IDEALS OF SMOOTH FUNCTIONS AND RESIDUE CURRENTS

MATS ANDERSSON

ABSTRACT. Let $f = (f_1, \dots, f_m)$ be a holomorphic mapping in a neighborhood of the origin in $\mathbb{C}^n$. We find sufficient condition, in terms of residue currents, for a smooth function to belong to the ideal in C^k generated by f. If f is a complete intersection the condition is essentially necessary. More generally we give sufficient condition for an element of class C^k in the Koszul complex induced by f to be exact. For the proofs we introduce explicit homotopy formulas for the Koszul complex induced by f.

1. INTRODUCTION

Let $f = (f_1, \dots, f_m)$ be a nontrivial holomorphic mapping at $0 \in \mathbb{C}^n$. It is wellknown, [10] and [11], that if f is a complete intersection, then a holomorphic function ϕ belongs to the ideal $(f) = (f_1, \dots, f_m)$ if and only if $\phi T^f = 0$, where T^f is the Coleff-Herrera current

$$T^f = \Big[\bar\partial \frac{1}{f_m} \wedge \dots \wedge \bar\partial \frac{1}{f_1}\Big].$$

Consider now the ideal $(f)_{\mathcal{E}}$ of smooth functions generated by f. If $\phi = \sum_j \psi_j f_j$, and $\partial^\alpha_{\bar z} = \partial^\alpha / \partial \bar z^\alpha$, then

$$\partial^\alpha_{\bar z}\phi = \sum_j (\partial^\alpha_{\bar z}\psi_j) f_j,$$

so if f is a complete intersection it follows that

$$(\partial^\alpha_{\bar z}\phi) T^f = 0 \tag{1.1}$$

for all multiindices α. Also the converse is true.

Theorem 1.1. *Let f be a complete intersection. A function $\phi \in \mathcal{E}$ is in the ideal $(f)_{\mathcal{E}}$ at $0 \in \mathbb{C}^n$ if and only if (1.1) holds for all α.*

This result follows from the theory of D-modules and Kashiwara's conjugation functor, using the fact that T^f is a regular holonomic current, [6] and [7]. We provide an explicit proof of Theorem 1.1 below,

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but our main interest is focused on conditions for lower regularity. For a complete intersection our result is

Theorem 1.2. *Let f be a complete intersection and let M be the order of the current T^f. There is a number c_n, only depending on n, such that if $\phi \in C^{c_n+2M+k}$ and (1.1) holds for $|\alpha| \le c_n + M + k$, then there are $u_j \in C^k$ such that $\sum f_j u_j = \phi$.*

The crucial point is the number of conditions (1.1); the extra differentiability assumption on ϕ is to ensure that (1.1) makes sense.

Remark 1. Assume that M is the order of the current T^f, and that $\phi = \sum f_j u_j$ for some $u_j \in C^{k+M}$. Then $\phi \in C^{k+M}$ and (1.1) holds for all $|\alpha| \le k$. Thus, asymtotically in k, Theorem 1.2 is sharp. □

The theorem is proved by integral formulas, but let us indicate a direct proof in the case when $m = 1$, i.e., when we have got only one generator f. If $[1/f]$ is the wellknown principal value current, see, e.g., [6] or [2], then $f[1/f] = 1$ and $f\bar\partial[1/f] = 0$. If $u = \phi[1/f]$, then the hypothesis about (1.1) implies that

$$\partial_{\bar z}^\alpha u = (\partial_{\bar z}^\alpha \phi)\left[\frac{1}{f}\right]$$

is a distribution of order at most M for $|\alpha| \le c_n + k + M$. If c_n is appropriately chosen, we can conclude that u is in C^k. (For instance, the assumption on $\bar\partial[1/f]$ implies that it belongs (locally) to some Sobolev space $W^{-M-c_n'}$; moreover if $\bar\partial\psi \in W^r$ then $\psi \in W^{r+1}$, and thus we obtain that $u \in W^{k+c_n''}$, which implies that $u \in C^k$.)

