Thesis for the Degree of Licentiate of Philosophy

Torsten Brodén and the Principles of Geometry

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Abstract

An article on the foundations of geometry from 1890 by the Swedish mathematician Torsten Brodén (1857-1931) is considered. His philosophical view on the nature of geometry is discussed and his thoughts on how to build up an axiomatic system are described. Thereafter his axiomatic system for Euclidean geometry is considered in detail and compared with his later work on the foundations of geometry. The two continuity axioms given are compared to and proven to imply Hilbert's two continuity axioms of 1903. A few of the criteria given for an axiomatic system to fulfill are considered in detail. Finally, possible influences upon Brodén are discussed.

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Chapter 1

Introduction

1.1 Historical Background

The most important event in the development of geometry, as we know it today, was Euclid's systematic treatment of the subject in the form of a uniform axiomatic-deductive system. His work entitled *Elements*,[1] written in Alexandria about 300 BC, still maintains its importance as one of the most valuable scientific books of all time. Influenced by the work of Aristotle, Euclid set himself the task of presenting geometry in the form of a logical system based on a number of definitions, postulates and common notions. It was believed that, in establishing this system, he was creating a sufficient foundation for the construction of geometry.

However, Euclid's *Elements* received a lot of criticism. One of the main issues concerned logical gaps in the proofs, where at some points assumptions that were not stated were used. This happened already in the proof of the first proposition, where an equilateral triangle is constructed.[2] To do this two circles are drawn through each others' centers. The corners of the triangle will now be in the centers of the two circles and in one of the points of intersection of the two circles. However, it does not follow from the postulates and common notions that such a point of intersection actually exists. If we, for example, consider the rational plane $\mathbb{Q}^2$, instead of the real plane $\mathbb{R}^2$, it is easily realised that there are no points of intersection in this case. Thus we could say that Euclid in the *Elements* assumed, without stating so, a continuity of the two circles, and in the same way a continuity of the straight line is assumed.

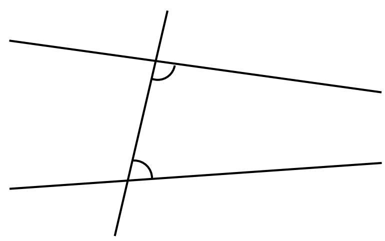
[1] For a complete treatement of the *Elements*, see Heath (1956). An overview of the history of geometry can be found in Eves (1990) and Kline (1972).

[2] Eves (1990) p. 38.

Another tacit assumption in the *Elements* is that the straight line has an infinite extent.[3] It is postulated that the line may be produced indefinitely, but this only implies that the line is endless, not that it is infinite in extent. For example a great circle on a sphere, i.e., a line in a spherical geometry, is endless but not infinite.

These defects are subtle ones, since we are not assuming something contrary to our experience. The tacit assumptions are so evident that there do not appear to be any assumptions. These gaps in Euclid's *Elements* were probably not considered to be of a very serious kind, since intuition could fill them in. Of particular interest was instead the problem whether or not Euclid's Fifth Postulate, also called the Parallel Axiom, is necessary for the construction of geometry, i.e., whether or not the Parallel Axiom is independent from the other postulates and common notions. The Parallel Axiom is formulated in the following way:[4]

> That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.



In the efforts to eliminate the doubts about the Parallel Axiom two approaches were followed. One was to replace it with a more self-evident statement. The other was to prove that it is a logical consequence of the remaining postulates, and that it therefore may be omitted without loss to the theory.

In spite of considerable efforts by several mathematicians for about two millenia, no one was able to do this. This is no wonder, since, as was eventually found out, the Parallel Axiom is independent of and thus cannot be derived from the other postulates and common notions, and also cannot be omitted. This observation was probably first made by Gauss, who claimed that he already in 1792, at the age of 15, had grasped the

[3]Ibid. p. 39.

[4]Heath (1956) p. 202. An equivalent formulation of the Parallel Axiom is Playfair's Axiom: "Through a given point only one parallel can be drawn to a given straight line." Ibid. p. 220.

idea that there could be a logical geometry in which the Parallel Axiom did not hold, i.e., a non-Euclidean geometry.[5] However, he never published anything of his work on the Parallel Axiom and non-Euclidean geometry.[6]

Generally credited with the creation of non-Euclidean geometry are Nikolai Ivanovich Lobatchevsky (1793-1856) and János Bolyai (1802-1860). Lobatchevsky published a first article on non-Euclidean geometry in 1829-1830 in the *Kasan Bulletin.*[7] Bolyai's article on non-Euclidean geometry was published in 1832.[8]

The realization that the Parallel Axiom could not be deduced from the other assumptions, and thus could be exchanged with a contradictory axiom, implied that Euclidean geometry was no longer the only possible geometry. Thus Euclidean geometry is not necessarily the geometry of physical space. The discovery of non-Euclidean geometry made mathematicians realize that the deficiencies in Euclid's *Elements* were a serious problem, and a reconstruction of the foundations of Euclidean geometry had to be made.[9]

The development of non-Euclidean geometry remained unknown to the general public until the 1860s.[10] Instead, because of its beauty and simplicity, projective geometry, which may be regarded as a non-metric geometry, since it ignores distances and sizes, received more attention.[11] In 1873 Felix Klein (1849-1925) proved that projective geometry is independent of the Parallel Axiom, and hence is valid in both Euclidean and non-Euclidean geometries.[12] Thus projective geometry can be considered to be more fundamental than these.

In 1882 Moritz Pasch (1843-1930) managed to develop a complete axiomatic system for projective geometry.[13] He explicitly formulated all primitive notions and axioms, and he understod the importance of a logical deduction of all the geometrical theorems from them. However, he did not manage to combine this position with his view of geometry as an empirical science.[14]

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[5]Gauss made this claim in letters to friends and colleagues, for example in a letter to Taurinus of November 8, 1824, and in a letter to Schumacher of November 28, 1846. For details, see Gauss (1973).

[6]Kline (1972) p. 871.

[7]Eves (1990) p. 62.

[8]The article was published as an abstract to his father Wolfgang Bolyai's book *Tentamen.* A translation into German can be found in J. Bolyai and W. Bolyai (1913).

[9]Kline (1972) p. 1007.

[10]Ibid. p. 879.

[11]Torretti (1978) p. 110.

[12]Klein (1873).

[13]Pasch (1882). The work can also be found in Pasch (1976), together with an appenix by Max Dehn. The axiomatic system is investigated in detail by Contro (1976).

[14]Contro (1970) p. 3.

Contro argues for two lines of development for research into the foundations of geometry after Pasch, one in Italy and another in Germany that was completed with the work of David Hilbert (1862-1943).[15]

The most complete of the Italian geometers is probably Mario Pieri (1860-1913), who focused on metamathematical issues while characterizing the nature of an axiomatic theory.[16] But his work was only the result of an Italian school that had been there for decades. Other important mathematicians who contributed to the field of geometry were Federigo Enriques (1871-1946), Gino Fano (1871-1952), Giuseppe Peano (1858-1932) and Giuseppe Veronese (1854-1917).

In Italy the formal and logical point of view regarding an axiomatic theory was mainly considered.[17] It seems like a complete and rigorous organisation of the foundations of geometry was achieved in Italy already in the 1890s. However, the inportance of question of foundations had a direct connection to issues arising from teaching, which obstructed the recognition of the subject as part of advanced mathematical research.[18] As a result, their work did not receive the attention abroad which it deserved, and became overshadowed by the work of Hilbert.

The axiomatic system of Euclidean geometry that gained most favour is due to Hilbert, who apparently did not know of the work of the Italians.[19] He published his first edition of *Grundlagen der Geometrie*[20] in 1899, but revised his system several times.[21] Compared to Pasch, Hilbert moves on a metageometrical level. His system is built up from undefined concepts, which he calles 'points', 'lines' and 'planes', but he does not assign an explicit meaning to them. The properties of the undefined concepts are specified by the axioms, which are independent of physical reality.[22] The axioms are no longer evident truths, and there is no sense in asking about their veracity anymore.[23]

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[15]Contro (1976) p. 291.

[16]Marchisotto (1993) p. 288.

[17]Contro (1976) p. 292.

[18]Avellone, Brigaglia and Zappulla (2002) p. 365.

[19]Kline (1972) p. 1010. However, Toepell disagrees with this remark, and claims that Hilbert know of Peano's and Veronese's work on the foundations of geometry. Toepell (1986) p. 57.

[20]A facsimile of the first edition of *Grundlagen der Geometrie* can be found in Sjöstedt (1968). A transcript can also be found in Hallett and Majer (2004).

[21]Hilbert published in all seven editions of *Grundlagen der Geometrie.* The last appeared in 1930. The eighth edition, revised by P. Bernays, appeared in 1956.

[22]Kline (1972) p. 1013.

[23]Freudenthal (1957) p. 111.

1.2 Main Questions

During the last couple of decades of the 19th century it seems like a very interesting discussion on the foundations of geometry took place in Germany and Italy. The mathematicians in Sweden do not seem to have taken part in this discussion. One exception, however, is Torsten Brodén. He wrote two articles on the foundations of geometry, one was published in 1890 and the other was presented at the second Scandinavian Mathematical Congress in 1911.[24] In the 1890 article he develops an axiomatic system for Euclidean geometry and in the Congress article he presents his earlier system again, but in a slightly revised form.

In this thesis I will consider the 1890 article and the Congress article in detail, with emphasis on the earlier one. With this as a starting point, I shall investigate what Brodén thought in general about science and mathematics. In particular I want to investigate his thoughts on geometry and its nature and what consequences his view has for how he proceeds in developing the axiomatic system. Further I want to examine what relation Brodén had to the age in which he lived, which mathematical texts he read, and who he was influenced by. What did he know about the development of the axiomatics of geometry that had taken place on the continent?

There was a gap of more than two decades between the appearence of Brodén's two articles. During this time much had happened in the foundational work on Euclidean geometry. Had Brodén's view on the foundations of geometry changed over this time? In particular the question of how to deal with the problem of continuity was not solved when Brodén's earlier article appeared, but the later article appeared after Hilbert had formulated the Completeness Axiom. Did this affect Brodén's axiomatic system in any way?

Finally, Brodén's 1890 article appears in a somewhat peculiar context. It is published in a pedagogical journal, and Brodén claims that his main motivation for the article is a pedagogical one. It would be interesting to look closer into Brodén's pedagogical view. However, in this thesis my emphasis is on Brodén's geometrical work.

One could question why one should research such an unknown mathematician as Brodén, whose work did not draw much attention from the wider mathematical community. I believe that it gives a better sense of the spirit of the mathematical society of the time to look into the work of one of the less known mathematicians.

[24] Brodén (1890); Brodén (1912).

Chapter 2

Torsten Brodén and His Work

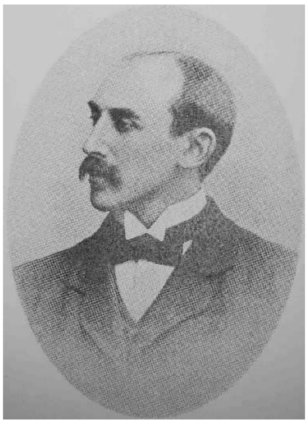
2.1 Biography

Torsten Brodén[1] was born on the 16th of December 1857 in Skara, Sweden. He began his studies at the University of Uppsala in 1877, but transferred two years later to the University of Lund. There he presented, in the spring of 1886, his Ph.D. thesis with the title *Om rotationsytors deformation till nya rotationsytor med särskildt afseende på algebraiska ytor* ('On the Deformation of Surfaces of Rotation to New Surfaces of Rotation with Special Attention to Algebraic Surfaces').[2] He continued teaching at the Mathematical Seminar in Lund and at secondary school before he in 1906 succeeded C.F.E. Björling (1839-1910) as a Professor of mathematics at the University of Lund. He retired as Professor Emeritus in 1922.

Brodén's mathematical activity was unusually many-faceted. He worked in such diverse fields as algebraic geometry, elliptic functions, Fuchsian differential equations, set theory and the logical foundations of mathematics.[3] Among Swedish mathematicians of his time he had an exceptional position because of his pronounced philosophical interest.[4] A characteristic of his work was his desire always to obtain full clarity regarding basic mathematical notions.[5] One of the first examples of this we can see in his 1890 article

[1] Biographical notes on Torsten Brodén can be found in Svenskt Biografiskt Lexikon (1925).
[2] Brodén (1886).
[3] Gårding (1994) p. 216.
[4] Zeilon (1931) p. 59*.
[5] Svenskt Biografiskt Lexikon (1925).

Om geometriens principer ('On the Principles of Geometry')[6] where he, several years before Hilbert's first attempt, develops an axiomatic system for Euclidean geometry.



Of great importance for his future career seems to be when Brodén in 1891 got a travelling scholarship, *Riksstatens mindre resestipendium*, and traveled to Germany and Austria for six months.[7] The purpose of this trip was, on the one hand, to study how mathematics was taught at the universities on the continent, and, on the other hand, to study mathematics and to do research. Brodén visited several universities, among others in Berlin, Heidelberg, München and Vienna. He stayed several months in Berlin, where he followed two courses given by Leopold Kronecker, (1823-1891), *Theorie der elliptischen Functionen zweier Paare reeler Argumente* and *Allgemeine Arithmetik, erster Theil*, and a course given by Lazarus Fuchs, (1833-1902), *Einleitung in die Theorie der Differentialgleichungen.* Brodén claims that he got ideas for further research in private conversations with Kronecker, but unfortunately Kronecker suddenly died at the end of the year.

