CHALMERS | GÖTEBORG UNIVERSITY

PREPRINT 2007:11

Sharp bounds on $2m/r$ of general spherically symmetric static objects

HÅKAN ANDRÉASSON

Department of Mathematical Sciences
Division of Mathematics
CHALMERS UNIVERSITY OF TECHNOLOGY
GÖTEBORG UNIVERSITY
Göteborg Sweden 2007

Preprint 2007:11

Sharp bounds on 2*m*/*r* of general spherically symmetric static objects

Håkan Andréasson

CHALMERS | GÖTEBORG UNIVERSITY

Department of Mathematical Sciences
Division of Mathematics
Chalmers University of Technology and Göteborg University
SE-412 96 Göteborg, Sweden

Göteborg, April 2007

Preprint 2007:11
ISSN 1652-9715

Matematiska vetenskaper
Göteborg 2007

Sharp bounds on $2m/r$ of general spherically symmetric static objects

Håkan Andréasson
Mathematical Sciences, Göteborg University
Mathematical Sciences, Chalmers University of Technology
S-41296 Göteborg, Sweden
email: hand@math.chalmers.se

April 4, 2007

Abstract

In 1959 Buchdahl [13] obtained the inequality $2M/R \leq 8/9$ under the assumptions that the energy density is non-increasing outwards and that the pressure is isotropic. Here M is the ADM mass and R the area radius of the boundary of the static body. The assumptions used to derive the Buchdahl inequality are very restrictive and e.g. neither of them hold in a simple soap bubble. In this work we remove both of these assumptions and consider *any* static solution of the spherically symmetric Einstein equations for which the energy density $\rho \geq 0$, and the radial- and tangential pressures $p \geq 0$ and p_T, satisfy $p + 2p_T \leq \Omega\rho$, $\Omega > 0$, and we show that

$$\sup_{r>0} \frac{2m(r)}{r} \leq \frac{(1+2\Omega)^2 - 1}{(1+2\Omega)^2},$$

where m is the quasi-local mass, so that in particular $M = m(R)$. We also show that the inequality is sharp. Note that when $\Omega = 1$ the original bound by Buchdahl is recovered. The assumptions on the matter model are very general and in particular any model with $p \geq 0$ which satisfies the dominant energy condition satisfies the hypotheses with $\Omega = 3$.

1 Introduction

The metric of a static spherically symmetric spacetime takes the following form in Schwarzschild coordinates

$$ds^2 = -e^{2\mu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

where $r \geq 0$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$. Asymptotic flatness is expressed by the boundary conditions

$$\lim_{r\to\infty} \lambda(r) = \lim_{r\to\infty} \mu(r) = 0,$$

and a regular centre requires $\lambda(0) = 0$. The Einstein equations read

$$e^{-2\lambda}(2r\lambda_r - 1) + 1 = 8\pi r^2 \rho, \tag{1.1}$$

$$e^{-2\lambda}(2r\mu_r + 1) - 1 = 8\pi r^2 p, \tag{1.2}$$

$$\mu_{rr} + (\mu_r - \lambda_r)(\mu_r + \frac{1}{r}) = 8\pi p_T e^{2\lambda}. \tag{1.3}$$

Here ρ is the energy density, p the radial pressure and p_T is the tangential pressure. If the pressure is isotropic, i.e., $p = p_T$, a solution will satisfy the well-known Tolman-Oppenheimer-Volkov equation for equilibrium

$$p_r = -\mu_r(p + \rho). \tag{1.4}$$

In the case of non-isotropic pressure this equation generalizes to

$$p_r = -\mu_r(p + \rho) - \frac{2}{r}(p - p_T). \tag{1.5}$$

Note that the radial pressure p is monotone in the isotropic case if $p + \rho \geq 0$ since $\mu_r \geq 0$, cf. (2.2). The quasi-local mass $m = m(r)$ is given by

$$m(r) = \int_0^r 4\pi\eta^2 \rho(\eta) d\eta, \tag{1.6}$$

and the ADM mass of a steady state for which the energy density has support in $[0, R]$ is thus given by $M = m(R)$.

Schwarzschild asked already in 1916 the question: How large can $2M/R$ possibly be? He gave the answer $2M/R \leq 8/9$ [25] in the special case of the Schwarzschild interior solution which has constant energy density and isotropic pressure. In 1959 Buchdahl [13] extended his result to isotropic solutions for which the energy density is non-increasing outwards and he showed that also in this case

$$2M/R \leq 8/9.$$

This is called the Buchdahl inequality and is included in most text books on general relativity in connection with the discussion of the interior solution by Schwarzschild, cf. e.g. [26] and [27]. The quantity $2m/r$ is fundamental for determining the spacetime geometry of a static spherically symmetric spacetime, cf. equations (2.1) and (2.4). A bound on $2M/R$ has also an immediate observational consequence since it limits the possible red shifts of spherically symmetric static objects.

The assumptions made by Buchdahl are extremely restrictive as pointed out by Guven and Ó Murchadha [17], e.g. neither of the assumptions hold in a simple soap bubble and they do not approximate any known topologically stable field configuration. Moreover, astrophysical models of stars are not unusually anisotropic. Lemaitre proposed a model of an anisotropic star already in 1933 [19], and Binney and Tremaine [9] explicitly allow for an anisotropy coefficient (cf. also [18] and the references therein).

One motivation for this study has its roots in the numerical investigation of the spherically symmetric Einstein-Vlasov (ssEV) system [7] which admits a very rich class of static solutions. The *overwhelming* number of these have neither an isotropic pressure nor a non-increasing energy density, but nevertheless $2M/R$ is always found to be less than $8/9$, cf. [7]. There are sometimes arguments which claim that the monotonicity of ρ is necessary for the stability of a steady state, cf. e.g. [27], but at least for Vlasov matter this is not the case by the results presented in [6].

In this work the problem of finding a sharp bound on $2m/r$ is solved in full generality in the class of matter models which satisfy

$$p + 2p_T \leq \Omega\rho, \text{ where } \Omega, p \text{ and } \rho \text{ are non-negative.} \tag{1.7}$$

We will show that

$$\sup_{r>0} \frac{2m(r)}{r} \leq \frac{(1+2\Omega)^2 - 1}{(1+2\Omega)^2},$$

for any static solution of the spherically symmetric Einstein equations which satisfies (1.7). The class of matter models defined by (1.7) is very general. Indeed, a realistic matter model should satisfy the dominant energy condition (DEC) which implies that $\rho \geq 0$ and that the inequality (1.7) holds with $\Omega = 3$. The remaining condition that p is non-negative is a standard assumption for most matter models in astrophysics. Moreover, Vlasov matter satisfies the conditions in (1.7) with $\Omega = 1$. An interesting feature of Vlasov matter, in comparison with a fluid model, is that no equation of state which relates the pressure and the energy density has to be specified. For Vlasov matter, ρ, p and p_T are all determined by a single density function on phase space, cf. [5] and [24] for more information on Vlasov matter and the EV system.

The bound that we obtain for $2m/r$ is sharp in the sense that an infinitely thin shell of matter, with $2m/r$ equal to the critical value, will satisfy a form of the generalized TOV equation which allows ρ and p_T to be measures ($p = 0$ here). This is described in detail in the next section. It should here be pointed out that for the ssEV system the results in [4] show that there exist *regular* static solutions with the property that $2M/R$ takes values arbitrary close to $8/9$ ($\Omega = 1$ for Vlasov matter). These solutions do approach the infinitely thin shell mentioned above as $2M/R \to 8/9$. In section 4 we give an analogy with a classical problem in electrostatics (or equivalently in Newtonian gravity) which shares the property that the maximizer is a measure at the boundary. In the work by Buchdahl, the solution that maximizes $2M/R$ is the Schwarzschild interior solution with constant energy density. This solution has the property that the pressure becomes unbounded as $2M/R \to 8/9$, and therefore the solution does not satisfy the DEC and is not a realistic steady state.

Before finishing this section with a review of previous results, let us point out that the original motivation for investigating the Buchdahl inequality in full generality comes from its possible role in understanding the formation of trapped surfaces. Christodoulou has obtained conditions which guarantee the formation of trapped surfaces in the case of a scalar field [14], and this result is crucial for his proof of the weak- and strong cosmic censorship conjectures [15]. For more information on this see the introduction in [4].

General investigations of the Buchdahl inequality have previously been undertaken by Baumgarte and Rendall [8] and Mars, Mercè Martín-Prats and Senovilla [21]. These studies concern very general matter models and they obtain the bound $2m/r < 1$. This bound gives little information on the spacetime geometry since $\lambda \to \infty$ as $2m/r \to 1$, and in particular it gives no bound on the red shift of a static body. In [3] shells supported in $[R_0, R_1]$ are considered and it is shown that if the support is narrow then a Buchdahl inequality holds (i.e. $2M/R < 1 - \epsilon,\ \epsilon > 0$). This result is superseded by the result presented here but some of the ideas in [3] play an essential role in this work. Guven and Ó Murchadha consider the general case in [17] and obtain a bound on $2m/r$ in terms of the ratio of the tangential- and the radial pressure, which they denote by γ. Their bound degenerates (i.e., $2m/r \to 1$) as $\gamma \to \infty$. (Cf. also [10] for a similar analysis which includes a cosmological constant.) It is interesting to note that $\gamma = \infty$ for the maximizing solution in our work since $p = 0$ and p_T is a Dirac measure at the boundary. Also note that $\gamma \to \infty$ for the sequence constructed in [4], which in the limit gives an infinitely thin shell with $p = 0$ and $2p_T = \rho$. In this context we mention the work [16] where an infinitely thin shell is studied. They obtain the bound

$2M/R \leq 24/25$. Note that this value agrees with our bound when $\Omega = 2$. This is not surprising since their infinitely thin shell satisfies the DEC and has $p = 0$ which in our terminology means that $\Omega = 2$. A similar study is carried out by Bondi [12] in the case $\Omega = 1$.

