic family of operators in the sector $\mathcal{A}_0$. Furthermore,

$$\begin{aligned} \sigma(\mathbf{H}_\theta(0)) &= \sigma(H_{11,\theta}) \cup \sigma(H_{22,\theta}) \\ &= \{\mu_1, \mu_2, \dots, \mu_N\} \cup \{\nu_1, \nu_2, \dots, \nu_M\} \\ &\quad \cup \{e^{-2\theta}\lambda \,:\, \lambda \in [0,\infty)\,\} \cup \{e^{-2\theta}\lambda + 1 \,:\, \lambda \in [0,\infty)\,\}. \end{aligned}$$

In particular, the eigenvalues embedded in $\sigma(\mathbf{H}(0))$ are discrete eigenvalues of $\mathbf{H}_\theta(0)$ for θ nonreal.

Henceforth E_0 denotes any of the embedded eigenvalues ν_m of $\mathbf{H}(0)$. Let $\mathbf{R}_0(\theta;\zeta)$ denote the resolvent of $\mathbf{H}_\theta(0)$. Since E_0 is an isolated eigenvalue of $\mathbf{H}_\theta(0)$, we may choose a contour Γ around E_0 such that Γ belongs to the resolvent set of $\mathbf{H}_\theta(0)$ and E_0 is the only eigenvalue of $\mathbf{H}_\theta(0)$ contained inside of Γ. Moreover, let $\mathbf{P}_\theta$ denote the eigenprojection associated with the eigenvalue E_0 and put

$$\mathbf{S}_\theta^{(p)} := \frac{1}{2\pi i}\int_\Gamma \frac{\mathbf{R}_0(\theta;\zeta)}{(E_0-\zeta)^p}\,d\zeta, \quad p \geq 1. \tag{6.6}$$

Then $\mathbf{P}_\theta = -\mathbf{S}_\theta^{(0)}$ and $\widehat{\mathbf{R}}_0(\theta;\zeta) := \mathbf{S}_\theta^{(1)}$ is the reduced resolvent of $\mathbf{H}_\theta(0)$ at the point ζ. Define

$$\mathbf{V}_\theta = \mathbf{S}_\theta \mathbf{V} \mathbf{S}_\theta^{-1} = \begin{pmatrix} V_{11,\theta} & V_{12,\theta} \\ V_{21,\theta} & V_{22,\theta} \end{pmatrix} \text{ with } V_{ij,\theta}(x) = V_{ij}(e^\theta x).$$

Then we have the following result.

Lemma 6.4. *Let Assumption 6.1 and Assumption 6.3 hold. Let Γ be the contour described above and let $\mathbf{S}_\theta^{(p)}$ be defined in (6.6).*
(i) If $\operatorname{Im}\theta \in (0,\alpha_0)$ then there exists a constant $C_\theta > 0$ such that

$$\max_{\zeta\in\Gamma} \|g\mathbf{V}_\theta \mathbf{R}_0(\theta;\zeta)\| \leq C_\theta |g|. \tag{6.7}$$

If ζ is replaced by $\zeta_0 = \min\{\mu_1,\nu_1\} - 1$ then the constant in (6.7) is independent of θ and the estimate holds for all $|\operatorname{Im}\theta| < \alpha_0$.
(ii) For $p \geq 0$ there exists a constant $C_\theta > 0$ such that

$$\|g\mathbf{V}_\theta \mathbf{S}_\theta^{(p)}\| \leq C_\theta \frac{|\Gamma|}{2\pi}\frac{|g|}{[\operatorname{dist}(E_0,\Gamma)]^p}.$$

Proof. The contour Γ is by assumption contained in the resolvent of $\mathbf{H}_\theta(0)$. Since $\mathbf{R}_0(\theta;\cdot)$ is bounded and continuous and Γ is compact, there exists a constant $\tilde{C}_\theta$ such that $\max_{\zeta\in\Gamma}\|\mathbf{R}_0(\theta;\zeta)\| \leq \tilde{C}_\theta$.

