On eigenvalues and eigensolutions of the Schrödinger equation on the complement of a set with classical capacity zero

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Abstract Let Γ be a closed subset of $\mathbb{R}^d$ with strictly positive c_2 – capacity. We provide a method to construct a large class of selfadjoint operators H^μ_α in $L^2(\mathbb{R}^d)$ which satisfy $H^\mu_\alpha f = -\Delta f$ for every smooth function with compact support away from Γ. The operators are parametrized via a positive real number α and a measure μ supported by Γ. We shall show that the negative number $-\beta$ is an eigenvalue of H^μ_α if and only if 0 is an eigenvalue of the operator $(-\Delta+\alpha)(-\Delta+\beta)-(\alpha-\beta)\mu$ and give a linear bijection between the corresponding eigenspaces. Moreover we shall derive estimates for the number, counting multiplicities, of negative eigenvalues of the operator H^μ_α.

1 Introduction

If the sum of two sequences of nonnegative real numbers converges to a real limit then both sequences are bounded; on the other hand the difference may converge to a real limit even if both sequences are unbounded.

This elementary fact explains why there is a striking difference between the shape of the limit of a sequence of Schrödinger operators with nonnegative potentials and the shape of the limit of certain sequences of Schrödinger operators with nonpositive potentials: Let H_0 be a nonnegative selfadjoint operator in the Hilbert space $\mathcal{H}$ and $\mathcal{E}_0$ the closed quadratic form uniquely associated to H_0 in the sense of Kato's representation theorem (cf. the appendix for various definitions). Let H_n be nonnegative selfadjoint operators in $\mathcal{H}$ such that the associated closed quadratic forms $\mathcal{E}_n$ admit a representation of the form

$$\mathcal{E}_n = \mathcal{E}_0 + \mathcal{P}_n, \quad n \in \mathbb{N},$$

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for suitably chosen nonnegative quadratic forms $\mathcal{P}_n$. Assume that the sequence (H_n) converges in the strong resolvent sense to a selfadjoint operator H with associated closed quadratic form $\mathcal{E}$. Let f be in the form domain $D(\mathcal{E}) = D(\sqrt{H})$ of H. Since the domain $D(H)$ of H is dense in the form domain $D(\mathcal{E})$ w.r.t. the graph norm of $\sqrt{H}$ and for selfadjoint operators convergence in the strong resolvent sense is equivalent to convergence in the strong graph sense ([29], theorem VIII 26), there exist $f_n \in D(H_n)$ such that

$$f_n \longrightarrow f, \text{ as } n \longrightarrow \infty \text{ in } \mathcal{H} \tag{1}$$

and

$$\mathcal{E}_0(f_n, f_n) + \mathcal{P}_n(f_n, f_n) \longrightarrow \mathcal{E}(f, f), \text{ as } n \longrightarrow \infty.$$

By the mentioned elementary fact about sequences of real numbers, this implies that $\sup_{n\in\mathbb{N}} \mathcal{E}_0(f_n, f_n) < \infty$. By (1) and the Banach – Alaoglu Theorem, it follows that $f \in D(\mathcal{E}_0)$ and a suitably chosen subsequence of (f_n) converges weakly w.r.t. the graph norm of $\sqrt{H_0}$ to f. Thus

$$D(\mathcal{E}) \subset D(\mathcal{E}_0) \tag{2}$$

and the closed quadratic form $\mathcal{E}$ associated to the limit H admits a representation of the form

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{P} \tag{3}$$

for some suitably chosen quadratic form $\mathcal{P}$.

One is strongly interested in the special case when H_0 equals the free (quantum mechanical) Hamiltonian, i.e. the selfadjoint operator $-\Delta$ in $L^2(\mathbb{R}^d)$ uniquely associated to the classical Dirichlet form $\mathcal{E}_0$, defined by

$$\begin{aligned} D(\mathcal{E}_0) &:= H^1(\mathbb{R}^d), \\ \mathcal{E}_0(f, f) &:= \int |\nabla f|^2 dx, \quad f \in H^1(\mathbb{R}^d), \end{aligned}$$

and $H_n = -\Delta + V_n$ for some measurable functions $V_n : \mathbb{R}^d \longrightarrow \mathbb{R}$, i.e. H_n is the selfadjoint operator in $L^2(\mathbb{R}^d)$ associated to the closed quadratic form $\mathcal{E}_n$, defined by

$$\begin{aligned} D(\mathcal{E}_n) &:= \{f \in H^1(\mathbb{R}^d) : \int |f|^2 |V_n| dx < \infty\}, \\ \mathcal{E}_n(f, f) &:= \int |\nabla f|^2 dx + \int |f|^2 V_n dx, \quad f \in D(\mathcal{E}_n). \end{aligned}$$

G. dal Maso and U. Mosco [13] have shown that the selfadjoint operator H in $L^2(\mathbb{R}^d)$ is the limit in the strong resolvent sense of a sequence of Schrödinger operators $-\Delta + V_n$ with nonnegative potentials $V_n : \mathbb{R}^d \longrightarrow [0,\infty)$ if and only if there exists a (nonnegative) measure μ on the Borel algebra of $\mathbb{R}^d$ such that $\mu(B) = 0$ for every Borel set B with c_1 – capacity zero and $H = -\Delta + \mu$, i.e., the perturbation term $\mathcal{P}$ in the representation (3) of the closed quadratic form $\mathcal{E}$ associated to the limit H is given by

$$\begin{aligned} D(\mathcal{P}) &= \{f \in H^1(\mathbb{R}^d) : \int |\tilde{f}|^2 d\mu < \infty\}, \\ \mathcal{P}(f,f) &= \int |\tilde{f}|^2 d\mu, \quad f \in D(\mathcal{P}). \end{aligned}$$

Here $\tilde{f}$ denotes any representative of f which is quasi – continuous w.r.t. the c_1 – capacity.

Due to the mentioned elementary fact about sequences of real numbers one may expect that an analogous result does not hold true for limits of Schrödinger operators with nonpositive potentials. In fact, C.N.Friedman ([17],[18]) has given measurable functions $V_n : \mathbb{R}^3 \longrightarrow [0,\infty)$ such that the sequence $(-\Delta - V_n)$ converges even in the norm resolvent sense to a lower semibounded selfadjoint operator $H \neq -\Delta$ whose associated closed quadratic form $\mathcal{E}$ has the following properties:

$$\begin{aligned} D(\mathcal{E}) &\supset D(\mathcal{E}_0) = H^1(\mathbb{R}^d), \quad D(\mathcal{E}) \neq D(\mathcal{E}_0), \\ \mathcal{E}(f,f) &= \mathcal{E}_0(f,f), \quad f \in H^1(\mathbb{R}^d). \end{aligned}$$

In particular, the closed quadratic form $\mathcal{E}$ associated to the limit H does not admit any representation of the form (3). Moreover, due to the fact that the classical capacity of a singleton in $\mathbb{R}^3$ equals zero, the limit H does not generate a submarkovian semigroup (cf. [20]) and the powerful probabilistic approach (cf. the monography [14] by M. Demuth and J.A. van Casteren and references given therein) does not work. Actually, it is an open and presumably a difficult problem to find an affirmative description of the set of limits in the strong resolvent sense of sequences of Schrödinger operators with nonpositive potentials.

The mentioned limits H in the work by C.N.Friedman are different from the free Hamiltonian $-\Delta$ but coincide with $-\Delta$ on the space $C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ of smooth functions with compact support away from 0. More generally, let

Γ be any closed subset of $\mathbb{R}^d$ and denote by $\mathcal{A}_\Gamma$ the set of all H with the following properties:

$$\begin{array}{l} H \text{ is a selfadjoint operator in } L^2(\mathbb{R}^d), \\ C_0^\infty(\mathbb{R}^d \setminus \Gamma) \subset D(H), \\ Hf = -\Delta f, \quad f \in C_0^\infty(\mathbb{R}^d \setminus \Gamma), \\ H \neq -\Delta. \end{array} \tag{4}$$