Furthermore, Bondi [11] investigates (non-rigorously) isotropic solutions which are allowed to have a non-monotonic energy density. He considers models for which $\rho \geq 0,\ \rho \geq p$, or $\rho \geq 3p$, and obtains bounds on $2M/R$ strictly less than one in the respectively cases. The isotropic condition is however crucial since these bounds are violated for strongly non-isotropic solutions as this work shows (cf. also [4], [16] and [12]).

The paper is organized as follows. In the next section we derive our basic inequality which only involves ρ and we formulate our main results. The main ideas of the paper are presented in section 3. In section 4 an electrostatic analogy (or equivalently a Newtonian analogy) is discussed and the proofs of the theorems are given in section 5.

2 Set up and main results

Let us collect a couple of facts concerning the system (1.1)-(1.3). A consequence of equation (1.1) is that

$$e^{-2\lambda} = 1 - \frac{2m(r)}{r}, \tag{2.1}$$

and from (1.2) it then follows that

$$\mu_r = (\frac{m}{r^2} + 4\pi r p)e^{2\lambda}. \tag{2.2}$$

Adding (1.1) and (1.2) and using the boundary conditions at $r = \infty$ gives

$$\mu(r) + \lambda(r) = -\int_r^\infty 4\pi\eta(\rho + p)e^{2\lambda}d\eta. \tag{2.3}$$

In particular if R is the outer radius of support of the matter then

$$e^{\mu(r)+\lambda(r)} = 1,$$

when $r \geq R$. Hence,

$$e^{\mu(r)} = e^{-\lambda(r)} = \sqrt{1 - \frac{2m(r)}{r}},\ r \geq R. \tag{2.4}$$

The generalized Tolman-Oppenheimer-Volkov equation (1.5) implies that a solution satisfies

$$(m + 4\pi r^3 p)e^{\mu+\lambda} = \int_0^r 4\pi\eta^2 e^{\mu+\lambda}(\rho + p + 2p_T)d\eta. \tag{2.5}$$

Indeed, let $S = (m + 4\pi r^3 p)e^{\mu+\lambda}$. Using (1.5) and (2.3) we get

$$\frac{dS}{dr} = 4\pi r^2(\rho + p + 2p_T)e^{\mu+\lambda},$$

and the claim follows since $S(0) = 0$.

Let us fix $r > 0$. Consider (2.5), using (2.3) we get

$$\begin{aligned}&(m + 4\pi r^3 p)e^{-\int_r^\infty 4\pi\eta(\rho+p)e^{2\lambda}d\eta}\\ &= e^{-\int_r^\infty 4\pi\sigma(\rho+p)e^{2\lambda}d\sigma}\int_0^r 4\pi\eta^2 e^{-\int_\eta^r 4\pi\sigma(\rho+p)e^{2\lambda}d\sigma}(\rho + p + 2p_T)d\eta,\end{aligned}$$

and we have

$$m + 4\pi r^3 p = \int_0^r 4\pi\eta^2 e^{-\int_\eta^r 4\pi\sigma(\rho+p)e^{2\lambda}d\sigma}(\rho + p + 2p_T)d\eta.$$

Since p is non-negative we obtain the inequality

$$m(r) \leq \int_0^r 4\pi\eta^2 e^{-\int_\eta^r 4\pi\sigma(\rho+p)e^{2\lambda}d\sigma}(\rho + p + 2p_T)d\eta.$$

Using again the non-negativity of p and the inequality (1.7) we obtain

$$m(r) \leq (1 + \Omega)\int_0^r 4\pi\eta^2 \rho e^{-\int_\eta^r 4\pi\sigma\rho e^{2\lambda}d\sigma}d\eta. \tag{2.6}$$

Note that only ρ, and not p and q, appears in this inequality in view of (2.1). This is our fundamental inequality.

Let $\mathcal{B}$ be the Borel $\sigma-$algebra of $\mathbb{R}_+$ and let $\mathcal{M}$ denote the space of non-negative $\sigma-$finite measures on $\mathcal{B}$ such that $2m(r)/r < 1$, where $m(r) = \int_{[0,r]} dh(\eta)$. Let $R > 0$ and define the operator $F_R : \mathcal{M} \to R_+$ by

$$F_R(h) = \int_{[0,R]} e^{-\int_{[r,R]} \frac{dh(\eta)}{\eta\left(1-\frac{2m(\eta)}{\eta}\right)}} dh(r). \tag{2.7}$$

With abuse of notation it will be understood that $F_R(u)$, where u is a function, is the value obtained by applying F to the measure ν where $d\nu = udr$. Now let $\bar{\rho} = 4\pi r^2\rho$, and note that the inequality (2.6) can be written

$$m(r) \leq (1 + \Omega)F_r(\bar{\rho}). \tag{2.8}$$

Furthermore, note that by taking $p = 0$ and $2p_T = \rho$ the inequalities above become equalities and we can for this special class of solutions define a form of the generalized TOV equation which is valid whenever $4\pi r^2 \rho = h \in \mathcal{M}$,

$$m(r) = (1 + \Omega) F_r(h). \tag{2.9}$$

This form of the TOV equation will be used to see that the infinitely thin shell which maximizes $2m/r$ satisfies the TOV equation in the sense of measures.

By a steady state we mean a solution of the Einstein equations (1.1)-(1.3) such that ρ, p and p_T are C^1 functions on $[0, \infty)$. A steady state of course satisfies the generalized Tolman-Oppenheimer-Volkov equation. For our purposes it is sufficient that the triplet (ρ, p, p_T) satisfies the integrated form (2.5) of the generalized TOV equation. We say that (ρ, p, p_T) is an *admissible triplet* if: each of these functions is in $L^1_{loc}([0, \infty); 4\pi r^2)$, where $4\pi r^2$ is the weight, equation (2.5) is satisfied a.e., and there is an $\Omega \geq 0$ such that (1.7) holds a.e. The following theorem is our main result.

Theorem 1 *Consider any admissible triplet (ρ, p, p_T). Then*

$$\sup_{r>0} \frac{2m(r)}{r} \leq \frac{(1 + 2\Omega)^2 - 1}{(1 + 2\Omega)^2}. \tag{2.10}$$

The arguments leading to Theorem 2 in [3] (and also the arguments in the proof of Theorem 1 above) imply that the bound (2.10) is sharp in the sense given by the theorem below. Before stating this theorem let us introduce the notation $\mathring{\nu}_R$ for the Dirac measure at $r = R$.

Theorem 2 *Take $R > 0$, and let*

$$M = \frac{R}{2} \frac{((1 + 2\Omega)^2 - 1)}{(1 + 2\Omega)^2}.$$

Let

$$\rho = \frac{M \mathring{\nu}_R}{4\pi R^2},$$

and let $p = 0$ and $2p_T = \Omega\rho$, then (2.9) holds with $h = 4\pi R^2 \rho$ and $r = R$.

3 Main ideas

The details of the proofs make the main ideas become less transparent so let us describe them in this section.

Given a steady state with support in $[0, R]$, $R > 0$, there is a smallest $r_* \in [0, R]$, with the property that

$$\frac{2m(r_*)}{r_*} = \sup_{r>0} \frac{2m(r)}{r}.$$

We will show that if

$$\frac{2m(r_*)}{r_*} > \frac{(1+2\Omega)^2 - 1}{(1+2\Omega)^2}, \tag{3.1}$$

then

$$F_{r_*}(\bar{\rho}) < \frac{m(r_*)}{1+\Omega}. \tag{3.2}$$

In view of (2.8) we thus obtain a contradiction and no steady state with the property (3.1) can exist. To show that (3.1) implies (3.2) is of course the main difficulty.

We will approximate the given steady state with a sum of step functions. The precise way this is done is left to the proof. Let r_* be as above and let

$$u(r) = \chi_{[r_0',r_1]} \frac{c_1 r_1}{r_1 - r_0'} + \chi_{[r_1',r_2]} \frac{c_2 r_2}{r_2 - r_1'} + ... + \chi_{[r_{N-1}',r_N]} \frac{c_N r_N}{r_N - r_{N-1}'}, \tag{3.3}$$

where $\{r_0, r_1, ..., r_N\}$ is a sub-division of the interval $[R_0, R_1]$, so that $r_0 = R_0$ and $r_N = r_*$, and where $r_k \leq r_k' < r_{k+1}$, and χ is the characteristic function. First we take $r_k' = r_k$ and choose the constants c_j, $j = 1, 2, ...$ so that u approximates $\bar{\rho}$ in sup norm. We will then admit the parameters r_k' to vary. Note that

$$\frac{m_u(r_k)}{r_k} = \frac{1}{r_k} \sum_{j=0}^{k} c_j r_j,$$

independently of the choices of r_j', where $m_u(r) := \int_0^r u dr$.

