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PREPRINT 2006:16

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Göteborg, August 2006

Preprint 2006:16
ISSN 1652-9715

Matematiska vetenskaper
Göteborg 2006

L^p-BOUNDEDNESS FOR ORTHOGONAL PROJECTIONS ONTO SPACES OF NEARLY HOLOMORPHIC FUNCTIONS AND OF VECTOR-VALUED HOLOMORPHIC FUNCTIONS

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Abstract. In this paper we establish L^p-boundedness criteria for orthogonal projections from $L^2(d\mu_\alpha)$ onto the discrete parts in the irreducible decomposition of $L^2(d\mu_\alpha)$ under the action of the Möbius group, where $d\mu_\alpha(z) = (1-|z|^2)^\alpha\, dm(z)$, $(\alpha > -1)$, and dm is the Lebesgue measure on the unit ball $\mathbb{B}$ in $\mathbb{C}^d$. These spaces can be realized as kernels of the power $\bar{D}^{l+1}$ of the invariant Cauchy-Riemann operator $\bar{D} = B(z,z)\bar{\partial}$ (where $B(z,z)^{-1}$ is the Bergman metric) and are therefore spaces of nearly holomorphic functions in the sense of Shimura. The operators $\bar{D}^l$ are intertwining operators from these spaces of nearly holomorphic functions into certain vector-valued Bergman-type spaces of holomorphic functions in $\mathbb{B}$. The orthogonal projections onto these spaces are given by matrix-valued Bergman-type kernels, and we study their L^p-boundedness properties for bounded symmetric domains of type I.

1. Introduction

Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^d$ with the Lebesgue measure dm. Consider the weighted L^2-space $L^2(d\mu_\alpha)$, where $d\mu_\alpha(z) = (1-|z|^2)^\alpha dm(z)$, $(\alpha > -1)$. The Möbius group of biholomorphic mappings of $\mathbb{B}$ acts on $L^2(d\mu_\alpha)$ as unitary (projective) representations. A weighted Plancherel formula was established by Peetre, Peng and Zhang in [PPZ] and Zhang [Z1], giving an explicit decomposition of the representation. There are continuous and disctrete parts in the decomposition. The discrete parts can be viewed as images of $L^2(d\mu_\alpha)$ under certain orthogonal projections. These spaces can be realized (see [Z2]) as the kernels of powers $\bar{D}^{m+1}$ of the invariant Cauchy-Riemann operator

Date: August 8, 2006.
2000 *Mathematics Subject Classification.* 32A25, 32A36, 32A37, 47B32.

$\bar{D} = B(z,z)\bar{\partial}$, where $B(z,z)^{-1}$ is the Bergman metric) and are therefore spaces of nearly holomorphic functions in the sense of Shimura (see [Sh1] and [Sh2]). Actually, for a certain k the operators $\bar{D}^l$, $l = 0, 1, \ldots, k$, are intertwining operators from the spaces of nearly holomorphic functions onto certain Bergman spaces of vector-valued holomorphic functions on $\mathbb{B}$ (see [PZ] and [EP]). We have the following diagram

$$\begin{array}{ccc} L^2(d\mu_\alpha) \cap C^\infty(\mathbb{B}) & \xrightarrow{\bar{D}^l} & L^2\left(\mathbb{B}, \odot^l \mathbb{C}^d, d\mu_\alpha\right) \cap C^\infty\left(\mathbb{B}, \odot^l \mathbb{C}^d\right) \\ \Big\downarrow{\scriptstyle P_l} & & \Big\downarrow{\scriptstyle P_{\nu,\,l}} \\ A_l^2(d\mu_\alpha) & \xrightarrow{\bar{D}^l} & L_a^2\left(\mathbb{B}, \odot^l \mathbb{C}^d, d\mu_\alpha\right) \end{array}$$