In [1] we introduced, for any nontrivial mapping f, a current R^f which coincides with the Coleff-Herrera current in the complete intersection case, and such that ϕ belongs to the ideal if ϕ annihilates R^f. To describe this current, let X be a neighborhood of $0 \in \mathbb{C}^n$ and let $E \to X$ be a trivial vector bundle with (holomorphic) frame $e_1, \ldots, e_m$ and let E^* be its dual bundle and $e_1^*, \ldots, e_m^*$ the dual frame. We consider f as the section $f = \sum f_j e_j^*$ to E^* and let δ_f denote interior multiplication with $2\pi i f$ so that $\delta_f\colon \mathcal{E}(X, \Lambda^{k+1}E) \to \mathcal{E}(X, \Lambda^k E)$, and $\delta_f^2 = 0$. The more general problem can be formulated: *Given $\phi \in \mathcal{O}(X, \Lambda^k E)$ such that $\delta_f\phi = 0$, find $\psi \in \mathcal{O}(X, \Lambda^{k+1}E)$ such that $\delta_f\psi = \phi$.* In case $k = 0$ this just means to solve $2\pi i \sum \psi_j f_j = \phi$.

Let $\sigma = \sum_1^m \bar f_j e_j / 2\pi i |f|^2$ outside

$$Y = \{z \in X;\ f(z) = 0\},$$

so that $\delta_f \sigma = 1$ there. We consider the exterior algebra of $E \oplus T^*(X)$, and therefore δ_f and $\bar\partial$ anticommute, and if

$$\nabla_f = \delta_f - \bar\partial$$

it follows that $\nabla_f^2 = 0$. If

$$u = \frac{\sigma}{\nabla_f \sigma} = \frac{\sigma}{1 - \bar\partial\sigma} = \sigma + \sigma \wedge (\bar\partial\sigma) + \sigma \wedge (\bar\partial\sigma)^2 + \ldots + \sigma \wedge (\bar\partial\sigma)^{m-1},$$

then $\nabla_f u = 1$ in $X \setminus Y$, since $\nabla_f^2 = 0$. The main result in [2] is

Theorem 1.3. *There is a current extension U of u across Y such that*

$$\nabla_f U = 1 - R^f, \tag{1.2}$$

where

$$R^f = R^f_{p,p} + \cdots + R^f_{m,m},$$

$R^f_{k,k}$ is a $(0,k)$-current with values in $\Lambda^k E$, and p is the codimension of Y.

Thus $R^f = R^f_{m,m}$ if Y is a complete intersection.

Theorem 1.4. *If f is a complete intersection, then*

$$R^f = T^f \wedge e_1 \wedge \ldots \wedge e_m.$$

This was first proved in [12]; a quite simple proof appeared in [2]. In [2] we also proved

Theorem 1.5. *Let ϕ be holomorphic in ΛE and $\delta_f \phi = 0$. If $\phi \wedge R^f = 0$, then (locally) ϕ is δ_f-exact.*

Remark 2. The condition $\phi \wedge R^f = 0$ is not necessary. More precisely it is shown in [2] that ϕ is δ_f-exact if and only if there is a smooth form w in a neighborhood of Y such that $\nabla_f(w \wedge R^f) = \phi \wedge R^f$, see Corollary 2.6 below. □

For a general holomorphic mapping f we have the following result.

Theorem 1.6. *Let f be any holomorphic mapping. Suppose that $\phi \in \mathcal{E}(X, \Lambda^r E)$ and that $\delta_f \phi = 0$. If*

$$(\partial_{\bar z}^\alpha \phi) \wedge R^f = 0 \tag{1.3}$$

for all α, then $\phi = \delta_f \psi$ for some $\psi \in \mathcal{E}(X, \Lambda^{r+1} E)$.

Let M be the order of R^f and U. There is an integer c_n only depending on n such that if $\phi \in C^{c_n+2M+k}(X, \Lambda^r E)$, $\delta_f \phi = 0$, and (1.3) holds for $|\alpha| \le c_n + M + k$, then $\phi = \delta_f \psi$ for some $\psi \in \mathbb{C}^k(X, \Lambda^{r+1} E)$.

Again (probably) the first part follows from Theorem 1.5 and Kashiwara's theorem. If the degree r of ϕ is larger than $m - codim(Y)$, then (1.3) is empty. In the other cases it is possible to be more precise and sharpen the statements by taking into account the degree of ϕ and the various orders of the components of U and R^f but we leave this to the interested reader.

In view of Theorem 1.4 it is clear that Theorem 1.6 implies (the if-part of) Theorem 1.1, since the order of U does not exceed the order of R^f (at least when f is a complete intersection).