First we consider the first two terms in (3.3) and perform the limit $r_0' \to r_1$ and $r_1' \to r_2$ so that the first two step functions become Dirac measures at $r = r_1$ and $r = r_2$. We then show that the operator F applied to the new measure is greater than $F(u)$. More precisely we show that

$$F_{r_*}(u) < F_{r_*}(\nu_2), \tag{3.4}$$

where

$$\nu_2 = c_1 r_1 \mathring{\nu}_{r_1} + c_2 r_2 \mathring{\nu}_{r_2} + u_2 dr,$$

and

$$u_2(r) = \chi_{[r_2',r_3]} \frac{c_3 r_3}{r_3 - r_2'} + \chi_{[r_3',r_4]} \frac{c_4 r_4}{r_4 - r_3'} + ... + \chi_{[r_{N-1}',r_N]} \frac{c_N r_N}{r_N - r_{N-1}'}. \tag{3.5}$$

Recall that $\mathring{\nu}_{r_j}$ is the Dirac measure at $r = r_j$. Clearly, a Dirac measure $\mathring{\nu}_{r_j}$, means that there is an infinitely thin shell at $r = r_j$ with unit ADM mass and we will call such a configuration a Dirac shell. The proof of (3.4) is a consequence of a crucial monotonicity property of F as $r_0' \to r_1$ and $r_1' \to r_2$.

The next step in our strategy is to show that

$$F_{r_*}(\nu_2) < F_{r_*}(\nu_2'),$$

where ν_2' is the measure obtained by *moving* the Dirac shell at $r = r_1$ to $r = r_2$, i.e.,

$$\nu_2' = (c_1 r_1 + c_2 r_2)\mathring{\nu}_{r_2} + u_2 dr.$$

It will be seen that the structure of F allows one to continue this process so that the next step is to replace the step function on the interval $[r_2', r_3]$ by a Dirac shell with weight $c_3 r_3$ at $r = r_3$ and again show that F applied to this measure increases the value. Then we move the Dirac shell with weight $c_1 r_1 + c_2 r_2$ at $r = r_2$ to $r = r_3$ and thus obtain a Dirac shell at $r = r_3$ with weight $c_1 r_1 + c_2 r_2 + c_3 r_3$. This measure thus takes the form

$$\nu_3' = (c_1 r_1 + c_2 r_2 + c_3 r_3)\mathring{\nu}_{r_3} + u_3 dr,$$

where

$$u_3(r) = \chi_{[r_3', r_4]} \frac{c_4 r_4}{r_4 - r_3'} + \chi_{[r_4', r_5]} \frac{c_5 r_5}{r_5 - r_4'} + \ldots + \chi_{[r_{N-1}', r_N]} \frac{c_N r_N}{r_N - r_{N-1}'}.$$

In this way we obtain the chain of inequalities

$$F_{r_*}(h_u) < F_{r_*}(\nu_2') < F_{r_*}(\nu_3') < \ldots < F_{r_*}(\nu_N'),$$

where

$$\nu_N' = \sum_{j=1}^{N} c_j r_j \; \mathring{\nu}_{r_*} =: m_* \mathring{\nu}_{r_*}. \tag{3.6}$$

Now

$$F_{r_*}(\nu_N') = \frac{2m_* \sqrt{1 - 2m_*/r_*}}{1 + \sqrt{1 - 2m_*/r_*}}, \tag{3.7}$$

which follows by using the method in [3], and also from the proof given in section 5. In view of (2.8) we thus obtain

$$m_* < \frac{2(1+\Omega) m_* \sqrt{1 - 2m_*/r_*}}{1 + \sqrt{1 - 2m_*/r_*}}, \tag{3.8}$$

and solving for $2m_*/r_*$ gives

$$\frac{2m_*}{r_*} < \frac{(1+2\Omega)^2 - 1}{(1+2\Omega)^2}. \tag{3.9}$$

4 An electrostatic analogy

A classical problem in electrostatics is the question how a unit amount of charge should be spread over a bounded set $E \in \mathbb{R}^3$ in order to minimize the Coulomb energy

$$\mathcal{E}(\rho) := \frac{1}{2}\int_E \int_E \rho(x)\rho(y)|x-y|^{2-n}dxdy.$$

Following the exposition in [20] the minimum energy is defined to be $\frac{1}{2}\mathrm{Cap}(E)^{-1}$, where $\mathrm{Cap}(E)$ is the capacity of E, i.e.,

$$\frac{1}{2\mathrm{Cap}(E)} := \inf\left\{\mathcal{E}(\rho) : \int_E \rho = 1\right\}. \tag{4.1}$$

A minimizing ρ does exist if E is a closed set. It is not a function but a measure (an equilibrium measure) concentrated on the surface of E. In particular, if E is a ball or a sphere of radius R then the optimum distribution for the charge will be

$$\rho = \frac{1}{4\pi R^2}\overset{\circ}{\nu}_R. \tag{4.2}$$

and

$$\mathrm{Cap}(B_R) = R. \tag{4.3}$$

Of course, this problem can equivalently be formulated as a variational problem for Newtonian gravity but since we wish to stress the relation to capacity theory which originates from the electrostatic problem we have preferred to use that formulation.

The analogy with our case should be clear in view of (4.2). Let us also note that capacity can equivalently be defined as the largest charge that can be carried by a body (e.g. a ball with radius R) if the voltage drops by at most one, cf. [2]. This formulation suggests that we in our situation define the capacity of a ball with radius R to be the largest ADM mass that a spherically symmetric static body with area radius R can have. Using this definition we then get in view of Theorem 2 that the capacity is given by

$$\frac{((1+2\Omega)^2-1)R}{2(1+2\Omega)^2}.$$

Of course, we could also introduce a similar definition as in (4.1) by using a variational formulation for F instead of $\mathcal{E}$. The following theorem, taken from [20], is an interesting feature of balls in $\mathbf{R}^n$ for the capacity in (4.1).

Theorem 3 *([20]) Let $E \subset \boldsymbol{R}^n$, $n \geq 3$, be a bounded set with Lebesgue measure $|E|$ and let B_E be the ball in $\boldsymbol{R}^n$ with the same measure. Then*

$$Cap(B_E) \leq Cap(E).$$

This theorem suggests that spherical symmetry might be an important case also for the compactness ratio "$2M/R$" (assuming one has a proper definition of such a quantity) of more general static objects.

5 Proofs

Proof of Theorem 1. Consider any admissible triplet, so that in particular $0 \leq 4\pi r^2\rho \in L^1_{loc}$, and let $f := 4\pi r^2\rho$. These are the only conditions of an admissible triplet needed in this section, the remaining conditions have already been invoked to derive the relations in section 2. We will show that (3.1) implies (3.2). Hence, assume that there is a $r_* > 0$ with the property that (3.1) holds. By continuity we can choose r_* so that $2m(r_*)/r_*$ is as close as we wish to the critical value and we choose r_* so that

$$\frac{(1+2\Omega)^2-1}{(1+2\Omega)^2} < \frac{2m(r_*)}{r_*} < \frac{1}{2}\Big(1+\frac{(1+2\Omega)^2-1}{(1+2\Omega)^2}\Big) := Q. \tag{5.1}$$

In what follows we use the notation $m_* := m(r_*)$. Fix $\epsilon > 0$. Let $\tilde{h}$ be such that $\tilde{h} = 0$ on $[0,\delta)$ and $\tilde{h} = f$ on $[\delta, r_*]$, $\delta > 0$. Obviously, for a sufficiently small $\delta > 0$ the difference $0 \leq m_f(r) - m_{\tilde{h}}(r)$ is arbitrary small and since the integration interval $[0, r_*]$ is finite it holds by a continuity argument that there is largest $\delta > 0$ such that $|F_{r_*}(\tilde{h}) - F_{r_*}(f)| < \epsilon/2$.