where P_l is the orthogonal projection from $L^2(d\mu_\alpha)$ onto the discrete part $A_l^2(d\mu_\alpha)$ of nearly holomorphic functions, $P_{\nu,\,l}$ ($\nu = \alpha + d + 1$) is the orthogonal projection from $L^2(\mathbb{B}, \odot^l \mathbb{C}^d, d\mu_\alpha)$ onto its holomorphic subspace and the L^2-norm (invariant under the action of the Möbius group) is given by

$$\|f\|_{l,\alpha,2} = \left(\int_{\mathbb{B}} \left\langle \otimes^l B(z,z)^{-1} f(z), f(z) \right\rangle \, d\mu_\alpha(z) \right)^{1/2},$$

where $\otimes^l B(z,z)^{-1}$ is the action on $\otimes^l \mathbb{C}^d$ induced by the action of $B(z,z)^{-1}$ on $\mathbb{C}^d$. This can be generalized into the setting of bounded symmetric domains [Z2].

The main objective of this paper is to establish the L^p-boundedness criteria for the orthogonal projections P_l onto the spaces of nearly holomorphic functions (Section 2) and also for the related Bergman-type projections $P_{\nu,\,l}$ onto the Bergman spaces of vector-valued holomorphic functions (Section 3) for the unit ball in $\mathbb{C}^d$; the question makes also sense for general bounded symmetric domains, and we study the Bergman-type projections $P_{\nu,\,l}$ for bounded symmetric domains of type I.

More concretely, if $\alpha > 2l - 1$ then, on one hand, Theorem 2.1 states that P_l is L^p-bounded if and only if

$$\frac{\alpha+1}{\alpha+1-1} < p < \frac{\alpha+1}{l}. \tag{1}$$

On the other hand, Theorem 3.6 states that $P_{\nu,\,l}$ is L^p-bounded if condition (1) is satisfied.

To find L^p-boundedness criteria for P_l we use concrete formulas, which can be found in [Z1]. In the more general setting of bounded symmetric domains we do not yet have such formulas. However, the generalizations of $P_{\nu,l}$ to bounded symmetric domains of type I is studied in Section 3. In this section, a sufficient conditions for these projections to be L^p-bounded is presented. Actually, this is a weak generalization of the corresponding result for the case of the unit ball in $\mathbb{C}^d$, weaker since the Forelli-Rudin type estimate is different in the general case (see [FK2] and [EZ]). The Bergman-type projections mentioned above are closely related to vector-valued Bergman-type projections studied in [Su1]. A weak generalization of the L^p-boundedness criteria for these projections is presented in Section 3.

Acknowledgements. I thank Henrik Seppänen for many illuminating discussions on bounded symmetric domains.

2. The projection operators onto nearly holomorphic functions

2.1. **The action of the Möbius group.** Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^d$, and let $G = Aut(\mathbb{B})$ be the group of holomorphic bijections on $\mathbb{B}$ with holomorphic inverse. An element $g \in G$, $g(z) = 0$, can be decomposed as $g = U\varphi_z$ where $U : \mathbb{C}^d \to \mathbb{C}^d$ is a unitary map and φ_z is a linear fractional map, taking 0 to z, see [Ru]. The complex Jacobian $J_{\varphi_z}(w)$ is given in [Su1] by

$$J_{\varphi_z}(w) = (-1)^d \frac{(1-|z|^2)^{(d+1)/2}}{(1-\langle w, z\rangle)^{d+1}},$$

where $\langle\cdot,\cdot\rangle$ is the scalar product on $\mathbb{C}^d$. Since G acts transitively on $\mathbb{B}$, see [Ru], we get $J_g(w)$ for any $g \in G$ in this way. Hence, we can define an action π_ν of G on $L^2(d\mu_\alpha)$ by

$$(\pi_\nu(g)f)(w) = f(g^{-1}(w)) \cdot J_{g^{-1}}(w)^{\nu/(d+1)}, \tag{2}$$

where $\nu = \alpha + d + 1$ and where we use the same convention as in [Su1] concerning the ambiguity of the definition of power. Then π_ν is a unitary projective representation of G.