The proof of Theorem 1.4 is based on an integral formula that represents the desired solution ψ. We first make a new construction of explicit integral operators T and S such that any holomorphic ϕ with values in ΛE can be written

$$\phi = \delta_f T\phi + T(\delta_f \phi) + S\phi, \tag{1.4}$$

where $T\phi$ and $S\phi$ are holomorphic in a neighborhood of $0 \in X$, and $S\phi$ only depends on $R^f \wedge \phi$. From this representation Theorem 1.5 immediately follows. We then elaborate the construction to provide a proof of Theorem 1.6. The idea is to consider a neighborhood $\tilde{X}$ of $X \sim \{(z, \bar{z})\}$ in $\mathbb{C}^{2n}$ and apply the formulas in $\tilde{X}$ to an almost holomorphic extension of ϕ to $\tilde{X}$.

Decompositions like (1.4) first occured in [3] for f with a regular singularity and in [11] and [4] for the case of a complete intersection, and functions ϕ. In (1.4), f can be any holomorphic mapping and ϕ taking values in $\Lambda^r E$.

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2. Explicit homotopy operators for the δ_f-complex

We first recall the construction of weighted representation formulas for holomorphic functions from [1]. Let X be an open set in $\mathbb{C}^n$, and let

$$\mathcal{L}^r(X) = \bigoplus_k \mathcal{E}_{k,k+r}(X).$$

Moreover, let $\delta_{\zeta-z}$ denote interior multiplication with the vector field

$$2\pi i \sum (\zeta_j - z_j) \frac{\partial}{\partial \zeta_j},$$

and let $\nabla_{\zeta-z} = \delta_{\zeta-z} - \bar{\partial}_\zeta$. Then $\nabla_{\zeta-z}$ maps $\mathcal{L}^r(X)$ into $\mathcal{L}^{r+1}(X)$ and $\nabla^2_{\zeta-z} = 0$. Moreover, the usual wedge product induces mappings

$$\mathcal{L}^r(X) \times \mathcal{L}^{r'}(X) \to \mathcal{L}^{r+r'}(X),$$

and $\nabla_{\zeta-z}$ is an antiderivation with respect to this product. We will use the following representation formula from [1].

Proposition 2.1. *Assume that $g = g_{0,0} + \ldots + g_{n,n} \in \mathcal{L}^0(X)$ is smooth and with compact support, z is a fixed point, $\nabla_{\zeta-z} g = 0$, and $g_{0,0}(z) = 1$. Then*

$$\phi(z) = \int g\phi = \int g_{n,n}\phi$$

for each function ϕ that is holomorphic in X.

It is possible to find such a g that depends holomorphically on z, locally.

Example 1. Let χ be a cutoff function in X which is 1 in a neighborhood of 0, and let s be any smooth $(1,0)$-form such that $\delta_\zeta s \neq 0$ on the support of $\bar{\partial}\chi$. Then also $\delta_{\zeta-z} s \neq 0$ for z in a small neighborhood of 0 and therefore $v = s/\nabla_{\zeta-z} s$ will be holomorphic in z in this neighborhood. Moreover, $\nabla_{\zeta-z} v = 1$ on the support of $\bar{\partial}\chi$, so we can take $g = \chi - \bar{\partial}\chi \wedge v$.

A more fancy choice (for z in the unit ball) is

$$g = \Big(1 - \nabla_{\zeta - z}\frac{\bar\zeta\cdot d\zeta}{1-\bar\zeta\cdot z}\Big)^{\ell+n} = \Big(\frac{1-|\zeta|^2}{1-\bar\zeta\cdot z} + \bar\partial\frac{\bar\zeta\cdot d\zeta}{1-\bar\zeta\cdot z}\Big)^{\ell+n}$$

for integers ℓ. It is $\mathcal{O}(|1-|\zeta|^2)^\ell)$ near the boundary and therefore at least of class $C^{\ell-1}$; this will do in this paper if ℓ is large enough. $\square$

