Thesis for the Degree of Licentiate of Philosophy

On the Density of Solutions to Diophantine Equations

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ABSTRACT

This thesis consists of two papers giving new upper bounds for the density of integral points on affine algebraic varieties, using a q-analogue of van der Corput's method of exponential sums first developed by Heath-Brown. In the first paper we consider complete intersections of r hypersurfaces of degree at least 3 in $\mathbb{A}^n$. In the second paper we iterate the van der Corput method twice to get upper bounds for the number of integral zeros of bounded height to a single polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$ of degree at least 4.

Keywords. Integral points, exponential sums, Weyl differencing, van der Corput's method.

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Introduction

In the study of Diophantine equations, one of the fundamental problems is to count the number of solutions $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ to a given equation

$$f(x_1, \ldots, x_n) = 0, \tag{1}$$

where $f \in \mathbb{Z}[x_1, \ldots, x_n]$. Sometimes it is possible to prove that there are only finitely many solutions. But if this is not the case, it is sensible to study the *density* of solutions, that is, for a given positive real number B we want to estimate the number of solutions satisfying $|\mathbf{x}| \leq B$, where $|\mathbf{x}| = \max_i |x_i|$. To this end we introduce the *counting function*

$$N(f, B) = \#\{\mathbf{x} \in \mathbb{Z}^n; f(\mathbf{x}) = 0, |\mathbf{x}| \leq B\}.$$

If the polynomial f has degree $d \leq n$, a heuristic upper bound for $N(f, B)$ is B^{n-d}. Indeed, the stronger statement that

$$N(f, B) \sim B^{n-d}$$

is predicted for a large class of polynomials by a conjecture of Manin [2] (see Remark 2 in Paper I).

By the *leading form* of the polynomial f we shall mean the homogeneous part of maximal degree. The best estimates for $N(f, B)$ can be obtained when the leading form F of f defines a non-singular hypersurface in $\mathbb{P}^{n-1}_{\mathbb{C}}$. We call F *non-singular* in this case.

It is often fruitful to study congruences

$$f(x_1, \ldots, x_n) \equiv 0 \pmod{m}$$

for different m, rather than the equation (1) itself. Thus, we introduce the counting functions

$$N(f, B, m) = \#\{\mathbf{x} \in \mathbb{Z}^n; f(\mathbf{x}) \equiv 0 \,(\mathrm{mod}\, m), |\mathbf{x}| \leq B\}.$$

Trivially, for any m, $N(f, B, m)$ is an upper bound for $N(f, B)$. In the study of congruences, exponential sums are fundamental. For example, the number of solutions $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{F}_p^n$ (p prime) to a congruence $f(u_1, \ldots, u_n) \equiv 0 \pmod{p}$ (where, by abuse of notation, the u_i:s denote residue classes as well as representatives thereof) is counted by the exponential sum

$$p^{-1} \sum_{a=1}^{p} \sum_{\mathbf{u} \in \mathbb{F}_p^n} e_p(af(\mathbf{u})),$$

where $e_p(x) = \exp(2\pi i x/p)$. In order to count only points of bounded height, and thus estimate $N(f, B, p)$, one can introduce a weight function into the sum. Using Deligne's bounds [1] for exponential sums over $\mathbb{F}_p$, Fujiwara [3] proved, for a non-singular form F of degree at least 2 in $n \geq 4$ variables, that

$$N(F, B) \ll B^{n-2+2/n}.$$

Heath-Brown [5] went further, proving that for a polynomial f of degree at least 3 such that the leading form F is non-singular, we have the estimate

$$N(f, B) \ll_F B^{n-3+15/(n+5)} \tag{2}$$

for $n \geq 5$. To prove this he started from $N(f, B, pq)$ for two primes $p < B < q$ and introduced a technique which might be regarded as a q-analogue of van der Corput's method with A- and B-processes. For an account of this method in its original context, the study of the Riemann zeta function, see [4]. For an introduction to Heath-Brown's q-analogue, see [7, §7.3]. The basis is still Deligne's bounds, but before these are applied, a rather elaborate differencing process relays the problem to the study of $N(f^{\mathbf{y}}, B, q)$ for a series of differenced polynomials $f^{\mathbf{y}}(\mathbf{x}) = f(\mathbf{x} + p\mathbf{y}) - f(\mathbf{x})$. The aim of this thesis is to extend the technique developed in [5] in two different directions.