Thus, $\max_{\zeta\in\Gamma}\|\mathbf{V}_\theta\mathbf{R}_0(\theta;\zeta)\| \leq \tilde{C}_\theta C_{\mathbf{V}}$, where $C_{\mathbf{V}}$ denotes a bound on the norm of $\mathbf{V}_\theta$, which is independent of θ by Assumption 6.3(b). This shows the first claim. Moreover, ζ_0 is to the left of the numerical range $\Theta(\mathbf{H}_\theta(0))$ of $\mathbf{H}_\theta(0)$ at the unit distance. Hence $\|\mathbf{R}_0(\theta;\zeta_0)\| = 1/[\operatorname{dist}(\zeta_0,\Theta(\mathbf{H}_\theta(0)))] = 1$. Therefore the constant $\tilde{C}_\theta$ in the above estimate may be replaced by 1. This verifies (i). The assertion (ii) follows immediately. □

Hence, provided g is small enough, it follows from Lemma 6.4(i) that $g\mathbf{V}_\theta$ is $\mathbf{H}_\theta(0)$-compact. The latter, in conjunction with [27, Lemma 1, page 16], implies that the perturbed operators $\mathbf{H}_\theta(g) = \mathbf{H}_\theta(0) + g\mathbf{V}_\theta$ are a type (A) analytic family of operators for $\theta \in \mathcal{A}_0$ and suitable small g. Since E_0 is an isolated, simple eigenvalue of $\mathbf{H}_\theta(0)$, the analyticity of $\mathbf{H}_\theta(g)$ allows us to apply regular perturbation theory. The next section is devoted to this task.

6.3. **Perturbation series and Fermi's golden rule.** Following Kato [13, Sections II.2 and VII.1] and using Lemma 6.4 we infer that $\mathbf{H}_\theta(g)$ has an eigenvalue near E_0 given by a convergent power series in g. The convergent series is given by

$$E(g) = E_0 + \sum_{j=1}^{\infty} E_j(g), \tag{6.8}$$

where

$$E_j(g) = \sum_{p_1+\cdots+p_j=j-1} \frac{(-1)^j}{j} \mathrm{tr} \prod_{i=1}^{j} g\mathbf{V}_\theta \mathbf{S}_\theta^{(p_i)} \tag{6.9}$$

In view of Lemma 6.4(ii) we see that $E_j(g) = O(g^j)$.

Let us compute the lowest-order terms of the series (6.8). Since $\mathrm{Rank}\,\mathbf{P}_\theta = 1$, $\mathbf{P}_\theta$ can be represented as

$$\mathbf{P}_\theta = \left\langle \cdot, \begin{pmatrix} 0 \\ \chi_m^\theta \end{pmatrix} \right\rangle \begin{pmatrix} 0 \\ \chi_m^\theta \end{pmatrix}$$

with $\chi_m^\theta := S_\theta \chi_m$, where χ_m is the eigenfunction associated with the eigenvalue E_0 of H_{22}. Indeed, $H_{22,\theta} = \chi_m^\theta = S_\theta H_{22} S_\theta^{-1} S_\theta \chi_m = E_0 S_\theta \chi_m = E_0 \chi_m^\theta$ and, consequently,

$$\mathbf{H}_\theta(0) \begin{pmatrix} 0 \\ \chi_m^\theta \end{pmatrix} = \begin{pmatrix} H_{11,\theta} & 0 \\ 0 & H_{22,\theta} \end{pmatrix} \begin{pmatrix} 0 \\ \chi_m^\theta \end{pmatrix} = E_0 \begin{pmatrix} 0 \\ \chi_m^\theta \end{pmatrix}.$$