Now, since the operator F consists of a composition of integrations, there is a natural number N, a sub-division $\{r_0, r_1, ..., r_N\}$, $r_j = \delta + j(r_* - \delta)/N$, of the interval $[\delta, r_*]$, and positive constants $\{c_1, c_2, ..., c_N\}$ such that the function $\bar{h}$ defined by

$$\bar{h}(r) = \chi_{[r_0,r_1]}\frac{c_1 r_1}{r_1 - r_0} + \chi_{[r_1,r_2]}\frac{c_2 r_2}{r_2 - r_1} + ... + \chi_{[r_{N-1},r_N]}\frac{c_N r_N}{r_N - r_{N-1}}, \tag{5.2}$$

satisfies $|F_{r_*}(\bar{h}) - F_{r_*}(\tilde{h})| < \epsilon/2$, and $|m_* - m_{\bar{h}}(r_*)| < \epsilon$. Here χ_S is the characteristic function, i.e., $\chi_S(r) = 1$ if $r \in S$, and $\chi_S(r) = 0$ if $r \notin S$. The condition that the constants c_j are positive is technical and it is not required that f must be positive, only non-negative, but since we only seek an approximation our positivity condition is easy to satisfy. For technical reasons we also require that N is taken large, i.e.,

$$N \geq \frac{10 r_*}{(1-Q)\delta}. \tag{5.3}$$

We now define

$$h(r) = \chi_{[r_0',r_1]}\frac{c_1 r_1}{r_1 - r_0'} + \chi_{[r_1',r_2]}\frac{c_2 r_2}{r_2 - r_1'} + ... + \chi_{[r_{N-1}',r_N]}\frac{c_N r_N}{r_N - r_{N-1}'}. \quad (5.4)$$

Here $r_j \leq r_j' < r_{j+1}$. Note that $h = \bar{h}$ if $r_j' = r_j$ for all $j \in \mathbb{N}$. Moreover note that

$$\int_0^{r_*} \bar{h} dr = \int_0^{r_*} h dr,$$

so that the quasi-local mass at $r = r_*$ given by the energy densities $\bar{\rho} = \bar{h}/(4\pi r^2)$, and $\rho = h/(4\pi r^2)$, are the same. The function h, will be the main object below. As explained in section 3 we will modify h, by varying the parameters r_j' and moving parts of the matter, and finally obtain the inequality

$$F_{r_*}(f) < F_{r_*}(\bar{h}) + \epsilon < F_{r_*}(\nu_N') + \epsilon, \quad (5.5)$$

where ν_N' is given by (3.6). The proof is split into four steps.

Step 1.

In the first step we will by a straightforward computation find an expression for $F_{r_*}(h)$. Since this computation is crucial and quite lengthy we will present the main steps. In what follows j and k will always be non-negative integers.

Let $c_0 = 0$, and let $k \geq 1$. From (5.4) we get

$$m(\sigma) = \sum_{j=0}^{j=k-1} c_j r_j + \frac{c_k r_k(\sigma - r_{k-1}')}{r_k - r_{k-1}'}, \text{ where } r_{k-1} \leq \sigma \leq r_k. \quad (5.6)$$

By defining

$$M_k := \sum_{j=0}^{j=k} c_j r_j, \ k \geq 1,$$

we thus get

$$m(\sigma) = M_{k-1} + \frac{c_k r_k(\sigma - r_{k-1}')}{r_k - r_{k-1}'}, \ r_{k-1} \leq \sigma \leq r_k. \quad (5.7)$$

Next we define

$$G[h](\eta) = \int_\eta^\infty \frac{h(\sigma)\, d\sigma}{\sigma\Big(1 - \frac{2m(\sigma)}{\sigma}\Big)}. \quad (5.8)$$

Note that this is the main expression in the operator F, cf. equation (2.7). From (5.4) it thus follows that for $r'_{j-1} \leq \eta \leq r_j,\ j \geq 1$,

$$G[h](\eta) = \int_\eta^{r_j} \frac{c_j r_j \, d\sigma}{(r_j - r'_{j-1})\sigma\Big(1 - \frac{2m(\sigma)}{\sigma}\Big)} + \int_{r'_j}^{r_{j+1}} \frac{c_{j+1} r_{j+1} \, d\sigma}{(r_{j+1} - r'_j)\sigma\Big(1 - \frac{2m(\sigma)}{\sigma}\Big)}$$
$$+ ... + \int_{r'_{N-1}}^{r_N} \frac{c_N r_N \, d\sigma}{(r_N - r'_{N-1})\sigma\Big(1 - \frac{2m(\sigma)}{\sigma}\Big)} =: \tilde{G}_j(\eta) + G_{j+1} + ... + G_N.$$

Here the twiddle over the first term emphasizes that it depends on η whereas the remaining ones do not. By inserting the expression (5.7) for m we get

$$\tilde{G}_j(\eta) = \int_\eta^{r_j} \frac{c_j r_j \, d\sigma}{(r_j - r'_{j-1})\sigma\Big(1 - \frac{2M_{j-1}}{\sigma} - \frac{2c_j r_j (\sigma - r'_{j-1})}{\sigma(r_j - r'_{j-1})}\Big)}$$
$$= \int_\eta^{r_j} \frac{c_j r_j \, d\sigma}{\Big(2c_j r_j r'_{j-1} - 2M_{j-1}(r_j - r'_{j-1}) - \sigma(2c_j r_j - r_j - r'_{j-1})\Big)}.$$

Note that the denominator in the integrand is positive in view of (5.8). Let $\Delta_j := r_j - r'_{j-1}$, we then get

$$\tilde{G}_j = \frac{-c_j r_j}{2c_j r_j - \Delta_j} \log\Big(\frac{2c_j r_j r'_{j-1} - 2M_{j-1}\Delta_j - r_j(2c_j r_j - \Delta_j)}{2c_j r_j r'_{j-1} - 2M_{j-1}\Delta_j - \eta(2c_j r_j - \Delta_j)}\Big). \tag{5.9}$$

Analogously we get for the η independent terms

$$G_j = \frac{-c_j r_j}{2c_j r_j - \Delta_j} \log\Big(\frac{2c_j r_j r'_{j-1} - 2M_{j-1}\Delta_j - r_j(2c_j r_j - \Delta_j)}{2c_j r_j r'_{j-1} - 2M_{j-1}\Delta_j - r'_{j-1}(2c_j r_j - \Delta_j)}\Big). \tag{5.10}$$

Let us now consider the operator F. From the expression (5.4) we have

$$\begin{aligned} F_{r_*}(h) = & \frac{1}{\Delta_1} \int_{r'_0}^{r_1} c_1 r_1 \exp\Big(- \tilde{G}_1(\eta) - \sum_{j=2}^N G_j\Big) d\eta \\ & + \frac{1}{\Delta_2} \int_{r'_1}^{r_2} c_2 r_2 \exp\Big(- \tilde{G}_2(\eta) - \sum_{j=3}^N G_j\Big) d\eta \\ & + \quad ... \\ & + \frac{1}{\Delta_N} \int_{r'_{N-1}}^{r_N} c_N r_N \exp\Big(- \tilde{G}_N(\eta)\Big) d\eta. \end{aligned} \tag{5.11}$$

Since the only dependence on η in the integrand is in $\tilde{G}_j$ we thus obtain

$$\begin{aligned} F_{r_*}(h) = & \frac{c_1 r_1 \exp\Big(-\sum_{j=2}^N G_j\Big)}{\Delta_1} \int_{r_0'}^{r_1} \exp\Big(-\tilde{G}_1(\eta)\Big) d\eta \\ & + \frac{c_2 r_2 \exp\Big(-\sum_{j=3}^N G_j\Big)}{\Delta_2} \int_{r_1'}^{r_2} \exp\Big(-\tilde{G}_2(\eta)\Big) d\eta \\ & + \quad ... \\ & + \frac{c_N r_N}{\Delta_N} \int_{r'_{N-1}}^{r_N} \exp\Big(-\tilde{G}_N(\eta)\Big) d\eta. \end{aligned} \tag{5.12}$$

The first two terms in this expression can be written as

$$\Big(\frac{c_1 r_1 e^{-G_2}}{\Delta_1} \int_{r_0'}^{r_1} e^{-\tilde{G}_1(\eta)} d\eta + \frac{c_2 r_2}{\Delta_2} \int_{r_1'}^{r_2} e^{-\tilde{G}_2(\eta)} d\eta\Big) e^{-\sum_{j=3}^N G_j}. \tag{5.13}$$

As explained in section 3 the idea is to show that $F_{r_*}(h)$ is dominated by $F_{r_*}(\nu_2)$, where ν_2 is the measure

$$\nu_2(r) = c_1 r_1 \mathring{\nu}_{r_1} + c_2 r_2 \mathring{\nu}_{r_2} + \chi_{[r_2', r_3]} \frac{c_3 r_3}{r_3 - r_2'} + ... + \chi_{[r'_{N-1}, r_N]} \frac{c_N r_N}{r_N - r'_{N-1}}, \tag{5.14}$$

and then to show that $F_{r_*}(\nu_2) < F_{r_*}(\nu_2')$ where

$$\nu_2'(r) = (c_1 r_1 + c_2 r_2) \mathring{\nu}_{r_2} + \chi_{[r_2', r_3]} \frac{c_3 r_3}{r_3 - r_2'} + ... + \chi_{[r'_{N-1}, r_N]} \frac{c_N r_N}{r_N - r'_{N-1}}. \tag{5.15}$$

The measure ν_2' can thus be thought of as a modified h where c_1 and c_2 have been replaced by $c_1' = 0$ and $c_2' = (c_1 r_1 + c_2 r_2)/r_2$ respectively, and where the limit $r_1' \to r_2$ has been carried out. Note that the quasi-local mass generated by ν_2' and h are the same, i.e., $m_{\nu_2'}(r_*) = m_h(r_*)$. In order to show that $F_{r_*}(h) < F_{r_*}(\nu_2')$, the terms in the bracket in (5.13) must be dominated by

$$\Big(\lim_{r_0' \to r_1} \frac{c_1 r_1 e^{-G_2}}{\Delta_1} \int_{r_0'}^{r_1} e^{-\tilde{G}_1(\eta)} d\eta + \lim_{r_1' \to r_2} \frac{c_2 r_2}{\Delta_2} \int_{r_1'}^{r_2} e^{-\tilde{G}_2(\eta)} d\eta\Big), \tag{5.16}$$

which in turn must be dominated by

$$\lim_{r_1' \to r_2} \frac{c_2' r_2}{\Delta_2} \int_{r_1'}^{r_2} e^{-\tilde{G'}_2(\eta)} d\eta. \tag{5.17}$$