In the first paper we extend Heath-Browns method to systems of equations

$$f_1(\mathbf{x}) = \cdots = f_r(\mathbf{x}) = 0$$

such that the leading forms $F_1, \ldots, F_r$ define a complete intersection in $\mathbb{P}^{n-1}$. Suppose that the degrees of the f_i are all at least 3, and that the F_i define a non-singular subvariety of $\mathbb{P}^{n-1}_{\mathbb{Q}}$. If we define a counting function

$$N(f_1, \ldots, f_r, B) = \#\{\mathbf{x} \in \mathbb{Z}^n; f_1(\mathbf{x}) = \cdots = f_r(\mathbf{x}) = 0, |\mathbf{x}| \le B\}.$$

in the same spirit as for a single polynomial, then the main result of this paper states that

$$N(f_1, \ldots, f_r, B) \ll_{F_1,\ldots,F_r} B^{n-3r+r^2(13n-5-3r)/(n^2+4nr-n-r-r^2)}(\log B)^{n/2}$$

for $n \ge 4r + 2$. The dependence upon the coefficients of the F_i is made explicit. For the base step, i. e. the counting of points on the varieties defined by the differenced polynomials, we use an exponential sum estimate by Katz [6]. On account of this, besides generalizing (2), we also get a slight improvement in the case of a hypersurface. Recently, however, this has been improved by Salberger [8] to

$$N(f, B) \ll_F B^{n-3+9/(n+2)}. \tag{3}$$

In the second paper we look at polynomials $f \in \mathbb{Z}[X_1, \ldots, X_n]$ of degree at least 4, which allows us to iterate the differencing step twice. For this approach, we start from the counting function $N(f, B, \pi p q)$, where π, p, q are three different primes. Again we assume that the leading form F is non-singular. For $n \ge 10$ we then prove that

$$N(f, B) \ll_F B^{n-4+36/(n+8)}.$$

This improves upon (3) for $n \ge 17$.

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THE DENSITY OF INTEGRAL POINTS ON COMPLETE INTERSECTIONS

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WITH AN APPENDIX BY PER SALBERGER

ABSTRACT. In this paper, an upper bound for the number of integral points of bounded height on an affine complete intersection defined over $\mathbb{Z}$ is proven. The proof uses an extension to complete intersections of the method used for hypersurfaces by Heath-Brown [10], the so called "q-analogue" of van der Corput's AB process.

1. INTRODUCTION

If X is an affine algebraic set defined by a set of equations

$$f_i(x_1,\dots x_n)=0, i=1,\dots,r$$

with integral coefficients, and if B is a box in $\mathbb{R}^n$ - that is, a product of closed intervals - then we define the quantity

$$N(X,\mathsf{B})=\#\left\{\mathbf{x}=(x_1,\dots,x_n)\in\mathbb{Z}^n; f_i(\mathbf{x})=0,\mathbf{x}\in\mathsf{B}\right\}.$$

If m is a positive integer, and if B is small enough as to contain at most one representative of each congruence class modulo m, then we define

$$N(X,\mathsf{B},m)=\#\left\{\mathbf{x}=(x_1,\dots,x_n)\in\mathbb{Z}^n; f_i(\mathbf{x})\equiv 0 \pmod{m},\mathbf{x}\in\mathsf{B}\right\}.$$

