CONVERGENCE OF A hp-STREAMLINE DIFFUSION SCHEME FOR VLASOV–FOKKER–PLANCK SYSTEM

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ABSTRACT. We analyze the hp-version of the streamline diffusion finite element method for the Vlasov–Fokker–Planck system. For this method we prove stability estimates and derive sharp a priori error bounds in a stabilization parameter $\delta \sim \min(h/p, h^2/\sigma)$, with h denoting the mesh size of the finite element discretization in phase-space-time, p the spectral order of approximation, and σ the transport cross-section. In our study we use some hp-techniques introduced by Houston, Schwab and Süli, see e.g. [11]–[13].

1. INTRODUCTION

We study stability and convergence of hp-version of the streamline diffusion (SD) finite element method for a deterministic model for the Vlasov-Fokker-Planck (VFP) system. The objective is to derive sharp *a priori* hp-error bounds for a SD scheme in some L_2 type norms.

The Vlasov-Poisson-Fokker-Planck (VPFP) system arising in the kinetic description of a plasma of Coulomb particles under the influence of a self–consistent internal field and an external force can be formulated as follows: given the initial distribution of particles $f_0(x, v) \geq 0$, in the phase–space variable $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, $d = 1, 2, 3$, and the physical parameters $\beta \geq 0$ and $\sigma \geq 0$, find the distribution function $f(x, v, t)$ for $t > 0$, satisfying the nonlinear system of evolution equations

$$(1.1)\qquad \begin{cases} \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v[(E - \beta v)f] = \sigma \Delta_v f, & \text{in } \mathbb{R}^{2d} \times (0, \infty), \\ f(x, v, 0) = f_0(x, v), & \text{for } (x, v) \in \mathbb{R}^{2d}, \\ E(x, t) = \dfrac{\theta}{|\mathcal{S}|^{d-1}} \dfrac{x}{|x|^d} *_x \rho(x, t), & \text{for } (x, t) \in \mathbb{R}^d \times (0, \infty), \\ \rho(x, t) = \displaystyle\int_{\mathbb{R}^d} f(x, v, t)\, dv, & E = \theta \tilde{E}, \text{ and } \theta = \pm 1, \end{cases}$$

where $x \in \mathbb{R}^d$ is the position, $v \in \mathbb{R}^d$ is the velocity, and $t > 0$ is the time, $\nabla_x = (\partial/\partial x_1, \cdots, \partial/\partial x_d)$, $\nabla_v = (\partial/\partial v_1, \cdots, \partial/\partial v_d)$, and $\cdot$ is the inner product in $\mathbb{R}^d$. In our studies the parameter σ being the transport cross-section is very small and decoupled from $\beta = \mathcal{O}(1)$. Otherwise β and σ are assumed to be the viscosity and the thermal diffusivity coefficients, respectively, which are related by $\sigma = \beta \kappa T_0/m$, with κ being the Boltzmann's constant, T_0 the temperature of the surrounding medium and m the mass of a particle, (thus for normal temperatures the physical parameter σ is very small). $|\mathcal{S}|^{d-1} \sim 1/\omega_d$ is the surface area of the

1991 *Mathematics Subject Classification.* 65M60, 82C80.

Key words and phrases. Vlasov–Poisson–Fokker–Planck, hp-version, streamline diffusion, stability, convergence.

[1,2] Partially supported by the EU contract ERB FMRX CT97 0157.

unit disc in $\mathbb{R}^d$. Finally $\rho(x,t)$ is the spatial density, and $*_x$ denotes the convolution in x. E and ρ can be interpreted as the electrical field and charge, respectively. The macroscopic force field E can be assumed to be of the form

$$E(x,t) = -\nabla_x\Big(\psi(x) + \phi(x,t)\Big), \tag{1.2}$$

with $\psi(x) \geq 0$ being an external potential force, and $\phi(x,t)$ the internal potential field. Then, for $\theta = 1$ the VPFP system models a gas of charged particles, with an external potential ψ, interacting through a mean electrostatic field $-\nabla_x\phi$, generated by their spatial density ρ. Whereas $\theta = -1$ corresponds to a VPFP system modelling particles under the effect of the gravitational potential ψ.

For a gradient field, when E is divergence free and with no viscosity, i.e. for $\beta = 0$, the first equation in (1.1), would become

$$\partial_t f + v\cdot\nabla_x f + E\cdot\nabla_v f = \sigma\Delta_v f, \tag{1.3}$$

which, with the rest of equations in (1.1), gives rise to a simplified VPFP system. When E is given (known), we refer to this system as the Vlasov–Fokker–Planck (VFP) system. For $\sigma = 0$, and with a zero external force, i.e. $\psi(x) \equiv 0$ and hence $E(x,t) = -\nabla_x\phi(x,t)$, we obtain the classical Vlasov–Poisson equation with an internal potential field $\phi(x,t)$ satisfying the Poisson equation

$$\Delta_x\phi(x,t) = -\theta\int_{\mathbb{R}^d} f(x,v,t)dv = -\theta\rho(x,t), \tag{1.4}$$

with the asymptotic boundary condition

$$\begin{cases} \phi(x,t) \to 0, & \text{for} \quad d > 2, \qquad \text{as} \quad |x| \to \infty, \\ \phi(x,t) = \mathcal{O}(\log|x|), & \text{for} \quad d = 2, \qquad \text{as} \quad |x| \to \infty. \end{cases} \tag{1.5}$$

For $\beta \neq 0$ (and $\psi(x) = 0$) we have the following (modified) version of the VPFP equation

$$\partial_t f + v\cdot\nabla_x f - \nabla_x\phi\cdot\nabla_v f = \nabla_v\cdot(\beta v f + \sigma\nabla_v f), \tag{1.6}$$

where ϕ is assumed to be the exact solution for the Poisson equation (1.4) given by

$$\phi(x,t) = \theta\int_{\mathbb{R}^{2d}} \mathcal{G}(x-y)f(y,v',t)dydv', \tag{1.7}$$

with $\mathcal{G}$ being the Green's function associated with the fundamental solution of the Laplace's operator $-\Delta_x$.

1.1. **The Continuous Problem.** The mathematical study of the VPFP/VFP system has been considered by several authors in various settings, see e.g., [9], [10], [18], and [21]. Below we summarize a common theoretical framework involving stability estimates in the deterministic case. These results are due to J.L. Lions [16] and P. Degond [10] and are stated for (1.3) version, (the corresponding studies for (1.6) version are similar but somewhat lengthy): given the electric field $E^n(x,t)$ and the initial data f_0, with certain regularities, find f^{n+1}, the solution of the Vlasov-Fokker-Planck system, satisfying

$$\begin{cases} \partial_t f^{n+1} + v\cdot\nabla_x f^{n+1} + E^n\cdot\nabla_v f^{n+1} - \sigma\Delta_v f^{n+1} = 0, \\ f^{n+1}(x,v,0) = f_0(x,v), \end{cases} \tag{1.8}$$

and then compute the charge density ρ^{n+1} and electrical field E^{n+1} according to

$$\rho^{n+1}(x,t)=\int_{\mathbb{R}^d} f^{n+1}(x,v,t)\,dv, \qquad E^{n+1}(x,t)=C_d\int_{\mathbb{R}^d}\frac{x-y}{|x-y|^d}\rho^{n+1}(y,t)\,dy.$$

Problem (1.8) has a unique solution f^{n+1} satisfying, positivity, L_1 and L_∞ stability estimates:

$$f^{n+1}\ge 0, \quad \|\rho^{n+1}(t)\|_1\le \|f^{n+1}(t)\|_1\le \|f_0\|_1, \quad \|f^{n+1}(t)\|_\infty\le \|f_0\|_\infty. \tag{1.10}$$

See [1], for the definitions of the norms and function spaces. Note that for $\sigma=0$, equation (1.8) becomes the classical linear transport equation, which can be solved, e.g. by the method of characteristics, and the stability properties (1.10) are evident.

For the linear Fokker–Planck equation:

$$f_t+v\cdot\nabla_x f+\mathbf{E}\cdot\nabla_v f-\sigma\Delta_v f=g, \quad f(x,v,0)=f_0(x,v), \tag{1.11}$$

where

$$\mathbf{E}=\Big(\mathbf{E}_i(x,v,t)\Big)_{i=1}^{d},$$

is a given vector field and $f_0(x,v)$ and $g(x,v,t)$ are given functions; existence, uniqueness, stability and regularity properties of the solution for the equation (1.11) are straightforward generalizations of the one–dimensional classical results due to Baouendi and Grisvard [8] for the degenerate type equations. These generalizations as well as coupling to the nonlinear problem are due to J. L. Lions [16] and require some regularity assumptions on the data: f_0, g and $\mathbf{E}$.

In our studies, assuming a continuous Poisson solver of type (1.7) for the equation (1.4), we focus on the numerical convergence analysis of a deterministic model problem for the VFP system in a bounded phase-space-time domain. This is a convection dominated convection-diffusion problem of degenerate type, (full convection, but only small diffusion in v), for which we study the hp-version of the *streamline-diffusion* (SD) finite element method and derive convergence rates, which are otherwise more involved using, e.g. particle methods; the most common discretization schemes for the Vlasov type equations. More specifically, for the locally regular solution f in the Sobolev class $H^{s_K+1}(K)$, we derive *optimal* a priori error estimates, basically of order $\mathcal{O}(\delta_K^{s_K+1/2})$, where $\delta_K\sim\min(h_K/p_K, h_K^2/\sigma)$, with h_K and p_K being the local mesh size and the local spectral order, respectively. A corresponding discontinuous Galerkin study as well as numerical implementations are the subject of a forthcoming paper.

In classical finite element method (h-version) convergence order improvement relies on mesh refinement while keeping the approximation order within the elements at a fixed low value (suitable for problems with highly singular solutions that require small mesh parameter). Some studies on the h-version of the SD finite element method can be found, e.g., in [14] for advection-diffusion, Nevier-Stokes and first order hyperbolic equations, in [15] for Euler and Navier-Stokes equations, in [2] for the Vlasov-Poisson and in [3], and [4] for the Fokker-Planck and Fermi equations. On the other hand in the spectral method the accuracy improvement is accomplished by raising the order of approximation polynomial rather than mesh refinement (advantageous in approximating smooth solutions). However, most realistic problems have local behavior (are locally smooth or locally singular), therefore a more realistic numerical approach would be a combination of mesh refinement in the vicinity of singularities (with lower order polynomial approximations), and higher order polynomial approximations in high regularity regions (with larger, non-refined, mesh parameter). This strategy, which can be viewed as a generalized adaptive approach, is the *hp*-version of the finite element method. For some basic *hp*-finite element studies see, e.g. [7], [19] and [20].

An outline of remaining part of this paper is as follows. In Section 2 we introduce some notation and also our approximation spaces. In section 3 we derive error estimates for projection operators useful in our final estimates. Section 4 is devoted to the study of stability estimates and proof of convergence rates for the *hp*-streamline diffusion approximation of the VFP system. Finally, in appendix, we sketch the proof of existence and give an argument for the uniqueness of the approximate solution.