Let f be a holomorphic mapping in X and consider f as a section to the dual bundle E^* of the (trivial) bundle $E \to X$. Moreover, let $\tilde E$ and $\tilde E^*$ denote copies of E and E^*, respectively, and let $\tilde f$ denote the corresponding section to $\tilde E^*$. Let $F(\zeta, z) = f(\zeta) + \tilde f(z)$, thinking of z as a parameter and ζ as a variable. Then $\delta_F = \delta_f + \delta_{\tilde f}$ is interior multiplication with $2\pi i F$ on $\Lambda(E\oplus\tilde E)$. One can find forms $h_j(\zeta,z)$ in $\mathcal{L}^0(X)$ (Hefer forms) such that

$$\nabla_{\zeta-z}h_j = f_j(\zeta) - f_j(z),$$

where $\nabla_{\zeta-z} = \delta_{\zeta-z} - \bar\partial$. If X is Stein we can even find holomorphic such h_j. We let

$$H = \sum_1^m h_j \wedge e_j^*.$$

We also let

$$\nabla = \nabla_F + \delta_{\zeta-z} = \nabla_{\zeta-z} + \delta_F = \delta_{\zeta-z} + \delta_f + \delta_{\tilde f} - \bar\partial_\zeta.$$

Notice that

$$\nabla(\tau + H) = 0. \tag{2.1}$$

In fact,

$$\delta_F\tau = \sum(f_j(\zeta) - f(z))e_j^* = -\delta_{\zeta-z}H, \tag{2.2}$$

from which (2.1) follows.

We consider the exterior algebra over the direct sum of every bundle in sight, i.e., E, $\tilde E$, $T^*(X)$ etc. For any form α we introduce the integral

$$\int_e \alpha,$$

which is defined as the unique form α' such that $\alpha'\wedge(\sum_j e_j^*\wedge e_j)^m/m!$ is the term of α which has full degree in both e_j and e_j^*. The integral is invariant, i.e., independent of the choice of frame e_j, linear and it acts fiber-wise. Let

$$\tau = \sum_1 e_j^*\wedge(e_j - \tilde e_j).$$

Lemma 2.2. *If α is any form with values in ΛE (i.e., no $\tilde e_j$ only e_j), then*

$$\int_e \tau_m\wedge\alpha = \tilde\alpha, \tag{2.3}$$

where $\tau_m = \tau^m/m!$ *and* $\tilde{\alpha}$ *is the corresponding form where* e_j *is replaced by* $\tilde{e}_j$.

Proof. We may assume, with no loss of generality, that $\alpha = e_1 \wedge \ldots \wedge e_p$. Then

$$\int_e \tau_m \wedge \alpha = \int_e e_1^* \wedge (e_1 - \tilde{e}_1) \wedge \ldots \wedge e_m^* \wedge (e_m - \tilde{e}_m) \wedge \alpha =$$

$$(-1)^p \int_e e_1^* \wedge \tilde{e}_1 \wedge \ldots e_p^* \wedge \tilde{e}_p \wedge e_{p+1}^* \wedge e_{p+1} \wedge \ldots e_m^* \wedge e_m \wedge e_1 \wedge \ldots \wedge \tilde{e}_p.$$

We now just have to interchange $\tilde{e}_1 \wedge \ldots \wedge \tilde{e}_p$ and $e_1 \wedge \ldots \wedge e_p$ and this gives rise to the factor $(-1)^p$. □

Our main result in this section is

Theorem 2.3. *Let* f *be any holomorphic mapping and let* U *and* R^f *be as in Theorem 1.3 above. Moreover, let* g *be a smooth weight with compact support as in Proposition 2.1, with respect to the point* z. *For any holomorphic* ϕ *with values in* ΛE *we have*

$$\tilde{\phi}(z) = \delta_{\tilde{f}} \int_e \int_X e^{\tau+H} \wedge U \wedge g \wedge \phi + \\ + \int_e \int_X e^{\tau+H} \wedge U \wedge g \wedge \delta_f \phi + \int_e \int_X e^{\tau+H} \wedge R^f \wedge g \wedge \phi. \tag{2.4}$$

It is natural to define

$$T\phi(z) = \int_e \int_X e^{\tau+H} \wedge U \wedge g \wedge \phi, \tag{2.5}$$

and

$$S\phi(z) = \int_e \int_X e^{\tau+H} \wedge R^f \wedge g \wedge \phi, \tag{2.6}$$

and we then have that

$$\phi = \delta_f T\phi + T\delta_f \phi + S\phi. \tag{2.7}$$

If H and g depend holomorphically on z locally it follows that $T\phi$ and $S\phi$ are holomorphic there.