Brodén died on the 6th of July, 1931. When his wife, Fanny Kallenberg, whom he had married in 1896, died in 1952, their effects were donated to the society *Kungliga Fysiografiska Sällskapet i Lund*, to establish a fund for their memory, *Torsten och Fanny Brodéns fond.*[8] Brodén had been elected a member of the society, whose main purpose was to support research, in 1894.[9] The fund still exists today, and pays out scholarships annually for young researchers at the university of Lund.

[6] Brodén (1890).
[7] Details about Brodén's journey can be found in Brodén (1892).
[8] F. Brodén (1950).
[9] Zeilon (1931) p. 59*.

Not much is known about Brodén's personality. Nils Zeilon described him as being amiably sarcastic, but added that under this exterior a peculiar idealist was hidden.[10]

During the 1890s Brodén associated with a circle around August Strindberg. They often met in the pub *Åge Hans*, where Brodén played classical music on the piano.[11] In a letter Strindberg claims that it was Brodén's interpretation of Beethoven's piano sonata number 17 in D minor that inspired him to write the play *Brott och Brott* ('Crime and Crime').[12]

2.2 Literature on Brodén

Torsten Brodén is today a relative unknown mathematician. Little is known of his life and not very much is written on his professional activity. However, he is not totally forgotten.

In his book on mathematics and mathematicians in Sweden,[13] Lars Gårding writes very briefly about Brodén. He mainly considers Brodén's work on Fuchsian differential equations and does not mention Brodén's philosophical work or pedagogical interest.

Dennis Hesseling mentiones Brodén in his book on the foundational crisis in mathematics.[14] This crisis had unfolded in the 1920s as a reaction to Brouwer's intuitionism. Brodén criticized the intuitionists as being primarily motivated by their fear of antinomies, and claimed these could instead be resolved in a different way.[15] However, nobody ever responded to his criticism.

In an article about the evolution of the function concept, N. Luzin claims that Brodén was one of the first to state his dissatisfaction with Dirichlet's definition of a function.[16]

In his book on the development of modern probability,[17] Jan von Plato discusses a debate between Brodén and his collegue Anders Wiman. von Plato claims that Brodén in his study of Gyldén's problem, i.e., the question of limiting distribution of integers in a continued fraction, was the first to apply measure theory to probabilistic purposes.[18]

Concerning Brodén's work on the axiomatization of geometry, it has attracted some earlier attention. In 1985 Walter Contro wrote a relatively

[10] Ibid. p. 61*.
[11] Gårding (1994) p. 217.
[12] Strindbergssällskapets skrifter (1972) p. 248.
[13] Gårding (1994).
[14] Hesseling (2003).
[15] Ibid. p. 175.
[16] Luzin (1998) p. 265.
[17] von Plato (1994).
[18] Ibid. p. 31.

brief but clear-sighted article[19] for the *Festschrift für Helmut Gericke* where he discusses and criticizes Brodén's axiomatic system. As a starting point Contro mentions Freudenthal's assertion that during the latter part of the 1880s all parts of geometrical axiomatics were treated and only had to be combined to a unit so that the modern axiomatic could arise. Contro claims that it is already well-known that this happened in Germany via Hilbert and in Italy via Peano and his school, and that Brodén's 1890 article shows that this also happened in Scandinavia.

Unfortunately, instead of studying Brodén's 1890 axiomatic system, Contro chooses to study Brodén's more clearly formulated axiomatic system from 1911. Contro is aware of and mentions some of the differences between the systems, but not all of them. Thus a revision of Contro's investigation has to be made.

[19] Contro (1985).

Chapter 3

Brodén's 1890 Axiomatic System

3.1 Brodén's Aim and Motivation

Brodén's first article on the axiomatization of geometry was published in *Pedagogisk Tidskrift*, a journal for Swedish secondary school teachers, in 1890.[1] In the article Brodén gives a philosophical and pedagogical discourse on geometry and develops an axiomatic system for Euclidean geometry. It seems that the article did not get a lot of attention, even though Brodén wrote a summary of the mathematical part of his work for the *Jahrbuch über die Fortschritte der Mathematik*.[2] A reason for this might be his choice of a pedagogical journal instead of a mathematical journal. Furthermore, the Swedish language was an obstacle for the international public.

Brodén's aim with his 1890 article appears to be to take part in an ongoing pedagogical debate on the problems in Swedish schools.[3] He points out that there are faults and defects in the teaching of geometry, but does not further discuss what these are and how to do something about them. His aim, he claims, is not to call for any major reforms in the immediate future. As a reason for this he refers to, among other things, the difficult nature of geometry and that a thorough judgement of the scientific aspect of geometry demands considerations of deep and disputed questions.

In the beginning of the article, Brodén discusses the often heard statement, that the value of geometry as a school subject lies in the possibility

[1] Brodén (1890).
[2] Brodén (1893).
[3] Brodén (1890) p. 217.

for it to be treated in a strictly 'scientific' way.[4] To decide if this statement is true, he seeks to investigate, on the one hand, what a strictly scientific geometry should look like, on the other hand, if such a scientific character is possible or suitable at the school level. He treats these two aspects in his article. His axiomatic system is the result of his investigation into what a scientific geometry should look like. His conclusion after carrying out this investigation is that a strictly scientific geometry should not be present undiluted in the school setting.[5] It is a difficult balancing act, he claims, between, on the one hand, keeping a scientific direction in the education and, on the other, taking into consideration the students' ability. Even though the value of geometry, as a school subject, is considered to lie in its' ability to be treated in a strictly scientific way, Brodén is of the opinion that understanding and simplicity should have priority.[6] He continues that it is a *practical*, rather than a *scientific*, teaching that should be aimed at, but at the same time, education in geometry should prepare the students for possibly more rigorous studies.[7]

In his remarks on the ontological status of geometry, one clearly sees the influence on Brodén of the ideas of Hermann von Helmholtz (1821-1894). Brodén claims that:[8]

> Geometry, if it should have some application to the objects of nature, has to be looked upon as a natural science, an empirical, inductive science.

But he does not consider geometry to be like any other science. Quoting Helmholtz, he states that geometry is "die erste und vollendetste der Naturwissenschaften".[9]

Despite the fact that Brodén considers geometry to be a natural science, he considers natural science to presuppose geometry[10] (that is why geometry is 'die erste'). He states the reason for this to be that natural science endeavours to reduce different phenomena to 'motion', but to comprehend motion we need the 'empty, stationary space' as a background. In this sense one may say that motion presupposes geometry.

Even though Brodén considers geometry to be an empirical science, he claims that it deals with ideal objects that are not revealed by the immedi-

[4]Ibid. p. 218.
[5]Ibid. p. 263.
[6]Ibid. p. 264.
[7]Ibid. p. 265.
[8]"Geometrien är, för så vidt den skall hafva någon användning på naturföremålen, själf att betrakta som en naturvetenskap, en empirisk, induktiv vetenskap". Ibid. p. 218.
[9]Ibid. p. 218. This quote comes from Helmholtz' article *Über den Ursprung und Sinn der geometrischen Sätze* (1882) p. 642.
[10]Brodén (1890) p. 218.

ate external experience.[11] This might seem odd at first, but illustrates an attitude which is typical of the last couple of decades of the 19th century, between Pasch and Hilbert. Brodén stands with one foot in the old Aristotelian approach that geometry is founded on empirical grounds, but at the same time he has a more modern approach towards its foundations. He does not consider these two opinions to be in conflict and draws parallels to attempts to systematize chemistry and physics, where the ideal objects correspond to 'atoms' and 'ether vibrations' respectively.[12] The empirical comprehension, he claims, should only be considered as a starting point, and experience can hardly lead to logical contradictions.

In spite of the starting point that it is to be considered as a natural science, Brodén wants to point out that geometry, as a logical possibility, can be independent of space and time. He maintains this since, referring to Cantor, arithmetic can be considered as a logical system independent of space and time, and:[13]

> [...] geometry is nothing but arithmetic, or can at least be totally dressed in an arithmetic costume.

In this way, Brodén claims, Euclidean geometry becomes an a priori possible logical form among many other geometries. Its' special importance, he continues, it first gains through reality.[14]

In this context Broodén also mentiones Immanuel Kant (1724-1804). Without going into any details, he claims that his view on the nature of geometry could be considered as a development of Kant's theories.[15]

Brodén wants to gain support for his views by carrying out a detailed examination of the foundations of geometry. He does this by first considering a few criteria which the basic notions and axioms for a scientific geometry should fulfill. Thereafter he explains how he picks out the basic notions and he carries out the axiomatization. Finally he gives a proof that his axioms are sufficient for establishing Euclidean geometry. I will, in this chapter, explain in detail how Brodén proceeds.

Brodén does not state his axioms in an immediately clear way. One can extract them from the text, but they are not written down explicitly. I try to adhere as closely as possible to Brodén's formulations, but of course I still rely on my own interpretation of them. But comparing his slightly different formulations in the 1890 article, the Congress article and the summary

[11] Ibid. p. 218.
[12] Ibid. p. 218 and p. 259.
[13] "[...] geometrien [...] är ingenting annat än aritmetik, eller kan åtminstone fullständigt klädas i aritmetisk dräkt". Ibid. p. 219.
[14] Ibid. p. 219.
[15] Ibid. p. 220.

in *Jahrbuch über die Fortschritte der Mathematik*, I think I do justice to Brodén's 1890 axiomatization of geometry.

It is also worth noting that Brodén in 1890 never uses the word 'axiom'. Instead he uses the Swedish word 'fundamentalsats' or shorter 'sats', which is probably a direct translation from the German word 'Fundamentalsatz' or 'Grundsatz'. In *Jahrbuch über die Fortschritte der Mathematik* he translates it into just 'Satz', and in the Congress article he uses the word 'axiom'. Since this is the word used today, I decided to translate into 'axiom'.

3.2 The Criteria

Brodén considers the goal of science to be to get a clear insight into the 'nature of objects'. To do this he wants to describe the inner structure of the concepts in a clear way.[16] To attain this goal, he claims, a scientific system should be built up from a number of undefined 'basic notions' and a number of unproven 'axioms'. He gives a number of criteria which these basic notions and axioms for a scientific geometry should fulfill:[17]

1. The notions should be reduced to the smallest possible number of undefined basic notions.
2. All theorems should be proven from a smallest possible number of unproven axioms.
3. The axioms should be stated in a clear way.
4. There should be the greatest possible degree of empirical evidence for the axioms.
5. The axioms should form a homogeneous system.
6. The sufficiency of the axioms for arranging geometry under certain logical forms, should be clear.
7. The axioms should be independent of one another.
8. The axioms should be as easy as possible to handle.

The seventh criterion considers the independence of the axioms. However, the meaning of the axioms in Brodén's system depends upon the preceeding ones. This suggests that he considers an axiom to be independent if it cannot be deduced from those previously stated.

The method for systematically studying the mutual independence of axioms is the method of contructing models: the model is shown to disagree

[16] Ibid. p. 258.
[17] Ibid. p. 220-221.

with one and to satisfy all the other axioms, and hence the one cannot be a consequence of the others. However, Brodén might not have been aware of this method. On a few occasions he carries out a proof of independence by the construction of a model, but to me it seems that he does this only to motivate each axiom in turn, not to prove its' independence of the previous ones. Brodén also adds that one must have moderate pretensions when it comes to guaranteeing independence of the axioms, since it might be hard to combine this with the other criteria.[18]

Considering the sixth criterion, Brodén does not specify what he means with 'sufficiency' or 'logical forms'. Probably he alludes to sufficiency in an intuitive sense; the axioms are sufficient if we through deduction from them will obtain what we consider to be Euclidean geometry. I will consider this criterion further in Section 3.5 and 5.2.

With the fifth criterion Brodén probably alludes to a homogenous ontology in the axiomatic system, i.e., a scientific system should be built up of similar components and one should only use objects from the same category. This criterion also seems to be mainly of aesthetic character.

In the fourth criterion the empirical view Brodén has of geometry shines through; it should be evident from the axioms that geometry after all is a natural science. Since, according to Brodén, our experience cannot lead to logical contradictions,[19] this criterion may imply some kind of consistency. I will discuss this further in Section 5.2. Brodén notes that, even if the axioms form the logical foundations for all the theorems, they don't have to exceed them in the degree of evidence.[20] He does not believe that it is possible to formulate such an axiomatic system, since "evidence is one thing, the logical relation is a different matter".[21]

With the third criterion Brodén probably wants to say that the axioms should be stated in such a way that they cannot be misinterpreted, and with the eighth criterion he probably wants to say that the axioms should be formulated in such a way that we can use them without difficulty.