If the c_1 – capacity of Γ equals zero then, by Hedberg's Theorem on the spectral synthesis in Sobolev spaces [21], the space $C_0^\infty(\mathbb{R}^d \setminus \Gamma)$ is dense in $H^1(\mathbb{R}^d)$. This implies that the closed quadratic form $\mathcal{E}$ associated to any lower semibounded $H \in \mathcal{A}_\Gamma$ is a proper extension of the classical Dirichlet form $\mathcal{E}_0$ [1]; in particular, $\mathcal{E}$ does not admit any representation of the form (3) and it is already an interesting task to find methods to "construct" elements of $\mathcal{A}_\Gamma$. S. Albeverio, J.E.Fenstad, R.Høegh – Krohn and T.Lindstrøm have found representations for the elements in various subclasses of $\mathcal{A}_\Gamma$, cf. their monograph [2] for a summary of their results on this topic; cf. the papers [5], [7], [10], [11], [15], [22], [23], [24], [26], [28], [32] by Y.Berezansky, J.F.Brasche, P. Exner, R. Figari, W.Karwowski, S.Kondej, V. Koshmanenko, P.Kurasov, S.T.Kuroda, H. Neidhardt, L. Nizhnik, S.Ôta, A.Posilicano, A.Teta, V.A. Zagrebnov and references given therein for further results. Moreover the fact that a representation of the form (3) is not possible makes it difficult to analyze the spectral and scattering properties of such operators. Actually far reaching results on the properties of operators in $\mathcal{A}_\Gamma$ have only been obtained in the case when H and the free Hamiltonian have a common restriction with finite deficiency indices, cf. the monography [25] by V. Koshmanenko and references given therein, for point interactions, i.e. in the case when the set Γ is discrete, cf. the monographies [3] and [4] by S.Albeverio, F. Gesztesy, R. Høegh – Krohn and H.Holden and by S.Albeverio and P.Kurasov, respectively, for a summary of important parts of this theory and a large list of references, and for regular curves Γ, cf. the article [16] by P. Exner and Y. Kazushi (cf. also [12]).

In the present note we shall give detailed results on the distribution of negative eigenvalues and the corresponding eigenfunctions for a large class of operators belonging to $\mathcal{A}_\Gamma$. Our methods even work in the difficult case when the c_1 – capacity of Γ equals zero. We only require that the c_2 – capacity of Γ is positive and, in particular, do not require any regularity of the closed set Γ.

Before we describe the mentioned class of operators let us recall some well known facts. It easily follows from Hedberg's Theorem on the spectral synthesis in Sobolev spaces that the set $\mathcal{A}_\Gamma$ is non – empty if and only if the c_2 – capacity of Γ is positive. If $c_2(\Gamma) > 0$, then, by a result by B.Fuglede ([19]) and N.G.Meyers ([27]), there exist positive Radon measures $\mu \neq 0$ such that $\mu(\mathbb{R}^d \setminus \Gamma) = 0$,

$$\mu(B) = 0, \text{ if } c_2(B) = 0, \tag{5}$$

and

$$\int |\tilde{f}|^2 d\mu < \infty, \quad f \in H^2(\mathbb{R}^d), \tag{6}$$

where here $\tilde{f}$ denotes any representative of f which is quasi continuous w.r.t. the c_2 – capacity.

Let μ be any positive Radon measure on $\mathbb{R}^d$ satisfying (5) and (6) and α any positive real number. It is easily verified that there exists a unique selfadjoint operator H^μ_α in $L^2(\mathbb{R}^d, \mu)$ such that $-\alpha$ belongs to the resolvent set $\rho(H^\mu_\alpha)$ of H^μ_α and

$$(H^\mu_\alpha + \alpha)^{-1} = G_\alpha + (J_\mu G_\alpha)^* J_\mu G_\alpha, \tag{7}$$

where $G_\alpha := (-\Delta + \alpha)^{-1}$ and the mapping $J_\mu : H^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d, \mu)$ is defined by

$$J_\mu f := \tilde{f}, \quad f \in H^2(\mathbb{R}^d). \tag{8}$$

Obviously the operator H^μ_α belongs to the set $\mathcal{A}_\Gamma$ provided $\mu(\mathbb{R}^d \setminus \Gamma) = 0$ and $\mu \neq 0$. Thus the resolvent formula (7) yields a large class of operators belonging to $\mathcal{A}_\Gamma$ whenever the set $\mathcal{A}_\Gamma$ is non – empty. In the next section we shall scetch a simple method to construct an even larger class of operators belonging to $\mathcal{A}_\Gamma$, cf. Remark 4.

We shall assume that the measure μ satisfies the following regularity and decay conditions: There exists an $s < 2$ such that

$$\mu(B) = 0, \text{ if } c_s(B) = 0, \tag{9}$$

$$\int |\tilde{f}|^2 d\mu < \infty, \quad f \in H^s(\mathbb{R}^d), \tag{10}$$

where here $\tilde{f}$ denotes any representative of f which is quasi continuous w.r.t. the c_s – capacity, and

$$\mu(\{y : |x - y| < 1\}) \longrightarrow 0, \text{ as } |x| \longrightarrow \infty. \tag{11}$$

It has been shown in [9], that the essential spectrum of H_α^μ equals $[0, \infty)$ provided (9) – (11) hold true.

It remains to analyze the distribution of the negative eigenvalues of H_α^μ. If $\mu(B) > 0$ for some Borel set B with c_1 – capacity zero then the closed quadratic form associated to H_α^μ does not admit any representation of the form (3) and the "usual techniques" for the investigation of the eigenvalue distribution can not be applied. The main goal of the present note is to present a "trick" which makes it possible to use "standard methods" in order to analyze the negative part of the spectrum of H_α^μ even if μ charges some set with c_1 – capacity zero:

We shall show that the number of negative eigenvalues of H_α^μ equals the number of negative eigenvalues of the fourth order operator

$$A_{\alpha 0} - \mu := \frac{-\Delta\,(-\Delta + \alpha)}{\alpha} - \mu.$$

Here every eigenvalue is counted as many times as its multiplicity of the respective operator. This makes it, in particular, possible to use the well known scaling properties of the free Hamiltonian in order to derive lower bounds for the number of negative eigenvalues of H_α^μ. E.g., trivially $A_{\alpha 0} - \mu$ has at least one negative eigenvalue provided the dimension d of $\mathbb{R}^d$ equals 1 or 2. Thus the same holds true for the operators H_α^μ. Since one gets the operator $A_{\alpha 0} - \mu$ via a form perturbation of an operator with well known resolvent, general results in [6] yield upper estimates for the number of its negative eigenvalues and therefore automatically for the number of negative eigenvalues of H_α^μ.

Actually we shall prove a stronger result. We shall show that the negative number $-\beta$ is an eigenvalue of the operator H_α^μ if and only if 0 is an eigenvalue of the operator

$$A_{\alpha\beta} - \mu := \frac{(-\Delta + \beta)(-\Delta + \alpha)}{\alpha - \beta} - \mu$$

and that there exists a linear bijective mapping from the kernel of $A_{\alpha\beta} - \mu$ onto the kernel of $H_\alpha^\mu - \beta$. We shall even give an explicit description of such a linear bijection.

We shall use both ideas from operator theory and potential theory. In the next section we shall present the ideas from operator theory and consider any nonnegative selfadjoint operator H in any Hilbert space $\mathcal{H}$ instead of the free Hamiltonian. The mentioned results on the operators H_α^μ will be derived in the last section. In the appendix various notions are explained and several well known results, used in the present note, are recalled for convenience of the reader.

The trick to investigate the eigenvalue distribution of an operator H with the aid of the eigenvalue distribution of another operator works within a wider framework, cf. Remark 4.

2 The operator theoretical framework

In this section $\mathcal{H}$ and $\mathcal{H}_{aux}$ denotes a Hilbert space with scalar product $(\cdot,\cdot)$ and $(\cdot,\cdot)_{aux}$, respectively, H a nonnegative selfadjoint operator in $\mathcal{H}$,

$$\| f \|_H := (\| Hf \|^2 + \| f \|^2)^{1/2}, \quad f \in D(H),$$

the graph norm of H and J a bounded linear mapping from $(D(H), \| \cdot \|_H)$ to $\mathcal{H}_{aux}$. For $-z \in \rho(H)$ we put

$$G_z := (H+z)^{-1}.$$

Sometimes G_z will be regarded as an operator from $\mathcal{H}$ into $(D(H), \| \cdot \|_H)$ and sometimes as an operator on $\mathcal{H}$; in any case it will be clear from the context what is meant.

The following lemma will be used in the next section in order to construct a large class of operators belonging to the set $\mathcal{A}_\Gamma$ described in the introduction (cf. (4)) as well as other classes of selfadjoint operators.