Corollary 2.4. *If* $\delta_f \phi = 0$ *and* $\phi \wedge R^f = 0$, *then* $\delta_f T\phi = \phi$.

Proof of Theorem 2.3. From (2.1) it follows that

$$(\nabla_{\zeta - z} + \delta_F)(e^{\tau+H} \wedge U) = e^{\tau+H} \wedge (1 - R^f). \tag{2.8}$$

We can rewrite this as

$$\delta_F(e^{\tau+H} \wedge U) + e^{\tau+H} \wedge R^f = e^{\tau+H} - \nabla_{\zeta - z}(e^{\tau+H} \wedge U).$$

Now,

$$\int_e\int_\zeta e^{\tau+H}\wedge g\wedge\phi-\int_e\int_\zeta\nabla_{\zeta-z}(e^{\tau+H}\wedge U)\wedge g\wedge\phi=$$

$$\int_e\int_\zeta \tau_m\wedge g\wedge\phi-\int_e\int_\zeta\nabla_{\zeta-z}\big(e^{\tau+H}\wedge U\wedge g\wedge\phi\big)=\int_e\tau_m\wedge\phi-0=\tilde\phi(z),$$

where we have used Proposition 2.1, Lemma 2.2, and Stokes' theorem. On the other hand it is easy to verify that

$$\int_e\int_\zeta\Big(\delta_F(e^{\tau+H}\wedge U)+e^{\tau+H}\wedge R^f\Big)\wedge g\wedge\phi$$

is equal to the right hand side of (2.4), and thus the theorem is proved. □

If ϕ is a section to Λ^pE it follows from degree considerations that $T\phi$ is a section to $\Lambda^{p+1}\tilde E$, whereas $S\phi$ is a section to $\Lambda^p\tilde E$. In fact, to begin with we need full degree in e_j^* so we must have from $e^{\tau+H}$ a factor like $\tau_{m-k}\wedge H_k$. To match the differentials in g we must then combine with $U_{k+1,k}$. If ϕ has degree p this gives us a total degree $n+1$ in $e,\tilde e$. After integration we are left with degree $p+1$ in $\tilde e$. The argument for $S\phi$ is goes along the same lines. It follows that

$$T\phi=\int_e\int_X\Omega\wedge g\wedge\phi$$

and

$$S\phi=\int_e\int_X W\wedge g\wedge\phi,$$

where

$$\Omega=\sum_{k=0}\tau_{m-k}\wedge H_k\wedge U_{k+1,k} \tag{2.9}$$

and

$$W=\sum_{k=1}\tau_{m-k}\wedge H_k\wedge R^f_{k,k}. \tag{2.10}$$

The first explicit solution formula for division problems appeared in [3] in the case when f has no zeros, or a regular zero set. Formulas with f being a complete intersection have been used by several authors starting with [11] and [4]; see [5] for more references. Formulas allowing ϕ to take values in Λ^*E have not appeared before as far as we know. Another novelty in this paper is that f may be any holomorphic mapping.

Remark 3. One can derive our division formula in an alternative way when f is nonvanishing. If

$$G=\big(\tau-\nabla_{\zeta-z}(H\wedge\sigma)\big)^m/m!,$$

then $G_{0,0}(z) = \tau_m$, and by Lemma 2.2 therefore

$$\tilde{\phi}(z) = \int_e \int_\zeta G \wedge g \wedge \phi. \tag{2.11}$$

Now,

$$\tau - \nabla_{\zeta - z}(H \wedge \sigma) = \tau + \delta_F \tau \wedge \sigma + H \wedge \bar{\partial}\sigma = \\ \delta_F(\tau \wedge \sigma + H \wedge \sigma \wedge \bar{\partial}\sigma) = \delta_F\big(\sigma \wedge (\tau + H \wedge \bar{\partial}\sigma)\big).$$

and since $\sigma \wedge \sigma = 0$ therefore

$$G = \delta_F\big(\sigma \wedge (\tau + H \wedge \bar{\partial}\sigma)\big)^m / m!\big) = \delta_F \Omega,$$

where Ω is the form in (2.9). It now follows from (2.11) that

$$\tilde{\phi}(z) = \delta_{\tilde{f}} \int_e \int_\zeta \Omega \wedge g \wedge \phi + \int_e \int_\zeta \Omega \wedge g \wedge \delta_f \phi$$

which is the same as (2.4) since $R^f = 0$.