2.3. **An L^p-boundedness criterion.** The orthogonal projection operators from $L^2(d\mu_\alpha)$ onto the discrete parts of the irreducible decomposition under the action (2) of $L^2(d\mu_\alpha)$ are given explicitly in [Z1] by P_l, for $l = 0, 1, \cdots, k = [(\alpha+1)/2]$ (α is not an odd integer), where

$$P_l f(z) = \langle f, K_l(\cdot, z)\rangle_\alpha = \int_{\mathbb{B}} f(w) K_l(z,w)\, d\mu_\alpha(w) \tag{3}$$

and

$$K_l(z,w) = c_l \frac{1}{(1-\langle z,w\rangle)^{\alpha+d+1}} \times \\ \sum_{i=0}^{l} \frac{(-l)_i (l-\alpha-1)_i (-1)^i}{(d)_i i!} \left(1 - \frac{|1-\langle z,w\rangle|^2}{(1-|z|^2)(1-|w|^2)}\right)^i,$$

where c_l is a normalization constant and $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochammer symbol. In the next theorem we present necessary and sufficient conditions on $1 < p < \infty$ to make the projection operators P_l bounded on $L^p(d\mu_\alpha)$.

Theorem 2.1. *If $l \in \{0, 1, 2, \cdots, k\}$, $k = [(\alpha+1)/2]$ (α is not an odd integer), then the orthogonal projection operator P_l, defined in (3), is bounded on $L^p(d\mu_\alpha)$ if and only if*

$$\frac{\alpha+1}{\alpha+1-l} < p < \frac{\alpha+1}{l}$$

when $l \neq 0$ and $1 < p < \infty$ when $l = 0$.

Proof. The case $l = 0$ is classical (see for instance Theorem 2.11 in [Zhu]). Assume now $l \neq 0$. We can write the reproducing kernel K_l as

$$K_l(z,w) = h_{l-1}(z,w) + c_l T_l(z,w)$$

where

$$T_l(z,w) = \frac{(1-|z|^2)^{-l}(1-|w|^2)^{-l}(1-\langle w,z\rangle)^l}{(1-\langle z,w\rangle)^{\alpha-l+d+1}}.$$

First we observe that there is a constant $C > 0$ such that

$$|K_l(z,w)| \leq C \cdot \frac{(1-|z|^2)^{-l}(1-|w|^2)^{-l}}{|1-\langle z,w\rangle|^{\alpha+1+d-2l}} = C \cdot T_l(z,w).$$

Hence,

$$|P_l f(z)| \le C \int_{\mathbb{B}} T_l(z,w)|f(w)|\, d\mu_\alpha(w)\,.$$

We claim that there are real numbers $M > 0$ and t such that the inequalities

$$\int_{\mathbb{B}} T_l(z,w)(1-|z|^2)^{pt}\, d\mu_\alpha(z) \le M(1-|w|^2)^{pt} \tag{4}$$

and

$$\int_{\mathbb{B}} T_l(z,w)(1-|w|^2)^{qt}\, d\mu_\alpha(w) \le M(1-|z|^2)^{qt} \tag{5}$$

hold for q with $1/q+1/p=1$. If the claim is true then P_l is bounded on $L^p(d\mu_\alpha)$, by Schur's test (see [HKZ]). By the same arguments as in the proof of Theorem 7.2 in [Su1] it follows that the claim is true if

$$\frac{\alpha+1}{\alpha+1-l} < p < \frac{\alpha+1}{l}\,.$$

Now we consider the cases when $1 < p \le (\alpha+1)/(\alpha+1-l)$ or $(\alpha+1)/l \le p < \infty$. Actually, for duality reasons we need only to consider the case when $(\alpha+1)/l \le p < \infty$. Let $\varepsilon > 0$ and define χ_ε to be the characteristic function on $\mathbb{B}_\varepsilon = \{z \in \mathbb{C}^d : |z| < \varepsilon\}$. If a is a positive real number, then

$$(1-\langle z,w\rangle)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!}\langle z,w\rangle^k\,.$$