2. Notation and Assumptions

The continuous problem (1.1), formulated in fully unbounded phase-space-time domain, is not appropriate for numerical considerations. Below we restate the problem (1.1) for $\sigma > 0$ and bounded polyhedral domains $\Omega_x \subset \mathbb{R}^d$ and $\Omega_v \subset \mathbb{R}^d$ associated with some boundary conditions. For simplicity we assume that $\Omega := \Omega_x \times \Omega_v$ is a slight deformation of a bounded canonical cubic domain $(-x_0, x_0)^d \times (-v_0, v_0)^d$, $d = 1, 2, 3$. We start with a, non-homogeneous, initial-boundary value problem for the Vlasov–Fokker–Planck system viz,

$$
(2.1) \quad \begin{cases} \partial_t f + G \cdot \nabla f - \sigma \Delta_v f - \operatorname{div}_v(\beta v f) = S, & \text{in } \Omega_T := \Omega \times (0, T), \\ f(x, v, 0) = f_0(x, v), & \text{in } \Omega_0 := \Omega \times \{0\}, \\ f(x, v, t) = g(x, v, t), & \text{in } \partial\Omega \times (0, T) := \Gamma \times (0, T), \end{cases}
$$

where $\nabla f := (\nabla_x f, \nabla_v f) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_d}, \frac{\partial f}{\partial v_1}, \dots, \frac{\partial f}{\partial v_d}\right)$, $d = 1, 2, 3$, and

$$
G(f) := (v, -\nabla_x \phi) = \left(v_1, \dots, v_d, -\frac{\partial \phi}{\partial x_1}, \dots, -\frac{\partial \phi}{\partial x_d}\right) = (G_1, \dots, G_{2d}).
$$

Here ϕ satisfies

$$
(2.2) \qquad -\Delta_x \phi(x, t) = \int_{\Omega_v} f(x, v, t)\, dv, \qquad (x, t) \in \Omega_x \times (0, T],
$$

where $\nabla_x \phi$ is uniformly bounded with $|\nabla_x \phi| \to 0$ as x approaches $\partial\Omega_x$. Note that G is divergent free

$$
(2.3) \qquad \operatorname{div} G(f) = \sum_{i=1}^{d} \frac{\partial G_i}{\partial x_i} + \sum_{i=d+1}^{2d} \frac{\partial G_i}{\partial v_{i-d}} = 0, \quad d = 1, 2, 3.
$$

For technical reasons we split the boundary into the in–(out) flow boundaries:

$$
(2.4) \qquad \Gamma^{-(+)} = \left\{ (x, v) \in \Gamma := \partial\Omega_x \times \partial\Omega_v \,\middle|\, G \cdot \mathbf{n} < 0 (\geq 0) \right\}, \quad \mathbf{n} = (\mathbf{n}_x, \mathbf{n}_v),
$$

where $\mathbf{n}_x$ and $\mathbf{n}_v$ are outward unit normals to $\partial\Omega_x$ and $\partial\Omega_v$, respectively, and $G := G(f)$. Note that (2.4) is more adequate when σ is negligible.

Our discretization scheme concerns the modified problem (2.1), formulated for the bounded domain Ω_T, and NOT! the original VPFP system stated in $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^+$ as in (1.1). In what follows C will denote a general constant independent of the involved parameters on estimations, unless otherwise explicitly specified.

Let us denote an approximate solution for (2.1) by $\tilde{f}$ and recall the usual general procedure of a numerical investigation by decomposing the error viz,

$$f-\tilde{f}=(f-\Pi f)-(\tilde{f}-\Pi f)\equiv\eta-\xi,$$

where Π is an appropriate projection/interpolation operator from the space of the continuous solution f into the (finite dimensional) space of approximate solution $\tilde{f}$. Considering a suitable norm, denoted by $|||\cdot|||$, the process of estimating the error is split into the following two steps: (i) first we use approximation theory results to derive sharp error bounds for $|||\eta|||$, and then (ii) establish

$$|||\xi|||\leq C|||\eta|||, \tag{2.5}$$

which rely on the stability estimates of bounding $|||\tilde{f}|||$ by the $|||\text{data}|||$. The former step has theoretical nature and is related to the character of the projection operator Π, whereas the latter depending on the structure of the $|||\cdot|||$-norm varies in the order of its difficulty.

Below we present some basic assumptions/notation necessary in hp-studies for the continuous approximations, (see, e.g. [11]): assume a partition $\mathcal{P}$ of $\Omega=\Omega_x\times\Omega_v$ into open patches P which are images of canonical 2, 4 or 6-dimensional "cube": $\hat{P}=(-1,1)^{2d}:=\hat{I}^{2d},\ d=1,2,3,\ \hat{I}=(-1,1)$, under smooth bijections F_P:

$$\forall P\in\mathcal{P}:\qquad P=F_P(\hat{P}).$$

A mesh $\mathcal{T}$ on Ω is constructed by subdividing the patches: For each P, first we subdivide $\hat{P}=(-1,1)^{2d}$, into $2d$-dimensional generalized quadrilateral elements ($2d$-dimensional prisms, i.e. generalized triangular elements would work as well) labeled $\hat{\tau}$ which are affine equivalent to $\hat{P}$, we call this mesh $\hat{\mathcal{T}}_P$ (on $\hat{P}$). On each $P\in\mathcal{P}$ we define a mesh $\mathcal{T}_P$ by setting

$$\forall P\in\mathcal{P}:\qquad \mathcal{T}_P:=\{\tau|\tau=F_P(\hat{\tau}),\ \hat{\tau}\in\hat{\mathcal{T}}_P\}.$$

Note that each $\hat{\tau}(\tau)$ is an image of the reference domain $\hat{P}$ under an affine mapping $A_{\hat{\tau}}:\hat{P}\to\hat{\tau}(F_\tau=F_P\circ A_{\hat{\tau}})$. Now $\mathcal{T}:=\cup_{P\in\mathcal{P}}\mathcal{T}_P$ is a mesh on Ω. we also define the function space

$$F_{\mathcal{P}}=\{F_P:P\in\mathcal{P}\},$$

and the polynomial space

$$\mathcal{A}_p=\operatorname{span}\{(\hat{x},\hat{v})^\alpha:0\leq\alpha_i\leq p,\ 1\leq i\leq 2d\},$$

where

$$(\hat{x},\hat{v})\in\hat{P}:=\{(\hat{x},\hat{v})\in\mathbb{R}^d\times\mathbb{R}^d:|\hat{x}_i|\leq 1\&|\hat{v}_i|\leq 1\}.$$

We let now $\mathbf{p}$ be a polynomial degree vector in $\mathcal{T}$,

$$\mathbf{p}=\{p_\tau:\tau\in\mathcal{T}\},$$

and define the continuous hp-finite element spaces

$$S^{\mathbf{p},k}(\Omega,\mathcal{T},F_{\mathcal{P}}):=\{f\in H^k(\Omega):f|_\tau\circ F_\tau\in\mathcal{A}_{p_\tau},\quad\tau\in\mathcal{T}\},\quad k=0,1,\dots,$$

for polynomials with degree vector $\mathbf{p}$, and

$$S^{p,k}(\Omega,\mathcal{T},F_{\mathcal{P}}):=\{f\in S^{\mathbf{p},k}(\Omega,\mathcal{T},F_{\mathcal{P}}):\mathbf{p}=(p,p,\dots,p)\},$$

for the uniform polynomial degree $p_\tau=p,\ \forall\tau,\ p>1$.

Finally We denote by $\|f\|_{k,\hat{I}}$ and $|f|_{k,\hat{I}}$ the $H^k(\hat{I})$ norm and seminorm on $\hat{I}$, respectively (we shall suppress $k=0$, corresponding to the L_2-norm). We also denote by $S^p(\hat{I})$ the set of polynomials of degree p on $\hat{I}$.

Remark: To invoke the time variable we shall, basically, use the same notation: we assume a partition $\mathcal{Q}$ of $\Omega_T = \Omega \times (0,T)$ into open patches Q which are images of canonical cube $\hat{Q} = (-1,1)^{2d+1}$ subdivided into elements $\hat{k} := \hat{\tau} \times \hat{\kappa}$, where each $\hat{\kappa}$ is affine equivalent to $\hat{I}$ corresponding to the time interval $(0,T)$. The exception is that the progress in the time direction is performed successively on the slabs $\Omega_m := \Omega \times (t_m, t_{m+1}),\ m = 0,\dots,M-1$, with $t_0 = 0$ and $t_M = T$, and may have jump discontinuities across the discrete time levels $t_m,\ m = 1,\dots,M-1$. A global mesh is now denoted by $\mathcal{K}$, and more specific notations are given in Section 4.

3. Approximation of the Projection Error

We recall that proving the stability estimate (2.5) and estimates for the projection error η, in some suitable norm, are the main objectives in our investigations. For our choice of the norm: $|||\cdot|||$, as we shall see in the stability estimates of the next section, the terms involved in projection error are, basically, $\|\eta\|$ and $\|\mathcal{D}\eta\|$, where $\mathcal{D} := (\nabla_x, \nabla_v, d/dt)$ denotes the total gradient operator. In this section we estimate these two quantities for our $2d+1$ dimensional problem.

To proceed we denote by $\pi_p^i f$ the 1-dimensional L_2-projection of f onto the polynomials of degree p in the i-th coordinate, where $1 \le i \le d$ would correspond to x_is for the spatial variable, $d+1 \le i \le 2d$ to v_is for the velocity, and $i = 2d+1$ for the time variable. We shall apply the tensor product in $2d+1$-dimensions to the following one and two dimensional results due to [20]:

Proposition 3.1. *Let $f \in H^{k+1}(\hat{I})$ for some $k \ge 0$. Then for every $p \ge 1$ there exists a projection $\pi_p f \in S^p(\hat{I})$ such that,*

$$\|f' - (\pi_p f)'\|_{\hat{I}}^2 \le \frac{(p-s)!}{(p+s)!} |f|_{s+1,\hat{I}}^2, \tag{3.1}$$

$$\|f - \pi_p f\|_{\hat{I}}^2 \le \frac{1}{p(p+1)} \frac{(p-s)!}{(p+s)!} |f|_{s+1,\hat{I}}^2, \tag{3.2}$$

for any $0 \le s \le \min(p,k)$. Moreover,

$$\pi_p f(\pm 1) = f(\pm 1). \tag{3.3}$$

In particular for any $f \in H^1(\hat{I})$ we have that,

$$\|(\pi_p f)'\|_{\hat{I}} \le 2\|f'\|_{\hat{I}}, \qquad \|\pi_p f\|_{\hat{I}} \le \|f\|_{\hat{I}} + \frac{1}{\sqrt{p(p+1)}} \|f'\|_{\hat{I}}. \tag{3.4}$$

Corollary 3.1. *Let $p \ge 1$ and assume that $\psi \in H^{k+1}(\hat{I}^2)$ for some $k \ge 1$. Then for each $i,\ j,\ 0 \le i,\ j \le 2d+1$, the projectors π_p^i, and π_p^j satisfy the following estimates:*

$$\|\psi - \pi_p^i \psi\|_{\hat{I}^2}^2 \le \frac{1}{p(p+1)} \frac{(p-s)!}{(p+s)!} \|\partial_i^{s+1} \psi\|_{\hat{I}^2}^2, \tag{3.5}$$

$$\begin{aligned} \|\pi_p^i(\psi - \pi_p^j \psi)\|_{\hat{I}^2}^2 \le\ & \frac{2}{p(p+1)} \frac{(p-s)!}{(p+s)!} \|\partial_j^{s+1} \psi\|_{\hat{I}^2}^2 \\ & + \frac{2}{p^2(p+1)^2} \frac{(p-(s-1))!}{(p+(s-1))!} \|\partial_i \partial_j^s \psi\|_{\hat{I}^2}^2, \end{aligned} \tag{3.6}$$

where we have identified $\hat{I}_i \times \hat{I}_j$ by $\hat{I}^2$, and π_p^0 by the identity operator.

