THESIS FOR THE DEGREE OF LICENTIATE OF ENGINEERING

Valuing Path-Dependent Options using the Finite Element Method and Duality Techniques

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Abstract

In this thesis we develop an adaptive finite element method for pricing several path-dependent options including barrier options, lookback options, and Asian options. The options are priced using the Black-Scholes PDE-model, and the resulting PDE:s are of parabolic type in one spatial dimension with different boundary conditions and jump conditions at monitoring dates.

The adaptive finite element method is based on piecewise polynomial approximation in space and time. We derive a posteriori estimates for the error in pointwise values of the solution and it's derivatives, using duality techniques. The estimates are used to determine suitable resolution in space and time. The suggested adaptive finite element method is stable and gives fast and accurate results.

Keywords: finite element method, Galerkin, duality, a posteriori error estimation, adaptivity, option pricing, Brownian motion, European option, barrier option, lookback option, Asian option, average option

This thesis consists of an introduction to option pricing and the following papers:

- **Paper I:** Valuing European, Barrier, and Lookback Options using the Finite Element Method and Duality Techniques
- **Paper II:** Valuing Asian options using the Finite Element Method and Duality Techniques

Contents

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1 A brief introduction to option pricing

This section gives a brief introduction to the theory of option pricing. A short background is presented and the mathematical model is explained, together with some useful tools for option pricing. For a more detailed discussion about option pricing we refer to Björk, [1], Borell, [2], or Wilmott, [6].

1.1 Background

A contingent claim, or a derivate, is a contract the value of which depends on the values of other assets. One of the most common derivates is the European call option. A European call option on a given stock with strike price K and maturity date T is the right, but not the obligation, for the holder of the option to buy one share of the stock at the price K at the time T. A European put option with strike price K and time of maturity T gives the holder the right, but not the obligation, to sell one share of the stock at the price K at maturity. The so called American option differs from the European option so that the holder can exercise the option at any time prior to the maturity date. Calls and puts are often called vanilla options.

Stocks and options have a long history. Stocks have existed for at least 750 years. Option contracts were used already during the MiddleAges. Valuing financial derivates in a theoretical convincing way has been difficult throughout history. A very important contribution was given in 1973 when Black and Scholes presented their solution to the valuation of the European call option, based on the assumption that the stock log-price is governed by a so called Brownian motion. Their solution was based on the Itô calculus on Brownian motion. The concept arbitrage, that is risk free profit, is very central here. The most difficult part in this area is to understand the price dynamics of the underlying contracts.

Another kind of option is the exotic option with a payoff which does not just depend on its value on the maturity date, but on the history of the underlying asset price. There are many different kinds of exotic options. Some of them are easy to price and analytical pricing formulas exist, but most of them are more difficult to value. The average option, or the so called Asian option is an example of an option without a (known) closed form price formula.

1.2 Underlying theory

Throughout this section we are working in the time interval $0 \leq t \leq T$. Let $B(t)$ denote the price of a risk free asset at time t governed by the equation

$B(t) = B(0)e^{rt}$, where r is the constant interest rate. A common hypothesis about the behavior of asset prices is that they are given by geometric Brownian motions which implies that the asset prices are log-normally distributed (see e.g. Duffie [3] or Björk [1]). The price $S(t)$ of an asset at time t, solves the following stochastic differential equation

$$\begin{aligned} dS(t) &= S(t)(\mu dt + \sigma dW(t)), \\ S(0) &= S_0, \end{aligned} \tag{1.1}$$

where σ is the volatility, $\mu \in \mathbb{R}$ and $W(t)$ is a normalized Wiener process. Here σ is assumed to be a positive real number. The solution of (1.1) is

$$S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}. \tag{1.2}$$

Now set

$$\tilde{W}(t) = \frac{\mu - r}{\sigma} t + W(t), \tag{1.3}$$

and note that

$$dS(t) = S(t)(rdt + \sigma d\tilde{W}(t)). \tag{1.4}$$

According to Cameron-Martin's theorem there exists another probability measure than the objective measure P, the risk neutral measure Q, such that $\tilde{W}$ is a Q-Wiener process. The solution of (1.4) equals

$$S(t) = S(0)e^{(r - \frac{\sigma^2}{2})t + \sigma \tilde{W}(t)}, \tag{1.5}$$

and the measures P and Q are equivalent. The existence of the risk neutral measure Q assures that the market is free of arbitrage possibilities.