By binomial expansion and orthogonality,

$$\begin{aligned}&\int_{\mathbb{B}_\varepsilon} \frac{(1-\langle w,z\rangle)^j}{(1-\langle z,w\rangle)^{\alpha-j+d+1}}\,(1-|w|^2)^{\alpha-j} dm(w)\\ &= \sum_{i=0}^{j} \binom{j}{i} \frac{(\alpha-j+d+1)_i}{i!}(-1)^i \int_{\mathbb{B}_\varepsilon} |\langle z,w\rangle|^{2i}\,(1-|w|^2)^{\alpha-j} dm(w)\end{aligned}$$

for all $j = 0,1,2,\cdots,l$. Clearly we can find a constant D_l such that

$$|\langle \chi_\varepsilon, h_{l-1}(\cdot,z)\rangle_\alpha| \le D_l(1-|z|^2)^{-l+1}\,. \tag{6}$$

Also,

$$(7)\quad |\langle \chi_\varepsilon, T_l(\cdot, z)\rangle_\alpha| \geq$$
$$(1-|z|^2)^{-l}\int_{\mathbb{B}_\varepsilon}(1-|w|^2)^{\alpha-l}dm(w)\left(1-\sum_{i=1}^{l}\binom{l}{i}\frac{(\alpha-l+d+1)_i}{i!}\varepsilon^{2i}\right).$$

Thus by (6) and (7), if we choose ε to be small enough and $K < 1$ large enough, there is a positive constant C_l such that

$$|\langle \chi_\varepsilon, K_l(\cdot, z)\rangle_\alpha| \geq |\langle \chi_\varepsilon, c_l T_l(\cdot, z)\rangle_\alpha| - |\langle \chi_\varepsilon, h_{l-1}(\cdot, z)\rangle_\alpha| \geq C_l(1-|z|^2)^{-l}$$

if $K < |z| < 1$. Hence,

$$(8)\int_{\mathbb{B}} |\langle \chi_\varepsilon, K_l(\cdot, z)\rangle_\alpha|^p \, d\mu_\alpha(z) \geq C_l^p \int_{K<|z|<1}(1-|z|^2)^{\alpha-pl}dm(z)$$

and the integral on the right side of the inequality (8) is infinite if $p \geq (\alpha+1)/l$. □

3. Bergman spaces of vector-valued holomorphic functions

3.1. **Bounded symmetric domains of type I.** Let $\mathcal{D}$ be a type I bounded symmetric domain, i.e., $\mathcal{D} = \{Z \in M_{m,n}(\mathbb{C}) : ZZ^* < I_m\}$ and let $dm(Z)$ be the Lebesgue measure on $\mathcal{D}$. By Theorem 4.3.1 in [H], the Bergman kernel $k(Z, W)$ is up to a constant $h(Z, W)^{-(m+n)}$ where $h(Z, W) = \det(I - ZW^*)$. If $g : \mathcal{D} \to \mathcal{D}$ is biholomorphic then, by Theorem 2.10 in [FK1],

$$(9)\qquad k(Z, W) = \det(dg(Z)) \cdot k(g(Z), g(W)) \cdot \overline{\det(dg(W))}$$

where $dg(Z) : T_Z(\mathcal{D}) \to T_{g(Z)}(\mathcal{D})$ is the differential map.

The Bergman operator defined for $Z, W \in \mathcal{D}$ is given in [L] by

$$B(Z, W)X = (I - ZW^*)X(I - W^*Z)$$

for matrices $X \in M_{m,m}(\mathbb{C})$. By Lemma 2.11 in [L],

$$(10)\qquad B(g(Z), g(W)) = dg(Z)B(Z, W)dg(W)^*.$$

3.2. **Values in tensor products of a tangent space.** Consider the measure $d\mu_\alpha(Z) = h(Z,Z)^\alpha dm(Z)$ for $\alpha > 2s-1$ and the corresponding L^2-space $L^2(\mathcal{D}, \odot^s V, d\mu_\alpha)$ where $V = M_{m,m}(\mathbb{C})$ (so that we can identify a tangent space on $\mathcal{D}$ with V) and $\odot^s V$ is the induced symmetric tensor product for s copies of V where s is a nonnegative integer. The functions in $L^2(\mathcal{D}, \odot^s V, d\mu_\alpha)$ are tensor-valued and the L^2-norm is given by