Here $\tilde{G'}_2$ denotes the $G-$function which corresponds to the measure ν_2'. The structure of $F(h)$ revealed in (5.12) then shows that this procedure can be continued: we define the measures ν_3 and ν_3' by

$$\nu_3(r) = (c_1r_1+c_2r_2)\mathring{\nu}_{r_2}+c_3r_3\mathring{\nu}_{r_3}+\chi_{[r_3',r_4]}\frac{c_4r_4}{r_4-r_3'}+...+\chi_{[r_{N-1}',r_N]}\frac{c_Nr_N}{r_N-r_{N-1}'},$$

$$\nu_3'(r) = (c_1r_1+c_2r_2+c_3r_3)\mathring{\nu}_{r_3}+\chi_{[r_3',r_4]}\frac{c_4r_4}{r_4-r_3'}+...+\chi_{[r_{N-1}',r_N]}\frac{c_Nr_N}{r_N-r_{N-1}'},$$

and we show that $F_{r_*}(\nu_2') < F_{r_*}(\nu_3) < F_{r_*}(\nu_3')$. In this way we obtain a chain of inequalities

$$F_{r_*}(\bar{h}) < F_{r_*}(\nu_2') < F_{r_*}(\nu_3') < ... < F_{r_*}(\nu_N'),$$

where ν_N' is the Dirac measure at $r = r_N = r_*$ with $m_{\nu_N'}(r_*) = m_{\bar{h}}(r_*)$. Let us now compute the sum of the two terms in the bracket in (5.13). We use the following notation

$$T_1 = \frac{c_1r_1e^{-G_2}}{\Delta_1}\int_{r_0'}^{r_1} e^{-\tilde{G}_1(\eta)}d\eta, \tag{5.18}$$

and

$$T_2 = \frac{c_2r_2}{\Delta_2}\int_{r_1'}^{r_2} e^{-\tilde{G}_2(\eta)}d\eta. \tag{5.19}$$

We have from (5.9)

$$
\begin{aligned}
\int_{r_0'}^{r_1} e^{-\tilde{G}_1(\eta)} d\eta &= \int_{r_0'}^{r_1} \Big(\frac{2c_1r_1r_0' - r_1(2c_1r_1 - \Delta_1)}{2c_1r_1r_0' - \eta(2c_1r_1 - \Delta_1)}\Big)^{\frac{c_1r_1}{2c_1r_1-\Delta_1}} d\eta \\
&= \frac{\Big(2c_1r_1r_0' - r_1(2c_1r_1 - \Delta_1)\Big)^{\frac{c_1r_1}{2c_1r_1-\Delta_1}}}{\Big(1 - \frac{c_1r_1}{2c_1r_1-\Delta_1}\Big)(2c_1r_1 - \Delta_1)} \\
&\quad \times \Big[- \Big(2c_1r_1r_0' - \eta(2c_1r_1 - \Delta_1)\Big)^{1-\frac{c_1r_1}{2c_1r_1-\Delta_1}} \Big]_{r_0'}^{r_1} \\
&= \frac{\Big(2c_1r_1r_0' - r_1(2c_1r_1 - \Delta_1)\Big)^{\frac{c_1r_1}{2c_1r_1-\Delta_1}}}{c_1r_1 - \Delta_1} \\
&\quad \times \Big\{ \Big(2c_1r_1r_0' - r_0'(2c_1r_1 - \Delta_1)\Big)^{1-\frac{c_1r_1}{2c_1r_1-\Delta_1}} \\
&\quad - \Big(2c_1r_1r_0' - r_1(2c_1r_1 - \Delta_1)\Big)^{1-\frac{c_1r_1}{2c_1r_1-\Delta_1}} \Big\} \\
&= \frac{\Big(2c_1r_1r_0' - r_1(2c_1r_1 - \Delta_1)\Big)}{c_1r_1 - \Delta_1} \\
&\quad \times \Big\{ \Big(\frac{2c_1r_1r_0' - r_0'(2c_1r_1 - \Delta_1)}{2c_1r_1r_0' - r_1(2c_1r_1 - \Delta_1)}\Big)^{\frac{c_1r_1-\Delta_1}{2c_1r_1-\Delta_1}} - 1 \Big\}. \qquad (5.20)
\end{aligned}
$$

Furthermore, from (5.10) we have

$$
e^{-G_2} = \Big(\frac{2c_2r_2r_1' - 2M_1\Delta_2 - r_2(2c_2r_2 - \Delta_2)}{2c_2r_2r_1' - 2M_1\Delta_2 - r_1'(2c_2r_2 - \Delta_2)}\Big)^{\frac{c_2r_2}{2c_2r_2-\Delta_2}}.
$$

The term T_1 can thus be written

$$
\begin{aligned}
T_1 = \frac{c_1r_1}{\Delta_1} & \frac{\Big(2c_1r_1r_0' - r_1(2c_1r_1 - \Delta_1)\Big)}{c_1r_1 - \Delta_1} \\
& \times \Big\{ \Big(\frac{2c_1r_1r_0' - r_0'(2c_1r_1 - \Delta_1)}{2c_1r_1r_0' - r_1(2c_1r_1 - \Delta_1)}\Big)^{\frac{c_1r_1-\Delta_1}{2c_1r_1-\Delta_1}} - 1 \Big\} \\
& \times \Big(\frac{2c_2r_2r_1' - 2M_1\Delta_2 - r_2(2c_2r_2 - \Delta_2)}{2c_2r_2r_1' - 2M_1\Delta_2 - r_1'(2c_2r_2 - \Delta_2)}\Big)^{\frac{c_2r_2}{2c_2r_2-\Delta_2}}. \qquad (5.21)
\end{aligned}
$$

A very similar calculation shows that

$$T_2 = \frac{c_2 r_2}{\Delta_2} \frac{\Big(2c_2 r_2 r_1' - 2M_1\Delta_2 - r_2(2c_2 r_2 - \Delta_2)\Big)}{c_2 r_2 - \Delta_2} \times \Big\{ \Big(\frac{2c_2 r_2 r_1' - 2M_1\Delta_2 - r_1'(2c_2 r_2 - \Delta_2)}{2c_2 r_2 r_1' - 2M_1\Delta_2 - r_2(2c_2 r_2 - \Delta_2)} \Big)^{\frac{c_2 r_2 - \Delta_2}{2c_2 r_2 - \Delta_2}} - 1 \Big\}. \quad (5.22)$$

The aim is to obtain the inequality $F_{r_*}(\bar{h}) \leq F_{r_*}(\nu_2')$. Since h and ν_2' are identical for $r \geq r_3$ it follows from (5.12) that it is sufficient to obtain the estimate $T_1 + T_2 \leq T_1^{\nu_2'} + T_2^{\nu_2'}$, where $T_1^{\nu_2'}$ and $T_2^{\nu_2'}$ are the corresponding terms for ν_2'. Clearly $T_1^{\nu_2'} = 0$ since $c_1' = 0$, and $T_2^{\nu_2'}$ is the expression (5.17) which in view of (5.22) and the fact that $M_1 = 0$ in this case since $c_1' = 0$ is given by

$$T_2^{\nu_2'} = \lim_{r_1' \to r_2} T_2', \quad (5.23)$$

where

$$T_2' = \frac{c_2' r_2}{\Delta_2} \frac{\Big(2c_2' r_2 r_1' - r_2(2c_2' r_2 - \Delta_2)\Big)}{c_2' r_2 - \Delta_2} \times \Big\{ \Big(\frac{2c_2' r_2 r_1' - r_1'(2c_2' r_2 - \Delta_2)}{2c_2' r_2 r_1' - r_2(2c_2' r_2 - \Delta_2)} \Big)^{\frac{c_2' r_2 - \Delta_2}{2c_2' r_2 - \Delta_2}} - 1 \Big\}. \quad (5.24)$$

Here $c_2' = (c_1 r_1 + c_2 r_2)/r_2$. The expressions for T_1, T_2 and T_2' will now be simplified. Let us introduce the notation

$$b_k = \frac{r_{k-1}'}{r_k}, \; k = 1, 2, ...$$

which implies that

$$\Delta_k = r_k(1 - b_k).$$

Let us consider the term T_2. By dividing both the numerator and the denominator by $2c_2 r_2^2$, the first factor in the expression (5.22) can be written

$$\begin{aligned} &\frac{c_2 r_2}{\Delta_2} \frac{\Big(2c_2 r_2 r_1' - 2M_1\Delta_2 - r_2(2c_2 r_2 - \Delta_2)\Big)}{c_2 r_2 - \Delta_2} \\ &= \frac{c_2 r_2}{1 - b_2} \frac{\Big(b_2 - \frac{M_1(1-b_2)}{c_2 r_2} - (1 - \frac{1-b_2}{2c_2})\Big)}{\frac{c_2 r_2}{2c_2 r_2} - \frac{1-b_2}{2c_2}} \\ &= \frac{c_2 r_2}{1 - b_2} \frac{\Big((1-b_2)(\frac{1}{2c_2} - \frac{c_1 r_1}{c_2 r_2} - 1)\Big)}{\frac{1}{2} - \frac{1-b_2}{2c_2}} = \frac{c_2 r_2 (1 - 2c_1 \frac{r_1}{r_2} - 2c_2)}{c_2 - (1 - b_2)}. \end{aligned} \quad (5.25)$$