With the first and second criteria Brodén probably wants to emphasize that the basic notions and axioms must be chosen in an 'intelligent' way, i.e., we should try to choose them in such a way that we need as few of them as possible. He claims that "a reduction to the smallest possible is the goal of science".[22] We see that a balance in the choice of axioms has to be maintained so that the second and fourth criteria are fulfilled; at the same time as the axioms are chosen in an 'intelligent' way, the empirical evidence should still be clear.

[18] Ibid. p. 261.
[19] Ibid. p. 218.
[20] Ibid. p. 260.
[21] "[...] evidens är en sak för sig, det logiska sammanhanget en annan". Ibid. p. 261
[22] "[...] en reduktion till det minsta möjliga är vetenskapens uppgift". Ibid. p. 260.

Contro interprets the second criterion to be the same as the seventh, i.e., he considers the reduction to the smallest possible number of axioms to be the same as an independence criterion.[23] However, I do not believe this is what Brodén meant with the second criterion, since then there would be no point in stating both these criteria.

3.3 The Basic Notions

The first thing Brodén has to do in establishing an axiomatic system for geometry is to determine the basic notions, i.e., to determine the undefined notions that are needed to formulate the axioms and to give further definitions. In doing this he continues to discuss motion in order to characterize it. Since he considers geometry to be a natural science and maintains that natural science endeavours to reduce all phenomena to motion, it follows that geometry must also endeavour to do so. This may seem contradictory, since he maintains that motion presupposes geometry.

Motion, Brodén claims, is a change in certain relations between objects, i.e., motion has to do with a collection of objects and a collection of relations between them.[24] Brodén reduces the concept 'collection of objects' to simple 'undivisible objects' that he calls 'points'. Motion is then considered to be a change in certain relations between points. But to be able to apprehend this motion a system of stationary points is required, i.e., an empty motionless space that forms the background for our comprehension of motion.[25]

The points in a rigid body are mutually at rest, Brodén continues, also when the body moves.[26] If two points A and B in a body coincide, at a given moment, with two points C and D in the stationary background space, and at another moment coincide with C' and D', we can say that the distance between C and D is equal to the distance between C' and D', i.e., $CD = C'D'$. This notion of equal distance he reduces further to the notion of 'equal distance from the same point', or 'immediate equality of distance', i.e., $AP = BP$.[27]

Brodén chooses to use these two notions, 'point' and 'immediate equality of distance', as basic notions in his system. He never discusses if his choice of basic notions fulfills the first criterion he considers a scientific system should fulfill. Contro claims that Brodén's choice gives a minimal set of

[23] Contro (1985) p. 627.
[24] Brodén (1890) p. 221.
[25] Ibid. p. 221.
[26] Ibid. p. 222.
[27] Ibid. p. 223.

basic notions that cannot be reduced further.[28] Thus the criterion should be fulfilled.

3.4 The Axioms

After establishing the two basic notions 'point' and 'immediate equality of distance', Brodén continues to establish the axioms from which Euclidean geometry should be built up. His system consists of 16 axioms, and with the help of these he can define notions like 'straight line', 'plane', 'between', 'bigger than', 'smaller than' et cetera. Here I will give a complete description of his system.[29]

In establishing the axiomatic system, Brodén wants first to completely determine the notion of a straight line, before he proceeds to introduce the plane. To do this he needs to establish a more general notion of equality of distance than the basic notion 'immediate equality of distance'. As a first axiom he introduces an axiom of transitivity of equal distances:

> **Axiom I** Those distances (from the same point) which equal one and the same distance, are equal to each other, i.e., if $AP = BP$ and $CP = BP$ then $AP = CP$.

To be able to define the straight line Brodén now discusses the motion that is still possible in space when two of its points are fixed. Next to these two points also other points are fixed, and the collection of all these fixed points must form a straight line. But Brodén is not satisfied with defining the line in this way. He introduces, referring to Wolfgang Bolyai, the notion of 'Einziges'.[30] A point P is *Einziges* to two points A and B if P does not have the same distances to A and B as any other point P'. With the help of this concept Brodén now states the axiom he needs to define the straight line:

> **Axiom II** Two arbitrary points unambiguously determine a system of points, which, with respect to any two points in the system, arbitrarily chosen, form the summary of all *Einziges* belonging to them.

[28] Contro (1985) p. 632.

[29] The axioms can be found in Brodén (1890) pp. 223–230, 236.

[30] Brodén probably read W. Bolyai's *Kurzer Grundriss eines Versuchs* (1851), where the foundations of geometry are considered and 'Einziges' is defined. W. Bolyai gives the same discussion in *Tentamen*, from 1832, where his son wrote the better known appendix on non-Euclidean geometry. Brodén probably didn't read *Tentamen* since it was written in Latin and not translated into German until 1913. The translation can be found in J. Bolyai and W. Bolyai (1913).

Definition Such a system of points is called a line.

With this, Brodén claims, the straight line is completely defined in the sense that no other system of points in the space has the same characteristics. However, this is not enough to completely characterize the straight line, since its remaining characteristics do not logically follow from the axioms mentioned so far. Axiom II gives some kind of symmetry on the straight line; for two arbitrarily chosen points on the line every other point on the line is the only point with given distances to the two chosen points. But Brodén also wants an inner symmetry on the line. To obtain this he formulates the following two axioms:

> **Axiom III** Every point P on a line defines an unambiguously symmetric correspondence between the points of the line, where the distances from two corresponding points to the point P are equal, the distances from non-corresponding points to P are not equal, and P is the only point corresponding to itself.

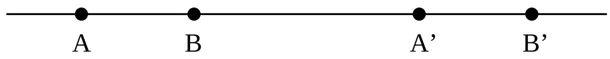
> **Axiom IV** Two arbitrarily chosen points define one and only one correspondence of that kind, such that they correspond to each other.

With Axiom III, a reflection in an arbitrarily chosen point is established on the line, and Axiom IV forces two arbitrarily chosen points to unambiguously determine such a reflection where these two points will correspond to each other. In this symmetrical reflection one and only one point will correspond to itself, and Brodén now can give the following definition:

> **Definition** The point corresponding to itself in the correspondence determined by two other points is called the midpoint of the two points.

With this definition, Brodén still cannot say anything about a point lying 'between' two other points, or a distance being 'bigger than' or 'smaller than' another distance. To do this he has to introduce an 'ordering axiom', but before he can do this he characterizes the notion of 'equal distance' on the line further. He gives a definition of equal distance, and a more general axiom on equality of distance:

> **Definition** The distance (on a line) between A and B equals the distance between A' and B' if there is a symmetric correspondence where A corresponds to A' and B corresponds to B' (or A corresponds to B' and B corresponds to A').



> **Axiom V** The distances (on a line), which equal one and the same distance, equal each other.

With this axiom Brodén can now compare arbitrary distances on the straight line in the sense of deciding whether they are equal or not, but he still cannot say anything about the distance between points not on the same line. Furthermore, the axioms stated so far do not suffice to characterize the inner structure of the line in Euclidean space. For example, there is still the possibility of finite geometries. Brodén gives a model (however he does not use the word 'model') of a finite geometry that fulfils all the axioms he has stated so far. A straight line in this geometry consists of the vertices of a regular polygon with an odd number of edges. It is easily checked that the first five axioms are fulfilled in this geometry.

To exclude finite geometries from his system of axioms, Brodén has to include further axioms to completely determine the line. But before he does this he first wants to show that, with the help of the axioms given so far, he can 'push the line' or give a 'transformation of the line upon itself', i.e., an unambigous and asymmetric correpondence in which corresponding distances are everywhere equal ($AB = A'B'$) and the distance between two corresponding points is constant ($AA' = BB'$). This means that we can push a line so that A ends up in A', B in B' and so on. I will discuss the construction he performs in Section 3.5.

This transformation of the line is not really needed in Brodén's construction of the axiomatic system. However, he needs it later, when he constructs a coordinate system and gives a proof that his axioms are sufficient for establishing Euclidean geometry. He also uses the notion of transformation of the line in the discussion that leads him to Axiom VI, which is an axiom which gives an ordering of certain points on the line.

To exclude finite geometries, Brodén has to include axioms which, together with the axioms already stated, imply that the line is an infinite continuum, i.e., that after the choice of a 'zero-point' (A) and a 'one-point' (B) the line will unambiguously correspond to the real numbers $\mathbb{R}$. The first obstacle in doing this is to determine points on the line corresponding to the natural numbers.

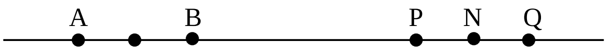
Brodén maintains that on the line there has to be a system of points with the characteristics that, if M is the midpoint of the point B and an arbitrary point P in the system, and if the point Q corresponds symmetrically to the point A with respect to M, then Q also belongs to the system, and each point in the system has the same relation to some other point in the system as Q has to P. Brodén calls Q the point 'immediately following' P, and P is the point 'immediately preceding' Q. With this construction Brodén can successively traverse a distance AB on the line, and he can now give the following axiom which excludes all finite geometries.

> **Axiom VI** On the straight line there is a system of points such that every point in the system has points in the system immediately following and immediately preceding it, with the single exception that the point A does not have a preceding point.

This axiom could be interpreted thus: if a length AB is successively traversed on the line, one does not come back to the starting point. With this axiom Brodén can characterize points on the line which correspond to the natural numbers. If the point A is the zero-point and B is the one-point, he can now successively traverse the distance one without coming back to the beginning and thus obtain all the natural numbers. By means of a symmetric correspondence with respect to the zero-point, he can also characterize the negative integers. Thus, with this axiom Brodén achieves an ordering of certain points on the line.

Now Brodén can define the notions 'between', 'bigger than' and 'smaller than', at least regarding the points in the system mentioned in Axiom VI. Brodén does not show how to do this; he just states that this can now easily be done.

However, Axiom VI is not enough to gain an unambiguous correspondence between all the points on the line and the real numbers, i.e., to get a continuous line. Brodén shows this by considering the two points P and Q, where Q is the point immediately following P, and N is the midpoint of Q and P. He claims that he can show, without difficulty, that N belongs to a system of positive integers, where A is chosen as zero-point and the midpoint between A and B is chosen as one-point, and P and Q are the points immediate preceding respective following N. It is clear, he further claims, that N cannot coincide with A, since then P should immediately precede A, which contradicts the properties of the system of positive integers.



This method, Brodén continues, can easily be generalized so that the midpoint between two arbitrary consecutive points in the original system of positive integers can not coincide with any point in this system. By everywhere taking midpoints of consecutive points, he maintains, nothing but new points are obtained, and together with the original points they form a new system of positive integers. By successively taking new midpoints new systems of positive integers are obtained. This leads, he continues, to a system of points that unambiguously is represented by all positive and negative integers and fractions with the denominator being a power of two.

However, as Brodén also points out, if one takes two different starting points A and B, for example the zero-point and the three-point instead of the zero-point and the one-point, then the new set of points obtained by successively taking midpoints does not contain all the points in the original set of points. So, if he does not want to impose further restrictions, Brodén continues, he has to allow 'different relations' among the points of the line. But since our experience does not give any indication of such a difference, Brodén realizes that he has to include a further axiom regarding the inner structure of the points of the line. With this axiom he wants to achieve a correspondence between every point on the line and the real numbers, i.e., he wants to obtain a continuity of the line. The idea behind the axiom is to successively take midpoints of smaller and smaller intervals and go to the limit. With a construction like this Brodén gets a bijection between the line and the real numbers. To be able to express this in an easier way he introduces the so-called c-system.

With the number system $\frac{a}{2^n}$ (a, n integers), i.e., the number system corresponding to the points of the line obtained by taking the midpoint a finite number of times, as basis, Brodén claims that all real numbers can be represented. He proceeds with the statement that, if n assumes all possible positive integer values, then

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n}$$

represents a system of points with the relation to the one-point that there are points in the system whose distance to it is smaller than any given distance, i.e., the one-point is a 'limit point' for the system. He claims that this in fact is the only limit point of the system, and that the one-point cannot be a limit point for any other infinite system

$$c_0 + \frac{c_1}{2} + \frac{c_2}{2^2} + \frac{c_3}{2^3} + \ldots + \frac{c_n}{2^n}$$

where c_0 is an integer or zero and c_i, $i \geq 1$, are equal to zero or one, but not all equal to zero after some given i. A system like this he refers to as a c-system in reduced form.[31] He now notices that not every c-system has a limit point in the system $\frac{a}{2^n}$. So to expand the point system on the line, Brodén simply wants every c-system to have a limit point. But he has

[31] Brodén notes that every infinite system $b_0 + \frac{b_1}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \ldots + \frac{b_n}{2^n}$ where b_i equals 0, +1 or -1, through the merging of the negative terms with the previous positive term, can be *reduced* to a c-system.

to express this in a different manner, since, if he goes outside the system $\frac{a}{2^n}$ the notions of 'bigger than' and 'smaller than' still does not have any meaning and thus the notion of 'limit point' cannot be used. To get around this problem he expresses the axiom in the following way:

> **Axiom VII** Between c-systems and the points of a line, a mutually unambiguous correspondence can be established so that to two arbitrary c-systems
>
> $$c_0 + \frac{c_1}{2} + \frac{c_2}{2^2} + \frac{c_3}{2^3} + ... + \frac{c_n}{2^n}$$
>
> and
>
> $$c'_0 + \frac{c'_1}{2} + \frac{c'_2}{2^2} + \frac{c'_3}{2^3} + ... + \frac{c'_n}{2^n}$$
>
> there correspond two points, whose distance to each other equals the distance from the zero-point to the point corresponding to the set
>
> $$c_0 - c'_0 + \frac{c_1 - c'_1}{2} + \frac{c_2 - c'_2}{2^2} + \frac{c_3 - c'_3}{2^3} + ... + \frac{c_n - c'_n}{2^n}$$
>
> or its reduced set, and the one-point corresponds to the set
>
> $$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^n} .$$

With this axiom Brodén directly obtains a bijection between the real numbers and the points on the line. Thus the geometry of the straight line is completely determined, and Brodén continues to determine the geometry of the plane. He does this in a very similar way as with the straight line, by considering symmetries. But first he wants to introduce an axiom which helps him to further determine the notion of equality of distance.