We compute $E_1(g)$:

$$\begin{aligned} E_1(g) &= \mathrm{tr}\,(g\mathbf{V}_\theta \mathbf{P}_\theta) = g \left\langle \begin{pmatrix} 0 \\ \chi_m^\theta \end{pmatrix}, \mathbf{V}_\theta \begin{pmatrix} 0 \\ \chi_m^\theta \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &= g \left\langle \begin{pmatrix} 0 \\ \chi_m \end{pmatrix}, \mathbf{V} \begin{pmatrix} 0 \\ \chi_m \end{pmatrix} \right\rangle_{\mathcal{H}} = g \langle \chi_m, V_{22} \chi_m \rangle_{L^2(\mathbb{R})}. \end{aligned} \tag{6.10}$$

We see that the first-order term is real and does not contribute to the resonance width. Next, we consider $E_2(g)$. According to (6.9),

$$E_2(g) = -g^2 \mathrm{tr}\,(\mathbf{P}_0 \mathbf{V}_\theta \widehat{\mathbf{R}}_0(\theta; E_0 - i0) \mathbf{V}_\theta \mathbf{P}_0).$$

Due to the standard constancy-in-θ argument (see e.g. [27, pages 55-56]), we may take the limit $\mathrm{Im}\,\theta \to 0$ and in this way we arrive at

$$\begin{aligned} E_2(g) &= -g^2 \mathrm{tr}\,(\mathbf{P}_0 \mathbf{V} \widehat{\mathbf{R}}_0(0; E_0 - i0) \mathbf{V} \mathbf{P}_0) \\ &= -g^2 \left\langle \begin{pmatrix} 0 \\ \chi_m \end{pmatrix}, \mathbf{V} \widehat{\mathbf{R}}_0(0; E_0 - i0) \mathbf{V} \begin{pmatrix} 0 \\ \chi_m \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &= -g^2 \langle \chi_m, V_{21} [(H_{11} - E_0 + i0)^{-1}]\widehat{\ } V_{12} \chi_m \rangle_{L^2} \\ &\quad -g^2 \langle \chi_m, V_{22} [(H_{22} - E_0 + i0)^{-1}]\widehat{\ } V_{22} \chi_m \rangle_{L^2} \\ &= -g^2 \langle V_{12} \chi_m, [(H_{11} - E_0 + i0)^{-1}]\widehat{\ } V_{12} \chi_m \rangle_{L^2} \\ &\quad -g^2 \langle V_{22} \chi_m, [(H_{22} - E_0 + i0)^{-1}]\widehat{\ } V_{22} \chi_m \rangle_{L^2}, \end{aligned} \tag{6.11}$$

where the notation $[(H_{jj}-E_0+i0)^{-1}]\hat{}$ refers to the reduced resolvent of H_{jj}, $j=1,2$.

We restrict our focus to the imaginary part of $E_2(g)$, which determines the resonance width to leading order. For this purpose we introduce

$$R_k = \left((-d^2/dx^2 + W_{kk}(x) - E_0 + \mathfrak{t}_k - i0)^{-1}\right)\hat{}\,, \quad k=1,2,$$

where $\mathfrak{t}_1 = 0$ and $\mathfrak{t}_2 = 1$ are the thresholds. Clearly,

$$\operatorname{Im} E_2(g) = -g^2 \sum_{k=1}^{2} \langle V_{k2}\chi_m (\operatorname{Im} R_k) V_{k2}\chi_m\rangle_{L^2(\mathbb{R})}. \tag{6.12}$$

Now, for $E>0$, the resolvent equation yields that

$$\begin{aligned}\operatorname{Im}(-d^2/dx^2 + W_{kk}(x) - E - i0)^{-1} &= \\ = t_k(E+i0)^* \operatorname{Im}(-d^2/dx^2 - E - i0)^{-1} t_k(E+i0),\end{aligned} \tag{6.13}$$

where

$$t_k(\zeta) = \left[I + |W_{kk}|^{1/2}(-\partial_x^2 - \zeta)^{-1}|W_{kk}|^{1/2}\operatorname{Sgn}(W_{kk})\right]^{-1}.$$