The second factor can be simplified in a similar way

$$\begin{aligned}
&\Big(\frac{2c_2r_2r_1' - 2M_1\Delta_2 - r_1'(2c_2r_2 - \Delta_2)}{2c_2r_2r_1' - 2M_1\Delta_2 - r_2(2c_2r_2 - \Delta_2)}\Big)^{\frac{c_2-(1-b_2)}{2c_2-(1-b_2)}} - 1 \\
&= \Big(\frac{b_2 - \frac{c_1r_1}{c_2r_2}(1-b_2) - b_2(1 - \frac{1-b_2}{2c_2})}{b_2 - \frac{c_1r_1}{c_2r_2}(1-b_2) - (1 - \frac{1-b_2}{2c_2})}\Big)^{\frac{c_2-(1-b_2)}{2c_2-(1-b_2)}} - 1 \\
&= \Big(\frac{(\frac{b_2}{2c_2} - \frac{c_1r_1}{c_2r_2})(1-b_2)}{(1-b_2)(\frac{1}{2c_2} - \frac{c_1r_1}{c_2r_2})}\Big)^{\frac{c_2-(1-b_2)}{2c_2-(1-b_2)}} - 1 \\
&= \Big(\frac{b_2 - 2c_1\frac{r_1}{r_2}}{1 - 2c_1\frac{r_1}{r_2} - 2c_2}\Big)^{\frac{c_2-(1-b_2)}{2c_2-(1-b_2)}} - 1.
\end{aligned} \tag{5.26}$$

In conclusion T_2 can be written

$$T_2 = \frac{c_2r_2(1 - 2c_1\frac{r_1}{r_2} - 2c_2)}{c_2 - z_2}\Big\{\Big(\frac{1 - 2c_1\frac{r_1}{r_2} - z_2}{1 - 2c_1\frac{r_1}{r_2} - 2c_2}\Big)^{\frac{c_2-z_2}{2c_2-z_2}} - 1\Big\}, \tag{5.27}$$

where we have introduced the notation

$$z_k = 1 - b_k,\ k = 1, 2, ...$$

Simplifying T_1 and T_2' in a similar way leads to the following expressions

$$T_1 = \frac{c_1r_1(1-2c_1)}{c_1 - z_1}\Big\{\Big(\frac{1-z_1}{1-2c_1}\Big)^{\frac{c_1-z_1}{2c_1-z_1}} - 1\Big\}\Big(\frac{1 - 2c_1\frac{r_1}{r_2} - 2c_2}{1 - 2c_1\frac{r_1}{r_2} - z_2}\Big)^{\frac{c_2}{2c_2-z_2}}, \tag{5.28}$$

and

$$T_2' = \frac{(c_1r_1 + c_2r_2)(1 - 2c_1\frac{r_1}{r_2} - 2c_2)}{c_2 + c_1\frac{r_1}{r_2} - z_2}\Big\{\Big(\frac{1 - z_2}{1 - 2c_1\frac{r_1}{r_2} - 2c_2}\Big)^{\frac{c_2 + c_1\frac{r_1}{r_2} - z_2}{2c_2 + 2c_1\frac{r_1}{r_2} - z_2}} - 1\Big\}. \tag{5.29}$$

Note that the expression for T_2' is obtained from (5.27) by putting $c_1 = 0$ and replacing c_2 by $c_2 + c_1r_1/r_2$ in accordance with the previous discussion.

Step 2.
In this step we show that $F_{r_*}(\bar{h}) < F_{r_*}(\nu_2)$ by showing that F is monotone as $r_0' \to r_1$ and $r_1' \to r_2$, i.e., as $z_1 \to 0$ and $z_2 \to 0$. Let us define

$$A(z, c) = \frac{1}{c - z}\Big\{\Big(\frac{1-z}{1-2c}\Big)^{\frac{c-z}{2c-z}} - 1\Big\}, \tag{5.30}$$

and

$$B(z,c) = \Big(\frac{1-2c}{1-z}\Big)^{\frac{c}{2c-z}}, \tag{5.31}$$

where $z \in [0, 1/10]$, and $c \in (0, Q/2)$. Recall the definition of Q in (5.1). These are the fundamental functions in the expressions for T_1 and T_2, namely

$$T_1 = c_1 r_1 (1-2c_1) A(z_1, c_1) B\Big(\frac{z_2}{1-2c_1 r_1/r_2}, \frac{c_2}{1-2c_1 r_1/r_2}\Big), \tag{5.32}$$

and

$$T_2 = \frac{c_2 r_2 (1-2c_1\frac{r_1}{r_2} - 2c_2)}{1-2c_1 r_1/r_2} A\Big(\frac{z_2}{1-2c_1 r_1/r_2}, \frac{c_2}{1-2c_1 r_1/r_2}\Big). \tag{5.33}$$

Let us now see that the domain of definition of the functions A and B is relevant. Since r_* is the smallest r with $2m_*/r_* = Q$ we have in view of (5.1) $c_1 r_1 + c_2 r_2 < r_2 Q/2$, and since $Q < 1$, $Qc_1 r_1 + c_2 r_2 < r_2 Q/2$, which implies that

$$c_2 r_2 < \frac{Q}{2} r_2 (1 - 2c_1 r_1/r_2),$$

and thus

$$c_2/(1-2c_1 r_1/r_2) < Q/2.$$

Since $c_1 < Q/2$ it follows that the second argument in A and B in (5.32) and (5.33) is less than $Q/2$, i.e., $c \in (0, Q/2)$. To see that the first argument in the functions A and B belong to $[0, 1/10]$ we first check that the condition (5.3) implies that

$$\frac{\delta + (r_* - \delta)/N}{\delta + 2(r_* - \delta)/N} \geq \frac{9+Q}{10}.$$

The inequality above can be written

$$\frac{(8+2Q)(r_* - \delta)}{10N} \leq \frac{(1-Q)\delta}{10},$$

which clearly is satisfied if

$$N \geq \frac{10 r_*}{(1-Q)\delta}.$$

In view of (5.3) we thus have for $j \geq 1$,

$$\frac{r_j'}{r_{j+1}} \geq \frac{r_j}{r_{j+1}} \geq \frac{r_1}{r_2} = \frac{\delta + (r_* - \delta)/N}{\delta + 2(r_* - \delta)/N} \geq \frac{9+Q}{10}, \tag{5.34}$$

so that $z_k = 1 - r'_{k-1}/r_k \le (1-Q)/10$ for all k. It follows that

$$\frac{z_2}{1 - 2c_1 r_1/r_2} < \frac{z_2}{1-Q} \le \frac{1}{10},$$

which proves our claim, i.e., $z \in [0, 1/10]$.

It is clear that these facts hold in general, i.e., not only for the terms T_1 and T_2 but at any step in our chain of inequalities since $\sum_{j=1}^{k} c_j r_j < r_k Q/2$. We can of course also express the term T'_2 in a similar way but it is not useful here. By construction the functions A and B are continuous in the domain of definition, in particular they are continuous along the lines $z = c$ and $z = 2c$.

Lemma 1 *For any $c \in (0, Q/2)$ the functions $A(\cdot, c)$ and $B(\cdot, c)$ are decreasing in z, $z \in [0, 1/10]$.*

Proof of Lemma 1. Monotonicity of A. Let us introduce the new variables

$$\beta = \frac{2c - z}{c} \text{ and } k = \frac{c}{1-2c}. \tag{5.35}$$

We thus have that

$$0 < k \le \frac{Q}{2(1-Q)}, \text{ and } \beta \le 2.$$

We now express A in terms of these variables and by abuse of notation we denote this function again by A. Since

$$\frac{1-z}{1-2c} = 1 + \frac{2c-z}{1-2c} = 1 + k\beta,$$

it follows that

$$A(\beta, k) = \frac{1+2k}{k}\Big\{(1+k\beta)^{\frac{\beta-1}{\beta}} - 1\Big\}. \tag{5.36}$$

We now want to show that $\partial_\beta A \ge 0$ since $\partial_z \beta$ is negative. A straightforward computation gives after some rearrangements

$$\begin{aligned} \partial_\beta A = \frac{(1+k\beta)^{\frac{\beta-1}{\beta}}}{(1-\beta)^2\beta^2(1+k\beta)}\Big[&-\beta^2(1+k\beta) + \beta^2(1+k\beta)^{\frac{1}{\beta}} \\ &+(\beta-1)(1+k\beta)\log(1+k\beta) + (1-\beta)^2 k\beta\Big]. \end{aligned} \tag{5.37}$$

Let us denote the factor in square brackets by Ψ. Adding the first and the last term in this expression gives

$$\Psi(\beta, k) = \beta^2(1+k\beta)^{\frac{1}{\beta}} + (\beta-1)(1+k\beta)\log(1+k\beta) - \beta^2 - k\beta(2\beta-1). \tag{5.38}$$