Proof. See [11]. □

We just generalize this procedure to arbitrary d (i.e. to $2d+1$ dimensions): To this approach we let $\Pi_p = \prod_{i=1}^{2d+1} \pi_p^i$ denote the tensor product projector and $\mathcal{D} = (\nabla_x, \nabla_v, d/dt)$ the $2d+1$ total derivative. We also define the binary multi-index $|m|_l \equiv \sum_{n=1}^{l} m_n$, with $m_n = 0$ or 1. Now we can formulate the main result in this section as:

Theorem 3.2. *Let $\hat{Q} := \hat{P} \times \hat{I}$, $p \geq 1$, $f \in H^{k+1}(\hat{Q})$ for some $k \geq 1$, and set $0 \leq s \leq \min(p,k)$. Then assuming $\Pi_p f = f$ at the vertices of $\hat{Q}$ and $\eta_p := f - \Pi_p f$, we have the following $\|\cdot\| \equiv \|\cdot\|_{L_2(\hat{Q})}$ estimates for η_p:*

$$\|\eta_p\|^2 \leq (2d+1) \sum_{i=1}^{2d+1} 2^{i-1} \sum_{|m|_{i-1} \leq i-1} \alpha_p^{|m|_{i-1}+1} \beta_{|m|_{i-1}} \|\partial^{|m|_{i-1}} \partial_i^{s-|m|_{i-1}+1} f\|^2,$$

and its total derivative $\mathcal{D}\eta_p = (\nabla_x, \nabla_v, d/dt)\eta_p$:

$$\|\mathcal{D}\eta_p\|^2 \leq \sum_{i=1}^{2d+1} (2d+1) \sum_{j=1}^{2d+1} \sum_{\substack{|m|_{j-1} \leq j-1 \\ m_i = 1}} 2^j \alpha_p^{|m|_{j-1}} \beta_{|m|_{j-1}} \|\partial^{|m|_{j-1}} \partial_j^{s-|m|_{j-1}+1} f\|^2,$$

where $\partial^{|m|_{i-1}} = \partial_1^{m_1} \partial_2^{m_2} \cdots \partial_{i-1}^{m_{i-1}}$, $\alpha_p = \frac{1}{p(p+1)}$, and $\beta_{|m|_k} = \frac{(p-s+|m|_k)!}{(p+s-|m|_k)!}$.

Proof. We may use the telescopic identity

$$f - \Pi_p f = \Big(f - \prod_{k=1}^{2d+1} \pi_p^k f\Big) = \sum_{i=1}^{2d+1} \Big(\prod_{j=0}^{i-1} \pi_p^j\Big)(f - \pi_p^i f),$$

to get the estimate

$$\|f - \Pi_p f\|^2 \leq (2d+1) \sum_{i=1}^{2d+1} \Big\| \Big(\prod_{j=0}^{i-1} \pi_p^j\Big)(f - \pi_p^i f)\Big\|^2. \tag{3.9}$$

By a straight forward calculus it is easy to show that,

$$\Big\| \Big(\prod_{j=0}^{n} \pi_p^j\Big)(f - \pi_p^{n+1} f)\Big\|^2 \leq \sum_{|m|_n \leq n} 2^n \alpha_p^{|m|_n+1} \beta_{|m|_n} \|\partial^{|m|_n} \partial_{n+1}^{s-|m|_n+1} f\|^2. \tag{3.10}$$

Note that for $n = 0$, and 1, (3.10) is just as (3.5) and (3.6), respectively. Further, since $\pi_p^0 = id$, we have by the second inequality in (3.4) and twice use of (3.6) that

$$\begin{aligned}
\|\pi_p^1 \pi_p^2 (f - \pi_p^3 f)\|^2 \leq& 2\|\pi_p^2 (f - \pi_p^3 f)\|^2 + \frac{2}{p(p+1)} \|\partial_1 \pi_p^2 (f - \pi_p^3 f)\|^2 \\
\leq& 2\frac{2}{p(p+1)} \frac{(p-s)!}{(p+s)!} \|\partial_3^{s+1} f\|^2 + 2\frac{2}{p^2(p+1)^2} \frac{(p-(s-1))!}{(p+(s-1))!} \|\partial_2 \partial_3^s f\|^2 \\
&+ \frac{2}{p(p+1)} \Big\{ \frac{2}{p(p+1)} \frac{(p-(s-1))!}{(p+(s-1))!} \|\partial_1 \partial_3^s f\|^2 \\
&\qquad + \frac{2}{p^2(p+1)^2} \frac{(p-(s-2))!}{(p+(s-2))!} \|\partial_1 \partial_2 \partial_3^{s-1} f\|^2 \Big\},
\end{aligned}$$

which gives (3.10) for $n = 2$. For the remaining values of n, i.e. for $3 \leq n \leq 2d+1$, (3.10) is justified by a similar, however lengthy and "induction-like" procedure which we omit.

The first assertion of the theorem now follows from (3.9) and (3.10). To show the second estimate we start by rewriting and subsequently simplifying, via (3.9), the total derivative $\mathcal{D}\eta_p$,

$$\|\mathcal{D}\eta_p\|^2 = \sum_{i=1}^{2d+1} \Big\|\partial_i\Big(f - \prod_{k=1}^{2d+1} \pi_p^k f\Big)\Big\|^2 \le \sum_{i=1}^{2d+1} (2d+1) \sum_{j=1}^{2d+1} \Big\|\partial_i\Big(\prod_{l=0}^{j-1} \pi_p^l\Big)(f - \pi_p^j f)\Big\|^2.$$

Below we split the estimate of $\|\partial_i(\prod_{l=0}^{j-1} \pi_p^l)(f - \pi_p^j f)\|_{\hat{Q}}^2$ into the following three possible cases:

Case I: $i \le j-1$. Using the first estimate in (3.4) we have

$$\Big\|\partial_i\Big(\prod_{l=0}^{j-1} \pi_p^l\Big)(f - \pi_p^j f)\Big\|^2 = \Big\|\partial_i \pi_p^i \Big(\prod_{\substack{l=0\\ l\ne i}}^{j-1} \pi_p^l\Big)(f - \pi_p^j f)\Big\|^2 \le 4\Big\|\partial_i \Big(\prod_{\substack{l=0\\ l\ne i}}^{j-1} \pi_p^l\Big)(f - \pi_p^j f)\Big\|^2.$$

Now since ∂_i is no longer in the direction of any of the remaining projections in the product, we can use the second estimate in (3.4) and (3.10) as

$$\Big\|\partial_i \Big(\prod_{\substack{l=0\\ l\ne i}}^{j-1} \pi_p^l\Big)(f - \pi_p^j f)\Big\|_{\hat{Q}}^2 \le \sum_{\substack{|m|_{j-1}\le j-1\\ m_i=1}} 2^{j-2}\alpha_p^{|m|_{j-1}}\beta_{|m|_{j-1}} \Big\|\partial^{|m|_{j-1}}\partial_j^{s-|\bar{m}|_{j-1}+1} f\Big\|_{\hat{Q}}^2,$$

where $|\bar{m}|_{j-1} = |m|_{j-1}$, with $m_k = 0$ or 1 for $k \ne i,\ 0 \le k \le j-1$, and $m_i \equiv 1$. In this way the contribution of ∂_i is included in the right hand side above. Hence, we have shown the second assertion of the theorem in case I.

Case II: $i = j$. Thus we can write

$$\Big\|\partial_i\Big(\prod_{l=0}^{j-1} \pi_p^l\Big)(f - \pi_p^j f)\Big\|_{\hat{Q}}^2 = \Big\|\partial_j\Big(\prod_{l=0}^{j-1} \pi_p^l\Big)(f - \pi_p^j f)\Big\|_{\hat{Q}}^2 = \|\partial_j(\mathcal{F} - \pi_p^j\mathcal{F})\|_{\hat{Q}}^2,$$

where $\mathcal{F} = (\prod_{l=0}^{j-1} \pi_p^l)f$. By (3.1) we have $\|\partial_j(\mathcal{F} - \pi_p^j\mathcal{F})\|_{\hat{Q}}^2 \le \beta_0\|\partial_j^{s+1}\mathcal{F}\|_{\hat{Q}}^2$. This quantity can now be estimated by a (repeated) use of the second estimate in (3.4):

$$\Big\|\Big(\prod_{l=0}^{j-1} \pi_p^l\Big)\varphi\Big\|_{\hat{Q}}^2 \le \sum_{|m|_{j-1}\le j-1} 2^{j-1}\alpha_p^{|m|_{j-1}}\|\partial^{|m|_{j-1}}\varphi\|_{\hat{Q}}^2,$$

so that, replacing φ by $\partial_j^{s+1}f$, we obtain the desired result also for the case $i = j$.