$$\|f\|_{s,\alpha,2} = \left(\int_{\mathcal{D}} \left\langle \otimes^s B(Z,Z)^{-1} f(Z), f(Z) \right\rangle \, d\mu_\alpha(Z) \right)^{1/2}$$

where $\langle X, Y\rangle = \mathrm{tr}(XY^*)$. The reproducing kernel is up to a constant

$$K_{\nu,s}(Z,W) = h(Z,W)^{-\nu} \otimes^s B(Z,W) \tag{11}$$

where $\nu = \alpha + m + n$. This can be proved by using the transformation properties (9) and (10) of $h(Z,W)$ and $B(Z,W)$ respectively (see e.g. [Su1] for the case of the unit ball and [ØZ] for similar results).

Lemma 3.1. *Let s be a nonnegative integer. Then there is a constant $C_s > 0$ such that*

$$\left\| \otimes^s \left(B(Z,Z)^{-1/2} B(Z,W) B(W,W)^{-1/2} \right) X \right\| \leq C_s \frac{|h(Z,W)|^{2s}}{h(Z,Z)^s h(W,W)^s} \|X\|$$

for all $X \in M_{m,m}(\mathbb{C})^s$.

Proof. The case $s = 0$ is trivial, so first we prove the case $s = 1$. It is clear for $W = 0$ since

$$\|B(Z,Z)^{-1/2} X\| \leq C \cdot h(Z,Z)^{-1} \cdot \|X\| .$$

Now, let g be a biholomorphic map on $\mathcal{D}$ such that $g(0) = W$ and $g^{-1} = g$. On one hand,

$$\|B(g^{-1}(Z), g^{-1}(Z))^{-1/2} X\| \leq C \cdot h(g^{-1}(Z), g^{-1}(Z))^{-1} \|X\|$$

On the other hand, by (9),

$$h(g^{-1}(Z), g^{-1}(Z)) = h(g(Z), g(Z)) = \frac{h(W,W) h(Z,Z)}{|h(Z,W)|^2} .$$

Hence

$$\|B(g^{-1}(Z), g^{-1}(Z))^{-1/2} X\| \leq C \cdot \frac{|h(Z,W)|^2}{h(Z,Z) h(W,W)} \cdot \|X\| . \tag{12}$$

Let $Y = dg(0)^* B(W,W)^{-1/2} X$. Then we can replace X by Y in the inequality (12). Also $\|Y\| = \|X\|$, so if we let $Z_0 = g^{-1}(Z)$ then by (10),

$$\begin{aligned}
& \left\| B(Z,Z)^{-1/2} B(Z,W) B(W,W)^{-1/2} X \right\|^2 \\
= & \left\| B(Z,Z)^{-1/2} dg(Z_0) Y \right\|^2 \\
= & \operatorname{tr}\left(B(Z,Z)^{-1/2} dg(Z_0) Y Y^* dg(Z_0)^* B(Z,Z)^{-1/2} \right) \\
= & \operatorname{tr}\left(dg(Z_0)^* B(Z,Z)^{-1} dg(Z_0) Y Y^* \right) \\
= & \operatorname{tr}\left(B(Z_0,Z_0)^{-1} Y Y^* \right) \\
\leq & C^2 \cdot \left(\frac{|h(Z,W)|^2}{h(Z,Z)h(W,W)} \right)^2 \cdot \|X\|^2 .
\end{aligned}$$

Hence, the lemma is proved for the case $s = 1$. Now, consider the case where $s = 2, 3, \ldots$ and let

$$A_{Z,W} = B(Z,Z)^{-1/2} B(Z,W) B(W,W)^{-1/2}$$

and

$$t_{Z,W} = \frac{|h(Z,W)|^2}{h(Z,Z)h(W,W)} .$$

We have proved that

$$A_{Z,W}^* A_{Z,W} \leq C^2 t_{Z,W}^2 I$$

so that

$$(\otimes^s A_{Z,W})^* \otimes^s A_{Z,W} = \otimes^s \left(A_{Z,W}^* A_{Z,W} \right) \leq C^{2s} t_{Z,W}^{2s} \otimes^s I$$

which proves the lemma. □

As a special case we get the following lemma.