Lemma 1 *Let B be a selfadjoint operator in $\mathcal{H}_{aux}$ such that the domain $D(B)$ of B contains the range ran(J) of J. Let $-\alpha$ be in the resolvent set $\rho(H)$ of H. Then the following holds true.*

If ran$((JG_\alpha)^*) \cap$ ran$(G_\alpha) = \{0\}$ or if $\alpha > 0$ and $B \geq 0$ then there exists a unique selfadjoint operator $H_\alpha^{J,B}$ in $\mathcal{H}$ such that $H_\alpha^{J,B} + \alpha$ is invertible and

$$(H_\alpha^{J,B} + \alpha)^{-1} = G_\alpha + (JG_\alpha)^* BJG_\alpha. \tag{12}$$

Moreover the number $-\alpha$ belongs to the resolvent set of $H_\alpha^{J,B}$. If, in addition, $\alpha > 0$ and $B \geq 0$ then there exists a number $c > -\alpha$ such that $H_\alpha^{J,B} \geq c$.

If $B \neq 0$ and ran(J) is dense in $\mathcal{H}_{aux}$ or $B \geq 0$ and $BJ \neq 0$ then $H_\alpha^{J,B} \neq H$.

Proof: Denote the operator on the right hand side of (12) by R_α. JG_α is a bounded everywhere defined operator. Thus it is closed and its adjoint $(JG_\alpha)^*$ is also a bounded everywhere defined operator. Since B is closed the operator BJG_α is closed. Since it is everywhere defined it is bounded. Since B is symmetric also the operator $(JG_\alpha)^* BJG_\alpha$ is symmetric and, due to the fact that it is everywhere defined, even selfadjoint. Since the inverse of a selfadjoint operator is also self – adjont the operator G_α is selfadjoint. Thus the operator R_α is bounded and selfadjoint.

Trivially the operator R_α is invertible if $\text{ran}((JG_\alpha)^*) \cap \text{ran}(G_\alpha) = \{0\}$. If $\alpha > 0$ and $B \geq 0$ then

$$(f, R_\alpha f) \geq (f, G_\alpha f) > 0, \quad f \neq 0,$$

and R_α is nonnegative and invertible. Since the inverse of a selfadjoint operator is selfadjoint the operator

$$H_\alpha^{J,B} := R_\alpha - \alpha$$

is also selfadjoint and, obviously, $-\alpha$ belongs to its resolvent set. If $\alpha > 0$ and $B \geq 0$ then $H_\alpha^{J,B} \geq -\alpha$ since $R_\alpha \geq 0$. Since $-\alpha$ belongs to the resolvent set of $H_\alpha^{J,B}$ this implies that there exists a $c > -\alpha$ such that $H_\alpha^{J,B} \geq c$.

Let ran(J) be dense in $\mathcal{H}_{aux}$ and $B \neq 0$. Since the operator B is closed the kernel of B is also closed and there exists an f such that $BJG_\alpha f \neq 0$. Since the range of JG_α is dense in $\mathcal{H}_{aux}$ the kernel of $(JG_\alpha)^*$ is trivial and $(JG_\alpha)^* BJG_\alpha f \neq 0$. Thus $R_\alpha \neq G_\alpha$. Thus $H_\alpha^{J,B} \neq H$.

If $BJ \neq 0$ then there exists an f such that $BJG_\alpha f \neq 0$. If, in addition, $B \geq 0$ then

$$(BJG_\alpha f, JG_\alpha f) > 0$$

since B is selfadjoint. Thus $R_\alpha \neq G_\alpha$ in this case, too. □

We have seen in the previous proof that the operator BJG_α is bounded. Moreover along with JG_α also its adjoint $(JG_\alpha)^*$ is compact. Thus the following corollary follows from (12) and Weyl's essential spectrum theorem:

Corollary 2 *Suppose that the hypothesis of the Lemma 1 is satisfied. Suppose, in addition, that the operator J from $(D(H), \|\cdot\|_H)$ to $\mathcal{H}_{aux}$ is compact. Then*

$$\sigma_{ess}(H_\alpha^{J,B}) = \sigma_{ess}(H),$$

i.e. the operator $H_\alpha^{J,B}$ has the same essential spectrum as H.

In what follows we shall consider the case when B equals the identity I on the Hilbert space $\mathcal{H}_{aux}$. We fix $\alpha > 0$ and put, for notational brevity,

$$H_\alpha := H_\alpha^{J,I},$$

i.e. denote by H_α the unique selfadjoint operator in $\mathcal{H}$ such that $-\alpha$ belongs to the resolvent set of H_α and

$$R_\alpha := (H_\alpha + \alpha)^{-1} = G_\alpha + (JG_\alpha)^* JG_\alpha. \tag{13}$$

For real $\beta \in \rho(H_\alpha) \cap \rho(H)$ we try to determine an operator B_β in $\mathcal{H}_{aux}$ such that

$$(H_\alpha + \beta)^{-1} = G_\beta + (JG_\beta)^* B_\beta JG_\beta. \tag{14}$$

It suffices to choose B_β such that $D(B_\beta) \supset \mathrm{ran}(J)$ and

$$R_\alpha - R_\beta = (\beta - \alpha) R_\alpha R_\beta, \tag{15}$$

where the operator R_β is defined by the right hand side of (14).

In fact, if (15) holds true then the range of R_β is contained in the range of R_α, i.e. the domain $D(H_\alpha)$ of H_α, and (13) and (15) imply that

$$(H_\alpha + \alpha) R_\beta f = f + (\alpha - \beta) R_\beta f, \quad f \in \mathcal{H}.$$

Thus

$$(H_\alpha + \beta) R_\beta f = f, \quad f \in \mathcal{H}.$$

Thus $R_\beta = (H_\alpha + \beta)^{-1}$.

It remains to determine the operator B_β with the required properties. Since

$$G_\alpha - G_\beta = (\beta - \alpha) G_\alpha G_\beta, \tag{16}$$

(15) holds true, provided

$$\begin{array}{rl} & (JG_\alpha)^* JG_\alpha - (JG_\beta)^* B_\beta JG_\beta \\ = & (\beta-\alpha) G_\alpha (JG_\beta)^* B_\beta JG_\beta + (\beta-\alpha)(JG_\alpha)^* JG_\alpha G_\beta \\ + & (\beta-\alpha)(JG_\alpha)^* JG_\alpha (JG_\beta)^* B_\beta JG_\beta. \end{array}$$

This can be rewritten as

$$\begin{array}{rl} & (JG_\alpha)^* JG_\alpha + (\alpha-\beta)(JG_\alpha)^* JG_\alpha G_\beta \\ = & (JG_\beta)^* B_\beta JG_\beta + (\beta-\alpha) G_\alpha (JG_\beta)^* B_\beta JG_\beta \\ + & (\beta-\alpha)(JG_\alpha)^* JG_\alpha (JG_\beta)^* B_\beta JG_\beta. \end{array} \tag{17}$$

By the first resolvent equation (16), the left hand side of (17) equals $(JG_\alpha)^* JG_\beta$. Thus (17) holds true provided

$$\begin{array}{rl} & (JG_\alpha)^* \\ = & \left((JG_\beta)^* + (\beta-\alpha) G_\alpha (JG_\beta)^* + (\beta-\alpha)(JG_\alpha)^* JG_\alpha (JG_\beta)^*\right) B_\beta. \end{array}$$

The last equality holds true if there exists an operator K_β in $\mathcal{H}_{aux}$ such that

$$B_\beta = (I - K_\beta)^{-1}, \quad D(B_\beta) = \mathcal{H}_{aux}, \tag{18}$$

and

$$\begin{array}{rl} & (JG_\alpha)^*(I - K_\beta) \\ = & (JG_\beta)^* + (\beta-\alpha) G_\alpha (JG_\beta)^* + (\beta-\alpha)(JG_\alpha)^* JG_\alpha (JG_\beta)^*. \end{array} \tag{19}$$

$G_\alpha (JG_\beta)^* = (JG_\beta G_\alpha)^*$ since G_α is a selfadjoint operator in $\mathcal{H}$. Together with the first resolvent identity this implies, that

$$(JG_\alpha)^* - (JG_\beta)^* - (\beta-\alpha) G_\alpha (JG_\beta)^* = 0,$$

and (19) can be rewritten as

$$(JG_\alpha)^* \left((\alpha-\beta) JG_\alpha (JG_\beta)^*\right) = (JG_\alpha)^* K_\beta.$$

This holds true if we define K_β by

$$K_\beta := (\alpha-\beta) JG_\alpha (JG_\beta)^*, \tag{20}$$

and we have proved one part of the following

Theorem 3 *Let $-\alpha, -\beta \in \mathbb{R} \cap \rho(H)$. Let H_α be the unique selfadjoint operator in $\mathcal{H}$ such that $-\alpha \in \rho(H_\alpha)$ and*

$$(H_\alpha + \alpha)^{-1} = G_\alpha + (JG_\alpha)^* JG_\alpha.$$

Assume in addition that the operator J from $(D(H), \| \cdot \|_H)$ to $\mathcal{H}_{aux}$ is compact and its range is dense in $\mathcal{H}_{aux}$. Then the following holds true:

$-\beta$ belongs to the resolvent set of H_α if and only if the operator $I-(\alpha-\beta)JG_\alpha(JG_\beta)^*$ in $\mathcal{H}_{aux}$ is invertible. In this case its inverse is bounded, the domain of its inverse equals $\mathcal{H}_{aux}$ and

$$(H_\alpha + \beta)^{-1} = G_\beta + (JG_\beta)^* \left(I - (\alpha - \beta) JG_\alpha (JG_\beta)^*\right)^{-1} JG_\beta. \tag{21}$$

In particular, the interval $(-\infty, -\alpha]$ is contained in the resolvent set of H_α if $\alpha > 0$.