Since $N(X,\mathsf{B})\le N(X,\mathsf{B},m)$ one can obtain upper bounds for $N(X,\mathsf{B})$ by considering $N(X,\mathsf{B},m)$ for suitably chosen m. If $\mathsf{B}=[-B,B]^n$ for some $B>0$ we write

$$N(X,B)=N(X,\mathsf{B}) \text{ and } N(X,B,m)=N(X,\mathsf{B},m).$$

Throughout this paper we shall be concerned with the case when X is a complete intersection, that is, when $\dim X=n-r$, where r is the number of equations defining X in $\mathbb{A}^n$. Our main concern shall be to find an upper bound for $N(X,B)$. One result in this direction is the following, by Fujiwara [5]: let X be a non-singular hypersurface in $\mathbb{A}^n$ defined by the vanishing of a polynomial f with integer coefficients, of degree at least 2. Then $N(X,B)\ll_{f,n} B^{n-2+2/n}$ for $n\ge 4$. Fujiwara proved this by exhibiting an asymptotic formula for $N(X,B,p)$ for primes p, the proof of which uses the estimates for exponential sums by Deligne [3] as a key tool. Heath-Brown [10] was able to sharpen the exponent to $n-2+2/(n+1)$ by averaging over primes in an interval. In the same paper he introduced a new technique, the so called q-analogue of van der Corput's method. He could then prove the bound

$$N(X,B)\ll_{f,n} B^{n-3+15/(n+5)} \tag{1}$$

for a non-singular hypersurface X defined by a polynomial f of degree at least 3 (Theorem 2 in [10]), by considering $N(X,B,pq)$ for two suitable primes p and q.

In this paper we will generalize the method of Heath-Brown to complete intersections of arbitrary codimension.

Notation. If X is a scheme over $\mathbb{Z}$ we write $X_{\mathbb{Q}} = X \times_{\mathrm{Spec}\ \mathbb{Z}} \mathrm{Spec}\ \mathbb{Q}$ and $X_q = X_{\mathbb{F}_q} = X \times_{\mathrm{Spec}\ \mathbb{Z}} \mathrm{Spec}\ \mathbb{F}_q$ for every prime q. When a norm $|\mathbf{x}|$ on $\mathbb{C}^n$ occurs we will always mean the maximum norm $|\mathbf{x}| = \max_{1\le i\le n} |x_i|$. For a polynomial $F \in \mathbb{C}[X_1,\dots,X_n]$ we define the height $\|F\|$ as the maximum modulus of the coefficients of F.

Theorem 1. *Let*

$$X = \mathrm{Spec}\ \mathbb{Z}[X_1,\dots,X_n]/(f_1,\dots,f_r),$$

where the leading forms $F_1,\dots,F_r$ of $f_1,\dots,f_r$ are of degree ≥ 3, and let

$$Z = \mathrm{Proj}\ \mathbb{Z}[X_1,\dots,X_n]/(F_1,\dots,F_r).$$

Assume that $Z_{\mathbb{Q}}$ is non-singular of codimension r in $\mathbb{P}^{n-1}_{\mathbb{Q}}$. Then, if $n \geq 4r+2$, we have for $B \geq 1$

$$N(X,B) \ll_{n,d} B^{n-3r+r^2\frac{13n-5-3r}{n^2+4nr-n-r-r^2}} (\log B)^{n/2} \left(\sum_{i=1}^{r} \log \|F_i\|\right)^{2r+1},$$

where $d = \max_i(\deg f_i)$.

Remark 1. The factor $(\log B)^{n/2}$ can in fact be disposed of, and we sketch in the end of Section 4 how this can be done.