Because the Wiener process is not differentiable in the usual sense, the equation (1.1) is interpreted in the sense of stochastic differential calculus initiated by K. Itô. The most fundamental tool in stochastic calculus, Itô's lemma is given below. But first we state a definition. If the stochastic process $(h(t))_{0 \le t \le T}$ is progressively measurable and

$$\int_0^T \mid h(t) \mid^p dt < \infty \text{ almost surely,} \tag{1.6}$$

for some $p \in [1, \infty[$, then we say that h belongs to the class $L^p_W[0, T]$.

Lemma 1.1 ***(Itô's lemma).*** *Let the function $u(t, x_1, \ldots, x_m)$ be two times continuously differentiable in $x_1, \ldots, x_m \in \mathbb{R}$ and one time continuously differentiable in $t \in [0, T]$. Suppose we have m stochastic differentials*

$$dX_i(t) = a_i(t)dt + \sum_{k=1}^{n} b_{ik}(t)dW_k(t), \tag{1.7}$$

dependent on n stochastic independent Wiener Processes $W_1, \ldots, W_n$. *Let* $\mathcal{F}_t = \sigma(W_1(\lambda), \ldots, W_n(\lambda), \lambda \leq t)$. *Let also the coefficients* $a_i(t), b_{ik}(t)$ *fulfil* $a_i(t) \in L^1_W[0,T]$, $b_{ik}(t) \in L^2_W[0,T]$ *and so, especially, for fixed t the processes are* $\mathcal{F}_t$*-measurable. Let also* $X(t) = (X_1(t), \ldots, X_m(t))$. *Then we have*

$$\begin{aligned} du(t, X(t)) &= \frac{\partial u}{\partial t}(t, X(t))dt + \sum_{i=1}^{m} \frac{\partial u}{\partial x_i}(t, X(t))dX_i(t) \\ &+ \frac{1}{2}\sum_{i,j=1}^{m} \frac{\partial^2 u}{\partial x_i \partial x_j}(t, X(t))dX_i(t)dX_j(t). \end{aligned} \tag{1.8}$$

Note that

$$\begin{aligned} &dtdt = 0, \quad dtdW_i(t) = 0, \\ &dW_i(t)dW_i(t) = dt, \quad dW_i(t)dW_j(t) = 0 \quad \text{if } i \neq j. \end{aligned}$$

1.3 Derivation of the Black-Scholes formula

Let $v(t, S(t))$ denote the value of the portfolio at time t, with the terminal condition $v(T, S(T)) = g(S(T))$, where the function g is piecewise continuous and fulfils

$$\sup_{x \in \mathbb{R}} (e^{-C|x|}|g(e^x)|) < \infty \tag{1.9}$$

for an appropriate constant $C > 0$. We then say that $g \in \mathcal{P}$. Suppose that the process $(v(t, S(t))_{0 \leq t \leq T}$ is the value process of a self-financing strategy $\big(h_S(t), h_B(t)\big)_{0 \leq t \leq T}$ in the stock and the risk free asset, that is

$$v(t, S(t)) = h_S(t)S(t) + h_B(t)B(t), \tag{1.10}$$
$$dv(t, S(t)) = h_S(t)dS(t) + h_B(t)dB(t). \tag{1.11}$$

By applying Ito's lemma and using (1.11) we get

$$\begin{aligned} dv(t, S(t)) &= v_t(t, S(t))dt + v_s(t, S(t))dS(t) + \frac{1}{2}v_{ss}(t, S(t))(dS(t))^2 \\ &= h_S(t)dS(t) + rh_B(t)B(t)dt. \end{aligned} \tag{1.12}$$

Identifying coefficients in (1.12) yields $h_S = v_s$. Rearranging the terms and using (1.10) we get the famous Black-Scholes differential equation

$$v_t(t, S(t)) + \frac{\sigma^2 S(t)^2}{2} v_{ss}(t, S(t)) + rS(t)v_s(t, S(t)) - rv(t, S(t)) = 0, \quad t < T,\ S(t) > 0. \tag{1.13}$$

Together with the terminal condition $v(T, S(T)) = g(S(T))$, equation (1.13) has the following solution,

$$v(t, S(t)) = e^{-r\tau} E\left[g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma W(\tau)})\right], \tag{1.14}$$

where $s = S(t)$ and $\tau = T - t$.

Remark 1.1 *Observe that (1.14) is independent of the drift coefficient μ.*

We thus have the following important result.

Theorem 1.1 *Let $g \in \mathcal{P}$. A simple European derivate with payoff $Y = g(S(T))$ at maturity T has the theoretical value $v(t, S(t))$ at time t, where*

$$v(t, S(t)) = e^{-r\tau} E\left[g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma W(\tau)})\right], \tag{1.15}$$

and $\tau = T - t$.