Case III: $i > j$. Here we can apply (3.10) directly since ∂_i and the projections in $\|\partial_i(\prod_{l=0}^{j-1} \pi_p^l)(f - \pi_p^j f)\|_{\hat{Q}}^2$, are decoupled and therefore

$$\Big\|\partial_i\Big(\prod_{l=0}^{j-1} \pi_p^l\Big)(f - \pi_p^j f)\Big\|_{\hat{Q}}^2 \le \sum_{|m|_{j-1}\le j-1} 2^{j-1}\alpha_p^{|m|_{j-1}+1}\beta_{|m|_{j-1}-1}\times$$
$$\times \|\partial_i\partial^{|m|_{j-1}}\partial_j^{s-|m|_{j-1}}f\|_{\hat{Q}}^2.$$

So, summing over i, we conclude that the second estimate of the theorem holds and the proof is complete. □

Remark. We can write the above estimates in a general setting for a partition $\mathcal{R}$ of a bounded, convex, curved polyhedral domain $D \subset \mathbb{R}^{\mathcal{N}}$: Let $R \in \mathcal{R}$ be an image of the $\mathcal{N}$-dimensional canonical hypercube $\hat{R} := (-1,1)^{\mathcal{N}}$, under the bijective map $G_R : R = G_R(\hat{R})$, and with a generalized $\mathcal{N}$-dimensional quadrilateral mesh $\mathcal{M}_R$. Then for a global generalized quadrilateral mesh $\mathcal{M} := \cup_{R\in\mathcal{R}}\mathcal{M}_R$ on D, the projection error estimates are obtained by change of variables and a simple scaling argument where we assume that the patch $\hat{R}$ is the canonical deformation of R with no significant rescaling. More specifically we assume that there are positive constants c_1 and c_2 such that

$$c_1 \le h_K/\hat{h}_K \le c_2, \qquad \forall K \in \mathcal{M}, \tag{3.16}$$

where $h_K = \text{diam}(K)$, $\hat{h}_K = \text{diam}(\hat{K})$, $K = G_R(\hat{K})$, and $\hat{K} \subset \hat{R}$ is a reference element in the mesh $\mathcal{M}_R$. All the corresponding notation such as the polynomial degree distribution $\mathbf{r} = \{r_K : K \in \mathcal{M} := \cup_{R\in\mathcal{R}}\mathcal{M}_R\}$, the affine mapping $A_{\hat{K}} : \hat{R} \to \hat{K}$, the patch-map vector $G_{\mathcal{R}} = \{G_R : R \in \mathcal{R}\}$, and the element map $G_K := G_R \circ A_{\hat{K}}$ with $K = G_K(\hat{R})$, as well as the function space $S^{\mathbf{r},k}(D,\mathcal{M},G_{\mathcal{R}})$ are defined correspondingly as in the Section 2. However, since in the streamline diffusion method we allow discontinuities in time, we formulate the generalization in fully discontinuous setting using a local version of $S^{\mathbf{r},k}(D,\mathcal{M},G_{\mathcal{R}})$ with, only, elementwise high regularity:

$$S^{\mathbf{r},\mathbf{k}}_{loc}(D,\mathcal{M},G_{\mathcal{R}}) := \{f \in S^{\mathbf{r},0}(D,\mathcal{M},G_{\mathcal{R}}) : f|_K \in H^{k_K+1}(K)\},$$

where $\mathbf{k} := \{k_K : K \in \mathcal{M}\}$, and we have the following general result:

Theorem 3.3. *Let $R \in \mathcal{R}$ and the polynomial degree distribution $\mathbf{r}$ be defined as above. $\forall K \in \mathcal{M}_R$, let $f|_K \in H^{k_K+1}(K)$ for some $k_K \ge 1$ and define $\Pi f \in S^{\mathbf{r},\mathbf{k}}_{loc}(D,\mathcal{M},G_{\mathcal{R}})$ elementwise by $(\Pi f)|_K \circ G_R := \Pi_{r_K}(f|_K \circ G_R)$, $\forall K \in \mathcal{M}_R$. Then, for $r_K \ge 1$ and for $0 \le s_K \le \min(r_K,k_K)$ we have the following estimates:*

$$\begin{aligned}
\|f - \Pi f\|_R^2 \le& C \sum_{K\in\mathcal{M}_R} \left(\frac{h_K}{2}\right)^{2s_K+2} \Phi_1(r_K,s_K)\|\hat{f}\|^2_{s_K+1,\hat{K}},\\
\|\mathcal{D}(f - \Pi f)\|_R^2 \le& C \sum_{K\in\mathcal{M}_R} \left(\frac{h_K}{2}\right)^{2s_K} \Phi_2(r_K,s_K)\|\hat{f}\|^2_{s_K+1,\hat{K}},
\end{aligned}$$

where $\hat{f} = f \circ G_R$, $K = G_R(\hat{K})$, $\|\cdot\|_{s_K+1,\hat{K}}$ is the Sobolev norm in $H^{s_K+1}(\hat{K})$ and

$$\Phi_1(p,s) = \mathcal{N}\sum_{i=1}^{\mathcal{N}} 2^{i-1} \sum_{|m|_{i-1}\le i-1} \alpha_p^{|m|_{i-1}+1}\beta_{|m|_{i-1}},$$

$$\Phi_2(p,s) = \mathcal{N}\sum_{i=1}^{\mathcal{N}}\sum_{j=1}^{\mathcal{N}} 2^{j} \sum_{\substack{|m|_{j-1}\le j-1\\ m_i=1}} \alpha_p^{|m|_{j-1}}\beta_{|m|_{j-1}}.$$

Proof. The proof is based on a scaling argument due to the use of a corresponding affine mapping $A_{\hat{K}}$, this time $A_{\hat{K}} : \hat{R} \to \hat{K}$, on the results of theorem 3.2 above. It is just a consequence of applying tensor product to the proof of Theorem 3.4 in [11]. □

4. The streamline diffusion method

The SD-method for (2.1) is based on using finite elements over the phase-space-time domain Ω_T. To define this method, following the notation in Section 2, we let $\mathcal{T}_h = \{\tau\}$ be a finite element subdivision of $\Omega = \Omega_x \times \Omega_v$ into open elements $\tau := F_P(\hat{\tau})$, where P corresponds to a patch in Ω, and let $0 = t_0 < t_1 < \cdots < t_M = T$ be a subdivision of the time interval $(0,T)$ into subintervals $I_m := (t_m, t_{m+1})$. Let $\mathcal{K}_h = \{K\}$ be the corresponding subdivision of Ω_T into elements $K = \tau \times I_m$ with h being piecewise constant mesh function defined by $h(x,v,t) := h_K = \text{diam}(K)$, $(x,v,t) \in K$. For each $m = 0,\ldots,M-1$, we denote the corresponding subdivision of $\Omega_m := \Omega \times I_m$ by $\mathcal{K}_{h,m} := \{K : \; K = \tau \times I_m, \; \tau \in \mathcal{T}_h\}$. Thus $\mathcal{K}_h = \cup_m \mathcal{K}_{h,m}$. We assume that the family of partitions $\{\mathcal{K}_h\}_{h>0}$ is and *shape regular* (quasi-uniform); i.e., for each $K \in \mathcal{K}_h$ there is an inscribed $(2d+1$ dimensional) sphere in K such that the ratio of the diameter of this sphere and the diameter of K is bounded below independent of K and h_K: there is a positive constant C_0, independent of h, such that

$$C_0 h_K^{2d+1} \le meas(K), \qquad \forall K \in \cup_h \mathcal{K}_h. \tag{4.1}$$

Now on each slab Ω_m we define a corresponding finite element space by

$$V_h^{\mathbf{p}_m} = \Big\{ f \in S^{\mathbf{p}_m,\mathbf{k}}(\Omega_m, \mathcal{K}_{h,m}) : \; f\Big|_K \in P_{p_K}(\tau) \times P_{p_K}(I_m); \quad \forall K = \tau \times I_m \Big\},$$

where $P_{p_k}(K)$ denotes the set of polynomials in x, v, and t of degree at most $p_k \ge 1$ on K. We let now $\mathbf{q} = (\mathbf{p}_0, \mathbf{p}_2, \ldots \mathbf{p}_{M-1})$ be the polynomial degree (multi-) vector in the mesh $\mathcal{K}_h$ for the Ω_T, and define

$$V_h^{\mathbf{q}} = \prod_{m=0}^{M-1} V_h^{\mathbf{p}_m},$$

to be a finite element space in the whole $\Omega_T = \Omega \times (0,T)$. Further, for convenience, we introduce the slabwise representations:

$$(f,g)_m = (f,g)_{\Omega_m}, \qquad \|g\|_m = (g,g)_m^{1/2},$$

and define the inner product and seminorm at the time level t_m by

$$\langle f,g \rangle_m = (f(\cdot,\cdot,t_m), (g(\cdot,\cdot,t_m))_\Omega, \quad |g|_m = \langle g,g \rangle_m^{1/2}.$$

We also present the jump term by,

$$[g] = g^+ - g^-,$$

where (to include also the case with $\sigma \equiv 0$),

$$g^\pm = \lim_{s\to 0\pm} g(x,v,t+s), \qquad \text{for } (x,v) \in (\text{Int } \Omega_x) \times \Omega_v, \qquad t \in I,$$

$$g^\pm = \lim_{s\to 0\pm} g(x+sv,v,t+s), \qquad \text{for } (x,v) \in \partial\Omega_x \times \Omega_v, \qquad t \in I,$$

and use the following notation for the boundary integrals

$$\langle f^\mp, g^\mp \rangle_{\Gamma^\pm} = \int_{\Gamma^\pm} f^\mp g^\mp \, \big|(G^h \cdot \mathbf{n})\big| \, d\nu, \qquad G^h := G(f^h) \equiv G(f_{SD}),$$

$$\langle f^\mp, g^\mp \rangle_{\lambda_m^\pm} = \int_{I_m} \langle f^\mp, g^\mp \rangle_{\Gamma^\pm} \, dt, \qquad \lambda_m^\pm := \Gamma^\pm \times I_m,$$

$$\langle f^\mp, g^\mp \rangle_{\Lambda^\pm} = \int_0^T \langle f^\mp, g^\mp \rangle_{\Gamma^\pm} \, dt, \qquad \Lambda^\pm := \Gamma^\pm \times (0,T).$$

4.1. **Stability of the time dependent SDM.** In the conventional h version of the SD-method for time dependent problems, assuming $\tilde{f}$ to be an approximate solution and using test functions of the form: $u+\delta\Big(u_t+G(\tilde{f})\cdot\nabla u\Big)$, where δ is a small parameter (normally $\delta\sim h$), would supply us with a necessary (missing) diffusion term of order δ in the direction of the streamlines: $(1,G(\tilde{f}))$: More specifically, in the stability estimates we will be able to control an extra term of the form $\delta\|u_t+G(\tilde{f})\cdot\nabla u\|\sim h\|u_t+G(\tilde{f})\cdot\nabla u\|$. In the hp studies, however, the choice of δ is somewhat more involved and depends on the equation type as well as the parameters h and p which are chosen locally (elementwise) in an optimal manner. Therefore, in our estimates, δ would appropriately appear as an elementwise (local) parameter. Below we formulate both global and local time-dependent SD-method for problem (2.1) and continue the analysis of hp-version for the local case. The SD-method for (2.1) can now be formulated as follows: *find* $f_{SD}\in V_h^{\mathbf{q}}$ *such that for* $m=0,\dots M-1$,

$$
\begin{aligned}
&\Big(f_t+G(\tilde{f})\cdot\nabla f,u+\delta(u_t+G(f_{SD})\cdot\nabla u)\Big)_m+\sigma\Big(\nabla_v f,\nabla_v u\Big)_m\\
&\qquad+\langle[f],u^+\rangle_m-\delta\sigma\Big(\Delta_v f,u_t+G(f_{SD})\cdot\nabla u\Big)_m\\
&\qquad-\Big(\nabla_v\cdot(\beta v f),u+\delta(u_t+G(f_{SD})\cdot\nabla u)\Big)_m\\
&\qquad=\Big(S,u+\delta(u_t+G(f_{SD})\cdot\nabla u)\Big)_m+\langle g^+,u^+\rangle_{\lambda_m^-}+\langle g^-,u^-\rangle_{\lambda_m^+}.
\end{aligned}\tag{4.4}
$$