Remark 2. Suppose, in the situation of Theorem 1, that all the f_i are homogeneous, i.e. $f_i = F_i$. Then the problem of measuring the density of integer points on X is equivalent to the corresponding problem for rational points on Z. To be more specific, each point $x \in \mathbb{P}^{n-1}(\mathbb{Q})$ can be represented uniquely up to sign by an n-tuple $\mathbf{x} \in \mathbb{Z}^n$ such that $\gcd(x_1,\dots,x_n) = 1$. We can then define a height function on $\mathbb{P}^{n-1}(\mathbb{Q})$ by $H(x) = |\mathbf{x}|$, and the corresponding counting function

$$\mathcal{N}(Z,B) = \#\left\{x \in Z \cap \mathbb{P}^{n-1}(\mathbb{Q}); H(x) \leq B\right\}.$$

Elementary considerations (see e.g. [11, Ex. F.16]) then show that $\mathcal{N}(Z,B) \ll B^{\theta}$, $\theta > 1$, if and only if $N(X,B) \ll B^{\theta}$. In the case where all $d_i = 3$, it is worth comparing Theorem 1 to a conjecture by Manin (see [4] or [2, Conj. A]) which predicts that

$$\mathcal{N}(Z,B) \ll_{F_i} B^{n-3r},$$

for $n \geq 3r+1$. (The $\log B$-factor present in the conjecture vanishes since $\mathrm{Pic}(Y) = \mathbb{Z}$ for all projective complete intersections Y of dimension ≥ 3 ([8, Cor. 3.2]).)

The estimate given by Theorem 1 in the case $r = 1$ is in fact slightly sharper than (1), owing to the use of estimates by Katz [14] on exponential sums modulo q. Theorem 1 is proven in Section 4, and is a corollary to the following theorem.

Theorem 2. *Let*

$$X = \text{Spec } \mathbb{Z}[X_1, \dots, X_n]/(f_1, \dots, f_r),$$

where $r < n$ and the leading forms $F_1, \dots, F_r$ of $f_1, \dots, f_r$ are of degree ≥ 3, and let

$$Z = \text{Proj } \mathbb{Z}[X_1, \dots, X_n]/(F_1, \dots, F_r).$$

Let B be a positive number, and let p and q be primes, with $2p < 2B+1 < q-p$, such that both Z_p and Z_q are non-singular of dimension $n-1-r$. Then we have

$$N(X, B, pq) = \frac{(2B+1)^n}{p^r q^r} + O_{n,d}\Big(B^{(n+1)/2} p^{-r/2} q^{(n-r-1)/4} (\log q)^{n/2} + B^{(n+1)/2} p^{(n-2r)/2} q^{-1/4} (\log q)^{n/2} + B^{n/2} p^{-r/2} q^{(n-r)/4} (\log q)^{n/2} + B^{n/2} p^{(n-r)/2} (\log q)^{n/2} + B^n p^{-(n+r-1)/2} q^{-r} + B^{n-1} p^{-r+1} q^{-r}\Big),$$

where $d = \max_i(\deg f_i)$.

The proof of Theorem 2 is carried out in Section 4 and more or less follows [10]. However, in contrast to Heath-Brown, we do not use Poisson summation, but a more direct approach.

We also prove, in Section 3, a generalization (and slight sharpening) of Theorem 3 in [10], a weighted asymptotic formula for the density of $\mathbb{F}_q$-points on affine complete intersections defined over $\mathbb{F}_q$. However, for the proof of Theorem 2, we will use an unweighted version of this result, proven by Salberger in an Appendix to this paper. This is because we desire an unweighted asymptotic formula in Theorem 2.

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2. Preliminary Results from Algebraic Geometry

We recall some facts from algebraic geometry that will provide helpful tools for proving our main results.

Definition. Let X be a scheme. A point $x \in X$ is a *singular point of X* if the local ring $\mathcal{O}_{X,x}$ is not a regular local ring. X is said to be *singular* if it has singular points, and *non-singular* if not. We denote the *singular locus of X* - the set of singular points - by Sing$X$.