If f has zeros, then G has no obvious meaning, whereas in the proof of Theorem 2.3 only the welldefined expressions U and R^f appear. □

Remark 4 (The case when $k = 0$). If ϕ is a function, i.e., $k = 0$, and again for simplicity f is nonvanishing, then we claim that (2.4) becomes

$$\phi(z) = \int_\zeta G \wedge g \wedge \phi, \tag{2.12}$$

where

$$G = 1 - \nabla_{\zeta - z} \frac{H \cdot \sigma}{1 + H \cdot \bar{\partial}\sigma},$$

if we use $\cdot$ for the natural pairing of E^* with E (and $\tilde{E}$). It is maybe worthwhile to point out that this is *not* the same formula as in [3]; in fact it could not be since in [3] only weights of the form (expressed in the notation from [1]) $1 + \nabla_{\zeta - z} q$, where q is a $(1,0)$-form, occur. The formula in [3] is defined by

$$G = (1 - \nabla_{\zeta - z} H \cdot \sigma)^\alpha$$

for an appropriate integer α, and this gives "unnecessary" factors $f(z)$.

We omit the tedious computation needed to verify the claim. Let us just indicate directly that (2.12) provides a division formula. To this end first notice that the very equality (2.12) holds in view of Proposition 2.6 (since $H \cdot \bar{\partial}\sigma$ has even degree the quotient makes sense). A simple computation shows that

$$G = \frac{f(z) \cdot \sigma}{1 + H \cdot \bar{\partial}\sigma} + f(z) \cdot \bar{\partial}\sigma \wedge \frac{H \cdot \sigma}{(1 + H \cdot \bar{\partial}\sigma)^2},$$

and thus "divisible" by $f(z)$. □

One can apply the operator S to any smooth form w defined in a neighborhood of Y with values in $\Lambda(E \oplus T^*_{0,1})$.

Theorem 2.5. *If w is a smooth form defined in a neighborhood of Y and with values in $\Lambda(E\oplus T^*_{0,1})$, then*

$$\delta_f Sw = S(\nabla_f w).$$

Proof. We have that

$$S\phi = \int_e\int_X e^{\tau+H}\wedge\nabla_F(R\wedge w)\wedge g = \int_e\int_X (\nabla_F+\delta_{\zeta-z})\left[e^{\tau+H}\wedge R\wedge w\wedge g\right] =$$
$$= \delta_{\tilde f} Sw + \int_e\int_X \nabla_{\zeta-z}(\qquad) + \int_e\int_X \delta_f(\qquad),$$

and both the last integrals vanish for degree reasons and Stokes' theorem. □

Corollary 2.6. *Let ϕ be holomorphic with values in $\Lambda^r E$. If there is a smooth form w defined in a neighborhood of Y such that $\nabla_f(w\wedge R^f)=\phi$, then ϕ is δ_f-exact, and a (holomorphic) solution is provided by*

$$\psi = T\phi + Sw.$$

Proof. Since ϕ is holomorphic, $\delta_f\phi = \nabla_f\phi = \nabla_f^2 w = 0$ close to Y and hence globally. Now, the corollary follows from Theorems 2.3 and 2.5. □

Example 2 (Interpolation). Let f be any nontrivial holomorphic mapping. We claim that if ϕ is any germ of a holomorphic function at Y, then $S\phi$ provides a holomorphic function in the whole domain (where it is holomorphic), such that $\phi - S\phi$ belongs to the ideal I^f. In fact, if Φ is any such extension, then it follows from Theorem 2.3 that

$$\Phi = \delta_f T\Phi + S\Phi = \delta_f T\Phi + S(\Phi-\phi) + S\phi.$$

Since $\phi - \Phi = \delta_f\psi$ for some holomorphic ψ, Theorem 2.5 now implies that $S(\Phi-\phi) = \delta_f S\psi$ and thus $S\phi-\phi$ belongs to the ideal as claimed.

For instance, let f be holomorphic in a neighborhood of the closed unit ball, If we take the weight g from Example 1 for the ball, with a sufficiently high power α, so that $g\wedge R^f\phi$ is welldefined, then $S\phi$ is a holomorphic extension to the entire ball. □

Further study and applications of this interpolation formula will be the topic of a forthcoming paper.