> **Axiom VIII** On every straight line through an arbitrary point P there exist points, whose distances from P equal the distance to P from an arbitrary point in space.

From previous axioms it follows that there exist two such points on the line whose distances from a point P on the line equal the distance from P to an arbitrary point in space.

Now Brodén can give a definition which helps him to compare two arbitrary distances.

Definition The distances AB and CD are equal if, on the straight line AC, the distances from A and C, which equal AB respectively CD, are also equal to each other.

With this Brodén can now add an axiom which gives a general notion of equality of distance.

Axiom IX Without exception it holds good that the distances that are equal to one and the same are equal to each other.

Definition Two systems of points are equivalent if an unambiguous mutual correspondence can be established, for which all corresponding distances are equal.

With the following axiom, Brodén wants to introduce the plane by the construction of a system of points. If there is such a system, he claims, it has to be generated by a straight line that rotates around a fixed point following a straight line. Our experience, he continues, tells us that a system of this kind arises, but for the sake of simplicity he chooses to formulate the axiom in the following way:

Axiom X There is a system of points, such that a straight line through two arbitrarily chosen points in the system completely belongs to the system, without filling the complete space.

Definition Such a system of points is called a plane.

After introducing the plane, it is now plausible for Brodén to seek analogies between the fundamental properties of the plane and of the straight line. He does this in the following three axioms, which correspond to Axiom III and Axiom IV. With these axioms he obtains a 'symmetrical equivalence' in the plane, which can be considered as a reflection of the plane in a straight line in the plane.

Axiom XI Every straight line in the plane unambiguously defines a symmetric equivalence, where every point on the line but no other point is self-corresponding.

Axiom XII Two arbitrarily chosen points unambiguously define such an equivalence, where they correspond to each other.

Axiom XIII The self-corresponding line is the complete locus for equal distance from two corresponding points.

Definition The self-corresponding line in a symmetrical equivalence is called the axis of symmetry.

However, these axioms are still not sufficient for the establishment of Euclidean geometry. Brodén points out that a so called 'pseudo-spherical' geometry, i.e., a hyperbolic geometry with constant negative curvature, is still possible. To exclude this he has to add an axiom, which is a version of the Parallel Axiom.

> **Axiom XIV** The complete locus for symmetrically corresponding points with the same mutual distance as two given points, forms two straight lines.

With the axioms stated so far, Brodén claims, Euclidean plane geometry appears. He gives a proof of sufficiency of his axioms, which I will discuss in the next section.

Now that the inner structure of the plane has been taken care of, Brodén proceeds to space and adds the final two axioms:

> **Axiom XV** Through three arbitrarily chosen points in space there goes a plane, and if the points are not in a straight line, there is only one such plane.

> **Axiom XVI** Two planes cannot have only one point in common.

The last axiom, Brodén claims, excludes a fourth dimension.[32] Thus, he continues, he now has all the requirements needed for establishing Euclidean three-dimensional geometry.

3.5 Brodén's Proof of Sufficiency

After presenting the axioms, Brodén wants to prove that they are sufficient to establish Euclidean geometry. He gives an explicit proof for the sufficiency of Axiom I to Axiom XIV for establishing plane Euclidean geometry, and, after adding Axiom XV and XVI, he claims that in a similar manner he can prove sufficiency for establishing three-dimensional Euclidean geometry. In this section I will present and explain Brodén's proof.[33]

However, it is a bit hard to grasp Brodén's proof of sufficiency, since it is quite long and he makes no effort to give an overview of his ideas. The entire proof is written as one long account without any explanatory pictures. To make Brodén's argumentation easier to read, I dissected it into several propositions with shorter proofs and also included some illustrations. In

[32] Axiom XVI and the problem of dimension will be discussed further in Section 5.3.

[33] The proof is found in Brodén (1890) pp. 230–236.

the proofs of the propositions I follow Brodén very closely. In between I try to give a more general overview of what he is doing. The reader can, without losing track of Brodén's main idea, skip the details in the proofs.

Brodén does not make it clear what axioms he uses in his proof. A reason for this might be the fact that his axioms are originally not numbered. However, throughout this section, symmetrical correspondence (Axiom III and IV) and symmetrical equivalence (Axiom XI, XII and XIII) play a decisive part.

The main idea in Brodén's proof is, with the help of his axioms, to construct a coordinate system in which he can denote a straight line through two arbitrary points by an equation. With this he can deduce the formula for calculating the distance between these two points. In the distance formula, Brodén claims, the whole plane Euclidean geometry lies embedded. This claim originates from his view that geometry is nothing but arithmetic.

To construct the coordinate system, Brodén first discusses perpendicular lines and shows that perpendicularity is a symmetrical relation (Proposition 1) and that through a given point there is one and only one line perpendicular to a given line (Proposition 2). Two lines perpendicular to each other will form the x- and y-axes in the coordinate system. Thereafter he proves that the position of a point in the plane can be unambiguously determined with coordinates (Proposition 3). He continues to discuss how to transform[34] a line (Proposition 4) to be able to do a coordinate transformation. With this he can easily determine the equation of the straight line. Finally, by rotating the line by performing two symmetrical equivalences, he can deduce the distance formula.

Brodén starts his discussion by claiming that, in a symmetrical equivalence in the plane, a straight line will correspond to another straight line (he does not prove this claim), and if a line goes through two points that correspond symmetrically to each other, then the straight line must correspond to itself. He now gives the definition of a line being 'perpendicular' to another line.

Definition The self-corresponding line in a symmetric equivalence is perpendicular to the axis of symmetry.

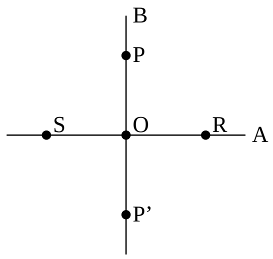
Brodén further claims (again without giving a proof) that, through a point not on a given line, there goes one and only one straight line perpendicular to the line. If the point lies on a line there is also one and only one

[34]When Brodén uses the word 'transformation' it is obvious that he means what we today would call 'translation'.

straight line through the point perpendicular to the given line. This last statement Brodén proves explicitly, but to be able to do this he first has to show that the notion of a line being perpendicular to another line is a symmetric relation.

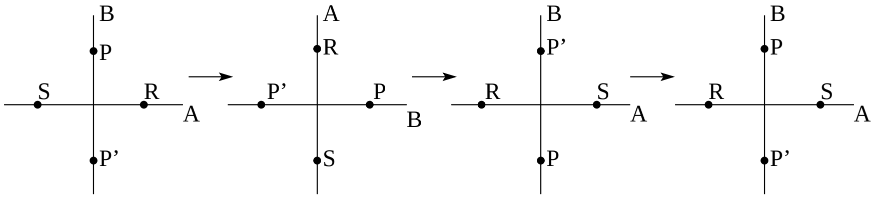
Proposition 1 If a straight line B is perpendicular to another straight line A then A is perpendicular to B.

Proof Suppose the straight line B is perpendicular to the straight line A, i.e., B is a self-corresponding line in the symmetric equivalence where A is the axis of symmetry. The two lines A and B have a common point ('point of intersection') namely the midpoint of two points on B that correspond to each other in the symmetric equivalence where A is the axis of symmetry. Let this point be O. Let P and P' be two arbitrary points on B that correspond symmetrically to each other, and let R and S be two points on A whose distance from O is equal to the distance OP (and consequently also equal to OP').



The points P and R determine a symmetric equivalence where the axis of symmetry goes through O. In the same way S and P determine a symmetric equivalence where the axis of symmetry goes through O. In the former equivalence, P and R correspond to each other, and, since O corresponds to itself, the line B and the line A correspond to each other, and the points P' and S correspond to each other. In the latter equivalence, the points S and P respectively R and P' correspond to each other.

If now the two equivalences are combined, an equivalence is obtained in which the lines A and B each correspond to themselves, but the point O is the only point corresponding to itself, and R and S correspond to each other. Thus two corresponding points on each line lie symmetrically to O.



If we now put this equivalence together with the original (the symmetric equivalence that had the line A as axis of symmetry) we get an equivalence in which every point on the line B corresponds to itself, and the line A connects points that correspond to each other. Thus A is a self-corresponding line in the symmetric equivalence where the line B is the axis of symmetry, i.e., the line A is perpendicular to the line B. □

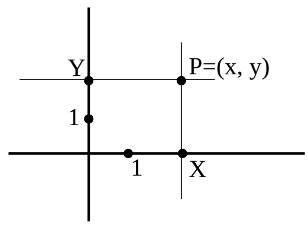
Now that Brodén has proven that the notion 'perpendicular' is a symmetrical relation, he claims that it is easy to see that through every point on a straight line passes just one perpendicular line. He gives the following proof of this:

Proposition 2 Through every point O on a straight line A goes one line B perpendicular to A.

Proof Choose two arbitrary points on the line A that symmetrically correspond to each other with respect to the point O. The symmetrical axis to A with respect to these two points goes through O. The line A is perpendicular to the line B, and thus the line B is perpendicular to the line A. But through O there can only be one line perpendicular to A, since, if there were more, A would be perpendicular to all of them, and then two symmetrical points on the line A would correspond to several different axes of symmetry. Thus there can only be one line B through O perpendicular to A. □

Brodén maintains that he can now, without any difficulty, unambiguously determine the position of a point in the plane. This he does by constructing a coordinate system where the position of each point is described by its coordinates. To construct this coordinate system Brodén chooses two arbitrary lines A and B that are perpendicular to each other and have the intersection point O. On each of the lines he chooses a 'one-point', both of which have the same distance from O, which in turn he chooses as the 'zero-point' of the two lines. The points of the lines are now (according to Axiom VII) unambiguously determined by real numbers.

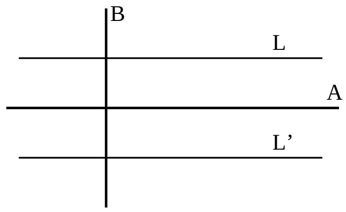
To determine an arbitrarily chosen point P in the plane, Brodén puts two lines through this point, perpendicular to the lines A and B, and intersecting these lines in the points X and Y. The two points X and Y are represented by the real numbers x and y. He assignes these two numbers to the point P.



Brodén now maintains that, because of the Parallel Axiom, i.e., Axiom XIV, every pair of values of x and y will determine one and only one point in the plane. He proves this explicitly:

Proposition 3 Every pair of values of x and y corresponds to one and only one point in the plane.

Proof Consider two straight lines L and L' that, with respect to the line A (the x-axis) as axis of symmetry, form a locus of symmetric points with the same mutual distance. These lines must intersect the line B (the y-axis), since on this line there are two points, symmetric with respect to O, with the same mutual distance as two arbitrarily given points.



It is possible to arbitrarily choose two symmetrical points since, according to the assumptions about the straight line, all straight lines are 'equivalent systems'.

The lines L and L' must be perpendicular to the y-axis, since their relation to the x-axis cannot change through some equivalence in which the x-axis corresponds to itself. Therefore, in such an equivalence, L and L' must either correspond to each other or correspond to themselves. The latter is valid, in particular, for the symmetry with the y-axis as axis of symmetry. Since L and L' intersect the axis of symmetry in different points, and in this symmetry cannot correspond to each other, L and L' must each correspond to themselves, i.e., be perpendicular to the y-axis. But the y-axis is an arbitrary line perpendicular to the x-axis. Thus it must hold that if two lines, with respect to a third (A) as axis of symmetry, form a 'locus for corresponding points with the same mutual distance', then these two lines must be perpendicular to every line that is perpendicular to A.

Conversely, it also holds that if a line (L) is perpendicular to another line (B) and this in turn is perpendicular to a third line (A), then the first line (L) together with its, with respect to A, symmetrically corresponding line (L'), forms a 'locus etc' in relation to A. This is so easily realized that a proof of it need not be written out.