Lemma 3.2. *If $\mathcal{D} = \mathbb{B}$, then for any nonnegative integer s, there is a constant $C_s > 0$ such that*

$$\left\| \otimes^s \left(B(z,z)^{-1/2} B(z,w) B(w,w)^{-1/2} \right) x \right\| \leq C_s \frac{|1 - \langle z, w \rangle|^{2s}}{(1-|z|^2)^s (1-|w|^2)^s} \|x\|$$

for all $x \in \otimes^s V$.

As a special case of Theorem 4.1 in [FK2] we have the following lemma.

Lemma 3.3. *Let $\beta - 1 > \alpha > -1$. Then there is a constant $C > 0$ such that*

$$\int_{\mathcal{D}} \frac{h(Z,Z)^{\alpha}}{|h(Z,W)|^{\beta+m+n}}\, dm(Z) \le C \cdot h(W,W)^{-(\beta-\alpha)}\,.$$

Remark 3.4. There is an orthogonal projection $P_{\nu,s}$, from the Hilbert space $L^2(\mathcal{D}, \odot^s V, d\mu_\alpha)$ into its holomorphic subspace, such that for any $f \in L^2(\mathcal{D}, \odot^s V, d\mu_\alpha)$ and any $X \in \odot^s V$ we have that

$$\begin{aligned}(13)\quad &\langle P_{\nu,s} f(Z), X\rangle \\ &\qquad = c \int_{\mathcal{D}} \big\langle \otimes^s B(W,W)^{-1} f(W), K_{\nu,s}(W,Z)X\big\rangle\, d\mu_\alpha(W)\,.\end{aligned}$$

Theorem 3.5. *Let $\alpha > 2s$ and let $P_{\nu,s}$ be the orthogonal projection operator, where $\nu = \alpha + m + n$. If*

$$\frac{\alpha+2}{\alpha+1-s} < p < \frac{\alpha+2}{s+1}\,,$$

then $P_{\nu,s}$ is bounded on $L^p(\mathcal{D}, \odot^s V, d\mu_\alpha)$.

Proof. The formula (13) can be rewritten as

$$P_{\nu,s} f(Z) = c \int_{\mathcal{D}} K_{\nu,s}(W,Z)^* \otimes^s B(W,W)^{-1} f(W)\, d\mu_\alpha(W)\,.$$

Let

$$T(Z,W) = \frac{h(Z,Z)^{-s} h(W,W)^{-s}}{|h(Z,W)|^{\nu-2s}}\,.$$

By the equality $K_{\nu,s}(W,Z)^* = K_{\nu,s}(Z,W)$ and Lemma 3.1 it follows that

$$\begin{aligned}&\big\|\otimes^s B(Z,Z)^{-1/2} P_{\nu,s} f(Z)\big\| \\ &\qquad \le C \int_{\mathcal{D}} T(Z,W)\, \big\|\otimes^s B(W,W)^{-1/2} f(W)\big\|\, d\mu_\alpha(W)\,.\end{aligned}$$

Now by Lemma 3.3, using the same techniques as in the proof of Theorem 7.2 in [Su1], it follows that there exists a real number t and a constant $M > 0$ such that

$$\int_{\mathcal{D}} T(Z,W) h(Z,Z)^{pt+\alpha}\, dm(Z) \le M \cdot h(W,W)^{pt}$$

and

$$\int_{\mathcal{D}} T(Z,W)h(W,W)^{qt+\alpha}\,dm(W) \le M \cdot h(Z,Z)^{qt}$$

where $1/p + 1/q = 1$. Namely, there exists such t if p and α satisfies the condition

$$\frac{\alpha+2}{\alpha+1-s} < p < \frac{\alpha+2}{s+1}\,.$$

So, with this condition for p and α it follows by Schur's test that

$$\int_{\mathcal{D}} \left\|\otimes^s B(Z,Z)^{-1/2} P_{\nu,s} f(Z)\right\|^p \, d\mu_\alpha(Z) \\ \le C \int_{\mathcal{D}} \left\|\otimes^s B(W,W)^{-1/2} f(W)\right\|^p \, d\mu_\alpha(W)\,.$$