If X is a scheme and x a point on X, then $\mathcal{O}_x$ is the local ring at x, $\mathfrak{m}_x$ its maximal ideal and $\kappa(x) = \mathcal{O}_x/\mathfrak{m}_x$ the residue field of x. If $X \to Y$ is a morphism of schemes, $\Omega_{X/Y}$ denotes the sheaf of relative differentials of X over Y, and we abbreviate $\Omega_{X/\text{Spec } R} = \Omega_{X/R}$.

We have the following characterization of singular points on a scheme.

Proposition 1. *Let X be a scheme of finite type over a perfect field k. Suppose that X is equidimensional of dimension n. Then for every point $x \in X$, the following conditions are equivalent:*

(i) x is a singular point of X;

(ii) $\dim_{\kappa(x)} \Omega_{X/k,x} \otimes_{\mathcal{O}_x} \kappa(x) > n$.

Proof. Since this is a local question, we can assume that $X = \operatorname{Spec} R$ with R equidimensional. Suppose $x = \mathfrak{p} \in \operatorname{Spec} R$. Then we have, by [18, Ex. 14.36],

$$\begin{aligned} n = & \operatorname{ht}\mathfrak{p} + \dim R/\mathfrak{p} \\ = & \dim \mathcal{O}_x + \text{tr.d.}\kappa(x)/k. \end{aligned} \tag{2}$$

By definition, x is singular if and only if

$$\dim_{\kappa(x)} \mathfrak{m}_x/\mathfrak{m}_x^2 > \dim \mathcal{O}_x.$$

Furthermore, by [9, Ex. II.8.1], we have an exact sequence of $\kappa(x)$-vector spaces

$$0 \to \mathfrak{m}_x/\mathfrak{m}_x^2 \to \Omega_{\mathcal{O}_x/k} \otimes_{\mathcal{O}_x} \kappa(x) \to \Omega_{\kappa(x)/k} \to 0.$$

Since $\Omega_{\mathcal{O}_x/k}$ is equal to the stalk $\Omega_{X/k,x}$ of the sheaf of relative differentials, and since $\dim_{\kappa(x)} \Omega_{\kappa(x)/k} = \text{tr.d.}\kappa(x)/k$ by [9, Thm. II.8.6A], this implies that

$$\dim_{\kappa(x)} \Omega_{X/k,x} \otimes_{\mathcal{O}_x} \kappa(x) = \dim_{\kappa(x)} \mathfrak{m}_x/\mathfrak{m}_x^2 + \text{tr.d.}\kappa(x)/k.$$

In view of (2) it follows that $x \in \operatorname{Sing}X$ if and only if

$$\dim_{\kappa(x)} \Omega_{X/k,x} \otimes_{\mathcal{O}_x} \kappa(x) > \dim \mathcal{O}_x + \text{tr.d.}\kappa(x)/k = n.$$

□

Remark 3. By [9, Ex. II.5.8] the function

$$\varphi(x) = \dim_{\kappa(x)} \Omega_{X/k,x} \otimes_{\mathcal{O}_x} \kappa(x)$$

is upper semicontinuous, so that in the situation described in the proposition, $\operatorname{Sing}X$ is a closed subscheme of X.

Remark 4. The proposition also shows that for X equidimensional and of finite type over a perfect field k, X is non-singular if and only if it is *smooth over* k (see [9, Ch. III.10]).

Remark 5. The particular case where we will use the proposition is for X a complete intersection of positive dimension in projective space over a perfect field. Such X are indeed equidimensional, since firstly, any local complete intersection is Cohen-Macaulay ([9, Prop. 8.23]) and thus locally equidimensional, and secondly, a complete intersection in $\mathbb{P}_k^n$ of dimension ≥ 1 is connected ([9, Ex. III.5.5]).