Since, as just pointed out, a line that belongs to such a locus, must intersect every line that is perpendicular to the axis of symmetry, it holds that two straight lines, each of which is perpendicular to one of two mutually perpendicular lines, have one (and of course only one) point in common. From this it follows that, to every pair of values of x and y, there corresponds one and only one point in the plane. □

Now Brodén gives the definition of two lines beeing parallel to each other:

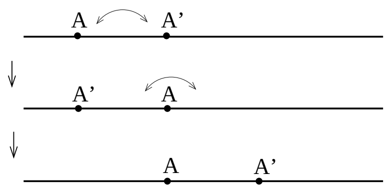
Definition Two lines, which are perpendicular to the same line, are *parallel.*

It remains for Brodén to determine the mutual position between points whose x- and y-values are given. For this he needs to be able to do a coordinate transformation. To do so he first defines the notion of 'transforming an object along a line', or 'pushing an object along a straight line'. This notion connects very closely to the notion 'pushing a straight line along itself', so Brodén only refers to the account of the latter. The only difference is that, instead of keeping to the points of the line as in the latter case, he in the former case has to consider the lines perpendicular to the line the object is pushed along.

With the notion 'pushing a straight line along itself', Brodén means the possibility of an unambiguous and asymmetric correspondence in which all corresponding distances are equal (i.e., if A corresponds to A' and B to B' then $AB = A'B'$) and the distance between two corresponding points is constant (i.e., $AA' = BB'$). To characterize the notion of pushing a line along itself, Brodén only has to use Axioms I to V, and in fact he did this already after stating Axiom V. He does it like this:

Proposition 4 There is an unambiguous and asymmetric correspondence on the line at which corresponding distances on the line are everywhere equal and the distance between two corresponding points is constant.

Proof Suppose we want to establish such a correspondence in which a given point A corresponds to another given point A'. First perform the symmetric correspondence in which A corresponds to A', and thereafter correspond symmetrically with respect to A'.

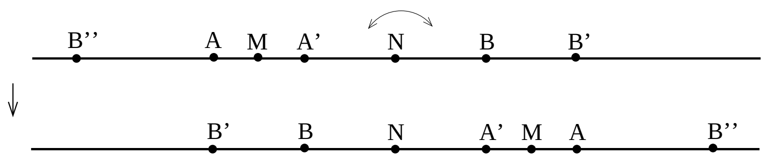


The result will be an asymmetric transformation that leaves all distances unchanged. That the distance between two corresponding, but otherwise arbitrarily chosen, points B and B' will equal the distance AA' is realized in the following way:

Let B'' correspond symmetrically to B with respect to the midpoint M of A and A', so that $MB'' = MB$. Then B'' and B' lie symmetrically with respect to A' (i.e., $B''A' = B'A'$). The midpoint N of B and A' cannot coincide with M. Take N as the center of symmetry. Then A' corresponds to B, and since $AB = A'B'' = A'B'$ the point A must correspond to either B'' or B'.[35] But A and B'' cannot correspond to each other, i.e., N cannot be their midpoint, since this midpoint must, when M is the center of symmetry, correspond to the midpoint of B and A', i.e., N, and thus

[35] Brodén here writes "B'' or B" instead of "B'' or B'", but this must be a misprint.

cannot coincide with N.



Thus, when N is the center of symmetry, A must correspond symmetrically to B'. Thus, $BB' = AA'$. □

Brodén maintains that the transformation of an object along a line does not presuppose the Parallel Axiom (Axiom XIV). But, he continues, if the Parallel Axiom holds, the transformation becomes simpler than would otherwise be the case, since then not only the line L along which the transformation is performed will correspond to itself, but every line parallel to L will do so.

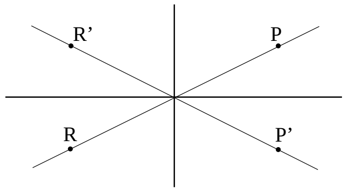
Furthermore, he continues, every line M perpendicular to L, and consequently perpendicular to every line parallel to L, will correspond to another line parallel to M. From this it follows that, since the 'perpendicular distance', i.e., the shortest distance, between two parallel lines is constant, along each line parallel to L the same transformation will be performed as along L.

Brodén claims that he can now easily show that an arbitrary transformation of an object in the plane, through the composition of transformations along two mutually perpendicular lines, is possible. From this it follows, he points out, that two straight lines are perpendicular to each other if they are each perpendicular to one of two mutually perpendicular lines. With this he can let every substitution $x = x_1 + h$, $y = y_1 + k$ represent a 'coordinate transformation'.

Now Brodén has constructed a coordinate system and he has shown how he can transform an object in this system. He proceeds to seek the arithmetic relation between the x- and y-values for points on a straight line. Brodén first remarks that the points (x_1, y_1) and $(-x_1, -y_1)$ are on the same straight line through O, and that this is independent of the Parallel Axiom. He shows this in the following way:

Proposition 5 The two points (x_1, y_1) and $(-x_1, -y_1)$ are in a straight line with O.

Proof The two points $P = (x_1, y_1)$ and $P' = (x_1, -y_1)$ are symmetrical with respect to the x-axis. Hence the lines through O and either P or P' will be symmetric with respect to the x-axis. Let the two points R and R' on these lines be symmetric to P respectively P', with respect to O (that $OR = OP = OP' = OR'$). Then R and R' also have to be symmetric to the x-axis.



But P and R', respectively P' and R, also correspond symmetrically to each other with respect to a line through O, different from the x-axis, as axis of symmetry. With respect to this line the midpoints to P and P', respectively R' and R, also form a symmetric pair. But these midpoints belong to the x-axis. Thus the x-axis must be perpendicular to the aforementioned axis of symmetry, which hence must coincide with the y-axis. Thus the points R and R' are $(-x_1, -y_1)$ and $(-x_1, y_1)$. In other words, (x_1, y_1) and $(-x_1, -y_1)$ are in a straight line with O. □

Now Brodén can present an equation for the straight line. To do this he once again considers the line ROP, and $P = (x_1, y_1)$, $R = (-x_1, -y_1)$, where P is in the first quadrant (i.e., $x_1 > 0$, $y_1 > 0$). He performs the coordinate transformation such that the point R is transformed to the point O. The coordinates for O then become (x_1, y_1) and for P, $(2x_1, 2y_1)$.

Brodén maintains that the coordinates of those two points are in the same proportion. He further claims that, by a simple reasoning, he can show that the same holds for all the points on the line whose abscissas (x-values) have the form $a/2^n$ (a and n integers),[36] i.e., the relation between y and x is constant for all the points on the line. He further asserts that, as long as he keeps to the mentioned abscissas, the equation of the line through O and $P = (x_1, y_1)$ becomes

$$y = \frac{y_1}{x_1} x \; .$$

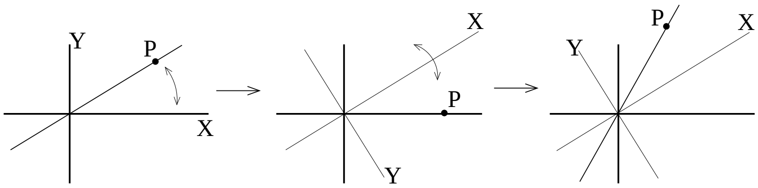
He proceeds that, when he returns to the original origin O, the equation of the line keeps the same form, and it can be proven that the same equation

[36] It doesn't seem clear to me why he restricts himself to x-values of the form $\frac{a}{2^n}$. Because of Axiom VII this should not be neccessary.

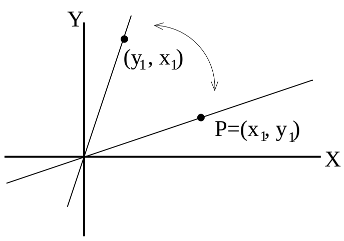
holds for all the points on the line. But for simplicity he ignores this proof, and only states that the equation for a line not passing through the origin is obtained through a coordinate transformation.

What now remains for Brodén to do is to determine the constant relation between the distance from a point on the line to the origin and the abscissa. To do this he considers the rotation of a line around a point. He determines the rotation around a point O as being the composition of two symmetric equivalences, whose axis of symmetry passes through O.

To obtain a rotation for which the positive part of the x-axis is transferred into that part of the line through O under consideration, which lies in the first quadrant, or as Brodén also expresses it, that the positive direction of the x-axis is transformed into the direction from O to P, he takes the symmetric equivalence in which these two directions correspond to each other, and thereafter takes the symmetric equivalence in where the new direction of the x-axis is the axis of symmetry. The result is a rotation of the coordinate system around the origin.



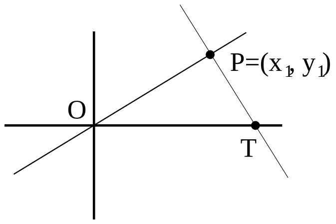
Brodén now supposes that in this rotation the direction OY is transferred into the direction OQ. In the same way as in the case of transforming a line along itself, he states that he can now show that an asymmetric equivalence can be established in which OX corresponds to OY and OP to OQ, i.e., he can establish a 90-degree rotation of the line. He then asserts that from this it follows that the equation for the line OQ must be either $x = \frac{y_1}{x_1}y$ or $x = -\frac{y_1}{x_1}y$. To decide which equation is valid, he considers the symmetric equivalence in which the directions OX and OY correspond to each other, and where the lines $x = x_1$ and $y = y_1$ correspond to $y = x_1$ and $x = y_1$ respectively, i.e., the point $P = (x_1, y_1)$ corresponds to the point (y_1, x_1) and the line OP (i.e., $y = \frac{y_1}{x_1}x$) to the line $y = \frac{x_1}{y_1}x$.



But the line $y = \frac{x_1}{y_1}x$ cannot coincide with the line OQ, and thus the line OQ must have the equation $y = -\frac{x_1}{y_1}x$. Thus, Brodén claims, the line through the origin perpendicular to the line $y = \frac{y_1}{x_1}x$ must be the line $y = -\frac{x_1}{y_1}x$.

It is now easy for Brodén to determine the distance OP. He considers the line through P perpendicular to OP, which intersects the x-axis in the point T. After a coordinate transformation, Brodén states that the equation for this line is

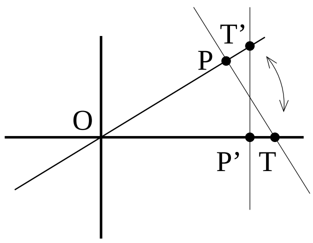
$$y - y_1 = -\frac{x_1}{y_1}(x - x_1) \ .$$



Brodén now lets $y = 0$ and obtaines the abscissa for the point T:

$$OT = \frac{x_1^2 + y_1^2}{x_1}$$

He further considers the symmetric equivalence that interchanges the directions OP and OX. With this equivalence, he says, P must correspond to a point P' on OX, and T to a point T' on OP. He claims that, since TP is perpendicular to OP, also $T'P'$ must be perpendicular to OX, and further $OP = OP'$ and $OT = OT'$.



Now Brodén claims that this, together with the fact that

$$\frac{OP}{x_1} = \frac{OT'}{OP'} \ ,$$

implies that

$$\frac{OP}{x_1} = \frac{OT}{OP} \ ,$$

and thus

$$OP^2 = x_1 \cdot OT = x_1^2 + y_1^2 \ .$$

Letting $OP = r$ and doing a coordinate transformation, Brodén now obtaines the formula for calculating the distance between two arbitrarily chosen points (x_1, y_1) and (x_2, y_2), which is:

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In this formula, Brodén claims, the Euclidean plane geometry lies embedded. Thus he asserts that he has proven that his first 14 axioms are sufficient for establishing plane Euclidean geometry. I will discuss sufficiency further in Section 5.2.

Upon adding Axioms XV and XVI, Brodén claims that every point can be unambiguously represented with the coordinates (x, y, z). In a similar manner as in the two-dimensional case, he claims that he can prove the sufficiency of the 16 axioms for establishing Euclidean three-dimensional geometry, by deducing the distance formula

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \ .$$

However, he does not carry out the proof.

Chapter 4

Brodén's Later Work on the Axiomatization of Geometry

4.1 The Scandinavian Congress

After the publication of the 1890 article, it seems that Brodén changed his field of interest. It was not untill 1911, when he went to the second Scandinavian Mathematical Congress in Copenhagen, that he resumed his work on the foundations of geometry.

During the end of the 19th century mathematics had gradually improved its' position in Scandinavia.[1] Of special importance during this period was the founding of *Acta Mathematica* by Gösta Mittag-Leffler in 1882, which from the outset became one of the leading international journals. As a result of the mathematical development in Scandinavia, Mittag-Leffler took the initiative to launch a Scandinavian Mathematical Congress. The first congress took place in Stockholm in 1909 and became a monument to the mathematical development that had been achieved.

The second Scandinavian Mathematical Congress was held from August 28 to 31, 1911. In all 93 mathematicians from Denmark, Norway and Sweden took part, and 23 lectures were given. Proceedings were printed the following year.[2] Two talks were given on the foundations of geometry. Johannes Hjelmslev (1873-1950), professor at the university of Copenhagen, gave a talk with the title *Nye Undersøgelser over Geometriens Grundlag*

[1] Nielsen (1912) p. XII.
[2] Ibid.

('New Investigations on the Foundations of Geometry'), and Brodén's talk was entitled *Ett axiomsystem för den euklidiska geometrien* ('An Axiomatic System for Euclidean Geometry'). Throughout this chapter I will consider the axiomatic system in Brodén's Congress article and highlight the differences between this and the 1890 system.