□

Theorem 3.6. *Let $\alpha > 2s-1$ and let $P_{\nu,s}$ be the orthogonal projection where $\nu = \alpha + d + 1$, i.e. $\mathcal{D} = \mathbb{B}$. If $s \neq 0$ and*

$$\frac{\alpha+1}{\alpha+1-s} < p < \frac{\alpha+1}{s}\,,$$

then $P_{\nu,s}$ is bounded on $L^p(\mathbb{B}, \odot^s V, d\mu_\alpha)$. If $s = 0$, then $P_{\nu,s}$ is bounded on $L^p(\mathbb{B}, \odot^s V, d\mu_\alpha)$ for any $1 < p < \infty$.

Proof. The case $s = 0$ is classical (see for instance Theorem 2.11 in [Zhu]). Assume now $s \neq 0$. By similar arguments as in the proof of Theorem 3.5, using Lemma 3.2, we get that

$$\left\|\otimes^s B(z,z)^{-1/2} P_{\nu,s} f(z)\right\| \le C \int_{\mathbb{B}} T(z,w) \left\|\otimes^s B(w,w)^{-1/2} f(w)\right\| \, d\mu_\alpha(w)$$

where

$$T(z,w) = \frac{(1-|z|^2)^{-s}(1-|w|^2)^{-s}}{|1-\langle z,w\rangle|^{\nu-2s}}\,.$$

Again, following the proof of Theorem 3.5, using Proposition 1.4.10 in [Ru] instead of Lemma 3.3, we get the desired result. □

3.3. **Values in tensor products of a cotangent space.** Once we have studied the L^p-boundedness for Bergman-type projections onto Bergman spaces with functions with values in symmetric tensor products of a tangent space, it is natural to do so even for the case of cotangent space. These Bergman-type projections are closely related to the Bergman-type projections studied in [Su1] and [Su2].

Let $\mathcal{D}$ be the type I bounded symmetric domain given in the previous subsection. Most notation are the same as in the previous subsection, only $\alpha > -1$ and $L^2(\mathcal{D}, \odot^s V, d\mu_\alpha)$ is replaced by $L^2(\mathcal{D}, \odot^s V', d\mu_\alpha)$ with norm

$$\|f\|'_{s,\alpha,2} = \left(\int_{\mathbb{B}} \langle \otimes^s B(Z,Z)' f(Z), f(Z) \rangle \, d\mu_\alpha(Z) \right)^{1/2} ,$$

where $B(Z,Z)'$ is the dual action of $B(Z,Z)$ acting on the dual space V'. Also $B(Z,Z)'$ may be identified with $B^t(Z,Z)$ where

$$B^t(Z,W)X = (I - ZW^*)^t X (I - W^*Z)^t$$

for matrices $X \in M_{m,m}(\mathbb{C})$ and where t is the transpose of a matrix. The reproducing kernel for $L^2(\mathcal{D}, \odot^s V', d\mu_\alpha)$ is given, up to a nonzero constant, by

$$K'_{\nu,s}(Z,W) = h(Z,W)^{-\nu} \otimes^s B^t(Z,W)^{-1}$$

where again $\nu = \alpha + m + n$. The orthogonal projection $P'_{\nu,s}$ in question from $L^2(\mathcal{D}, \odot^s V', d\mu_\alpha)$ onto its holomorphic subspace, is defined in the following way. For any $f \in L^2(\mathcal{D}, \odot^s V', d\mu_\alpha)$ and any $X \in \odot^s V'$ we have that

$$(14) \quad \left\langle P'_{\nu,s} f(Z), X \right\rangle = c' \int_{\mathcal{D}} \left\langle \otimes^s B^t(W,W) f(W), K'_{\nu,s}(W,Z) X \right\rangle \, d\mu_\alpha(W) \,.$$

Hence, if we can find a result similar to Lemma 3.1 then we can use the same arguments as in the proof of Theorem 3.5 to find criteria for the projections $P'_{\nu,s}$ to be bounded on $L^p(\mathcal{D}, \odot^s V', d\mu_\alpha)$.