The problem (4.4) is equivalent to: *find* $f_{SD}\in V_h^{\mathbf{q}}$ *such that,*

$$
B_\delta(G(f_{SD});f_{SD},u)-J_\delta(f_{SD},u)=L_\delta(u)\qquad\forall u\in V_h^{\mathbf{q}},\tag{4.5}
$$

where for a given appropriate function $\tilde{f}$, the trilinear form B_δ is defined as

$$
\begin{aligned}
B_\delta(G(\tilde{f});f,u)=&\sum_{m=0}^{M-1}\Big[\Big(f_t+G(\tilde{f})\cdot\nabla f,u+\delta(u_t+G(f_{SD})\cdot\nabla u)\Big)_m\\
&+\sigma\Big(\nabla_v f,\nabla_v u\Big)_m-\delta\sigma\Big(\Delta_v f,u_t+G(f_{SD})\cdot\nabla u\Big)_m\Big]\\
&+\sum_{m=1}^{M-1}\langle[f],u^+\rangle_m+\langle f^+,u^+\rangle_{\Lambda^-}+\langle f^-,u^-\rangle_{\Lambda^+}+\langle f^+,u^+\rangle_0,
\end{aligned}
$$

the bilinear form J_δ by,

$$
J_\delta(f,u)=\sum_{m=0}^{M-1}\Big(\nabla_v\cdot(\beta v f),u+\delta(u_t+G(f_{SD})\cdot\nabla u)\Big)_m,
$$

and finally the linear form L_δ is given by,

$$
L_\delta(u)=\sum_{m=0}^{M-1}\Big(S,u+\delta(u_t+G(f_{SD})\cdot\nabla u)\Big)_m+\langle f_0,u^+\rangle_0+\langle g^+,u^+\rangle_{\Lambda^-}+\langle g^-,u^-\rangle_{\Lambda^+}.
$$

Note that both B_δ and J_δ depend implicitly on f_{SD} through the term $G(f_{SD})$.

In the sequel we let the parameter $\beta=\mathcal{O}(1)$, and relate the cross-section σ to the element size h_K by assuming that $\sigma\le\min_K h_K,\ K\in\mathcal{K}_h$. Note also that the discrete version of (2.3) takes now the following form:

$$\text{div } G(f_{SD}) = 0. \tag{4.8}$$

Stability and convergence estimates for (4.5) are derived in a triple norm defined by

$$|||u|||^2 = \frac{1}{2}\Big[2\sigma\|\nabla_v u\|^2_{\Omega_T} + |u|^2_M + |u|^2_0 + \sum_{m=1}^{M-1} |[u]|^2_m + 2\delta\|u_t + G(f_{SD})\cdot\nabla u\|^2_{\Omega_T} + \\ + 3\int_{\partial\Omega\times I} u^2 \big|G^h\cdot\mathbf{n}\big|\, d\nu\, ds\Big].$$

Let now $(\cdot,\cdot)_K$ denote the L_2-inner product over K and define the nonnegative piecewise constant function δ by

$$\delta|_K = \delta_K, \qquad \text{for} \quad K \in \mathcal{K}_h,$$

where δ_K is a nonnegative constant on element K. To formulate the local version of (4.5) we replace in the definitions for B_δ, J_δ and L_δ the inner products $(\cdot,\cdot)_m$, over the slab Ω_m by the corresponding sum: $\sum_{K\in\mathcal{K}_{h,m}}(\cdot,\cdot)_K$, and all δ by δ_K. Thus, more specifically we have the problem (4.5), with the trilinear form B_δ defined as:

$$\begin{aligned} B_\delta(G(\tilde f); f, u) = \sum_{m=0}^{M-1}\sum_{K\in\mathcal{K}_{h,m}} \Big[\Big(f_t + G(\tilde f)\cdot\nabla f, u + \delta_K(u_t + G(f_{SD})\cdot\nabla u)\Big)_K \\ + \sigma\Big(\nabla_v f, \nabla_v u\Big)_K - \delta_K\sigma\Big(\Delta_v f, u_t + G(f_{SD})\cdot\nabla u\Big)_K\Big] \\ + \sum_{m=1}^{M-1} \langle [f], u^+\rangle_m + \langle f^+, u^+\rangle_{\Lambda^-} + \langle f^-, u^-\rangle_{\Lambda^+} + \langle f^+, u^+\rangle_0, \end{aligned}$$

the bilinear form J_δ as,

$$J_\delta(f,u) = \sum_{m=0}^{M-1}\sum_{K\in\mathcal{K}_{h,m}} \Big(\nabla_v\cdot(\beta v f), u + \delta_K(u_t + G(f_{SD})\cdot\nabla u)\Big)_K,$$

and the linear form L_δ given by,

$$\begin{aligned} L_\delta(u) = \sum_{m=0}^{M-1}\sum_{K\in\mathcal{K}_{h,m}} \Big(S, u + \delta_K(u_t + G(f_{SD})\cdot\nabla u)\Big)_K \\ + \langle f_0, u^+\rangle_0 + \langle g^+, u^+\rangle_{\Lambda^-} + \langle g^-, u^-\rangle_{\Lambda^+}. \end{aligned}$$

Note that in the h version of the SD approach for the time dependent problems we interpret $(\cdot,\cdot)_{\Omega_T}$ as $\sum_{m=0}^{M-1}(\cdot,\cdot)_m$ and, assuming discontinuities in the time variable, include jump terms in the time direction. Thus we estimate sum of the norms over slabs Ω_m as well as the contributions from the jumps over time levels t_m, $m = 1,\dots M-1$. Whereas in hp version we have, in addition to slabwise estimates, a further step of identifying $(\cdot,\cdot)_m$ by $\sum_{K\in\mathcal{K}_{h,m}}(\cdot,\cdot)_K$ counting for the local character of the parameters h_K, p_K and δ_K, and consequently replacing some of the terms of the form $(\cdot,\cdot)_m$ and $\|\cdot\|_m$ (e.g., those involving δ_K), by the equivalent ones: $(\cdot,\cdot)_m = \sum_{K\in\mathcal{K}_{h,m}}(\cdot,\cdot)_K$ and $\|\cdot\|_m = \sum_{K\in\mathcal{K}_{h,m}}\|\cdot\|_K$, respectively.

In the remaining part of this section we prove stability estimates and derive convergence rates for the error in $|||\cdot|||$.

Proposition 4.1. *We assume that the mesh $\mathcal{K}_h$ consists of shape-regular elements K and the SD-parameter $\delta_K (:= \delta|_K)$ on K satisfies $0 \le \delta_K \le \min\left(\frac{h_K^2}{\sigma C_I^2}, \frac{h_K}{p_K C_I^2}\right)$, with $C_I = C(C_{inv}, C_0)$, where C_{inv} is the constant in an inverse estimate and C_0 is as in (4.1). then the trilinear form $B_\delta(G(f_{SD}), \cdot, \cdot)$ is coercive on $V_h^{\mathbf{q}} \times V_h^{\mathbf{q}}$:*

$$B_\delta(G(f_{SD}); u, u) \ge \frac{1}{2} |||u|||^2, \qquad \forall u \in V_h^{\mathbf{q}}$$

Further, for any constant $C_1 > 0$ we have for any $u \in V_h^{\mathbf{q}}$,

$$\|u\|_{\Omega_T}^2 \le \left[\frac{1}{C_1}\|u_t + G(f_{SD}) \cdot \nabla u\|_{\Omega_T}^2 + \sum_{m=1}^{M} |u^-|_m^2 + \int_{\partial\Omega \times I} u^2 |G^h \cdot \mathbf{n}| \, d\nu \, ds\right] \delta e^{C_1 \delta}.$$

Proof. Starting from our trilinear form,

$$\begin{aligned} B_\delta(G(f_{SD}); u, u) =& \big(u_t, u\big)_{\Omega_T} + \langle u^+, u^+ \rangle_0 - \sigma \sum_{K \in \mathcal{K}_h} \delta_K \Big(\Delta_v u, u_t + G(f_{SD}) \cdot \nabla u\Big)_K \\ &+ \sum_{K \in \mathcal{K}_h} \delta_K \|u_t + G(f_{SD}) \cdot \nabla u\|_K^2 + \sigma \|\nabla_v u\|_{\Omega_T}^2 + \sum_{m=1}^{M-1} \langle [u], u^+ \rangle_m \\ &+ \sum_{m=0}^{M-1} \Big[\Big(G(f_{SD}) \cdot \nabla u, \, u\Big)_m + \langle u^+, u^+ \rangle_{\lambda_m^-} + \langle u^-, u^- \rangle_{\lambda_m^+}\Big]. \end{aligned}$$

We work separately on pieces of this form. Integrating by parts,

$$(u_t, u)_{\Omega_T} + \langle u^+, u^+ \rangle_0 + \sum_{m=1}^{M-1} \langle [u], u^+ \rangle_m = \frac{1}{2}\Big[|u|_M^2 + |u|_0^2 + \sum_{m=1}^{M-1} |[u]|^2\Big]. \tag{4.12}$$

To estimate the term involving $\delta_K \sigma$ we apply Cauchy-Schwartz and the inverse inequalities, and use the assumption on δ_K, to get

$$\begin{aligned} &\delta_K \sigma \Big(\Delta_v u, u_t + G(f_{SD}) \cdot \nabla u\Big)_K \le \\ &\le \frac{1}{2} C_I h_K^{-1} \sqrt{\sigma \delta_K} \Big[\sigma \|\nabla_v u\|_K^2 + \delta_K \|u_t + G(f_{SD}) \cdot \nabla u\|_K^2\Big] \\ &\le \frac{1}{2}\Big[\sigma \|\nabla_v u\|_K^2 + \delta_K \|u_t + G(f_{SD}) \cdot \nabla u\|_K^2\Big], \end{aligned} \tag{4.13}$$

where, as we mentioned earlier, the inverse inequality C_I depends on the constant in inverse estimate and the shape-regularity of the triangulation $\mathcal{K}_h$.