We can simplify (1.15) using the risk neutral measure Q (see Geman, Karoui and Rochet [4], for a detailed discussion about changes of probability measure).

Theorem 1.2 *The value $v(t, S(t))$ is equal to*

$$e^{-r\tau} E^Q[g(S(T)) \mid \mathcal{F}_t].$$

Proof. According to (1.5) we have $S(T) = S(t)e^{(r-\frac{\sigma^2}{2})\tau+\sigma(\tilde{W}(T)-\tilde{W}(t))}$ and hence

$$E^Q[g(S(T)) \mid \mathcal{F}_t] = E^Q\left[g(S(t)e^{(r-\frac{\sigma^2}{2})\tau+\sigma(\tilde{W}(T)-\tilde{W}(t))}) \mid \mathcal{F}_t\right]. \tag{1.16}$$

But since $(\tilde{W}(T) - \tilde{W}(t))$ and $\mathcal{F}_t$ are stochastic independent and $\tilde{W}$ is a Q-Brownian motion, the right hand side of (1.16) becomes

$$E\left[g(se^{(r-\frac{\sigma^2}{2})\tau+\sigma(W(T)-W(t))})\right]_{|s=S(t)} = e^{r\tau} v(t, S(t)),$$

□

We now state the famous Black-Scholes formula which gives the value of a European call option with payoff $Y = \max(0, S(T) - K)$ at maturity T.

Theorem 1.3 ***(Black-Scholes formula).*** *A European call option with maturity date T and strike price K has the value $c(t, S(t), K)$ at time $t < T$ where*

$$c(t,s,K) = s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2), \tag{1.17}$$
$$d_1 = \frac{\ln\frac{s}{K} + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{\tau},$$

and where Φ is the probability distribution function for a $N(0,1)$ distributed stochastic variable.

Proof. Theorem 1 gives that

$$c(t,s,K) = e^{-r\tau} E\left[\max\left(0, se^{(r-\frac{\sigma^2}{2})\tau - \sigma\sqrt{\tau}G} - K\right)\right],$$

where $G \in N(0,1)$. From this it follows that

$$\begin{aligned} c(t,s,K) &= e^{-r\tau} E\left[se^{(r-\frac{\sigma^2}{2})\tau-\sigma\sqrt{\tau}G} - K; \quad G \le \frac{\ln\frac{s}{K} + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\right] \\ &= e^{-r\tau}\left(E\left[se^{(r-\frac{\sigma^2}{2})\tau-\sigma\sqrt{\tau}G}; \quad G \le d_2\right] - K\Phi(d_2)\right). \end{aligned}$$

Here

$$\begin{aligned} e^{-r\tau} E\left[se^{(r-\frac{\sigma^2}{2})\tau-\sigma\sqrt{\tau}G}; \quad G \le d_2\right] &= s\int_{x\le d_2} e^{\frac{-\sigma^2}{2}\tau - \sigma\sqrt{\tau}x - \frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ = s\int_{x \le d_2} e^{\frac{-(\sigma\sqrt{\tau}+x)^2}{2}} \frac{dx}{\sqrt{2\pi}} &= s\Phi(\sigma\sqrt{\tau} + d_2) = s\Phi(d_1), \end{aligned}$$

which proves the theorem. □

The price of the European put option can be derived in the same manner as the call price. Alternatively to derive the European put price one can use the so called call-put parity relation.

Theorem 1.4 ***(Call-put parity).*** *Let c and p be the value of an European call and put option respectively. Then we have*

$$p(t,s,K,T) = Ke^{-r\tau} - s + c(t,s,K,T). \tag{1.18}$$

Using Theorems 3 and 4 we can easily calculate the price of an European put option, p(t, s, K, T).

$$\begin{aligned} p(t,s,K,T) &= Ke^{-r\tau} - s + s\Phi(d_1) - Ke^{-r\tau}\Phi(d_2) \\ &= Ke^{-r\tau}\Phi(-d_2) - s\Phi(-d_1). \end{aligned} \tag{1.19}$$

1.4 General derivate valuation formula

To be able to handle more complex derivates we extend the previous valuation formula in Theorem 2 to European derivates with payoff $X \in L^2(Q)$ and state the following theorem (for a more detailed discussion see Borell [2]).