4.2 The Revised Axiomatic System

The most striking difference between Brodén's 1890 and Congress articles is that the latter is considerable clearer and briefer in its presentation. Brodén is now distinct in formulating the axioms, but unfortunately something goes missing in this more concise format. He does not give any aim, or motivation as to why he wants to give an axiomatic system for Euclidean geometry, but considering the context in which this article was presented this might not have been necessary. In 1911 Hilbert's work was well known and most certainly no mathematician at the Scandinavian congress would question research on the foundations of geometry and it must have been of special interest since Brodén claimed the system to have been constructed already in 1890. But the lack of background discussion results in that Brodén's way of proceeding in creating his axiomatic system becomes unclear. For example he does not say anything about why he chooses 'point' and 'immediate equality of distance' as the two basic notions, and the whole idea of geometry as a science that wants to reduce different phenomena to motion is absent. This results in that it is unclear why he chooses the axioms he does.

Brodén also gives a very meager discussion on how a scientific axiomatic system should be built up, i.e., what criteria the basic notions and axioms should fulfill. He just mentions that, to the greatest extent possible, the axioms should be empirically evident, and the whole system of axioms should be simple, natural and homogeneous.[3] With simple and natural he probably means that the axioms should be formulated in such a way that they are easy to understand and can be used without difficulty. At the end of the article he also brings up the sufficiency of the axioms and he discusses their necessity, i.e., their independence from each other.[4] With these latter additions, the criteria for an axiomatic system are basically the same in the two articles. However it is not as clearly discussed in the later one.

After the brief comment on the criteria for an axiomatic system, Brodén proceeds with stating the axioms. Here one immediately sees the influence

[3] Brodén (1912) p. 123.
[4] Ibid. p. 133.

of Hilbert. Brodén, just as Hilbert did in 1899, chooses to sort the axioms in different groups. He does not say anything about why he chooses to do this, but he probably wants to indicate similarities between his and Hilbert's systems, similarities that are not very prominent when one looks closer at the axioms.

Here follows a translation of Brodén's 1911 axiomatic system:[5]

I. Fundamental axiom

Axiom 1 If $AP = BP$ and $CP = BP$ then $AP = CP$, or, in words: with respect to immediate equality of distance, those distances which are equal to one and the same distance, are equal to one another.

II. Axioms that make the general concept of equality of distance possible

Axiom 2 The locus of a point P such that $PA = PB$, where A and B are two given points, consists of more than one point.

Definition This locus is called a plane.

Axiom 3 A corresponding set within a plane consists of more than one point.

Definition This set of points is called a straight line.

Axiom 4 The corresponding set within a straight line consists of a single point which is distinct from both A and B.

Definition This point is called the midpoint for A and B.

Definition On a straight line, $AB = CD$ if the pairs A, D and B, C or A, C and B, D have the same midpoint.

Axiom 5 On a straight line those distances are equal that equal one and the same distance.

Axiom 6 Through two arbitrarily chosen points there is always at least one straight line (and hence also at least one plane).

Axiom 7 If P is a point on a straight line and A is a point outside the line, then there is at least one point B on the line such that $BP = AP$.

Definition Let two pairs of points, A, B and C, D, be given and let a point in the first pair be connected with a point in the

[5] Ibid. pp. 124–128.

second pair (for example A and C) by a straight line. Take two points H and K on the line such that $HA = BA$ and KC=DC. If HA=KC then also AB=CD.

Axiom 8 Without exception it holds good that distances that are equal to one and the same are equal to each other.

III. Axioms for characterising a straight line and a plane

Axiom 9 Through two (different) points there is never more than one straight line.

Axiom 10 The straight line that goes through two points in a plane lies completely in the plane.

Axiom 11 Through three points not on a line there is always one and only one plane.

IV. Axioms of symmetry

Axiom 12 Each point M on a straight line uniquely determines a symmetric correspondence in which corresponding distances are equal and M is the only point corresponding to itself.

Axiom 13 Every straight line in a plane uniquely determines a symmetric correspondence of points in which corresponding distances are equal and every point on the line but no other point is self-corresponding.

V. Axioms of continuity

Axiom 14 By means of successive traversal of equally long segments, one never returns to the point of departure.

Axiom 15 Completeness Axiom.

VI. Parallel axiom

Axiom 16 By means of the symmetric equivalence in the plane equidistant symmetric pairs form two straight lines.

The main difference between Brodén's two axiomatic systems is that in the version of the Congress article the concept of symmetry is not as striking as in the 1890 version. In the 1890 system, symmetry was used to characterize the straight line and the plane and, with the help of symmetry, Brodén could extend the notion of immediate equality of distance. In the Congress article the concept of symmetry is not used to the same extent. The group of axioms concerning symmetry includes only two axioms, and they are introduced quite late, just before the two continuity axioms and the Parallel Axiom. In 1890 five axioms had a direct connection to symmetry (Axioms III, IV, XI, XII and XIII). In the Congress article Brodén, with Axioms 12 and 13, only retains Axiom III and a slightly stronger version of Axiom XI.

The reason for not having to use symmetry to the same extent is Brodén's choice to introduce the line and the plane in a different manner. In 1890 he introduces the line with the help of two points on it, and characterizes it completely before he introduces the plane. In the Congress article he claims that he uses Leibniz' definitions and introduces the line with the help of the concept of the plane. He does not explain why he relinquishes his former idea to build up the geometry from point to line to plane.

However, it seems that Brodén has not thought through this idea completely. As Contro remarks,[6] there will be a problem later in the system when Brodén introduces Axiom 10, saying that the line through two points in a plane completely lies in the plane. Since the line and the plane have already been introduced, this axiom should be proven from the other axioms, or at least reduced to a simpler form.

Axiom XVI from 1890, saying that two planes cannot have only one point in common, is not retained in the Congress article. As discussed further in Section 5.3, it seems to me like this axiom is not independent of the others. Probably Brodén realized that it can be derived from the other axioms, and thus removes it in the Congress article.

Axiom 14 is formulated in a different manner but has the same meaning as Axiom VI from 1890. In the Congress article Brodén refers to this axiom as an Archimedean Axiom. However, Axiom 14 is weaker than the Archimedean Axiom and should only be considered as an axiom of ordering. This is discussed more precisely in Section 5.1.

To obtain continuity of the straight line, Brodén does not go through the complicated construction using c-systems to establish a bijection between the real numbers and the points of the line, as he did in 1890. Instead he just states "Completeness Axiom", referring to Hilbert. He mentions that he already gave a formulation in 1890, but Hilbert only gave it in his second edition of *Grundlagen der Geometrie* in 1903.[7] It seems that Brodén wants to indicate that he was far in advance of Hilbert in realising the necessity of

[6]Contro (1985) p. 632.
[7]Brodén (1912) p. 128.

a Completeness Axiom, and at the same time he wants to pretend that he had succeeded in formulating this axiom at the same abstract level, which was not the case. The Completeness Axiom is discussed further in Section 5.1, in connection with the Archimedean Axiom.

4.3 Sufficiency and Necessity

After stating the axioms, Brodén gives a proof of sufficiency by deriving the distance formula. He carries through this proof in exactly the same way as in 1890, with the only exception being that he does not perform the construction of the transformation of the line along itself. It is notable that even if symmetry plays a far less important role in the axiomatic system than in 1890, it still has the same significance in the construction of the coordinate system.

After proving sufficiency of the axioms, Brodén discusses their necessity, i.e., if they are independent from each other. This is, he claims, an incomparably more complicated question than proving the sufficiency of the axioms.[8] He does not carry out a proof of independence of all the axioms, but only considers the special question whether the two axioms of continuity, Axioms 14 and 15, are independent from the others. He proves this explicitly by formulating a model in which the remaining axioms are fulfilled, but Axioms 14 and 15 are not. Since his model is plane he leaves out the axioms considering the space.

The model is a finite geometry consisting of nine points. When arranged in a 3×3 matrix and letting the distance be a between two points in the same row or column and b if not, three points will form a line if in the same row, column or element of the determinant. In this model Brodén can easily check that all the axioms, except Axioms 14 and 15, are fulfilled.

Brodén also briefly discusses some differences between Hilbert's axiomatic system and his own.[9] He asks whether his model would satisfy all of Hilbert's axioms, except the Archimedean and Completeness Axioms. This is not the case, he concludes, since Hilbert's axioms already have as a consequence that a line has infinitely many points. The reason for this is that, in Hilbert's system, the notion 'between' plays the role of a basic notion. In Brodén's system the notion 'between' cannot be defined until after Axiom 14 has been introduced. Thus, at least as long as we stay within the plane, Hilbert's axioms, excluding the two concerning continuity, contain something more than Brodén's corresponding axioms.

[8] Ibid. p. 133.
[9] Ibid. p. 134.

Chapter 5

Discussion

5.1 The Axioms of Continuity

One of the most intricate questions regarding the axiomatization of Euclidean geometry concerns the principle of continuity. As mentioned in Section 1.1, one of the main defects in Euclid's *Elements* was that continuity of the line was assumed intuitively and not postulated. This problem was eventually solved by Hilbert, who included two continuity axioms, the Archimedean Axiom and the Completeness Axiom, in his second edition of *Grundlagen der Geometrie* from 1903.[1]

In the Congress article Brodén gives two continuity axioms, Axioms 15 and 16.[2] In the 1890 article he gives two axioms, Axioms VI and VII,[3] which are basically the same as the two axioms in the Congress article, at least for Brodén. In this section I will discuss Brodén's two versions of these axioms in relation to Hilbert's continuity axioms and related principles.

In his first edition of the *Grundlagen der Geometrie*, Hilbert gives only one continuity axiom. This is the so-called Archimedean Axiom,[4] which he formulates in the following manner:[5]

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[1]Hilbert (1903) p. 16.

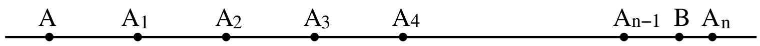
[2]See Section 4.2.

[3]See Section 3.4.

[4]Otto Stoltz (1842-1905) was probably the first to refer to this axiom as the Archimedean Axiom. Stoltz (1883) p. 504. Archimedes explicitly formulated an axiom that agrees with this, but it was probably used even earlier.

[5]"Es sei A_1 ein beliebiger Punkt auf einer Geraden zwischen den beliebig gegebenen Punkten A und B; man construire dann die Punkte A_2, A_3, A_4,..., so dass A_1 zwischen A und A_2, ferner A_2 zwischen A_1 und A_3, ferner A_3 zwischen A_2 und A_4 u. s. w. liegt und überdies die Strecken AA_1, A_1A_2, A_2A_3, A_3A_4,... einander gleich sind: dann giebt es in der Reihe der Punkte A_2, A_3, A_4,... stets einen solchen Punkt A_n, dass B zwischen A und A_n liegt". Quoted from Sjöstedt (1968) p. 884.

> Let A_1 be an arbitrarily chosen point on a line between the arbitrarily chosen points A and B; if then the points A_2, A_3, A_4,... are constructed, such that A_1 lies between A and A_2, A_2 lies between A_1 and A_3, A_3 lies between A_2 and A_4 and so on, and the segments
>
> $$AA_1, A_1A_2, A_2A_3, A_3A_4, ...$$
>
> are equal to each other then, in the sequence A_2, A_3, A_4,..., there is a point A_n such then B lies between A and A_n.



This axiom corresponds to the process of estimating the distance between two points on a line by using a measuring stick. If we start at one point and successively traverse equal distances along the line towards the second point, the axiom guarantees that we will eventually pass the second point. Euclid's theory of proportion and the entire theory of measurements depend on this axiom.[6]

In all editions of the *Grundlagen der Geometrie* Hilbert includes the Archimedean Axiom. He slightly changes the formulation in later editions, however, they are all equivalent.

In the Congress article Brodén claims that, with Axiom 15, which is equivalent to his 1890 Axiom VI, he has a version of the Archimedean Axiom.[7] However, this statement is not true. Brodén's axiom gives an ordering of certain points of the line, in the sense that he stepwise can walk along the line, or successively traverse equally long segments along the line, without coming back to the point of departure. The axiom implies that the line can be extended indefinitely and consists of at least countably many points. But it does not imply that it is always possible to pass an arbitrarily chosen point on the line, and thus it does not imply the Archimedean Axiom.

One could say that Brodén's Axiom VI bounds the line from below, in the sense that it forces the line to consist of at least countably many points and to be extended to infinity. On the other hand, the Archimedean Axiom in some sense bounds the line from above, forcing every point of the line to be reachable.

The axioms Hilbert gave in 1899 are not enough to guarantee the continuity of the line, i.e., that the line is homeomophic to the real numbers

[6]Eves (1990) p. 86.
[7]Brodén (1912) p. 127.

$\mathbb{R}$. To complete the line he includes in the second edition of the *Grundlagen der Geometrie*[8] from 1903, a second axiom of continuity, the so-called Completeness Axiom,[9] which was formulated in the following manner:[10]

> The elements (points, lines, planes) of geometry constitute a system of objects which, if we assume the foregoing axioms, does not admit any extension.