Further using Green's formula and (2.3) we have

$$\begin{aligned} &(G(f_{SD}) \cdot \nabla u, u)_\Omega + \langle u^+, u^+ \rangle_{\Gamma^-} + \langle u^-, u^- \rangle_{\Gamma^+} = \\ &= \frac{1}{2} \int_{\partial\Omega} u^2 |G(f_{SD}) \cdot \mathbf{n}| \, d\nu + \langle u^+, u^+ \rangle_{\Gamma^-} + \langle u^-, u^- \rangle_{\Gamma^+} \\ &= \frac{3}{2} \int_{\partial\Omega} u^2 |G(f_{SD}) \cdot \mathbf{n}| \, d\nu. \end{aligned} \tag{4.14}$$

Now summing (4.13) over K, integrating (4.14) over I_m, summing over m and combining with (4.12) gives the first assertion of the proposition. For the second part we apply (4.14) and Grönwall's inequality on $\|u\|_{\Omega_T}^2$ following Lemma 3.2 of [2]. □

Proposition 4.2. *Let* $f_{SD} \in V_h^{\mathbf{q}}$ *and write* $f - f_{SD} = \eta - \xi$, *where* $\eta = f - \Pi_p f$, $\xi = f_{SD} - \Pi_p f$ *and* $\Pi_p f \in V_h^{\mathbf{q}}$ *is defined as in Section 3. Further assume that*

$$\|\nabla f\|_\infty + \|G(f)\|_\infty + \|\nabla \eta\|_\infty \le C, \tag{4.15}$$

then we have the following estimate:

$$\begin{aligned}
\big|B_\delta(G(f); f, \xi) - B_\delta(G(f_{SD}); \Pi_p f, \xi)\big| &\le \frac{1}{8}|||\xi|||^2 + C\int_{\partial\Omega\times I} \eta^2 |G(f_{SD})\cdot \mathbf{n}|\, d\nu\, ds \\
&+ C\sum_{K\in\mathcal{K}_h}\Big[h_K^{-1}\Big(\|\eta\|_K^2 + (\|\eta_t\|_K + \|\nabla\eta\|_K)^2\Big) + h_K(\|\xi\|_K + \|\eta\|_K)^2\Big] \\
&+ C(\|\xi\|_{\Omega_T} + \|\eta\|_{\Omega_T})\|\xi\|_{\Omega_T} + \sum_{m=1}^{M} |\eta_-|_m^2,
\end{aligned}$$

Proof. Using the definition of η and ξ we write

$$\begin{aligned}
B_\delta(G(f); f, \xi) - B_\delta(G(f_{SD}); \Pi_p f, \xi) &= \\
&= B_\delta(G(f_{SD}); \eta, \xi) + B_\delta(G(f); f, \xi) - B_\delta(G(f_{SD}); f, \xi) \\
&:= T_1 + T_2 - T_3.
\end{aligned}$$

Now we estimate the terms T_1 and $T_2 - T_3$ separately. Starting with T_1, we have

$$\begin{aligned}
T_1 &= B_\delta(G(f_{SD}); \eta, \xi) = \\
&= \big(\eta_t, \xi\big)_{\Omega_T} + \langle \eta_+, \xi_+\rangle_0 - \sigma\sum_{K\in\mathcal{K}_h}\delta_K\Big(\Delta_v\eta, \xi_t + G(f_{SD})\cdot\nabla\xi\Big)_K \\
&+ \sum_{K\in\mathcal{K}_h}\delta_K\Big(\eta_t + G(f_{SD})\cdot\nabla\eta, \xi_t + G(f_{SD})\cdot\nabla\xi\Big)_K + \sigma\big(\nabla_v\eta, \nabla_v\xi\big)_{\Omega_T} \\
&+ \sum_{m=1}^{M-1}\langle[\eta], \xi_+\rangle_m + \sum_{m=0}^{M-1}\Big(G(f_{SD})\cdot\nabla\eta, \xi\Big)_m + \langle \eta_+, \xi_+\rangle_{\Lambda^-} + \langle \eta_-, \xi_-\rangle_{\Lambda^+}.
\end{aligned}$$

From the inverse inequality and the assumptions on σ and δ_K we have the estimates:

$$\begin{aligned}
\sigma\Big|\Big(\nabla_v\eta, \nabla_v\xi\Big)_K\Big| &\le \sigma\|\nabla_v\eta\|_K\|\nabla_v\xi\|_K \le Ch_K^{-1}\|\eta\|_K\sigma\|\nabla_v\xi\|_K \\
&\le Ch_K^{-1}\|\eta\|_K^2 + \frac{1}{8h_K}\sigma^2\|\nabla_v\xi\|_K^2 \le Ch_K^{-1}\|\eta\|_K^2 + \frac{\sigma}{8}\|\nabla_v\xi\|_K^2
\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}
\delta_K\sigma\Big|\Big(\Delta_v\eta, \xi_t + G(f_{SD})\cdot\nabla\xi\Big)_K\Big| &\le \delta_K\sigma\|\Delta_v\eta\|_K\|\xi_t + G(f_{SD})\cdot\nabla\xi\|_K \\
&\le C_I\delta_K\sigma h_K^{-2}\|\eta\|_K\|\xi_t + G(f_{SD})\cdot\nabla\xi\|_K \\
&\le C_I\delta_K h_K^{-1}\|\eta\|_K\|\xi_t + G(f_{SD})\cdot\nabla\xi\|_K \\
&\le C_I\sqrt{\delta_K h_K^{-1}}\Big[h_K^{-1}\|\eta\|_K^2 + \frac{\delta_K}{8}\|\xi_t + G(f_{SD})\cdot\nabla\xi\|_K^2\Big] \\
&\le C_p\Big[h_K^{-1}\|\eta\|_K^2 + \frac{\delta_K}{8}\|\xi_t + G(f_{SD})\cdot\nabla\xi\|_K^2\Big],
\end{aligned} \tag{4.17}$$

where by assumption on δ_K we have $C_p := C_I\sqrt{\delta_K h_K^{-1}} \le p_K^{-1/2}$. Then integrating by parts on the remaining terms, using (4.8), and a similar argument as in the proof of Proposition 4.1 we get,

$$\sum_{K\in\mathcal{K}_h}\Big(\eta_t+G(f_{SD})\cdot\nabla\eta,\xi+\delta_K(\xi_t+G(f_{SD})\cdot\nabla\xi)\Big)_K$$
$$+\sum_{m=1}^{M-1}\langle[\eta],\xi\rangle_m+\langle\eta_+,\xi_+\rangle_0+\langle\eta_+,\xi_+\rangle_{\Lambda^-}+\langle\eta_-,\xi_-\rangle_{\Lambda^+}$$
$$=-\big(\eta,\xi_t+G(f_{SD})\cdot\nabla\xi\big)_{\Omega_T}+\langle\eta_-,\xi_-\rangle_M+C\int_{\partial\Omega\times I}\eta\xi\big|G^h\cdot\mathbf{n}\big|\,d\nu\,ds$$
$$-\sum_{m=1}^{M-1}\langle\eta_-,[\xi]\rangle_m+\sum_{K\in\mathcal{K}_h}\delta_K\Big(\eta_t+G(f_{SD})\cdot\nabla\eta,\xi_t+G(f_{SD})\cdot\nabla\xi\Big)_K$$

which using Cauchy-Schwartz inequality together with (4.16) and (4.17) gives

$$\begin{aligned}|T_1|\le\frac{1}{8}|||\xi|||^2+C\Big[\int_{\partial\Omega\times I}\eta^2\big|G^h\cdot\mathbf{n}\big|\,d\nu\,ds+\sum_{K\in\mathcal{K}_h}h_K^{-1}\|\eta\|_K^2+\\ +\sum_{m=1}^{M}|\eta_-|_m^2+\sum_{K\in\mathcal{K}_h}\delta_K\|\eta_t+G(f_{SD})\cdot\nabla\eta\|_K^2\Big].\end{aligned}\tag{4.18}$$

Using basic properties on solutions of Poisson equations and the definition of G we now bound the last term on the right hand side of (4.18) (see [2] for details),

$$\|G(f_{SD})-G(f)\|_{\Omega_T}\le C\|f-f_{SD}\|_{\Omega_T}\le C(\|\xi\|_{\Omega_T}+\|\eta\|_{\Omega_T}),$$

which gives

$$\begin{aligned}\|\eta_t+G(f_{SD})\cdot\nabla\eta\|_{\Omega_T}\le\|\eta_t\|_{\Omega_T}+\|G(f)\|_\infty\|\nabla\eta\|_{\Omega_T}+\\ +C\|\nabla\eta\|_\infty(\|\xi\|_{\Omega_T}+\|\eta\|_{\Omega_T}).\end{aligned}\tag{4.20}$$

To estimate the term (T_2-T_3), we follow a similar argument as in [2] and get

$$\begin{aligned}|T_2-T_3|\le&C(\|\xi\|_{\Omega_T}+\|\eta\|_{\Omega_T})\|\nabla f\|_\infty\|\xi\|_{\Omega_T}\\ &+C\|\nabla f\|_\infty^2\sum_{K\in\mathcal{K}_h}h_K(\|\xi\|_K+\|\eta\|_K)^2\\ &+\frac{1}{8}\sum_{K\in\mathcal{K}_h}\delta_K\|\xi_t+G(f_{SD})\cdot\nabla\xi\|_K^2.\end{aligned}\tag{4.21}$$

Now combining the estimates (4.18)–(4.21), using assumption (4.15) and hiding the term $\frac{1}{8}\sum_{K\in\mathcal{K}_h}\delta_K\|\xi_t+G(f_{SD})\cdot\nabla\xi\|_K^2$ in $|||\xi|||$ the proof is complete. □

Proposition 4.3. *Under the assumptions of Proposition 4.2 we have*

$$\big|J_\delta(f_{SD},\xi)-J_\delta(f,\xi)\big|\le\frac{1}{8}|||\xi|||^2+C\|\xi\|_{\Omega_T}^2+C\sum_{K\in\mathcal{K}_h}h_K^{-1}\|\eta\|_K^2.$$

Proof. Using the definition of ξ and η, we have the identity

$$J_\delta(f_{SD},\xi)-J_\delta(f,\xi)=J_\delta(\xi,\xi)-J_\delta(\eta,\xi)\;:=\;J_1-J_2.$$

Below we bound the terms J_1 and J_2, separately. For the first term, using integration by parts, boundedness of Ω_v and the fact that $\xi\equiv0$, on $\partial\Omega\times(0,T)$ (ξ is the difference of two functions in $V_h^{\mathbf{q}}$, which on the boundary $\partial\Omega\times(0,T)$ are coinciding with the projection of g on the finite element space), we can easily show that

$$
\begin{aligned}
|J_1| =& \Big|\sum_{K\in\mathcal{K}_h}\Big(\nabla_v\cdot(\beta v\xi), \xi+\delta_K(\xi_t+G(f_{SD})\cdot\nabla\xi)\Big)_K\Big| \\
\le& \beta\sum_{K\in\mathcal{K}_h}\Big|\Big(d\xi+v\cdot\nabla_v\xi, \xi+\delta_K(\xi_t+G(f_{SD})\cdot\nabla\xi)\Big)_K\Big| \\
\le& C\beta d\|\xi\|_{\Omega_T}^2+\beta\sum_{K\in\mathcal{K}_h}\Big[|v|_{L_\infty(K)}^2\|\nabla_v\xi\|_K^2+\delta_k^2\|\xi_t+G(f_{SD})\cdot\nabla\xi\|_K^2\Big] \\
\le& C\beta d\|\xi\|_{\Omega_T}^2+\beta\sum_{K\in\mathcal{K}_h}\Big[h_K^2\|\nabla_v\xi\|_K^2+\delta_k^2\|\xi_t+G(f_{SD})\cdot\nabla\xi\|_K^2\Big].
\end{aligned}
\tag{4.23}
$$