When working in a projective space $\mathbb{P}^n$ with homogeneous coordinates $x_0, \ldots, x_n$ we denote by $\check{\mathbb{P}}^n$ the dual projective space with homogeneous coordinates $\xi_0, \ldots, \xi_n$. For a point $\mathbf{a} = (a_0, \ldots, a_n)$ in $\check{\mathbb{P}}^n$ we will let $H_{\mathbf{a}}$ denote the hyperplane defined in $\mathbb{P}^n$ by the equation $\mathbf{a} \cdot \mathbf{x} = a_0x_0 + \ldots + a_nx_n = 0$. We begin by proving the following corollary to Bertini's Theorem. By convention, the dimension of the empty set is defined to be -1.

Lemma 1. *Let k be an algebraically closed field. Let X be a non-empty complete intersection in $\mathbb{P}_k^n$. Suppose that*

$$\dim \operatorname{Sing}X = s.$$

Then there is a hyperplane H such that $\dim(X \cap H) = \dim X - 1$ *and*

$$\dim \operatorname{Sing}(X \cap H) < \max(s, 0).$$

Proof. The case $s = -1$ follows immediately from Bertini's Theorem [13, Cor 6.11(2)]. (X is then smooth over k by Remark 4.) If $s \geq 0$, let $Y = X \setminus \mathrm{Sing}X$, so that Y is smooth. Then, by Bertini's Theorem, there exists a non-empty Zariski open subset U of $\check{\mathbb{P}}^n_k$ such that for hyperplanes $H_{\mathbf{a}}$ parametrized by closed k-points $\mathbf{a}$ in U, $Y \cap H_{\mathbf{a}}$ is smooth and thus non-singular by Remark 4. Hence, for $\mathbf{a} \in U(k)$ we have

$$\mathrm{Sing}(X \cap H_{\mathbf{a}}) \subseteq \mathrm{Sing}X \cap H_{\mathbf{a}}. \tag{3}$$

Furthermore, there are non-empty open sets U', U'' such that for all closed k-points $\mathbf{a}$ of U', no irreducible component of $\mathrm{Sing}X$ of dimension s is contained in $H_{\mathbf{a}}$, and for $\mathbf{a} \in U''(k)$ no irreducible component of X is contained in $H_{\mathbf{a}}$. Then we have, for $\mathbf{a} \in U \cap U' \cap U''(k)$, that $\dim(X \cap H_{\mathbf{a}}) = \dim X - 1$ and $\dim \mathrm{Sing}(X \cap H_{\mathbf{a}}) < s$. □

Remark 6. For any hyperplane H such that $\dim X \cap H = \dim X - 1$, $\dim \mathrm{Sing}(X \cap H) \geq \dim \mathrm{Sing}X - 1$ (see [14, Lemma 3]).

The next lemma is an "effective" version of Bertini's Theorem. For a more explicit result of the same type, see [1].

Lemma 2. *Let $n, r, d_1, \ldots, d_r$ be natural numbers, and let $F_1, \ldots, F_r$ be forms in $X_0, \ldots, X_n$ with integer coefficients, and with $\deg F_i = d_i$. Let $V = \mathrm{Proj}\ \mathbb{Z}[X_0, \ldots, X_n]/(F_1, \ldots, F_r)$, and suppose that $V_{\mathbb{Q}}$ has dimension $n - r \geq 0$. Then for every prime q such that V_q has dimension $n - r$, there is a non-zero form $\Phi_q \in \mathbb{F}_q[\xi_0, \ldots, \xi_n]$ with degree bounded in terms of n and $d_1, \ldots, d_r$ only, such that for every point $\mathbf{a} = (a_0, \ldots, a_n) \in \check{\mathbb{P}}^n_{\mathbb{F}_q}$ satisfying $\Phi_q(a_0, \ldots, a_n) \neq 0$ we have*

(i) $\dim \mathrm{Sing}(V_q \cap H_{\mathbf{a}}) = \max(-1, \dim \mathrm{Sing}V_q - 1)$
(ii) $\dim V_q \cap H_{\mathbf{a}} = \dim V_q - 1$.

In particular, for each $q \geq q_0 = q_0(n, d_1, \ldots, d_r)$ there is an $\mathbf{a} \in \check{\mathbb{P}}^n_{\mathbb{F}_q}$ with the properties (i) and (ii).