If the operator $I - (\alpha - \beta)JG_\alpha(JG_\beta)^*$ is not invertible then $-\beta$ is an isolated point in the spectrum of H_α and an eigenvalue with finite multiplicity and the mapping

$$h \mapsto (JG_\beta)^* h \tag{22}$$

is a linear bijection from the kernel of $I - (\alpha - \beta)JG_\alpha(JG_\beta)^*$ onto the kernel of $H_\alpha + \beta$

Proof: Complexifying, if necessary, we may assume that the Hilbert space $\mathcal{H}$ is complex. The mapping

$$-z \mapsto K_z := (\alpha - z) JG_\alpha (JG_{\bar{z}})^*$$

is analytic on the resolvent set of H. Moreover the operators K_z are compact, by the hypothesis about J, and the operator $I - K_\alpha = I$ is invertible. Thus, by the analytic Fredholm theorem, there exists a discrete subset D of the resolvent set of H such that the operator $I - K_z$ is invertible, its inverse is bounded and $(D(I - K_z)^{-1}) = \mathcal{H}_{aux}$ for every $z \in \rho(H) \setminus D$. Moreover the operator $I - K_z$ is not invertible for any $z \in D$. By the considerations preceding the statement of the theorem, the real number $-\beta$ belongs to the resolvent set of H_α and the formula for the resolvent in the theorem holds true if $-\beta \in \rho(H) \setminus D$.

By the first resolvent identity,

$$\begin{aligned}(JG_\alpha(JG_\beta)^*h,h)_{aux} &= ((JG_\beta)^*h,(JG_\beta)^*h)_{aux} \\ &+ (\beta-\alpha)((JG_\beta)^*h,G_\alpha(JG_\beta)^*)_{aux}\end{aligned}$$

for every $h \in \mathcal{H}_{aux}$. Thus $JG_\alpha(JG_\beta)^*$ is a nonnegative selfadjoint operator and $I-(\alpha-\beta)JG_\alpha(JG_\beta)^*$ is invertible, provided $0<\alpha<\beta$. Thus the interval $(-\infty,-\alpha]$ belongs to the resolvent set of H_α, if $\alpha>0$.

Let $-\beta \in \rho(H)\cap D$. The following implications are obvious:

$$\begin{array}{ll} f \in D(H_\alpha) \text{ and } (H_\alpha+\beta)f=0 & \Longleftrightarrow \\ f \in D(H_\alpha) \text{ and } (H_\alpha+\alpha)f=(\alpha-\beta)f & \Longleftrightarrow \\ f=(\alpha-\beta)(H_\alpha+\alpha)^{-1}f & \Longleftrightarrow \\ f=(\alpha-\beta)\left(G_\alpha f+(JG_\alpha)^*JG_\alpha f\right) & \end{array} \tag{23}$$

Let

$$h-(\alpha-\beta)JG_\alpha(JG_\beta)^*h=0.$$

Then

$$\begin{aligned} & (\alpha-\beta)G_\alpha(JG_\beta)^*h \\ = \; & (J(\alpha-\beta)G_\alpha G_\beta))^*h \\ = \; & (JG_\beta)^*h-(JG_\alpha)^*h \\ = \; & (JG_\beta)^*h-(\alpha-\beta)(JG_\alpha)^*JG_\alpha(JG_\beta)^*h. \end{aligned}$$

By (23), this implies that

$$f:=(JG_\beta)^*h \in D(H_\alpha) \text{ and } (H_\alpha+\beta)f=0.$$

Thus $h \mapsto (JG_\beta)^*h$ really defines a mapping from the kernel of $I-(\alpha-\beta)JG_\alpha(JG_\beta)^*$ into the kernel of $H_\alpha+\beta$. Obviously this mapping is linear and, by the hypothesis that $\mathrm{ran}(J)$ is dense, also injective.

Let $(H_\alpha+\beta)f=0$. Put

$$h:=JG_\alpha f.$$

It remains to prove that h belongs to the kernel of $I-(\alpha-\beta)JG_\alpha(JG_\beta)^*$ and $f=(\alpha-\beta)(JG_\beta)^*h$.

We have

$$
\begin{array}{rl}
& (JG_\beta)^* h \\
= & (JG_\alpha)^* JG_\alpha f + (J(G_\beta - G_\alpha))^* JG_\alpha f \\
= & \frac{f}{\alpha-\beta} - G_\alpha f + (\alpha-\beta) G_\beta (JG_\alpha)^* JG_\alpha f \\
= & \frac{f}{\alpha-\beta} - G_\beta f + (\alpha-\beta) G_\beta G_\alpha f + (\alpha-\beta) G_\beta (JG_\alpha)^* JG_\alpha f \\
= & \frac{f}{\alpha-\beta} - G_\alpha \left(f - (\alpha-\beta) G_\alpha f - (\alpha-\beta)(JG_\alpha)^* JG_\alpha f\right) \\
= & \frac{f}{\alpha-\beta}.
\end{array} \tag{24}
$$

Here we used several times the first resolvent equation and at the third and last step the implications (23). By (24),

$$(\alpha-\beta) JG_\alpha (JG_\beta)^* h = JG_\alpha f.$$

Since $h = JG_\alpha f$ the theorem is proved. □

Remark 4 If one replaces the operator $(\alpha-\beta)JG_\alpha(JG_\beta)^*$ in the resolvent formula (21) by any bounded selfadjoint operator $\tilde{K}_\beta$ such that

$$\tilde{K}_\beta - \tilde{K}_\alpha = J(J(G_\beta - G_\alpha))^*,$$

then one gets again the formula for the resolvent of a selfadjoint operator, i.e. there is a canonical generalization of the Theorem 3; we refer to [2], chapter 6.2 and [28], Theorem 2.1 for related results.

If the operator J is not only bounded w.r.t. the graph norm of H but even w.r.t the graph norm of $\sqrt{|H|}$ with bound less than one then one simply can put

$$\tilde{K}_\beta := J(JG_\beta)^*$$

and get the formula for the resolvent of the selfadjoint operator which is associated to the closure of the quadratic form $(f, Hf) - \parallel Jf \parallel^2_{aux}$, cf. [8], (27) and the proof of Lemma 3 in [8]. We shall essentially use this result at the last part of this section. If the operator J is not bounded w.r.t. the graph norm of $\sqrt{|H|}$ then the operator $J(JG_\beta)^*$ is not bounded and the mentioned simple approach does not work.

In the remaining part of this section we fix $\alpha > 0$. We shall introduce operators $A^J_{\alpha\beta}$ which store the complete information about the negative eigenvalues and corresponding eigenvectors of H_α. Let $0 \le \beta < \alpha$. Let

$$A_{\alpha\beta} := \frac{(H+\alpha)(H+\beta)}{\alpha-\beta}. \tag{25}$$

$A_{\alpha\beta}$ is a nonnegative selfadjoint operator in $\mathcal{H}$, $D(A_{\alpha\beta}^{1/2}) = D(H)$ and the graph norm of $A_{\alpha\beta}^{1/2}$ is equivalent to the graph norm of H.

In what follows we assume, in addition, that there exist $a < 1$ and $b < \infty$ such that

$$\| Jf \|_{aux}^2 \leq a \| A_{\alpha 0}^{1/2} f \|^2 + b \| f \|^2, \quad f \in D(H). \tag{26}$$

Obviously (26) also holds true if we replace $A_{\alpha 0}$ by $A_{\alpha\beta}$. Thus the KLMN - theorem implies that the quadratic form $\mathcal{E}_{\alpha\beta}^J$ in $\mathcal{H}$, defined by

$$\begin{array}{rcl} D(\mathcal{E}_{\alpha\beta}^J) & := & D(A_{\alpha\beta}^{1/2}), \\ \mathcal{E}_{\alpha\beta}^J(f,f) & := & \| A_{\alpha\beta}^{1/2} f \|^2 - \| Jf \|_{aux}^2, \quad f \in D(\mathcal{E}_{\alpha\beta}^J), \end{array} \tag{27}$$

is lower semibounded and closed. We denote by $A_{\alpha\beta}^J$ the lower semibounded selfadjoint operator in $\mathcal{H}$ associated to $\mathcal{E}_{\alpha\beta}^J$ in the sense of Kato's representation theorem.