Theorem 1.5 *A European derivate with payoff $X \in L^2(Q)$ at maturity T has the theoretical value*

$$v(t) = e^{-r\tau} E^Q[X \mid \mathcal{F}_t]. \tag{1.20}$$

1.5 Hedging and the greeks

Hedging is the reduction of the sensitivity of a portfolio to the movement of an underlying asset by taking opposite positions in different financial instruments. One simple way to hedge is the so called delta-hedging. With $V(S(t),t) = V(s,t)$ denoting the value of a portfolio or derivative, using $It\hat{o}'s$ lemma, (1.8), we have that

$$dV = \sigma s \frac{\partial V}{\partial s} dW + \left(\mu s \frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + \frac{\partial V}{\partial t} \right) dt. \tag{1.21}$$

Note that V must at least have one t derivative and two s derivatives. Let Π be a portfolio consisting of one option and $-\Delta$ number of the underlying assets,

$$\Pi = V - \Delta s. \tag{1.22}$$

Then

$$d\Pi = dV - \Delta ds, \tag{1.23}$$

which together with (1.21) and (1.1) gives that

$$d\Pi = \sigma s \left(\frac{\partial V}{\partial s} - \Delta \right) dW + \left(\mu s \frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + \frac{\partial V}{\partial t} - \mu \Delta ds \right) dt. \tag{1.24}$$

By choosing $\Delta = \frac{\partial V}{\partial s}$ we eliminate the randomness

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} \right) dt. \tag{1.25}$$

Delta hedging is a dynamic hedging strategy, that is, it must be continuously rebalanced to be a perfect hedge. Transaction costs makes this impossible in practice. When delta-hedging one eliminates the largest random part of the portfolio. One can also hedge away smaller effects due to, such as for instance, the curvature of the portfolio value with respect to the underlying asset. Then one needs the so called *gamma*, defined as

$$\Gamma = \frac{\partial^2 V}{\partial s^2}. \tag{1.26}$$

The decay of value in time of a portfolio is represented by the *theta*, where

$$\Theta = -\frac{\partial V}{\partial t}. \tag{1.27}$$

Sensitivity to volatility called the *vega* and is defined by

$$\frac{\partial V}{\partial \sigma}, \tag{1.28}$$

and sensitivity to interest rate is called *rho*, defied as

$$\rho = \frac{\partial V}{\partial r}. \tag{1.29}$$

1.6 Dividends

Many assets, such as equities, pay out dividends. These dividends affect the prices of options. There are several ways to model dividends. Dividends may be deterministic or stochastic, and may be made continuously or at discrete times. We will consider only deterministic dividends, whose amount and timing is known prior to the start of the option's life. This is a reasonable assumption if the options lifetime is not too long, since many companies have a similar payment from year to year. There are several ways to incorporate dividends into the Black-Sholes model. In this section we show how this is done in the simplest case, when we have a continuous and constant dividend yield. This is a good model for index options, where the many discrete dividends can be approximated by a continuous yield without serious error. The model is also applicable to options on foreign currencies, though only for short dated options. For stocks, the dividends are often made at discrete times, and consequently this model is not suitable for stocks. In [5], Večeř shows how to include discrete dividend payments, for the path-dependent Asian option, studied later in this thesis, in a very simple manner.

Suppose that the underlying pays out a dividend $D_0 s dt$ during the time dt, where D_0 is a constant. The dividend yield is then defined as the ratio of the dividend payment to the asset price. Thus the dividend $D_0 s dt$ represents a continuous constant dividend yield. Arbitrage considerations show that the asset price must fall the amount of the dividend payment, that is, the stock price stochastic differential equation (1.1) is modified to

$$dS(t) = S(t)((\mu - D_0)dt + \sigma dW(t)), \tag{1.30}$$

But as noted before, (1.1), the Black-Scholes equation is independent by the drift-coefficient μ in the stochastic differential equation. What changes is that we must now include the change due to dividends in our self-financing portfolio dynamics (1.11). Since we receive $D_0 S dt$ for every asset held and since we hold h_S number of the underlying, the change in value of our self-financed portfolio now reads

$$dv(t, S(t)) = h_S(t)dS(t) + h_S D_0 S(t)dt + h_B(t)dB(t). \tag{1.31}$$

The analysis proceeds exactly as before, but with new term arising from the dividend, and we find that the value of our portfolio solves the following equation

$$v_t + \frac{\sigma^2 S(t)^2}{2} v_{ss} + (r - D_0)S(t)v_s - rv = 0, \quad t < T,\ S(t) > 0. \tag{1.32}$$

We see that using a continuous dividend yield only corresponds to adjusting one coefficient in the partial differential equation.

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