Proof. We let $\mathbb{P}_i$, for each $i = 1, \ldots, r$, be the projective space over $\mathbb{Z}$ parametrizing all hypersurfaces in $\mathbb{P}^n_{\mathbb{Z}}$ of degree d_i (as a Hilbert scheme), and work in the large multiprojective space $\mathbf{P} = \mathbb{P}_1 \times \ldots \times \mathbb{P}_r$. For a k-point in $\mathbf{P}$ representing a tuple $(F_1, \ldots, F_r)$ we write $V(F_1, \ldots, F_r)$ for the intersection of the corresponding r hypersurfaces in $\mathbb{P}^n_k$. Let $W \subseteq \mathbf{P} \times \check{\mathbb{P}}^n_{\mathbb{Z}} \times \mathbb{P}^n_{\mathbb{Z}}$ be defined as the closed set of points $P \in \mathbf{P} \times \check{\mathbb{P}}^n_{\mathbb{Z}} \times \mathbb{P}^n_{\mathbb{Z}}$ representing $(F_1, \ldots, F_r, \mathbf{a}, \mathbf{x})$ that satisfy

$$\mathbf{x} \in V(F_1, \ldots, F_r) \cap H_{\mathbf{a}}.$$

Let

$$\pi : W \to \mathbf{P}' := \mathbf{P} \times \check{\mathbb{P}}^n_{\mathbb{Z}}$$

be the projection. The function $\varphi(P) := \dim_{\kappa(P)} \Omega_{W/\mathbf{P}',P}$ is upper semicontinuous (see Remark 3), so the set

$$S = \{P \in W; \varphi(P) \geq n - r\}$$

is closed. Now, let $\tilde{\pi} : S \to \mathbf{P}'$ be the restriction of π to S, and let for every $s \in \{-1, 0, 1, \ldots, n\}$

$$A_s = \left\{Q \in \mathbf{P}'; \dim \tilde{\pi}^{-1}(Q) \geq s\right\}.$$

By Chevalley's Semicontinuity Theorem [7, Cor 13.1.5], A_s is closed in $\mathbf{P}'$, as is the set

$$D = \left\{Q \in \mathbf{P}'; \dim \pi^{-1}(Q) \geq n - r\right\}.$$

For each $s \in \{-1, 0, \dots, n\}$, let $T_s = D \cup A_s$. Then T_s is closed as well, so there exist multihomogeneous forms $H_1^s, \dots, H_t^s$ over $\mathbb{Z}$ that define T_s.

For a closed k-point $P \in W$ representing $(F_1, \dots, F_r, \mathbf{a}, \mathbf{x})$ we have an isomorphism of stalks $\Omega_{W/\mathbf{P}',P} \cong \Omega_{Y/k,\mathbf{x}}$, where

$$Y = V(F_1, \dots, F_r) \cap H_{\mathbf{a}} \subseteq \mathbb{P}_k^n.$$

Thus, for each tuple $(F_1, \dots, F_r, \mathbf{a})$ such that both $V = V(F_1, \dots, F_r)$ and $V \cap H_{\mathbf{a}}$ are complete intersections of codimension r and $r+1$, respectively, the fiber $\tilde{\pi}^{-1}(F_1, \dots, F_r, \mathbf{a})$ is precisely $\mathrm{Sing}(V \cap H_{\mathbf{a}})$ by Proposition 1. For every other point $(F_1, \dots, F_r, \mathbf{a})$ we have $\tilde{\pi}^{-1}(F_1, \dots, F_r, \mathbf{a}) = \mathbb{P}_k^n$. We conclude that T_s, for each s, is the set of tuples $(F_1, \dots, F_r, \mathbf{a})$ such that $V(F_1, \dots, F_r) \cap H_{\mathbf{a}}$ either has codimension $\leq r$ or has a singular locus of dimension at least s. In particular, if we have a closed k-point $Q \in \mathbf{P}$ representing $(F_1, \dots, F_r)$ such that $V = V(F_1, \dots, F_r)$ satisfies