By (26), both in [6] and [8] we can replace the operators H, $-J$ and A by $A_{\alpha\beta}$, J and $-I$, respectively. Here I denotes the identity in the Hilbert space $\mathcal{H}_{aux}$. Thus part (i) and (ii) of the following corollary are immediate consequences of formula (27) in [8] and the proof of Lemma 3 in [8] and part (iii) is an immediate consequence of the proof of Lemma 1 in [6].

Corollary 5 *([8], [6])*
Let $0 \leq \beta < \alpha$. Suppose that the operator J from $(D(H), \|\cdot\|_H)$ into $\mathcal{H}_{aux}$ is compact. Let $-\gamma \in \mathbb{R} \cap \rho(A_{\alpha\beta})$. Then the following holds:

(i) $A_{\alpha\beta}^J$ and $A_{\alpha\beta}$ have the same essential spectrum.

(ii) $-\gamma$ belongs to the resolvent set of $A_{\alpha\beta}^J$ if and only if the operator $I - J(JG_{\alpha\beta\gamma})^*$ in $\mathcal{H}_{aux}$ is invertible, where

$$G_{\alpha\beta\gamma} := (A_{\alpha\beta} + \gamma)^{-1}.$$

In this case the inverse of $I - J(JG_{\alpha\beta\gamma})^*$ is bounded and everywhere defined and

$$(A_{\alpha\beta}^J)^{-1} = G_{\alpha\beta\gamma} + (JG_{\alpha\beta\gamma})^* \left(I - J(JG_{\alpha\beta\gamma})^*\right)^{-1} JG_{\alpha\beta\gamma}. \tag{28}$$

(iii) If the operator $I - J(JG_{\alpha\beta\gamma})^*$ *is not invertible then*

$$f \mapsto Jf$$

defines a linear bijective mapping from the kernel of $A^J_{\alpha\beta} + \gamma$ *onto the kernel of* $I - J(JG_{\alpha\beta\gamma})^*$.

After this preparation we can describe the negative eigenvalues and corresponding eigenvectors of H_α with the aid of the operators $A^J_{\alpha\beta}$:

Lemma 6 *Let* $\alpha > \beta > 0$. *Let* H_α *be the unique selfadjoint operator in* $\mathcal{H}$ *such that* $-\alpha$ *belongs to the resolvent set of* H_α *and*

$$(H_\alpha + \alpha)^{-1} = G_\alpha + (JG_\alpha)^* JG_\alpha.$$

Suppose, in addition, that the operator J *from* $(D(H), \|\cdot\|_H)$ *into* $\mathcal{H}_{aux}$ *is compact and (26) holds true. Then* $-\beta$ *is an eigenvalue of* H_α *if and only if* 0 *is an eigenvalue of the unique selfadjoint operator* $A^J_{\alpha\beta}$ *in* $\mathcal{H}$ *associated to the lower semibounded closed quadratic form* $\mathcal{E}^J_{\alpha\beta}$ *defined by (27). In this case*

$$f \mapsto (JG_\beta)^* Jf$$

defines a linear bijective mapping from the kernel of $A^J_{\alpha\beta}$ *onto the kernel of* $H_\alpha + \beta$.

Proof: Consider the special case $\gamma = 0$ in Corollary 5. Obviously $G_{\alpha\beta 0} = (\alpha - \beta) G_\beta G_\alpha$. Since G_α is a selfadjoint operator in $\mathcal{H}$ this implies that $(JG_{\alpha\beta 0})^* = G_\alpha (JG_\beta)^*$. Thus the lemma follows immediately from Theorem 3 and Corollary 5 (with $\gamma = 0$). □

We shall show that the number of negative eigenvalues of H_α equals the number of negative eigenvalues of $A^J_{\alpha 0}$; here every eigenvalue is counted as many times as its multiplicity as an eigenvalue of the respective operator. In the proof we shall use the following simple convergence result:

Lemma 7 *Let* $0 \le \beta_0, \beta < \alpha$. *Then*

$$A^J_{\alpha\beta} \longrightarrow A^J_{\alpha\beta_0}, \text{ as } \beta \longrightarrow \beta_0,$$

in the norm resolvent sense.

Proof: Let $E_H(\cdot)$ be the spectral family of the selfadjoint operator H. By (26), the definition (27) of $\mathcal{E}^J_{\alpha\beta}$ and the spectral calculus for the selfadjoint operator H,

$$\begin{array}{rl} & \mathcal{E}^J_{\alpha\beta}(f,f) + (b+1) \parallel f \parallel^2 \\ \geq & (1-a) \parallel A^{1/2}_{\alpha 0} f \parallel^2 + \parallel f \parallel^2 \\ = & \int \left((1-a) \frac{\lambda(\lambda+\alpha)}{\alpha} + 1 \right) d(E_H(\lambda) f, f) \end{array} \tag{29}$$

for every $f \in D(\mathcal{E}^J_{\alpha\beta})$ and every $0 \leq \beta < \alpha$.

By the definitions (27) and (25) and the spectral calculus for H,

$$\begin{array}{rl} & |\mathcal{E}^J_{\alpha\beta}(f,f) - \mathcal{E}^J_{\alpha\beta_0}(f,f)| \\ = & | \parallel A^{1/2}_{\alpha\beta} f \parallel^2 - \parallel A^{1/2}_{\alpha\beta_0} f \parallel^2 | \\ = & | \int \left(\frac{(\lambda+\alpha)(\lambda+\beta)}{\alpha-\beta} - \frac{(\lambda+\alpha)(\lambda+\beta_0)}{\alpha-\beta_0} \right) d(E_H(\lambda) f, f)| \end{array}$$

for every $f \in D(\mathcal{E}^J_{\alpha\beta})$. By (29), this implies that for every compact subset K of $[0,\alpha)$ there exists a finite constant $c_{\beta_0 K}$ such that for every $f \in D(\mathcal{E}^J_{\alpha\beta_0})$ and every $\beta \in K$

$$|\mathcal{E}^J_{\alpha\beta}(f,f) - \mathcal{E}^J_{\alpha\beta_0}(f,f)| \leq c_{\beta_0 K} |\beta - \beta_0| \left(\mathcal{E}^J_{\alpha\beta_0}(f,f) + (b+1) \parallel f \parallel^2 \right).$$

By [29], theorem VIII.25, this implies that

$$A^J_{\alpha\beta} \longrightarrow A^J_{\alpha\beta_0}, \text{ as } \beta \longrightarrow \beta_0, \tag{30}$$

in the norm resolvent sense. □

We shall also use the well known min – max principle: Let A be a lower semibounded selfadjoint operator in $\mathcal{H}$. For every subset $\mathcal{M}$ of $\mathcal{H}$ we put

$$U_A(\mathcal{M}) := \inf\{\mathcal{E}(f,f) : f \in D(\mathcal{E}) \cap \mathcal{M}^\perp, \parallel f \parallel = 1\},$$

where $\mathcal{E}$ denotes the closed quadratic form in $\mathcal{H}$ which is associated to A. For every $n \in \mathbb{N}$ we put

$$\mu_n(A) := \sup\{U_A(\mathcal{M}) : \mathcal{M} \subset \mathcal{H}, \operatorname{card}(\mathcal{M}) \leq n-1\}.$$

Obviously we have

$$\mu_1(A) \leq \mu_2(A) \leq \ldots \leq \mu_n(A) \leq \ldots$$

and the following implication holds true:

$$A \leq \tilde{A} \Longrightarrow \mu_n(A) \leq \mu_n(\tilde{A}), \quad n \in \mathbb{N}.$$

Moreover it is known that for every $n \in \mathbb{N}$ the following holds (min – max principle, cf. [30], theorems XIII.1,2):

Either the number (counting multiplicities) of eigenvalues of A below the essential spectrum of A is larger than or equal to n and $\mu_n(A)$ is the n–th (multiplicities are counted) eigenvalue of A when the eigenvalues are counted in increasing order or $\mu_n(A)$ is the minimum of the essential spectrum of A,

$$\mu_n(A) = \mu_{n+1}(A) = \mu_{n+2}(A) = \dots,$$

and the number (counting multiplicities) of A below the essential spectrum of A is less than or equal to $n-1$.