Lemma 3.7. *Let s be a nonnegative integer. Then*

$$\left\| \otimes^s \left(B^t(Z,Z)^{1/2} B^t(Z,W)^{-1} B^t(W,W)^{1/2} \right) X \right\| \le \|X\|$$

for all $X \in M_{m,m}(\mathbb{C})^s$.

Proof. By the definition of the Bergman operator it follows that

$$\left\|B^t(Z,Z)^{1/2}X\right\| \leq \|X\| \tag{15}$$

Actually, if $\mathcal{D}$ is not the unit ball in $\mathbb{C}^d$ then we can find $Z \in \mathcal{D}$ such that $B^t(Z,Z)X = X$ for all $X \in M_{m,m}(\mathbb{C})$ and therefore (15) is actually the best estimate we can get in the general case. Now, given $W \in \mathcal{D}$, choose g as in the proof of Lemma 3.1. Then

$$\left\|B^t(Z_0,Z_0)^{1/2}X\right\| \leq \|X\|,$$

if $g(Z_0) = Z$. Since

$$\left(dg(Z_0)^t\right)^{-1} : T_{Z_0}(\mathcal{D})' \to T_Z(\mathcal{D})'$$

is an isometry then

$$\left\|B^t(Z,Z)^{1/2}\left(dg(Z_0)^t\right)^{-1}X\right\| \leq \|X\|.$$

Hence

$$\begin{aligned}\left\|B^t(Z,Z)^{1/2}B^t(Z,W)^{-1}B^t(W,W)^{1/2}X\right\| \\ = \left\|B^t(Z,Z)^{1/2}\left(dg(Z_0)^t\right)^{-1}Y\right\| \leq \|Y\|\end{aligned}$$

where $Y = ((dg(0)^*)^t)^{-1}B^t(W,W)^{1/2}X$. Also, $\|Y\| = \|X\|$ which follows in the same way as in the proof of Lemma 3.1. Thus, the lemma is proved for the case when $s = 1$ and the proof of the general case is done in exactly the same way as in the proof of Lemma 3.1. □

As we could see in the proof of the lemma above we need to treat the particular case $\mathcal{D} = \mathbb{B}$ separately. The following lemma can be proved by using the same techniques as in the proof of Lemma 3.1 and in Lemma 3.7. However, the same result can also be found in [Su1].

Lemma 3.8 (Lemma 7.1 in [Su1])**.** *If $\mathcal{D} = \mathbb{B}$, then for any nonnegative integer s, there is a constant $C_s > 0$ such that*

$$\begin{aligned}\left\|\otimes^s\left(B^t(z,z)^{1/2}B^t(z,w)^{-1}B^t(w,w)^{1/2}\right)x\right\| \\ \leq C_s\frac{(1-|z|^2)^{s/2}(1-|w|^2)^{s/2}}{|1-\langle z,w\rangle|^s}\|x\|\end{aligned}$$

for all $x \in \otimes^s V'$.

Now we can get the desired boundedness condition. This result is a weaker generalization of Theorem 7.2 in [Su1].

Theorem 3.9. *Let $\alpha > 0$ and let $P'_{\nu,s}$ be the orthogonal projection operator, where $\nu = \alpha + m + n$. If*

$$\frac{\alpha+2}{\alpha+1} < p < \alpha + 2\,,$$

then $P'_{\nu,s}$ is bounded on $L^p(\mathcal{D}, \odot^s V', d\mu_\alpha)$.

Proof. The result follows by exactly the same arguments as we used to prove Theorem 3.5. □

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