Let

$$\gamma = \frac{\log(1 + k\beta)}{\beta}, \tag{5.39}$$

which is well defined also when $\beta = 0$ since $\lim_{\beta\to 0}\gamma = k$. Since

$$k\beta = \frac{2c - z}{1 - 2c},$$

it follows that $k\beta < 1/(1-Q)$, and since $k\beta$ is positive as long as $2c \geq 1/10$ a rough estimate gives

$$k\beta \geq -1/10, \tag{5.40}$$

by the condition that $z \leq 1/10$. We will below distinguish between the two cases $0 \leq \beta \leq 2$, and $\beta < 0$. In both cases $\gamma > 0$, or more precisely, in the former case we have $\gamma \in [\log(1 + 2k)/2, k]$, and in the latter case $\gamma \in (0, k]$. By using the relation

$$k\beta = e^{\gamma\beta} - 1,$$

Ψ takes the form

$$\Psi(\beta, \gamma) = -1 + 2\beta + \beta^2(e^{\gamma} - 1) + e^{\gamma\beta}[(\beta - 1)\beta\gamma - 2\beta + 1]. \tag{5.41}$$

By expanding the exponential functions using the formula $e^x = 1 + x/1! + x^2/2! + ...$ and collecting the terms corresponding to different powers in γ gives

$$\Psi(\beta, \gamma) = \beta^2 \sum_{j=3}^{\infty} \Big[\frac{1}{j!} - \beta^{j-2}\Big(\frac{1}{(j-1)!} - \frac{1}{j!}\Big) + \beta^{j-1}\Big(\frac{1}{(j-1)!} - \frac{2}{j!}\Big)\Big]\gamma^j. \tag{5.42}$$

Note that the lower orders of γ vanish. We denote the factors in square brackets by Φ_j, and these can thus be written as

$$\Phi_j(\beta) = \frac{1}{j!}\Big(1 - (j-1)\beta^{j-2} + (j-2)\beta^{j-1}\Big). \tag{5.43}$$

We now claim that

$$\Phi_j(\beta) = \frac{(1-\beta)^2}{j!}(1 + 2\beta + 3\beta^2 + ... + (j-2)\beta^{j-3}),\ j \geq 3. \tag{5.44}$$

This statement is easily shown by an induction argument. First, if $j = 3$ we have from (5.43) that

$$\Phi_3 = \frac{1}{3!}(1 - 2\beta + \beta^2) = \frac{1}{3!}(1-\beta)^2,$$

so the claim is true for $j = 3$. Assume now that for any positive integer $P \geq 2$,

$$(1 - \beta^P(P+1) + \beta^{P+1}P) = (1-\beta^2)(1 + 2\beta + 3\beta^2 + ... + P\beta^{P-1}). \quad (5.45)$$

We then have by (5.45)

$$\begin{aligned}
&(1 - \beta^{P+1}(P+2) + \beta^{P+2}(P+1)) \\
&= (1-\beta^2)(1 + 2\beta + 3\beta^2 + ... + P\beta^{P-1}) \\
&\quad + \beta^P(P+1) - 2(P+1)\beta^{P+1} + \beta^{P+2}(P+1) \\
&= (1-\beta^2)(1 + 2\beta + 3\beta^2 + ... + P\beta^{P-1}) + \beta^P(P+1)(1-\beta)^2 \\
&= (1-\beta^2)(1 + 2\beta + 3\beta^2 + ... + (P+1)\beta^P),
\end{aligned}$$

and the claim (5.44) follows. In conclusion we have shown

$$\partial_\beta A = (1 + k\beta)^{\frac{-1}{\beta}} \sum_{j=3}^{\infty} \frac{1}{j!}[1 + 2\beta + 3\beta^2 + ... + (j-2)\beta^{j-3}]\gamma^j. \quad (5.46)$$

Note that the lower orders of γ have vanished. Now, $1 + k\beta > 0$, since $k\beta \geq -1/10$, and $\gamma > 0$, so in the case $\beta \geq 0$, it follows immediately that $\partial_\beta A \geq 0$. Let us therefore consider the remaining case $\beta < 0$. First we note that $\beta < 0$ implies that $z > 2c$. Now, since $z \leq 1/10$ this means that β is only negative if c is small, i.e., $c < 1/20$. Therefore, since $\gamma \leq k$ we get

$$\gamma \leq k = \frac{c}{1-2c} < 1/18. \quad (5.47)$$

From the inequality (cf. [1])

$$|\log(1-x)| < \frac{3x}{2}, \quad 0 < x \leq 1/2,$$

we have

$$|\gamma\beta| = |\log(1 - k|\beta|)| < 3k|\beta|/2 \leq 3/20, \quad (5.48)$$

where the last inequality followed from (5.40). Let us now estimate the sum in (5.46). For this we use that

$$\frac{1}{4!} + \frac{\gamma}{5!} + \frac{\gamma^2}{6!} + ... < \frac{1}{4!}(1 + \gamma + \gamma^2 + ...) = \frac{1}{4!}\frac{1}{(1-\gamma)} < \frac{1}{20},$$

by (5.47), together with the formula

$$1 + 2x + 3x^2 + ... = \frac{1}{(1-x)^2}, \quad -1 < x < 1.$$

We drop the non-negative terms except for the first one and obtain

$$\begin{aligned}
&\frac{1}{3!}+\frac{1}{4!}(1+2\beta)\gamma+\frac{1}{5!}(1+2\beta+3\beta^2)\gamma^2+\frac{1}{6!}(1+2\beta+3\beta^2+4\beta^3)\gamma^3+...\\
&\geq\frac{1}{3!}-\Big[2\gamma|\beta|\Big(\frac{1}{4!}+\frac{\gamma}{5!}+\frac{\gamma^2}{6!}\Big)+4(\gamma|\beta|)^3\Big(\frac{1}{6!}+\frac{\gamma}{7!}+\frac{\gamma^2}{8!}+...\Big)+...\Big]\\
&\geq\frac{1}{3!}-\frac{1}{20}\Big[2\gamma|\beta|+4(\gamma|\beta|)^3+6(\gamma|\beta|)^5...\Big]\\
&\geq\frac{1}{3!}-\frac{1}{20}\Big[1+2\gamma|\beta|+3(\gamma|\beta|)^2+4(\gamma|\beta|)^3+...\Big]\\
&=\frac{1}{3!}-\frac{1}{20(1-\gamma|\beta|)^2}\geq\frac{1}{3!}-\frac{20^2}{20\cdot17^2}>0. \qquad (5.49)
\end{aligned}$$

In the second last inequality we used (5.48) and in the last (5.40). Thus $\partial_\beta A>0$ also in the case when $\beta<0$, and the monotonicity of $A(\cdot,c)$ follows. Let us now turn to the monotonicity of $B(\cdot,c)$.

Monotonicity of B. We express B in the variables k and β and, by abuse of notation, get

$$B(\beta,k)=(1+k\beta)^{\frac{-1}{\beta}}.$$

As in the case of the function A the claimed monotonicity follows if we can show

$$\partial_\beta B\geq 0.$$

We have

$$\partial_\beta B=\frac{(1+k\beta)^{\frac{-1}{\beta}}}{\beta^2(1+k\beta)}[(1+k\beta)\log(1+k\beta)-k\beta].$$

Using the variable γ defined in (5.39) we can write the factor in square brackets as

$$[(1+k\beta)\log(1+k\beta)-k\beta]=\gamma\beta e^{\gamma\beta}-(e^{\gamma\beta}-1).$$

By letting $a=\gamma\beta$ we have a function of one variable and it is elementary to show the non-negativity of this expression for any a. This completes the proof of the lemma.

□

Step 3.
In this step we show that $F(\nu_2)<F(\nu_2')$. Hence, we want to show that

$$0\leq\lim_{z_2\to0}T_2'-\lim_{z_2\to0}\Big(\lim_{z_1\to0}T_1\Big)-\lim_{z_2\to0}T_2=:\bar{T}_2'-\bar{T}_1-\bar{T}_2.$$

We have from (5.27)-(5.29)

$$\bar{T}_1 = r_1\Big(\sqrt{\frac{1-2c_1}{1-2c_1r_1/r_2}} - \frac{1-2c_1}{\sqrt{1-2c_1r_1/r_2}}\Big)\sqrt{1-2c_2'},$$

$$\bar{T}_2 = r_2\Big(\sqrt{1-2c_1r_1/r_2} - \sqrt{1-2c_2'}\Big)\sqrt{1-2c_2'},$$

and

$$\bar{T}_2' = r_2\Big(1-\sqrt{1-2c_2'}\Big)\sqrt{1-2c_2'}.$$

Hence

$$\begin{aligned}\bar{T}_2' - \bar{T}_1 - \bar{T}_2 = \sqrt{1-2c_2'}\Big[&r_2(1-\sqrt{1-2c_1r_1/r_2}) \\ &-r_1\sqrt{\frac{1-2c_1}{1-2c_1r_1/r_2}}(1-\sqrt{1-2c_1})\Big].\end{aligned} \tag{5.50}$$