It easily follows that the following implication holds true:

$$\begin{aligned} &A_k \longrightarrow A, \text{ as } k \longrightarrow \infty, \text{ in the norm resolvent sense } \Longrightarrow \\ &\mu_n(A_k) \longrightarrow \mu_n(A), \text{ as } k \longrightarrow \infty, \quad n \in \mathbb{N}. \end{aligned} \tag{31}$$

In fact, assume that $\mu_n(B_k) \leq c < \mu_n(A)$, $k \in \mathbb{N}$, for some $n \in \mathbb{N}$, some subsequence (B_k) of (A_k) and some constant c. Take b such that $c < b < \mu_n(A)$. Since the sequence (B_k) converges to A in the norm resolvent sense the spectral projectors $E_{B_k}(b)$ converge to the spectral projector $E_A(b)$ in the operator norm sense (cf. [29], theorem VIII.23). Since $\mu_n(B_k) < b$ and $\mu_n(A) > b$ the min – max principle implies that the dimension of the range of $E_{B_k}(b)$, $k \in \mathbb{N}$, is larger than or equal to n and the dimension of the range of $E_A(b)$ is less than n. Since the orthogonal projections $E_{B_k}(b)$ converge to the orthogonal projection $E_A(b)$ in the operator norm sense, this is impossible. The assumption that $\mu_n(B_k) \geq c > \mu_n(A)$, $k \in \mathbb{N}$, for some $n \in \mathbb{N}$, some subsequence (B_k) of (A_k) and some constant c leads to a contradicition in a similar way. Thus the implication (31) holds true.

After these preparations we can easily prove the mentioned result on the number of negative eigenvalues:

Corollary 8 *Under the hypothesis of the Lemma 6 the number of negative eigenvalues of H_α equals the number of negative eigenvalues of $A^J_{\alpha 0}$. Here every eigenvalue is counted as many times as its multiplicity as an eigenvalue of H_α and $A^J_{\alpha 0}$, respectively.*

Proof: For $n \in \mathbb{N}$ and $0 \leq \beta < \alpha$ we put

$$\mu_n(\beta) := \mu_n(A^J_{\alpha\beta}).$$

Obviously the mapping $\beta \mapsto A^J_{\alpha\beta} - \frac{\alpha\beta}{\alpha-\beta}$ is nondecreasing. Thus

$$\mu_n(\cdot) \text{ is strictly increasing }, n \in \mathbb{N}, \tag{32}$$

$$\mu_n(\beta) \uparrow \infty, \text{ as } \beta \uparrow \alpha, \tag{33}$$

and

$$\{n : \mu_n(\beta) = 0\} \cap \{n : \mu_n(\beta_1) = 0\} = \emptyset, \text{ if } \beta \neq \beta_1. \tag{34}$$

The following statements are equivalent

$$\begin{array}{ll} \text{(i)} & \mu_n(0) < 0 \\ \text{(ii)} & \text{there exists a } \beta \in (0, \alpha) \text{ such that } \mu_n(\beta) = 0. \end{array} \tag{35}$$

In fact, if $0 < \beta < \alpha$ and $\mu_n(\beta) = 0$ then $\mu_n(0) < 0$ since the mapping $\mu_n(\cdot)$ is strictly increasing. Conversely let $\mu_n(0) < 0$. By (30) and (31), the mapping $\mu_n(\cdot)$ is continuous. It follows now from (32) and (33) that there exists a β such that $0 < \beta < \alpha$ and $\mu_n(\beta) = 0$.

By the Lemma 6 and the min – max principle,

$$\dim\ker(H_\alpha + \beta) = \dim\ker(A^J_{\alpha\beta}) = \operatorname{card}(\{n : \mu_n(\beta) = 0\}).$$

By (34) and (35), this implies that the number, counting multiplicities, of eigenvalues of H_α in the interval $(-\alpha, 0)$ equals the number of $n \in \mathbb{N}$ such that $\mu_n(0) < 0$. By the min – max principle and since $(-\infty, -\alpha] \subset \rho(H_\alpha)$ (cf. Theorem 3), the corollary is proved □

3 Singular perturbations of the free Hamiltonian

In this section we shall consider the case when H equals the free Hamiltonian $-\Delta$ and J the mapping J_μ defined by (8) where throughout this section μ

denotes a positive Radon measure on $\mathbb{R}^d$ satisfying (9) – (11). We fix an $\alpha > 0$ and for arbitrary $\beta \geq 0$ we put

$$G_\beta := (-\Delta + \beta)^{-1}.$$

It is well known that G_β is an integral operator with nonnegative kernel $g_\beta(x - y)$ provided $\beta > 0$ or $d \geq 3$. Moreover for $\beta > 0$

$$\begin{aligned} g_\beta(x) &= \tfrac{1}{2\beta} \exp(-\sqrt{\beta}|x|), \quad x \in \mathbb{R},\ d = 1, \\ g_\beta(x) &= \tfrac{1}{(2\pi)^{d/2}} \left(\tfrac{|x|}{\sqrt{\beta}}\right)^{1-d/2} K_{d/2-1}(-\sqrt{\beta}|x|), \quad x \in \mathbb{R}^d \setminus \{0\}, \quad d \geq 2. \end{aligned} \tag{36}$$

Here K_a denotes the modified Bessel function of the second kind. Note that

$$g_\beta(x) \uparrow g_0(x), \text{ as } \beta \downarrow 0, \quad x \in \mathbb{R}^d \setminus \{0\},$$

provided $d \geq 3$, while $g_\beta(x) \uparrow \infty$, as $\beta \downarrow 0$, if $d = 1$ or $d = 2$. Note also that $J_\mu G_\beta$ is the integral operator from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d, \mu)$ with kernel $g_\beta(x-y)$, i.e. $J_\mu G_\beta$ has the same kernel as G_β but it is regarded as an operator from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d, \mu)$. Since g_β is real and $g_\beta(x - y) = g_\beta(y - x)$ for all x, y, it follows that the adjoint $(J_\mu G_\beta)^*$ equals the integral operator from $L^2(\mathbb{R}^d, \mu)$ to $L^2(\mathbb{R}^d)$ with kernel $g_\beta(x - y)$, i.e. we have the same kernel but this time one integrates w.r.t. the measure μ and not w.r.t. the Lebesgue measure dx.

By (9) – (11), the operator J_μ from the Sobolev space $H^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d, \mu)$ is compact (cf. [8], Lemma 19). By Lemma 1, there exists a unique selfadjoint operator H_α^μ in $L^2(\mathbb{R}^d)$ such that $-\alpha$ belongs to the resolvent set of H_α^μ and the resolvent formula (7) holds true. Since J_μ is compact H_α^μ has the same singular sequences as $-\Delta$ and, in particular, the essential spectrum of H_α^μ equals $[0, \infty)$.

Since every sequence (f_n) converging to f in the Sobolev space $H^s(\mathbb{R}^d)$ has a subsequence (g_n) such that $\tilde{g}_n \longrightarrow \tilde{f}$ c_s – quasi everywhere, the mapping $f \mapsto \tilde{f}$ from $H^s(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d, \mu)$ is closed. Since, by (10), this mapping is everywhere defined, it is also bounded and there exists a finite constant c such that

$$\int |\tilde{f}|^2 d\mu \leq c \parallel f \parallel_{H^s}^2, \quad f \in H^s(\mathbb{R}^d).$$

Let $\varepsilon > 0$ be arbitrary. Since $s < 2$, there exists a finite constant $b = b(s, c, \alpha, \varepsilon)$ such that

$$c(1 + x^2)^{s/2} \leq \varepsilon x^2 + b, \quad x \in \mathbb{R}^d.$$

Thus the following inequality holds for every $\beta \geq 0$:

$$\| J_\mu f \|_{L^2(\mathbb{R}^d,\mu)} \leq \varepsilon \| A_{\alpha\beta}^{1/2} f \|^2 + b \| f \|^2, \quad f \in H^2(\mathbb{R}^d), \tag{37}$$

where

$$A_{\alpha\beta} := \frac{(-\Delta+\beta)(-\Delta+\alpha)}{\alpha}.$$

Thus all assumptions made about H and J in the previous section are satisfied in the special case when $H = -\Delta$ and $J = J_\mu$. Thus also all conclusions about these operators hold true. In particular, for every $0 \leq \beta < \alpha$ there exists a unique selfadjoint operator $A_{\alpha\beta} - \mu$ in $L^2(\mathbb{R}^d,\mu)$ satisfying

$$\begin{aligned} D(A_{\alpha\beta}-\mu) &\subset H^2(\mathbb{R}^d), \\ (f,(A_{\alpha\beta}-\mu)g) &= (\left(\tfrac{(-\Delta+\beta)(-\Delta+\alpha)}{\alpha-\beta}\right)^{1/2} f, \left(\tfrac{(-\Delta+\beta)(-\Delta+\alpha)}{\alpha-\beta}\right)^{1/2} g) \\ &\quad -\int \bar{\tilde{f}}\tilde{g}\,d\mu, \quad f \in H^2(\mathbb{R}^d), g \in D(A_{\alpha\beta}-\mu). \end{aligned} \tag{38}$$

Moreover we have proved the following

Theorem 9 *Let μ be a positive Radon measure on $\mathbb{R}^d$ satisfying (9) – (11) and $\alpha > 0$. Then the following holds true:*

(i) There exists a unique selfadjoint operator H_α^μ in $L^2(\mathbb{R}^d,\mu)$ such that $-\alpha$ belongs to the resolvent set of H_α^μ and the resolvent of H_α^μ satisfies the equation (7).