The quantities $t_k(E+i0)$ are well-defined in view of Assumption 6.1. Furthermore, again for $E>0$,

$$\operatorname{Im}(-d^2/dx^2 - E - i0)^{-1} = \frac{\pi}{2\sqrt{E}} \sum_{\sigma=\pm} (\gamma_E^\sigma)^* \gamma_E^\sigma, \tag{6.14}$$

where $\gamma_E^\sigma : H^1 \to \mathbb{C}$ is the trace operator which acts on the first Sobolev space $H^1(\mathbb{R})$ as follows (see, e.g., [18, Section IV.1]),

$$\gamma_E^\sigma \phi := \widehat{\phi}(\sigma\sqrt{E}), \;\; \sigma = \pm, \;\; E>0.$$

Here, as usual, $\widehat{\phi}$ denotes the Fourier transform of ϕ. Using (6.13) and (6.14) we can rewrite the expression (6.12) in the following way,

$$\begin{aligned}\operatorname{Im} E_2(g) &= -g^2 \sum_{k=1}^{2} \langle V_{k2}\chi_m, (\operatorname{Im} R_k) V_{k2}\chi_m\rangle_{L^2} \\ &= -g^2 \sum_{k=1}^{2} \langle V_{k2}\chi_m, t_k(E_0 - \mathfrak{t}_k + i0)^* \\ &\times \operatorname{Im}(-d^2/dx^2 - E_0 + \mathfrak{t}_k - i0)^{-1} t_k(E_0 - \mathfrak{t}_k + i0) V_{k2}\chi_m\rangle_{L^2}\end{aligned}$$

$$= -g^2 \sum_{k=1}^{2} \langle t_k(E_0 - \mathfrak{t}_k + i0)V_{k2}\chi_m, \mathrm{Im}\,(-d^2/dx^2 - E_0 + \mathfrak{t}_k - i0)^{-1}$$
$$\times t_k(E_0 - \mathfrak{t}_k + i0)V_{k2}\chi_m\rangle_{L^2}$$
$$= -g^2 \sum_{k=1}^{2} \sum_{\sigma=\pm} \frac{\pi}{2\sqrt{E_0 - \mathfrak{t}_k}} \langle \gamma^{\sigma}_{E_0-\mathfrak{t}_k} t_k(E_0 - \mathfrak{t}_k + i0)V_{k2}\chi_m, \gamma^{\sigma}_{E_0-\mathfrak{t}_k}$$
$$\times t_k(E_0 - \mathfrak{t}_k + i0)V_{k2}\chi_m\rangle_{\mathbb{C}}$$
$$= -g^2 \sum_{k=1}^{2} \sum_{\sigma=\pm} \frac{\pi}{2\sqrt{E_0 - \mathfrak{t}_k}} |\gamma^{\sigma}_{E_0-\mathfrak{t}_k} t_k(E_0 - \mathfrak{t}_k + i0)V_{k2}\chi_m|^2. \tag{6.15}$$

In this way we have established the following result.

Theorem 6.5. *Let Assumption 6.1 and Assumption 6.3 hold. Let ν_m be a simple eigenvalue of the operator H_{22} defined in (6.2) giving rise to the eigenvalue $E_0 = \nu_m$ embedded in the continuous spectrum of $\mathbf{H}(0)$. Let E_0 satisfy Assumption 6.2. For a small enough coupling constant g, the eigenvalue E_0 of $\mathbf{H}(0)$ turns into a resonance, i.e. $E_0 \notin \sigma(\mathbf{H}(g))$. The coordinates of its corresponding pole is given by (6.9)-(6.11). In particular, Fermi's golden rule takes the explicit form (6.15).*

Remark 6.6. If Assumption 6.2 is not fulfilled, i.e. we have an eigenvalue of H_{22} at the threshold point 0 of the continuous spectrum of $\mathbf{H}(0)$, complex dilation breaks down. An insight into this problem was established in [12]. For abstract Hamiltonians $\mathbf{H}(g)$ having the structure found in (6.1), it was shown that under small off-diagonal perturbations this eigenvalue never moves into the continuous spectrum.

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