With this is meant that a proper extension in which all the axioms remain true is not possible. If a point, before the extension, lies between two other points, it should still do so afterwards, and congruent lines and angles should stay congruent.[11] This axiom, together with the axioms it depends on, immediately implies that the set of all points lying on a given line is homeomorphic to the real numbers $\mathbb{R}$, the set of all points of a plane is homeomorphic to $\mathbb{R}^2$, and the set of all points in space is homeomorphic to $\mathbb{R}^3$.

In the seventh edition of the *Grundlagen der Geometrie*,[12] from 1930, Hilbert gives a weaker version of the Completeness Axiom, since he realized that it is enough to determine the continuity of the line with an axiom to be able to prove the original Completeness Axiom:[13]

> The system of points on a straight line, with its relationships of order and congruence, cannot be extended in such a way that the relationship between these elements and also the characteristics of the Axioms I-III of linear order and congruence, and of Axiom V1,[14] remain preserved.

Richard Baldus refers to the Completeness Axiom as Hilbert's most original achievement in the development of axiomatics.[15] The character of the Completeness Axiom differs from those of the other axioms, in that it

[8] Hilbert (1903).

[9] The 'Completeness Axiom' should not be confused with a 'complete axiomatic system'. This is just an unfortunate choice of words.

[10] "Die Elemente (Punkte, Geraden, Ebenen) der Geometrie bilden ein System von Dingen, welches bei Aufrechterhaltung sämtlicher genannten Axiome keiner Erweiterung mehr fähig ist". Hilbert (1903) p. 16.

[11] Hilbert (1909) p. 22.

[12] Hilbert (1930).

[13] "Das System der Punkte einer Geraden mit seinen Anordnungs- und Kongruenzbeziehungen ist keiner solchen Erweiterung fähig, bei welcher die zwischen den vorigen Elementen bestehenden Beziehungen sowie auch die aus den Axiomen I-III folgenden Grundeigenschaften der linearen Anordnung und Kongruenz, und V1 erhalten bleiben". Ibid. p. 22.

[14] Axioms I-III refers to the groups of axioms concerned with connection, order and congruence, and Axiom V1 refers to the Archimedean Axiom.

[15] Baldus (1928) p. 322.

does not state new relations between the basic notions, but says something about the relation between the axiomatic system and the objects which may conceivably satisfy it. The Completeness Axiom is a metatheoretical statement, though in a peculiar sense, since the theory with which it is concerned includes the axiom itself.[16] However, the axiom can be expressed in a different and more transparent manner. In 1930 Baldus showed that the Cantorean Axiom[17] gives the full import of the Completeness Axiom:[18]

> If, on a straight line, there is an infinite sequence of segments $A_\nu B_\nu$ such that each of these segments has its endpoints within the previous one and such that there is no segment on the line inside all the segments $A_\nu B_\nu$, then there is a point within all the segments $A_\nu B_\nu$.

With Axiom VII, Brodén already had some type of Completeness Axiom in 1890. Here he makes a construction to obtain a one-to-one correspondence between all real numbers and the points on the straight line. However, Brodén's axiom does not have the metatheoretical character of Hilbert's version. Instead it involves a messy construction of the points of the line, and assumes the existence of the points constructed in an infinite process.

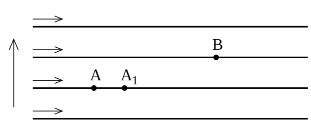
We can easily realize that Axiom VII implies Baldus' version of the Cantorean Axiom, and thus Hilbert's Completeness Axiom. But the Cantorean Axiom does not imply Brodén's Axiom VII. We can see this since Brodén's axiom immediately implies a correspondence between every point of the line and every real number, but the Cantorean Axiom does not neccessarily imply that every point on the line has a corresponding real number.[19]

[16]Torretti (1978) p. 234.

[17]The Cantorean Axiom is usually refered to as the Nested Intervals Theorem. Cantor uses the principle in his first proof of the nondenumerability of the reals. Cantor (1874).

[18]"Liegt in einer Geraden eine unendliche Folge von Strecken $A_\nu B_\nu$ derart, daβ jede dieser Strecken ihre Endpunkte innerhalb der vorhergehenden hat und daβ es keine Strecke auf der Geraden gibt, die innerhalb aller Strecken $A_\nu B_\nu$ liegt, dann gibt es einen Punkt, der innerhalb aller Strecken $A_\nu B_\nu$ liegt". Baldus (1930) p. 12.

[19]Veronese gives the following model of a non-Archimedean geometry, where the Cantorean Axiom is fulfilled: In the Euclidean plane there is an infinite sequence of equidistant parallel lines. If we assume that every Euclidean line is run through from the left to the right, and the sequence of Euclidean lines is run through from bottom to top, we can consider the collection of Euclidean lines to form a line in the new geometry. If we compare the segment AB with the segment AA_1 according to the illustration, we see that



the Archimedean Axiom is not fulfilled. It is easily seen that the Cantorean Axiom is fulfilled, but Brodén's Axiom VII is obviously not. This example also shows that the

Brodén needs Axiom VI to be able to to construct the points of the line corresponding to the integers. These are neccesary for him in order to formulate Axiom VII. With Axiom VII the line corresponds to the real numbers, and the Archimedean Axiom can be proven. However, Brodén does not do this, and may not even be aware of the importance of the Archimedean Axiom.

In the Congress article Brodén does not formulate a Completeness Axiom, but only states:[20]

Axiom 16: Completeness Axiom.

He is probably referring to Hilbert's formulation of the axiom in the second or third edition of the *Grundlagen der Geometrie*.[21] This is a mistake that leads to the most serious defect in the Congress article. Hilbert's Completeness Axiom is weaker than Brodén's original formulation in Axiom VII of 1890. When Brodén, in the Congress article, chooses to use Hilbert's Completeness Axiom instead of his own version and at the same time does not give a stronger formulation of Axiom 15, the Archimedean Axiom can no longer be proven and a non-Archimedean geometry is still possible. Thus Brodén's axiomatic system in the Congress article is not complete. Brodén probably does not realize this in his eager efforts to point out the similarities between his and Hilbert's axiomatic systems, and thus by mistake introduces this defect.

5.2 The Proof of Sufficiency

One of the criteria Brodén gives which a scientific system of geometry should fulfill, is that the sufficiency of the axioms for arranging geometry under certain logical forms should be clear. Since Brodén does not specify what he means with 'sufficiency' or 'logical forms', it is difficult to interpret this criterion in a reliable manner. As already mentioned in Section 3.2, I believe that Brodén considers his axioms to be sufficient if, through deduction from them, he will obtain what we consider to be Euclidean geometry.

In both of his articles Brodén, in the two-dimensional case, carries out what he calls a 'proof of sufficiency'. I considered the proof in detail in Section 3.5. Considering this proof it is possible to further investigate the meaning of the sufficiency criterion.

Archimedean Axiom is independent of the Cantorean Axiom. Veronese (1894) p. 184. See also Baldus (1930) p. 5 and Enriques (1907) p. 37.

[20] Brodén (1912) p. 128.

[21] Hilbert (1903) p. 16; Hilbert (1909) p. 22.

In the proof Brodén, with the help of his axioms, constructs a coordinate system and derives the equation for a straight line through two arbitrarily chosen points. Thereafter he deduces the formula for computing the distance between these two arbitrary points. In this formula, Brodén claims, the entire plane Euclidean geometry lies embedded, in the sense that 'everything' can be derived from this formula, after the required notions have been defined in a suitable way.

By 'everything can be derived' Brodén probably implies that every statement we intuitively consider to be a true statement of Euclidean geometry, can be derived from the axioms. This would give support to my interpretation of the sufficiency criterion.

Brodén's statement that Euclidean geometry lies embedded in the distance formula probably originates from his view on geometry. According to him geometry, like any other science, seeks to reduce all phenomena to motion, and motion is just a change in certain relations between objects. With the distance formula all the changes in the relations between the objects can be described. In this sense it should be enough to deduce the distance formula to be able to describe the entire Euclidean geometry, or in Brodén's words, to derive everything.

If the axioms are chosen in such a way that every statement we intuitively consider should be true in Euclidean geometry actually can be proven to be true, we could also consider the sufficiency of the axioms to imply some type of completeness requirement. It could be compared to what we today would call 'completeness of an axiomatic system', i.e., that every statement that can be formulated within the system can be proven either true or false. Thus, in this sense, it seems like Brodén, already in 1890, had some kind of intuitive feeling for the need of a completeness requirement for an axiomatic system.

Contro claims that Brodén implicitly proves consistency in the 1890 article, but he does not develop this statement further.[22] In fact, Brodén never, not even in the Congress article, discusses consistency. But from our point of view, Brodén's proof of sufficiency could possibly be interpreted as some kind of consistency proof, as I will try to explain:

Since Brodén has an empirical view of geometry and claims that experience cannot lead to logical contradictions, he must consider his system to be without contradictions. Thus we could consider his system to be consistent. In the proof of sufficiency of the axioms he has implicity shown that from them he can construct a coordinate system, i.e., Cartesian geometry. But, since his system is consistent and Cartesian geometry can be deduced from it, he has proven consistency of Cartesian geometry.

However, I do not believe that Brodén had a general concept of consis-

[22]Contro (1985) p. 632.

tency, for instance the concept later developed by Hilbert. Hilbert wanted to change the concept of truth in mathematics, so that the notion that an object exists is the same thing as saying that it is consistent with the rest of the system, i.e., that there are no logical contradictions.[23]

In 1890, when Brodén wrote his paper, empiricist views were not uncommon, in that mathematical truth was established by reference to 'reality'. There were no contradictions in a theory conforming to reality, because this has no inherent contradictions. However it was difficult to have both this concept of truth and also a view of mathematics as free, in the sense that there are a lot of logically possible geometries.[24] What about the consistency of these other possibilities? Could there be contradictions? This is a tension building up toward the end of the 19th century, and Hilbert's main object is to try to resolve it.

Hilbert's school in the foundations of mathematics is called formalism, but there is an older tradition of formalists from the 19th century: Erdmann, Heine (who wrote *Die Elemente der Functionslehre*), Hankel, Baumann, Grassmann (who wrote *Lehrbuch der Mathematik für Höheren Lehranstalten*) and Schröder (who wrote *Algebra der Logik*). It would be interesting to examine their connection with the ideas of Brodén in a future study.

5.3 The Problem of Dimension

In his 1890 article, Brodén claims that he has to introduce Axiom XVI, saying that two planes cannot have only one point in common, to exclude the fourth dimension.[25] Later he makes a brief comment that it is not the task of mathematics to investigate why space should have three dimensions.[26] These are the only times he mentions the concept of dimension in the article.

To me it seems that Brodén has to introduce Axiom XVI since he does not presuppose the space to have three dimensions. But he probably thinks of the plane as being two-dimensional. This, and the fact that Axiom XVI does not appear in the Congress article, made me think further about the problem of dimension, i.e., whether Brodén's system actually forces geometry to be three-dimensional and whether Axiom XVI is necessary.

In 1890 Brodén introduces, with Axiom II, the line with the help of the concept of *Einziges*.[27] If the geometry should in any sense be Euclidean,

[23] von Wright (1993)
[24] Compare to the discussion in Brodén (1890) pp. 218–220.
[25] Brodén (1890) p. 236.
[26] Ibid. p. 260.
[27] Ibid. p. 224.

this forces the line to be of dimension one. However, in Axiom X, Brodén introduces the plane to be a system of points, such that a straight line through two of its points completely belongs to the system, without filling it entirely. With this axiom the plane could be a hyperplane of any dimension greater than or equal to two.

If, for example, we think of the plane as the hyperplane of dimension three, there will not be a problem when we introduce Axioms XI, XII and XIII, concerning symmetric equivalence. We could think of the symmetric equivalence as a 180-degree rotation of the hyperplane around the line determined by the two points in question.

However, there will be a problem in introducing Axiom XIV, the Parallel Axiom, if the plane is a hyperplane of dimension three or greater. This axiom is crucial, since the complete locus of symmetrically corresponding points with the same mutual distance as the distance between two given points must form two lines. This is not the case in the hyperplane version. If the plane would be of three dimensions, then the complete locus would form a cylinder instead. This implies that the plane must be of two dimensions.

Still, it might be possible that the entire space is of dimension four or higher. But, if this were the case, and we chose three arbitrary points in space, then there would be infinitely many planes through these three points, also when they are not in a straight line, and hence Axiom XV would not be fulfilled. Thus the plane must be two-dimensional and embedded in a three-dimensional space, just as we would wish.

So, the first 15 axioms in the 1890 article force space to be of three dimensions, and hence it is unnecessary to introduce Axiom XVI to exclude a fourth dimension. The axioms already given will imply that two planes, of dimension two and imbedded in three dimensions, cannot have only one point in common, since they are complete. Thus Axiom XVI is dependent on the previous axioms, and should be excluded.

In the Congress article Brodén removed Axiom XVI. However, this does not necessarily mean that he realized that the axiom was superfluous in the 1890 axiomatic system, since in the Congress article the problem of dimension does not depend on the Parallel Axiom. Brodén chooses to introduce the plane and the line in a different way, so the problem of dimension does not arise.