The term J_2 is estimated using the integration by parts, boundedness of Ω_v^h, and that $\xi\equiv 0$ on $\partial\Omega\times(0,T)$ viz,

$$
\begin{aligned}
|J_2| =& \Big|\sum_{K\in\mathcal{K}_h}\Big(\nabla_v\cdot(\beta v\eta), \xi+\delta_K(\xi_t+G(f_{SD})\cdot\nabla\xi)\Big)_K\Big| \\
=&\beta\Big|\big(d\eta+v\cdot\nabla_v\eta,\xi\big)_{\Omega_T}+\sum_{K\in\mathcal{K}_h}\delta_K\Big(d\eta+v\cdot\nabla_v\eta,\xi_t+G(f_{SD})\cdot\nabla\xi\Big)_K\Big| \\
=&\beta\Big|d(\eta,\xi)_{\Omega_T}-\big(\eta, v\cdot\nabla_v\xi\big)_{\Omega_T}+\sum_{K\in\mathcal{K}_h}\delta_K\Big(d\eta+v\cdot\nabla_v\eta,\xi_t+G(f_{SD})\cdot\nabla\xi\Big)_K\Big| \\
\le& \beta(d+1)\delta^{-1}\|\eta\|_{\Omega_T}^2+\frac{\beta}{4}\delta\Big(\|\xi\|_{\Omega_T}^2+|v|_\infty^2\|\nabla_v\xi\|_{\Omega_T}^2\Big) \\
&+\beta\sum_{K\in\mathcal{K}_h}\delta_K\Big(d\|\eta\|_K^2+|v|_{L_\infty(K)}^2\|\nabla_v\eta\|_K^2+\frac{1}{2}\|\xi_t+G(f_{SD})\cdot\nabla\xi\|_K^2\Big) \\
\le& C\beta\Big[\delta^{-1}\|\eta\|_{\Omega_T}^2+\delta\|\xi\|_{\Omega_T}^2+C_v\delta\|\nabla_v\xi\|_{\Omega_T}^2 \\
&+\sum_{K\in\mathcal{K}_h}\delta_K\Big(\|\eta\|_K^2+C_v\|\eta\|_{1,K}^2+\|\xi_t+G(f_{SD})\cdot\nabla\xi\|_K^2\Big)\Big].
\end{aligned}
\tag{4.24}
$$

Combining these two estimates, recalling the assumption on β, and δ_K and hiding the terms $\sum_{K\in\mathcal{K}_h}\delta_K\|\xi_t+G(f_{SD})\cdot\nabla\xi\|_K^2$ and $\sum_{K\in\mathcal{K}_h}\delta_K\|\nabla_v\xi\|_K^2$ in $|||\xi|||^2$ we get the desired result. □

Note that in the above estimate for J_2 we may use the element-size and inverse to write $|v|_{L_\infty(K)}^2\|\nabla_v\eta\|_K^2\le h_K^2h_K^{-2}\|\eta\|_K^2$. Thus, in the last step, we can replace $\|\eta\|_{1,K}^2$ by $h_K^2h_K^{-2}\|\eta\|_K^2=\|\eta\|_K^2$ and hence get a gradient-free estimate.

We will now derive a stability estimate underlying our main convergence result.

Lemma 4.1. *For ξ and η as above, there exist a constant $C>0$ such that,*

$$
|||\xi|||^2\le C\Big[\int_{\partial\Omega\times I}\eta^2\big|G^h\cdot\mathbf{n}\big|\,d\nu\,ds+\delta^{-1}\|\eta\|_{\Omega_T}^2+\sum_{m=1}^M|\eta_-|_m^2+\delta\|\eta\|_{1,Q}^2+\sum_{m=1}^M|\xi_-|_m^2\delta\Big].
$$

Proof. The exact solution f satisfies (4.5), i.e.

$$
B_\delta(G(f);f,u)-J_\delta(f,u)=L_\delta(u)\qquad\forall u\in V^{\mathbf{q}}.
$$

The coercivity result: Proposition 4.1 yields

(4.27)
$$\begin{aligned}\frac{1}{2}|||\xi|||^2 &\le B_\delta(G(f_{SD}); f_{SD} - \Pi f, \xi) = L_\delta(\xi) + J_\delta(f_{SD}, \xi) - B_\delta(G(f_{SD}); \Pi f, \xi)\\ &= B_\delta(G(f); f, \xi) - B_\delta(G(f_{SD}); \Pi f, \xi) + J_\delta(f_{SD}, \xi) - J_\delta(f, \xi)\\ &:= \Delta B_\delta + \Delta J_\delta.\end{aligned}$$

Now we use Propositions 4.2 and 4.3 to bound the terms ΔB_δ and ΔJ_δ. Further using the second result in Proposition 4.1 we estimate $\|\xi\|^2_{\Omega_T}$ and $\|\eta\|^2_{\Omega_T}$ with sufficiently large C_1. Combining all these estimates we obtain the desired result and the proof is complete. □

4.2. **Convergence.** We now put together all of the previously established results and prove our main convergence estimate. Recalling our previous notation $e := f - f_{SD} = f - \Pi f + \Pi f - f_{SD} := \eta - \xi$, we show that:

Theorem 4.1. *If $f_{SD} \in V^{\mathbf{q}}$ satisfies (4.5) and $\delta_K = \min\left(\frac{h_K^2}{\sigma C_I^2}, \frac{h_K}{p_K C_I}\right)$ for each $K \in \mathcal{T}$, then there is a constant $C > 0$ such that,*

$$|||f - f_{SD}|||^2 \le C \sum_{K \in \mathcal{M}_R} h_K^{2s_k+1} \frac{\Phi(p_K, s_K)}{p_K} \|\hat{f}\|^2_{s_K+1,\hat{K}}, \tag{4.28}$$

where $\Phi(p_K, s_K) = \max(\Phi_1(p_K, s_K), \Phi_2(p_K, s_K))$ as defined in Theorem 3.3.

Proof. We split the right hand side of the estimation in Lemma 4.1 and rewrite it consicely as

$$|||\xi|||^2 \le C(A_1 + A_2 + A_3), \tag{4.29}$$

with

$$\begin{aligned}A_1 &:= \sum_{K \in \mathcal{K}_h} \delta_K^{-1} \|\eta\|^2_K + \delta \|\eta\|^2_{1,\Omega_T},\\ A_2 &:= \int_{\partial\Omega \times I} \eta^2 \left|G^h \cdot \mathbf{n}\right| d\nu\, ds + \sum_{m=1}^{M} |\eta_-|^2_m,\\ A_3 &:= \sum_{m=1}^{M} |\xi_-|^2_m \sum_{K \in \mathcal{K}_{h,m}} h_K \delta_K.\end{aligned}$$

Below We estimate each A_i separately: As for A_1 we have using Theorem 3.3,

$$A_1 \le \sum_{K \in \mathcal{K}} \left(\frac{h_K}{2}\right)^{2s_K} \Phi(p_K, s_K)(\delta_K^{-1} h_K^2 + \delta_K) \|\hat{f}\|^2_{s_K+1,\hat{K}} \tag{4.30}$$

To get an estimate for A_2 we use trace estimate combined with an inverse inequality, see [1], to get

$$\|\eta\|^2_{\partial K} \le C(\|\nabla \eta\|_K \|\eta\|_K + h_K^{-1} \|\eta\|^2_K), \qquad \forall K \in \mathcal{M}_R, \tag{4.31}$$

which gives,

$$\begin{aligned}A_2 \le\, &C \sum_{K \in \mathcal{K}} \Big[\left(\frac{h_K}{2}\right)^{s_K} \Phi_2^{1/2}(p_K, s_K) \left(\frac{h_K}{2}\right)^{s_K+1} \Phi_1^{1/2}(p_K, s_K)\\ &+ h_K^{-1} \left(\frac{h_K}{2}\right)^{2s_K+2} \Phi_1(p_K, s_K)\Big] \|\hat{f}\|^2_{s_K+1,\hat{K}},\end{aligned} \tag{4.32}$$

where both terms in A_2 are estimated combining (4.31) with Theorem 3.3 and using $\delta_K = \min\left(\frac{h_K^2}{\sigma C_I^2}, \frac{h_K}{p_K C_I}\right)$. Summing up we can now rewrite (4.29) as,

$$|||\xi|||^2 \leq C\Big[\sum_{K\in\mathcal{K}} h_K^{2s_k+1}\frac{\Phi(p_K,s_K)}{p_K}\|\hat{f}\|^2_{s_K+1,\hat{K}} + \sum_{m=1}^{M}|\xi_-|_m^2 \sum_{K\in\mathcal{K}_{h,m}} h_K\delta_K\Big]. \tag{4.33}$$

To proceed we need to estimate also the A_3 term. To this approach we use the following discrete Grönwalls type estimate as, e.g. in [2]: If

$$y(\cdot,t_m) \leq C + C_1\sum_{j\leq m}|y(\cdot,t_j)|\sum_{K\in\mathcal{K}_{h,m}} h_K\delta_K, \tag{4.34}$$

then,

$$y(t_m) \leq Ce^{C_1 t} \leq Ce^{C_1 T}. \tag{4.35}$$

Note that (4.33) also implies,

$$|\xi_-|_m^2 \leq C\Big[\sum_{K\in\mathcal{K}} h_K^{2s_k+1}\frac{\Phi(p_K,s_K)}{p_K}\|\hat{f}\|^2_{s_K+1,\hat{K}} + \sum_{m=1}^{M}|\xi_-|_m^2 \sum_{K\in\mathcal{K}_{h,m}} h_K\delta_K\Big], \tag{4.36}$$

which gives using (4.34), (where we interpret the term under $\sum_{\mathcal{K}}$ as a new constant depending on f, $\mathcal{K}$ and $\mathbf{q}$), that

$$|\xi_-|_m^2 \leq C\sum_{K\in\mathcal{K}} h_K^{2s_k+1}\frac{\Phi(p_K,s_K)}{p_K}\|\hat{f}\|^2_{s_K+1,\hat{K}}e^{CT}. \tag{4.37}$$

Thus we now also have an estimate for A_3, which together with (4.30), (4.32), $\delta|_K := \delta_K$, gives the desired result. See also [2] and [15]. □

Remark. One can show that the convergence rate (4.28): $h_K^{2s_k+1}\frac{\Phi(p_K,s_K)}{p_K}$ is of order $\delta_K^{2s_K+1}$. However this remaining part is basically, similar to the type of estimates derived in [12] in their full details and therefore are omitted in here.