Define

$$\kappa := 1 - \frac{r_1}{r_2}, \text{ so that } \kappa \in (0,1),$$

then

$$\sqrt{1-2c_1r_1/r_2} = \sqrt{1-2c_1}\sqrt{1+\frac{2c_1\kappa}{1-2c_1}}.$$

The factor in square brackets above can be written

$$\begin{aligned}\Gamma :&= r_2\Big(1-\sqrt{1-2c_1\frac{r_1}{r_2}}\Big) - r_1\sqrt{\frac{1-2c_1}{1-2c_1\frac{r_1}{r_2}}}(1-\sqrt{1-2c_1}) \\ &= r_2\Big(1-\sqrt{1-2c_1}\sqrt{1+\frac{2c_1\kappa}{1-2c_1}}\Big) \\ &\quad -r_1\frac{1}{\sqrt{1+\frac{2c_1\kappa}{1-2c_1}}}(1-\sqrt{1-2c_1}) \\ &= \frac{r_2}{\sqrt{1+\frac{2c_1\kappa}{1-2c_1}}}\Big[\sqrt{1+\frac{2c_1\kappa}{1-2c_1}} - \sqrt{1-2c_1}\Big(1+\frac{2c_1\kappa}{1-2c_1}\Big) \\ &\quad -(1-\kappa)(1-\sqrt{1-2c_1})\Big] \\ &= \frac{r_2}{\sqrt{1+\frac{2c_1\kappa}{1-2c_1}}}\Big[\sqrt{1+\frac{2c_1\kappa}{1-2c_1}} - \frac{\kappa}{\sqrt{1-2c_1}} + \kappa - 1\Big] \\ &= \frac{r_2}{\sqrt{1-2c_1(1-\kappa)}}\Big[\sqrt{1-2c_1(1-\kappa)} - \kappa - \sqrt{1-2c_1}(1-\kappa))\Big].\end{aligned}$$

Let us introduce the notation

$$\Gamma(c_1,\kappa) := \sqrt{1-2c_1(1-\kappa)} - \kappa - \sqrt{1-2c_1}(1-\kappa)), \tag{5.51}$$

so that

$$\bar{T}_2' - \bar{T}_1 - \bar{T}_2 = r_2\sqrt{\frac{1-2c_2'}{1-2c_1(1-\kappa)}}\Gamma(c_1,\kappa).$$

We want to show that the right hand side is non-negative for any admitted choice of the parameters c_1, c_2 and κ. Since $\Gamma(0,\kappa)=0$ the statement follows since $\partial_{c_1}\Gamma > 0$. Indeed, we have

$$\frac{\partial\Gamma}{\partial c_1} = (1-\kappa)\Big[\frac{1}{\sqrt{1-2c_1}} - \frac{1}{\sqrt{1-2c_1(1-\kappa)}}\Big],$$

which is positive since $\kappa \in (0,1)$. Hence $\bar{T}_2' - \bar{T}_1 - \bar{T}_2 > 0$.

Step 4.
At this stage it is clear that by repeating the arguments we obtain

$$F_{r_*}(\bar{h}) < F_{r_*}(\nu_2') < F_{r_*}(\nu_3') < \ldots < F_{r_*}(\nu_N'),$$

where ν_N' is the Dirac measure at $r = r_N = r_*$ with $m_{\nu_N'}(r_*) = m_h(r_*)$. An appropriate method for computing $F_{r_*}(\nu_N')$ is given in [3]. However, we can also use the formula (5.29) with $c_1 = 0$, $c_2 = m_h(r_*)/r_*$ and $z_2 = 0$, and we get with $m_*' := m_h(r_*)$

$$F_{r_*}(\nu_N') = r_*(1-\frac{2m_*'}{r_*})\Big\{\frac{1}{\sqrt{1-\frac{2m_*'}{r_*}}} - 1\Big\} = \frac{2m_*'\sqrt{1-\frac{2m_*'}{r_*}}}{1+\sqrt{1-\frac{2m_*'}{r_*}}}. \tag{5.52}$$

The inequalities (2.8) and (5.5) then gives

$$m_* < \frac{2(1+\Omega)m_*'\sqrt{1-2m_*'/r_*}}{1+\sqrt{1-2m_*'/r_*}} + \epsilon, \tag{5.53}$$

Using that $|m_* - m_*'| < \epsilon$ we get

$$m_* < \frac{2(1+\Omega)m_*\sqrt{1-2m_*/r_*}}{1+\sqrt{1-2m_*/r_*}} + o(\epsilon), \tag{5.54}$$

and solving for $2m_*/r_*$ gives

$$\frac{2m_*}{r_*} < \frac{(1+2\Omega)^2 - 1}{(1+2\Omega)^2} + o(\epsilon). \tag{5.55}$$

Since $\epsilon > 0$ is arbitrary this contradicts our assumption on $2m_*/r_*$, which completes the proof of Theorem 1.

□

Proof of Theorem 2. The proof is a direct consequence of the discussion leading to (2.9) and the formula (5.52), cf. also [3]. Indeed, let

$$\frac{2M}{R} = \frac{(1+2\Omega)^2 - 1}{(1+2\Omega)^2}.$$

The formula (5.52) with $r_* = R$ and $m_* = M$ gives

$$F_R(\nu_N') = \frac{2M}{(1+2\Omega)\Big(1+\frac{1}{1+2\Omega}\Big)} = \frac{M}{(1+\Omega)}, \tag{5.56}$$

and the proof of Theorem 2 is complete.

□

Acknowledgement

I want to thank Alan Rendall for some comments on the manuscript.

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions,* Dover Publications, New York 1972.

[2] H. Aikawa and M. essén, *Potential theory – selected topics,* Lecture Notes in Mathematics 1633. Springer-Verlag Berlin Heidelberg 1996.

[3] H. Andréasson, On the Buchdahl inequality for spherically symmetric static shells. To appear in *Commun. Math. Phys.*

[4] H. Andréasson, On static shells and the Buchdahl inequality for the spherically symmetric Einstein-Vlasov system. To appear in *Commun. Math. Phys.*

[5] H. Andréasson, The Einstein-Vlasov System/Kinetic Theory, *Living Rev. Relativ.* **8** (2005).

[6] H. Andréasson, G. Rein, A numerical investigation of the stability of steady states and critical phenomena for the spherically symmetric Einstein-Vlasov system. *Class. Quantum Grav.* **23**, 3659-3677 (2006).

[7] H. Andréasson, G. Rein, On the steady states of the spherically symmetric Einstein-Vlasov system. To appear in *Class. Quantum Grav.*

[8] T.W. Baumgarte, A.D. Rendall, Regularity of spherically symmetric static solutions of the Einstein equations. *Class. Quantum Grav.* **10**, 327–332 (1993).

[9] J. Binney and S. Tremaine *Galactic Dynamics,* Princeton University Press 1987.

[10] C.G. Böhmer and T. Harko, Bounds on the basic physical parameters for anisotropic compact general relativistic objects. Preprint gr-qc/0609061.

[11] H. Bondi, Massive spheres in general relativity. *Proc. R. Soc. A* **282**, 303–317 (1964).

[12] H. Bondi, Anisotropic spheres in general relativity. *Mon. Not. Roy. Astr. Soc.* **259**, 365 (1992).

[13] H.A. Buchdahl, General relativistic fluid spheres. *Phys. Rev.* **116**, 1027–1034 (1959).

[14] D. Christodoulou, The formation of black holes and singularities in spherically symmetric gravitational collapse. *Commun. Pure Appl. Math.* **44**, 339–373 (1991).

[15] D. Christodoulou, The instability of naked singularities in the gravitational collapse of a scalar field, *Ann. Math.* **149**, 183–217, (1999).

[16] J. Fraundiener, C. Hoenselaers and W. Konrad, A shell around a black hole, *Class. Quantum Gravity* **7**, 585–587 (1990).

[17] J. Guven, N. Ó Murchadha, Bounds on $2m/R$ for static spherical objects. *Phys. Rev. D* **60**, 084020 (1999).

[18] L. Herrera and N.O. Santos, Local anisotropy in self-gravitating systems. *Phys. Rep.* **286**, 53 (1997).

[19] G. Lemaitre, L'univers en expansion. *Ann. Soc. Sci. (Bruxelles) A* **53**, 51 (1993).

[20] E. Lieb and M. Loss, *Analysis,* Graduate Studies in Mathematics, Vol 14. American Mathematical Society 1997.

[21] M. Mars, M. Mercè Martín-Prats and J.M.M. Senovilla, The $2m \leq r$ property of spherically symmetric static spacetimes, *Phys. Lett. A* **218**, 147 (1996).

[22] G. Rein, Static shells for the Vlasov-Poisson and Vlasov-Einstein systems, *Indiana University Math. J.* **48**, 335–346 (1999).

[23] G. Rein, A. D. Rendall, Compact support of spherically symmetric equilibria in non-relativistic and relativistic galactic dynamics, *Math. Proc. Camb. Phil. Soc.* **128**, 363–380 (2000)

[24] A. D. Rendall, An introduction to the Einstein-Vlasov system, *Banach Center Publ.* **41**, 35–68 (1997).

[25] K. Schwarzschild, Über das Gravitationsfeld einer Kugel aus inkompressibler Flussigkeit nach der Einsteinschen Theorie, *Sitz. Preuss. Akad. Wiss. Berlin* Kl. Math-Phys. Tech. 424-434 (1916).

[26] R.M. Wald, *General Relativity,* The University of Chicago Press 1984.

[27] S. Weinberg, *Gravitation and cosmology,* John Wiley and Sons, Inc. 1972.