If we assume space is of dimension n, and introduce the plane with Axiom 2, then the plane must necessarily be of dimension $n-1$. Axiom 3 introduces the line in such a way that it necessarily must be of dimension $n-2$ and, in a similar manner, Axiom 4 introduces the midpoint that must be of dimension $n-3$. But Axiom 4 also states that the midpoint must, obviously, consist of only one point, and thus be of dimension 0. Hence the space is of dimension three, and Brodén does not have to worry about the dimension anymore. Thus, he does not have to include Axiom XVI in the Congress article.

5.4 Influences Upon Brodén

In both his articles Brodén refers to a number of works by other mathematicians and philosophers. In this section I will discuss some of the references Brodén gives, and by whom he might have been influenced.

In the 1890 article, most of the references are given in the beginning, where Brodén treats philosophical questions regarding the nature of geometry. In his view on the status of geometry as a natural science, it seems that Brodén is influenced by Helmholtz. He refers, in particular, to two of Helmholtz' articles: *Über den Ursprung und die Bedeutung der Geometrischen Axiome*, from 1876, and *Über den Ursprung und Sinn der geometrischen Sätze*, from 1882.[28] With these two articles Helmholtz took part in a debate with Kantian philosophers about the epistemological status of non-Euclidean geometry. He argued that, in general, geometry derives from physical measurements, rather than from a priori features of our spatial intuition.[29] This implies that Euclidean geometry only represents one possible outcome of our spatial measurements, and therefore it is an empirical choice between it and various non-Euclidean geometries. Helmholtz understood geometry to be an empirical science, but he also recognized its' status as a formal deductive structure that stands independently of its' intutive or sensory content.[30]

We can recognize this view when we read Brodén's 1890 article. Brodén names empirical evidence as one of the criteria for how his system should be built up.[31] When it comes to the choice between Euclidean and non-Euclidean geometry, Brodén talks about the 'approximative' validity of the Parallel Axiom, Axiom XIV.[32] It is not impossible, he claims, that this axiom is not true. If it is not true, a 'pseudo-spheric', i.e., hyperbolic, geometry is obtained, where there are infinitely many lines through a given point outside a given line, that do not meet this line. Again referring to Helmholtz, Brodén questions whether Euclidean geometry is the only possible geometry in which we live, but until further notice he admits the validity of Euclidean geometry, since no measurements have so far been able to demonstrate something else.[33] These statements on the nature of

[28] Helmholtz (1876); Helmholtz (1882).
[29] DiSalle (1993) p. 498.
[30] Ibid. p. 500.
[31] Brodén (1890) p. 220.
[32] Ibid. p. 257.
[33] Ibid. p. 258.

geometry suggest that Brodén had read and was influenced by Helmholtz.

In addition to the references made to Helmholtz, Brodén also refers to Richard Dedekind (1831-1916) and Georg Cantor (1845-1918).[34] Specifically, he refers to Dedekind's article *Stetigkeit und irrationale Zahlen*,[35] where the theory of Dedekind cuts for defining the real numbers is developed. Another article referred to is *Was sind und was sollen die Zahlen.*[36] Here Dedekind, by using set-theoretic ideas, gives a theory of the integers. Furthermore, Brodén mentiones two articles by Cantor, *Über die verschiedenen Standpunkte in bezug auf das aktuale Unendliche* and *Beiträge zur Lehre vom Transfiniten.*[37] Regarding the latter, he probably gave the wrong title and actually meant *Mitteilungen zur Lehre vom Transfiniten.* Both these articles present discussions of philosophical questions concerning the infinite. It is not immediately clear why Bodén chooses to refer to them.

Brodén refers to Cantor when he claims that arithmetic is independent of space and time.[38] In the same discussion Brodén claims that geometry can be expressed by arithmetic. This idea he probably attributes to Dedekind. Brodén's conclusion is that geometry is independent of space and time, and thus becomes a possible logical form among many others, whose special importance is gained through reality.

In this discussion, Brodén takes Cantor as an authority to criticize Kant's view of geometry as being the result of pure intuition of space and time. Cantor is a platonist who considers mathematical truths to exist a priori, independent of us.[39] But at the same time, Brodén considers his own view on the nature of geometry to be a development of Kant's theories. This shows perhaps a lack of deeper thought behind Brodén's philosophical discussion. On the one hand, he is clearly influenced by Helmholtz' empirical view of geometry, and, on the other, he appeals to Cantor who does not consider geometry to be an empirical science.

Brodén is influenced by ideas about how geometry and arithmetic can be detached from space and time, and that there are several geometries that are logically possible. But the problem with his 'formalism' (recall the end of Section 5.2) is that he also appeals to reality in a highly eclectic move: Reality is free from contradictions and this is the basis for determining whether his axiomatization is correct.

When we consider the mathematical part of Brodén's 1890 article, we see further traces of possible influence from Cantor. If we compare Brodén's

[34] Ibid. (1890) p. 220.
[35] Dedekind (1872).
[36] Dedekind (1888).
[37] These articles can be found in Cantor (1932).
[38] Brodén (1890) p. 219. See also Section 3.1 for a further discussion on this.
[39] Dauben (1979) p. 83.

Axiom VII, where he gets, through a construction, a one-to-one correspondence between the real numbers and the points of the line, to Cantors' theory of the real numbers, we can see some similarities. Cantor constructs the real numbers from the rationals by considering Cauchy sequences of rational numbers, i.e., sequences with the property that for any given ε, all the members of the sequence except a finite number differ from each other by less than ε.[40] Each such sequence is a real number. Brodén transfers this idea to the straight line, where he considers a sequence of binary fractions corresponding to bisected distances. However, he does not give any specific references to Cantor regarding this.

Brodén begins his Congress article by claiming that his 1890 axiomatic system exhibits similarities with those of Hilbert, Veronese and Pieri, among others. The interesting thing is that they published their work on the foundations of geometry after 1890, and thus Brodén cannot, at least in his 1890 work, have been influenced by them. However, it is also unlikely that they have been influenced by Brodén. Hilbert and Pieri might, of course, have read the summary of Brodén's 1890 article in *Jahrbuch über die Fortschritte der Mathematik*,[41] where he gives the basic mathematical ideas behind his system, but they most certainly did not read the whole article, since it was published in Swedish.

Brodén does not refer to any specific articles, but concerning Hilbert it is obvious that he is referring to the second or third edition of *Grundlagen der Geometrie*,[42] considering that the first edition did not include a Completeness Axiom. With this reference, Brodén probably wants to point out the importance of his work and stress, in particular, the early appearance of his first article. This is a very intriguing comment, since, in the 1890 article, he claims that his attempt to axiomatize geometry should in no way be considered original.[43]

Furthermore, the similarities between Brodén's 1890 axiomatic system and that of Hilbert are not as prominent upon closer inspection as one might first think. Hilbert's system is at a more abstract level. He conceives three different sets of things, that might be called 'points', 'lines' and 'planes', but he does not assign an explicit meaning to them. They stand in certain mutual relations whose exact description is given by the axioms, which are independent of physical reality.[44] This differs from Brodén's way of proceeding. He has not, even in the Congress article, freed himself from the empirical view of geometry.[45]

[40] Kline (1972) p. 984.
[41] Brodén (1893).
[42] Hilbert (1903); Hilbert (1909).
[43] Brodén (1890) p. 217.
[44] Kline (1972) p. 1013.
[45] Brodén (1912) p. 123.

Brodén mentions Veronese and Pieri at the end of the Congress article in a short discussion on the choice to reduce the notions to the two basic notions 'point' and 'immediate equality of distance'.[46] He claims that Veronese and Pieri have expressed the possibility of constructing an axiomatic system for Euclidean geometry with these two basic notions, but that they, as far as he knows, have not carried through this thought. Contro claims that this statement shows that Brodén did not know of Pieri's *La geometria elementare istituita sulle nozioni di punta e sfera* from 1908, where he does exactly this.[47] Probably Brodén read of Pieri's work in Enriques' article in *Enzyklopädie der Mathematischen Wissenschaften*,[48] which was written the year before Pieri's article, and thus only refers to his earlier work. Considering Veronese, Brodén probably refers to *Grundzüge der Geometrie*, from 1894, which was a translation of his 1891 book in Italian. It is not possible that Brodén and Veronese were influenced by each other; Veronese's Italian edition appeared after Brodén's 1890 article, which he most certainly did not know of, but before Brodén's summary in *Jahrbuch über die Fortschritte der Mathematik* appeared. Thus it seems like Brodén was unaware of the development in Italy when he wrote his 1890 article.

It is remarkable that Brodén in neither of his articles refers to Pasch, whom he should have known of if he was seriously working on the foundations of geometry. Of course it does not necessarily follow that Brodén did not know of Pasch, just because he did not refer to him. But Brodén's way of building up his axiomatic system renders unlikely any knowledge of Pasch's work, or at least any influence of his work upon Brodén. For Pasch, the concept 'between' was of great importance in building up projective geometry. Brodén cannot define 'between' until after Axiom VI, and he does not have to use the concept at all throughout his system.

Characterizing Brodén's axiomatic system is his use of symmetries, the symmetric correspondence in the line and the symmetric equivalence in the plane. I have not found any traces of by whom Brodén might thus have been influenced. Since Brodén, in 1890, gives careful references concerning his discussion on the more philosophical questions regarding the nature of geometry, but does not give any references concerning his axiomatic system, and in particular his use of symmetries, this suggests that the latter was his own idea. The fact that, in the Congress article, he does not give any references to material that preceeded his earlier work further supports this claim.

[46] Brodén (1912) p. 134.
[47] Contro (1985) p. 634.
[48] Enriques (1907).

Chapter 6

Conclusions

In this thesis I have studied an article written by Torsten Brodén and published in *Pedagogisk Tidskrift* in 1890. In the article Brodén considers the foundations of geometry from a pedagogical and philosophical point of view. I have analysed the article in detail with respect to the philosophical and mathematical content and considered it in connection with Brodén's later work on the foundations of geometry and with respect to the time in which he lived.

To motivate his axiomatic system, Brodén gives a philosophical discussion on the nature of geometry. I found that, in many aspects, there are similarities between the conceptions of Brodén and Helmholtz. Brodén has an empirical view of geometry, and wants to obtain a theoretical basis for the fact that the external reality as described by Euclidean geometry corresponds to experience. This idea agrees with Helmholtz' understanding of Euclidean geometry as representing the only possible outcome of our spatial measurements. But, at the same time, appealing to Cantor, Brodén considers geometry to be an a priori possible logical form. Brodén's discourse shows him to be an eclectic philosopher; he borrows points of view from various schools of thought.

Brodéns aim is probably a pedagogical, rather than a mathematical or philosophical one. He claims that there is nothing new in his work from a mathematical point of view. Rather, he asserts that he wants to give a clear axiomatization of Euclidean geometry. But he seems to have too much trouble giving a philosophical justification for his axiomatization, which perhaps blurs his aims a little. He wants to claim that his system is correct by referring to the inherent consistency of reality. At the same time he declares that there can be many possible geometries. It is a little difficult for the reader to understand what status his particular axiomatization has among these many other possibilities.

Brodén gives a number of criteria which a scientific system of geometry should fulfill. Among other things, he gives a criterion of independence of the axioms from one another. However, the meaning of each his axioms depends in turn upon preceeding ones. This suggests that Brodén considers an axiom to be independent if it cannot be deduced from those previously stated. Implicity he performs a proof of independence of a subset of his axioms by constructing a model of a finite geometry.

Brodén also gives a criterion of empirical evidence for the axioms, which further suggests the influence of Helmholtz. Since Brodén considers experience not to lead to logical contradictions, this could in a weak sense be considered as some type of consistency requirement, as later developed by Hilbert, even though Brodén does not have such a concept.

Another criterion given is that of sufficiency of the axioms. Considering the proof of sufficiency, which Brodén performs in the latter part of the article, I interprete this criterion to assert the possibility of proving every statement one intuitively considers as belonging to the geometry. From our point of view one could possibly claim that this implies some kind of completeness requirement.

Brodén's view of geometry as being a science that seeks to reduce different relations to motion, guides him in his choice to reduce the notions of geometry to 'point' and 'immediate equality of distance'. With these he obtains a minimal set of basic notions that cannot be reduced further. A similar choice of basic notions was made by Pieri in 1908, but I have not found any trace of influence from him, or anyone else, on Brodén regarding this matter.

With the help of the basic notions, Brodén develops the system of axioms. Characterizing the system is the use of symmetries on the line and in the plane. Brodén does not give any indications of by whom he may have been influenced in this use of symmetries, which helps him in a beautiful way to develop his system.

In addition to the axioms of symmetry, Brodén gives two continuity axioms. Here we can see the further influence of Cantor upon Brodén. Brodén tranfers Cantor's theory of the real numbers to the line, where a Cauchy sequence of rational numbers correponds to a successive bisection of a segment. From this he obtains a bijection between the real numbers and the points of the line, which has been one of the main obstacles throughout the history of the axiomatization of geometry. Furthermore, these two axioms imply Hilbert's two continuity axioms, the so-called Archimedean and Completeness Axioms.

To conclude, it could be said that, even though Brodén is eclectic in his philosophy of geometry and did not possess the general concepts of a formal system that would later appear with Hilbert, he still manages to present an axiomatic system of Euclidean geometry that is in many ways remarkable.

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