(ii) The essential spectrum of H_α^μ equals $[0,\infty)$ and the interval $(-\infty,-\alpha]$ belongs to the resolvent set of H_α^μ.

(iii) Let $0 < \beta < \alpha$. $-\beta$ is an eigenvalue of H_α^μ if and only if 0 is an eigenvalue of the unique selfadjoint operator $A_{\alpha\beta} - \mu$ in $L^2(\mathbb{R}^d)$ satisfying (38). Moreover

$$f \mapsto \int g_\beta(\cdot - y)\tilde{f}(y)\mu(dy),$$

where the function g_β is defined by (36), defines a linear bijective mapping from the kernel of $A_{\alpha\beta} - \mu$ onto the kernel of $H_\alpha^\mu + \beta$.

(iv) If $\mu \neq 0$ and $\mu(\mathbb{R}^d \setminus \Gamma) = 0$ then the operator H_α^μ belongs to the set $\mathcal{A}_\Gamma$ defined in (4).

As a corollary to Theorem 9 and Corollary 8 we get

Corollary 10 *Under the hypothesis of the Theorem 9 the number of negative eigenvalues of the operator H^μ_α equals the number of negative eigenvalues of $A_{\alpha 0} - \mu$. Here every eigenvalue is counted as many times as its multiplicity as an eigenvalue of the respective operator.*

The corollary makes it possible to use the well know scaling properties of the free Hamiltonian in order to get lower bounds for the number of negative eigenvalues of H^μ_α. For instance take any smooth function $g : [0, \infty) \longrightarrow \mathbb{R}$ such that

$$g(r) = \begin{cases} 1, & r \leq 2, \\ 0, & r \geq 4, \end{cases}$$

and put

$$f_n(x) := g(|x|^{1/n}), \quad x \in \mathbb{R}^d, n \in \mathbb{N}.$$

A short computation yields that

$$(f_n, -\Delta(-\Delta + \alpha) f_n) \longrightarrow 0, \quad n \longrightarrow \infty,$$

provided the dimension d of $\mathbb{R}^d$ equals 1 or 2. On the other hand

$$\liminf_{n \longrightarrow \infty} \int |f_n|^2 d\mu \geq \mu(\mathbb{R}^d).$$

Thus there exists an f such that $(f, (A_{\alpha 0} - \mu) f) < 0$ provided $d = 1, 2$ and $\mu \neq 0$. Thus the operator $A_{\alpha 0} - \mu$ has at least one negative eigenvalue in this case. By Corollary 10, the same holds true for the operator H^μ_α:

Corollary 11 *Under the hypothesis of Theorem 9 the operator H^μ_α has at least one negative eigenvalue provided $d < 3$ and $\mu \neq 0$.*

By using the scaling properties of the free Hamiltonian, we can easily derive conditions which are sufficient in order that the operator $A_{\alpha 0} - \mu$ has infinitely many negative eigenvalues. For instance take any smooth function $g : [0, \infty) \longrightarrow \mathbb{R}$ and put

$$f_{R,y}(x) := g(|\frac{x - y}{R}|), \quad x \in \mathbb{R}^d,$$

for every $y \in \mathbb{R}^d$, $R > 0$. A short computation shows that there exists a finite constant $c = c(\alpha, d)$ such that

$$(f_{R,y}, -\Delta(-\Delta+\alpha) f_{R,y}) \leq c R^{d-2}$$

for every $y \in \mathbb{R}^d$ and $R \geq 1$. Thus there exist infinitely many functions f with pairwise disjoint support such that $(f, (A_{\alpha 0} - \mu) f) < 0$ provided there exist sequences (x_n) in $\mathbb{R}^d$ and (R_n) in $[1, \infty)$ satisfying

$$\limsup_{n \longrightarrow \infty} \frac{\mu(\{x \in \mathbb{R}^d : |x - x_n| < R_n\})}{R_n^{d-2}} = \infty \tag{39}$$

and

$$\{x \in \mathbb{R}^d : |x - x_n| < R_n\} \cap \{x \in \mathbb{R}^d : |x - x_j| < R_j\} = \emptyset, \quad n \neq j. \tag{40}$$

Thus the operator $A_{\alpha 0} - \mu$ has infinitely many negative eigenvalues in this case and the same holds true for the operator H_α^μ:

Corollary 12 *Suppose that there exist sequences (x_n) in $\mathbb{R}^d$ and (R_n) in $[1, \infty)$ satisfying (39) and (40). Suppose, in addition, that the hypothesis of the Theorem 9 is satisfied. Then the operator H_α^μ has infinitely many negative eigenvalues.*

By using the fact, that we get the operator $A_{\alpha 0} - \mu$ via a form perturbation of an operator with a well known resolvent, and general results from [6] we can also derive upper bounds for the number of its negative eigenvalues: For $\beta' \geq 0$ denote by $N_0(\mu, \alpha, \beta')$ the number, counting multiplicities, of negative eigenvalues of the operator $A_{\alpha\beta'} - \mu$. Let $\beta > 0$. By (37), the hypothesis of Theorem 5 in [6] is satisfied in the special case when we replace in [6] the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_{aux}$, the operators H, J and A and the number E by the Hilbert spaces $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d, \mu)$, the operators $A_{\alpha\,\beta}$, J_μ and $-I$ and the number 0, respectively. Here I denotes the identity in $L^2(\mathbb{R}^d, \mu)$. Moreover the considerations before the statement of Lemma 1 in [6] yield immediately that in this case the operator U_0 in [6] has to be replaced by $(J_\mu A_{\alpha\beta}^{-1})^*$. Now Theorem 5 in [6] yields the following upper bounds for the number of negative eigenvalues of the operator $A_{\alpha\beta} - \mu$:

$$N_0(\mu, \alpha, \beta) \leq \| J_\mu (J_\mu A_{\alpha\beta}^{-1})^* \|_{B_p}^p, \quad 1 \leq p < \infty.$$

Here $\| C \|_{B_p}$ denotes the the norm of C in the p – th Schatten class, with the convention that $\| C \|_{B_p}$ equals infinity if the operator C does not belong to this class. By (31), Lemma 7 and the min – max – principle,

$$N_0(\mu, \alpha, \beta) \longrightarrow N_0(\mu, \alpha, 0), \quad \beta \downarrow 0.$$

Thus

$$N_0(\mu, \alpha, 0) \leq \sup_{\beta > 0} \| J_\mu (J_\mu A_{\alpha\beta}^{-1})^* \|_{B_p}^p, \quad 1 \leq p < \infty.$$

By Corollary 10, the number (counting multiplicities) $N_0(\mu, \alpha)$ of negative eigenvalues of the operator H_α^μ equals $N_0(\mu, \alpha, 0)$. Thus

$$N_0(\mu, \alpha) \leq \sup_{\beta > 0} \| J_\mu (J_\mu A_{\alpha\beta}^{-1})^* \|_{B_p}^p, \quad 1 \leq p < \infty. \tag{41}$$

By the considerations at the beginning of this section, $A_{\alpha\beta}$ is an integral operator with kernel $k_{\alpha\beta}(x, y)$ where for $\beta' \geq 0$

$$k_{\alpha\beta'}(x, y) := \alpha \int g_{\beta'}(x - y') g_\alpha(y' - y) dy', \quad x, y \in \mathbb{R}^d, \tag{42}$$

and it easily follows that $J_\mu (J_\mu A_{\alpha\beta}^{-1})^*$ is the integral operator in $L^2(\mathbb{R}^d, \mu)$ with the same kernel, i.e.