3. Division formulas for smooth functions

The definitions (2.5) and (2.6) make sense even if Φ is a smooth form in $\mathcal{L}^0(X)$ with values in ΛE, and if $\nabla_{\zeta-z}\Phi = 0$ for some z, then

$$\Phi_{0,0}(z) = \delta_f(T\Phi) + T(\delta_f\Phi) + S\Phi. \tag{3.1}$$

This follows from precisely the same argument as in the holomorphic case. Therefore, if $\delta_f\Phi = 0$ and $\Phi\wedge R^f = 0$, then $\Phi_{0,0}(z) = \delta_{f(z)}(T\Phi)$.

A first attempt to find such a Φ^z for the point z would be to take

$$\Phi^z = \phi - v^z \wedge \bar\partial\phi,$$

where $v^z(\zeta)$ is a (scalar-valued) current such that $\nabla_{\zeta-z} v^z = 1 - [z]$ for each z; e.g., the Bochner-Martinelli form

$$v^z = \frac{b}{\nabla_{\zeta-z} b} = b + b\wedge\bar\partial_\zeta b + \cdots + b\wedge(\bar\partial_\zeta b)^{n-1},$$

where $b = \partial_\zeta|\zeta-z|^2/2\pi i|\zeta-z|^2$, cf., [1]. Then $\nabla_{\zeta-z}\Phi^z = 0$, but unfortunately Φ^z is not smooth. Therefore, although (3.1) holds for each z outside Y it will not hold across Y.

Remark 5. If we could find, given a smooth function ϕ, with $\delta_f\phi = 0$, (with values in ΛE), a smooth form Φ^z for each z, depending smoothly on z, such that $\nabla_{\zeta-z}\Phi^z = 0$ and $\Phi^z_{0,0} = \phi$, then it would follow from (3.1) of course that ϕ is smoothly δ_f-exact. On the other hand, then $\bar\partial\phi(\zeta) = \bar\partial_\zeta\Phi^z_{0,0}(\zeta) = \delta_{\zeta-z}\Phi^z_{1,1}(\zeta,z)$, and taking $z=\zeta$ we find that $\bar\partial\phi(z) = 0$. Since z is arbitrary it follows that in fact ϕ is holomorphic then. $\square$

Instead we identify X with the set $\{(\zeta,\bar\zeta)\in\mathbb{C}^{2n};\ \zeta\in X\}$ and let $\widetilde X$ be an open neighborhood of X in $\mathbb{C}^{2n}$. If ϕ is a smooth function (with values in ΛE) on X, then we consider the following almost holomorphic extension to $\widetilde X$,

$$\tilde\phi(\zeta,\omega) = \sum_\alpha (\partial^\alpha_{\bar\zeta}\phi)(\zeta)\frac{(\omega-\bar\zeta)^\alpha}{\alpha!}\chi(\lambda_{|\alpha|}(\omega-\bar\zeta)), \tag{3.2}$$

where χ is a cutoff function in $\mathbb{C}^n$ which is 1 in a neighborhood of 0, and λ_k are positive numbers. If $\lambda_k\to\infty$ fast enough, the series converges to a smooth function in $\widetilde X$ such that

$$\tilde\phi(\zeta,\bar\zeta) = \phi(\zeta),$$

and

$$\bar\partial\tilde\phi(\zeta,\omega) = \mathcal{O}(|\omega-\bar\zeta|^\infty).$$

If ϕ is realanalytic one can take $\lambda_k = 1$ for all k and then $\tilde\phi$ is the holomorphic extension of ϕ. If ϕ is in C^{c_n+2M+k} as in the second half of Theorem 1.6, then we take instead just

$$\tilde\phi(\zeta,\omega) = \sum_{|\alpha|\le c_n+M+k} (\partial^\alpha_{\bar\zeta}\phi)(\zeta)\frac{(\omega-\bar\zeta)^\alpha}{\alpha!}, \tag{3.3}$$

which is then of class C^M in $\widetilde X$; again $\tilde\phi(\zeta,\bar\zeta) = \phi(\zeta)$, and at least

$$\bar\partial\tilde\phi(\zeta,\omega) = \mathcal{O}(|\omega-\bar\zeta|^{c_n+M+k}). \tag{3.4}$$