5. Appendix: A sketch of existence and uniqueness

Proposition 5.1. *Let f_0 and $S \in L_2(\Omega_T)$, then for any $h > 0$ the problem (4.5) admits at least one solution.*

Proof. We apply a version of Brouwer's fix point theorem to prove that for each $m = 0,\dots,M-1$, given $f_{SD}^-(\cdot,t_m)$ the problem (4.4) admits a solution. To this approach we define $\mathcal{G}^m : V_h^{\mathbf{P}_m} \to V_h^{\mathbf{P}_m}$ by

(5.1)

$$\begin{aligned}
[\mathcal{G}^m f, u]_\delta = \sum_{K\in\mathcal{K}_{h,m}} &\Big[\Big(f_t + G(\tilde{f})\cdot\nabla f, u + \delta_K(u_t + G(f_{SD})\cdot\nabla u)\Big)_K \\
&- \Big(\sigma\Delta_v f + \nabla_v\cdot(\beta v f) + S, u + \delta_K(u_t + G(f_{SD})\cdot\nabla u)\Big)_K\Big] \\
&+ \langle f^+, u^+\rangle_m - \langle f_{SD}^-, u^+\rangle_m - \langle g^+, u^+\rangle_{\lambda_m^-} - \langle g^-, u^-\rangle_{\lambda_m^+} \quad \forall u \in V_h^{\mathbf{P}_m},
\end{aligned}$$

where

$$[f,u]_\delta = \langle f^-, u^-\rangle_{m+1} + \langle f^+, u^+\rangle_m + (f,u)_m.$$

Note that (for a fixed h) $\mathcal{G}^m$ is well defined and continuous on $V_h^{\mathbf{P}_m}$ with the norm $|[\cdot]|_\delta$ given by the scalar product $[\cdot,\cdot]_\delta$. Further $\mathcal{G}^m f = 0$ if and only if f satisfies

(4.4). Now using inverse estimate, recalling that $\sigma \leq \min_K h_K$ and following the same procedure as in the stability proofs we have that

$$
\begin{aligned}
[\mathcal{G}^m f, f]_\delta \geq & \frac{1}{2}\Big[|f^-|^2_{m+1} + |f^+|^2_m + 2\sigma\|\nabla_v f\|^2_m + 3\int_{\partial\Omega\times I_m} f^2|G(\tilde{f})\cdot\mathbf{n}|\,d\nu + \\
& + 2\sum_{K\in\mathcal{K}_{h,m}} \delta_K\|f_t + G(f_{SD})\cdot\nabla f\|^2_K\Big] \\
& - |f^-_{SD}|_m|f^+|_m - |g^+|_{\lambda^-_m}|f^+|_{\lambda^-_m} - |g^-|_{\lambda^+_m}|f^-|_{\lambda^+_m} \\
& - \Big[d\beta\|f\|_m + \beta\|v\cdot\nabla_v f\|_m + \sigma h^{-1}\|\nabla_v f\|_m + \|S\|_m\Big]\times \\
& \Big(\|f\|_m + \sum_{K\in\mathcal{K}_{h,m}} \delta_K\|f_t + G(f_{SD})\cdot\nabla f\|_K\Big),
\end{aligned} \tag{5.2}
$$

so that using the second estimate of proposition (4.1) we get

$$
\begin{aligned}
[\mathcal{G}^m f, f]_\delta \geq C|[f]|^2_\delta - C|[f]|_\delta\Big[&\|S\|_m + |f^-_{SD}|_m + |g^-|_{\lambda^+_m} + |g^+|_{\lambda^-_m} \\
&+ \|\nabla_v f\|_m + \beta(d\|f\|_m + \|v\cdot\nabla_v f\|_m)\Big].
\end{aligned} \tag{5.3}
$$

Hence $[\mathcal{G}^m f, f]_\delta \geq 0$ if $|[f]|_\delta = \gamma$ is large enough and by Brouwer's fix point theorem (see [17]) it follows that there exists a $f \in V_h^{\mathbf{P}_m}$ with $|[f]|_\delta \leq \gamma$ such that $\mathcal{G}^m f \equiv 0$ and the proof is complete. □

Proposition 5.2. *As a consequence of regularity assumption (4.15) in the Proposition (4.2) and the convergence Theorem 4.1 we can show that for sufficiently small h the solutions of the discrete problem (4.5) are unique, i.e., for each* $m = 0, \dots M-1$ *there is a unique solution* $f\Big|_{\Omega_m}$.

Proof. Suppose that f_{SD} and $\tilde{f}_{SD}$ are two solutions of (4.5) with corresponding velocities $G(f_{SD})$ and $G(\tilde{f}_{SD})$. By subtraction we then have for any m, writing $u = f_{SD} - \tilde{f}_{SD}$ and assuming $u(\cdot, t_m)^- \equiv u\Big|_{\partial\Omega_m} \equiv 0$,

$$
\begin{aligned}
&\sum_{K\in\mathcal{K}_{h,m}} \delta_K\Big(S, (G(f_{SD}) - G(\tilde{f}_{SD}))\cdot\nabla u\Big)_K = \\
&\quad = (u_t, u)_m + |u^+|^2_m + \Big(G(f_{SD})\cdot\nabla u, u\Big)_m + \sigma\|\nabla_v u\|^2_m \\
&\quad + \sum_{K\in\mathcal{K}_{h,m}} \Big((G(\tilde{f}_{SD}) - G(f_{SD}))\cdot\nabla\tilde{f}_{SD}, u + \delta_K(u_t + G(f_{SD})\cdot\nabla u)\Big)_K \\
&\quad + \sum_{K\in\mathcal{K}_{h,m}} \delta_K\|u_t + G(f_{SD})\cdot\nabla u\|^2_K - \beta\Big(d\|u\|^2_m + (v\cdot\nabla_v u, u)_m\Big) \\
&\quad - \beta\sum_{K\in\mathcal{K}_{h,m}} \delta_K\Big(du + v\cdot\nabla_v u, u_t + G(f_{SD})\cdot\nabla u\Big)_K \\
&\quad - \sigma\sum_{K\in\mathcal{K}_{h,m}} \delta_K\Big(\Delta_v u, u_t + G(f_{SD})\cdot\nabla u\Big)_K \\
&\quad + \sum_{K\in\mathcal{K}_{h,m}} \delta_K\Big(\tilde{f}_{SD,t} + G(\tilde{f}_{SD})\cdot\nabla\tilde{f}_{SD}, (G(\tilde{f}_{SD}) - G(f_{SD}))\cdot\nabla u\Big)_K.
\end{aligned} \tag{5.4}
$$

By Theorem 4.1 and inverse estimate it is easy to see that

$$\delta^\gamma \|\tilde{f}_{SD}\|_{1,\infty} + \|G_{SD}(\tilde{f}_{SD})\|_\infty \le C,$$

with $\gamma = 1/2$ if $k = 1$ and $\gamma = 0$ if $k \ge 2$. Taking account in the previous relation, this proves that

$$|u^-|^2_{m+1} + |u^+|^2_m + \sigma\|\nabla_v u\|^2_m + \sum_{K\in\mathcal{K}_{h,m}} \delta_K \|u_t + G(f_{SD})\cdot\nabla u\|^2_K \le Ch^{-\gamma}\|u\|^2_m,$$

by virtue of the fact that

$$\|G(\tilde{f}_{SD}) - G(f_{SD})\|_m \le \|u\|_m.$$

However by argument in the proof of second relation in proposition (4.2), see [2] we have

$$\|u\|_m \le C \sum_{K\in\mathcal{K}_{h,m}} \Big[\delta_K^2 \|u_t + G(f_{SD})\cdot\nabla u\|^2_K + \delta_K |u^+|^2_K + \sigma\|u\|^2_K\Big], \tag{5.5}$$

and thus

$$\delta^{-1}\|u\|^2_m \le C\delta^{-\gamma}\|u\|^2_m,$$

which shows that $u = 0$ if δ is sufficiently small, and the uniqueness follows. □

Remark. A more realistic streamline diffusion mesh structure, for time-dependent convection dominated convection-diffusion problem, (i.e. in our case $\sigma/|G(f)|$ small), would be obtained via characteristic streamline diffusion (CSD) method: The idea is for each $m = 0, \dots, M$ let $\mathcal{T}_h^m$ be a subdivision of $\Omega\times\{t_m\}$ into open elements τ_m with mesh parameter $h_m = h_m(x,v) := \operatorname{diam}(\tau_m)$, $(x,v)\in\tau_m$, and set $I_m := (t_m, t_{m+1})$ with $|I_m| = \kappa_m = t_{m+1} - t_m$, (that is on each slab we have a decoupled phase-space and time mesh). On each slab Ω_m, a function space $\mathcal{V}^{\mathbf{P}_m}_{h_m,\kappa_m}$, replaces the previously defined $V_h^{\mathbf{P}_m}$ and induced a mesh $\mathcal{T}_m^{h,\kappa}$ on $\Omega\times\{t_m\}$. Now let $G_m^h(f)\in\mathcal{V}^{\mathbf{P}_m}_{h_m,\kappa_m}$ denote a nodal interpolant of $G_m(f) = G(f(\cdot, t_{m-1}))$ and introduce a map $\mathcal{F}_m : \Omega_m \to \Omega_m$ with the "local" tilting velocity G_m^h. Applying $\mathcal{F}_m$ would result to new function space $\tilde{\mathcal{V}}^{\mathbf{P}_m}_{h_m,\kappa_m}$ and mesh $\tilde{\mathcal{T}}_m^{h,\kappa}$ on Ω_m. In this way there are two phase-space meshes associated on each interior time level t_m, $m = 1, \dots, M-1$: The mesh $\mathcal{T}_m^h$ associated to $\Omega\times\{t_m\}$, that is the "bottom mesh" on the slab Ω_m and

$$\tilde{\mathcal{T}}_m^{h,-} := \{\mathcal{F}_{m-1}(\tau_{m-1}\times\{t_{m-1}\}) : \tau_{m-1}\in\mathcal{T}^h_{m-1}\},$$

that is the "top mesh" on the previous slab Ω_{m-1} which is obtained by letting the previous "bottom mesh" $\mathcal{T}^h_{m-1}$ be transported by the flow. Usually the two meshes do not coincide and it is necessary to use L_2-projection to interpolate a function on $\mathcal{T}_m^{h,-}$ into $\mathcal{V}^{\mathbf{P}_m}_{h_m}$. In this way then we can define the streamline diffusion approximation, f_{SD}, of f by

$$f_{SD} := \sum_{m=1}^{M} \zeta_m(t)\Xi_m(x,v,t),$$

where $\zeta_m(t)$ and $\Xi_m(x,v,t)$ are basis functions in t and (x,v), respectively, with Ξ functions moving in the direction of characteristics. We have studied this method in a simpler setting for the Fermi pencil beam equation in [6].

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