$$J_\mu (J_\mu A_{\alpha\beta}^{-1})^* h(x) = \int k_{\alpha\beta}(x, y) h(y) \mu(dy) \quad \mu\text{– a.e.}, \quad h \in L^2(\mathbb{R}^d, \mu). \tag{43}$$

Thus the Hilbert – Schmidt norm of the operator $J_\mu (J_\mu A_{\alpha\beta}^{-1})^*$ is given by

$$\| J_\mu (J_\mu A_{\alpha\beta}^{-1})^* \|_{B_2}^2 = \int \int k_{\alpha\beta}(x, y)^2 \mu(dx) \mu(dy). \tag{44}$$

Let $d > 2$. Then, by (36) and (42),

$$0 \leq k_{\alpha\beta}(x, y) \leq k_{\alpha 0}(x, y), \quad x, y \in \mathbb{R}^d. \tag{45}$$

By (41), (44) and (45), the following theorem holds:

Theorem 13 *Let $d \geq 3$. Suppose that the hypothesis of Theorem 9 is satisfied. Denote by $N_0(\mu, \alpha)$ the number, counting multiplicities, of negative eigenvalues of the operator H^μ_α. Then*

$$N_0(\mu, \alpha) \leq \int\int k_{\alpha 0}(x, y)^2 \mu(dx)\mu(dy),$$

where the kernel $k_{\alpha 0}(x, y)$ is defined by (42).

Note that the kernel $k_{\alpha 0}(x, y)$ is bounded when $d \leq 3$. Thus Theorem 13 implies the following

Corollary 14 *Suppose that the hypothesis of Theorem 9 is satisfied, the measure μ is finite and $d = 3$. Then the number, counting multiplicities, of negative eigenvalues of the operator H^μ_α is finite.*

The kernel $k_{\alpha 0}(x, y)$ has a logarithmic singularity for $x - y \sim 0$, if $d = 4$, and $k_{\alpha 0}(x, y) \sim |x - y|^{4-d}$ for $x - y \sim 0$, if $d = 5$. Thus it may happen that the measure μ is finite and the hypothesis of Theorem 9 is satisfied but

$$\int\int k_{\alpha 0}(x, y)^2 \mu(dx)\mu(dy) = \infty$$

and Theorem 13 only yields the trivial upper bound for the number of negative eigenvalues of H^μ_α. On the other hand, the considerations preceeding the statement of Theorem 13 admit to work with Schatten classes of order higher than 2. By a result by M. Solomyak ([31]) on Schatten class norms of integral operators and (43),

$$\| J_\mu (J_\mu A^{-1}_{\alpha\beta})^* \|_{B_p}, \leq \mu(\mathbb{R}^d)^{\frac{q-1}{q}} \left(\sup_{x \in \mathbb{R}^d} \int k_{\alpha\beta}(x, y)^q \mu\,(dy) \right)^{\frac{1}{q}}, \quad 1 < q < 2, \tag{46}$$

where $p = q/(q-1)$. By (41), (45) and (46), the following theorem holds:

Theorem 15 *Let $d \geq 3$. Suppose that the hypothesis of Theorem 9 is satisfied. Let $N_0(\mu, \alpha)$ be the number, counting multiplicities, of the operator H^μ_α. Then*

$$N_0(\mu, \alpha) \leq \mu(\mathbb{R}^d) \left(\sup_{x \in \mathbb{R}^d} \int k_{\alpha 0}(x, y)^q \mu\,(dy) \right)^{\frac{1}{q-1}},$$

where the function $k_{\alpha 0}(x, y)$ is defined by (42).

Appendix

Closed forms and associated selfadjoint operators

Let $\mathcal{D}$ be a linear subspace of the Hilbert space $\mathcal{H}$ and $\mathcal{E} : \mathcal{D} \times \mathcal{D} \longrightarrow \mathbb{C}$ a hermitean sesquilinear form. Assume that there exists a real number c such that $\mathcal{E}(f, f) \geq c \parallel f \parallel^2$ for every $f \in \mathcal{D}$. We choose any $\alpha > -c$ and denote by $\mathcal{E}_\alpha$ the scalar product $\mathcal{E}_\alpha(f, g) := \mathcal{E}(f, g) + \alpha(f, g)$, $f, g \in \mathcal{D}$, on $\mathcal{D}$. The mapping $\mathcal{D} \longrightarrow \mathbb{R}$, $f \mapsto \mathcal{E}(f, f)$ is called a (lower semibounded) closed quadratic form in $\mathcal{H}$ if and only if $(\mathcal{D}, \mathcal{E}_\alpha)$ is complete. Often the domain $\mathcal{D}$ of this quadratic form is denoted by $D(\mathcal{E})$.

Let H be a selfadjoint operator in $\mathcal{H}$ and assume that there exists a real number c such that $H \geq c$, i.e. $(f, Hf) \geq c \parallel f \parallel^2$ for every f in the domain $D(H)$ of H. Denote the spectral family of H by $E_H(\cdot)$. Then

$$\begin{array}{rcl} \mathcal{D} & := & \{f \in \mathcal{H} : \int \lambda \, d \parallel E_H(\lambda) f \parallel^2 < \infty\}, \\ \mathcal{E}(f, f) & := & \int \lambda \, d \parallel E_H(\lambda) f \parallel^2, \quad f \in \mathcal{D}, \end{array} \tag{47}$$

defines a closed quadratic form in $\mathcal{H}$. $\mathcal{E}$ is referred to as the closed quadratic form associated to H.

Vice versa let $\mathcal{E} : \mathcal{D} \longrightarrow \mathbb{R}$ be a closed quadratic form in $\mathcal{H}$ and assume that $\mathcal{D}$ is dense in $\mathcal{H}$. Then, by Kato's representation theorem, there exists a unique lower semibounded selfadjoint operator H in $\mathcal{H}$ such that (47) holds true. H is referred to as the selfadjoint operator associated to $\mathcal{E}$.

Capacities and quasi continuity

$L^2(\mathbb{R}^d)$ denotes the space of (equivalence classes) of functions which are square integrable w.r.t. the Lebesgue measure dx and $\hat{f}$ the Fourier transform of f. Let $s > 0$. $H^s(\mathbb{R}^d)$ denotes the Sobolev space of all $f \in L^2(\mathbb{R}^d)$ such that

$$\parallel f \parallel_{H^s} := \left(\int (1 + x^2)^{s/2} |\hat{f}(x)|^2 dx \right)^{1/2} < \infty.$$

The c_s – capacity of the compact set $K \subset \mathbb{R}^d$ is defined by

$$c_s(K) := \inf \parallel f \parallel_{H^s}^2,$$

where the infimum is taken over all f in the space $C_0^\infty(\mathbb{R}^d)$ of smooth functions f with compact support support satisfying $f(x) \geq 1$ for all $x \in K$. The

c_s – capacity of an arbitrary Borel set B is defined by

$$c_s(B) := \sup c_s(K),$$

where the supremum is taken over all compact subsets of K. The capacity is σ – subadditive, i.e.

$$c_s(\bigcup_{n\in\mathbb{N}} B_n) \leq \sum_{n\in\mathbb{N}} c_s(B_n).$$

The function $g : \mathbb{R}^d \longrightarrow \mathbb{C}$ is quasi continuous w.r.t. the c_s – capacity if and only if for every $\varepsilon > 0$ there exists an open subset O_ε of $\mathbb{R}^d$ such that

$$c_s(O_\varepsilon) < \varepsilon$$

and the restriction of g to the complement $\mathbb{R}^d \setminus O_\varepsilon$ is continuous. Every $f \in H^s(\mathbb{R}^d)$ has a representative $\tilde{f}$ which is quasi continuous w.r.t. the c_s – capacity. If $\tilde{f}$ and f° are representatives of $f \in H^s(\mathbb{R}^d)$ and quasi continuous w.r.t. the c_s – capacity then the c_s – capacity of the set $\{x \in \mathbb{R}^d : \tilde{f}(x) \neq f^\circ(x)\}$ equals zero. In the present note $\tilde{f}$ denotes any repesentative of $f \in H^s(\mathbb{R}^d)$ which is quasi continuous w.r.t. the c_s – capacity; this notation does not indicate which s is meant, but this will always be clear from the context.

Suppose now that the sequence (f_n) converges to f in the Sobolev space $H^s(\mathbb{R}^d)$. Then there exists a subsequence (g_n) of (f_n) such that $\tilde{f}_n \longrightarrow \tilde{f}$ c_s – quasi everywhere, i.e. the c_s – capacity of the set

$$\{x \in \mathbb{R}^d : (\tilde{f}_n(x)) \text{ does not converge to } \tilde{f}(x)\}$$

equals zero.

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