Proposition 3.1. *Let ϕ be a form in X, let v^z denote the Bochner-Martinelli form in $\tilde X$ with respect to the point $(z,\bar z)$, and let*

$$\Phi^z(\zeta,\omega) = \tilde\phi(\zeta,\omega) - \bar\partial\tilde\phi\wedge v^z.$$

If ϕ is smooth (and $\tilde\phi$ as in (3.2)*) then Φ^z is smooth in ζ,ω,z. If ϕ is in C^{c_n+M+k} (and $\tilde\phi$ as in* (3.3)*), then Φ^z is of class C^M in ζ,ω even after taking up to k derivatives with respect to z. In any case $\nabla_{(\zeta,\omega)-(z,\bar z)}\Phi^z=0$.*

Moreover, if $\delta_f\phi=0$, then $\delta_f\Phi^z=0$ and if (1.3) *holds (for all α in the smooth case, for all $|\alpha|\le c_n+M+k$ in the differentiable case), then $\Phi^z\wedge(R^f\otimes 1)=0$.*

Proof. Since

$$v^z=\frac{b}{\nabla_{(\zeta,\omega)-(z,\bar z)}b},$$

where $b=\sum_1^n(\zeta_j-z_j)d\zeta_j+\sum_1^n(\omega_j-\bar z_j)d\omega_j$, we have that

$$\Phi^z(\zeta,\omega)=\tilde\phi(\zeta,\omega)+\sum_{\ell=1}^{2n}\frac{\mathcal{O}(|\omega-\bar\zeta|^\infty)}{(|\zeta-z|^2+|\omega-\bar z|^2)^{\ell-1/2}},$$

if ϕ is smooth, and thus Φ^z is smooth. In the differentiable case, (3.4) ensures that one can take up to k derivatives with respect to z and still remain in $C^M(\widetilde X)$.

If $\delta_{f(\zeta)}\phi(\zeta)=0$, we have that $\delta_{f(\zeta)}(\partial^\alpha_{\bar\zeta}\phi)(\zeta)=0$ for all α (all $|\alpha|\le M+c_n+k$ in the differentiable case) and therefore $\delta_{f(\zeta)}\tilde\phi(\zeta,\omega)=0$. In the same way, $\delta_{f(\zeta)}(\bar\partial\tilde\phi)(\zeta,\omega)=0$. Finally, if $(\partial^\alpha_{\bar\zeta}\phi)\wedge R^f=0$ for all α (for $|\alpha|\le c_n+M+k$), then also $(\partial^\alpha_{\bar\zeta}\bar\partial\phi)\wedge R^f=0$ for all α (for $|\alpha|\le c_n+M+k-1\sim c_n+M+k$, with a small redefinition of c_n) and therefore $\tilde\phi\wedge R^f$ and $\bar\partial\tilde\phi\wedge R^f=0$. □

Proof of Theorem 1.6. Consider $\tilde f(\zeta)$ in $\tilde X$, and notice that the corresponding current $R^{\tilde f}$ in $\tilde X$ is just the tensor product $R^f\otimes 1$. If now T and S denote the operators from the previous section but in $\tilde X$ instead of X, we have that

$$\Phi_{0,0}=\delta_f T\Phi+T(\delta_f\Phi)+S\Phi$$

if Φ is any smooth form such that $\nabla_{(\zeta,\omega)-(z,\bar z)}=0$; in fact since U and R^f have order M it is enough that Φ is in C^M. We can thus apply to the forms Φ^z from the proposition and get

$$\phi(z)=\tilde\phi(z,\bar z)=\delta_f T\Phi^z+T(\delta_f\Phi^z)+S\Phi^z. \tag{3.5}$$

If also the other assumptions on ϕ are fulfilled, it follows from the proposition that

$$\phi(z)=\delta_{f(z)}T\Phi^z;$$

thus $\psi(z)=T\Phi^z(z)$ is a smooth solution to $\delta_f\psi=\phi$ if ϕ is smooth, and a solution in C^k if $\phi\in C^{c_n+M+k}$. Thus the proof is complete. □

Notice that the final division formula depends on the almost holomorphic extension $\tilde\phi$ and it is thus not linear. However, for ϕ in some given differentiable or ultradifferentiable class one can use the same λ_k, and therefore get a linear formula.

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Department of Mathematics, Chalmers University of Technology and Göteborg University, S-412 96 GÖTEBORG, SWEDEN

E-mail address: `matsa@math.chalmers.se`