$$\dim V = n - r, \quad \dim \mathrm{Sing} V = s, \tag{4}$$

and if $\pi_s : T_s \to \mathbf{P}$ is the projection, then the fiber $\pi_s^{-1}(Q)$ is the closed set of points $\mathbf{a} \in \check{\mathbb{P}}_k^n$ such that either $\dim \mathrm{Sing}(V \cap H_{\mathbf{a}}) \geq \dim \mathrm{Sing} V$ or $\dim(V \cap H_{\mathbf{a}}) = \dim V$.

Now let $F_1, \dots, F_r$ be forms as in the hypothesis, and let q be a prime such that (4) is satisfied for $Q \in \mathbf{P}$ representing the tuple of (mod q)-reductions $((F_1)_q, \dots, (F_r)_q)$. Then $\pi_s^{-1}(Q)$ is defined in $\mathbb{P}_k^n$, where $k = \kappa(Q) = \mathbb{F}_q$, by the specializations $H_i^s|_Q$ of the multihomogeneous forms H_i^s. Applying Lemma 1 we get that $\pi_s^{-1}(Q) \times \mathrm{Spec}\, \bar{k}$ is a proper closed subset of $\mathbb{P}_{\bar{k}}^n$ (where $\bar{k}$ is an algebraic closure of k). Therefore one of the forms $H_i^s|_Q \in k[\xi_0, \dots, \xi_n]$ must be non-zero, so the form

$$\Phi_q(\xi_0, \dots, \xi_n) = H_i^s|_Q\,(\xi_0, \dots, \xi_n)$$

has the desired properties.

The last assertion of the lemma follows from the easy observation that a polynomial of degree at most q cannot vanish at every point of $\mathbb{P}_{\mathbb{F}_q}^n$. □

The following lemma explores the new geometry arising from the Weyl differencing in Section 4. For a polynomial $f(X_1, \dots, X_n)$ we denote by ∇f the gradient $\left(\frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_n}\right)^t$ and by $\nabla^2 f$ the Hessian matrix $\left(\frac{\partial f}{\partial X_i \partial X_j}\right)_{i,j}$.

Lemma 3. *Let $G_1, \dots, G_r$ be homogeneous polynomials in $\mathbb{Z}[X_1, \dots, X_n]$ of degrees $d_1, \dots, d_r$, and let*

$$V = \mathrm{Proj}\ \mathbb{Z}[X_1, \dots, X_n]/(G_1, \dots, G_r).$$

Let q be a prime such that $q \nmid d_i$ for all $i = 1, \dots, r$ and suppose that V_q is a non-singular complete intersection of codimension r in $\mathbb{P}_{\mathbb{F}_q}^{n-1}$.

(i) Let

$$S = \Big\{(\mathbf{x},\mathbf{y}) \in \mathbb{P}^{n-1}_{\mathbb{F}_q} \times \mathbb{P}^{n-1}_{\mathbb{F}_q};\ \mathbf{y}\cdot\nabla G_i(\mathbf{x}) = 0,\ i=1,\dots,r, \\ \operatorname{rank}\big(\mathbf{y}\cdot\nabla^2 G_i(\mathbf{x})\big)_{1\le i\le r} < r\Big\}.$$

Then $\dim S \le n-2$.

(ii) For $\mathbf{y} \in \mathbb{P}^{n-1}_{\mathbb{F}_q}$, let

$$S_{\mathbf{y}} = \Big\{\mathbf{x} \in \mathbb{P}^{n-1}_{\mathbb{F}_q};\ \mathbf{y}\cdot\nabla G_i(\mathbf{x}) = 0,\ i=1,\dots,r, \\ \operatorname{rank}\big(\mathbf{y}\cdot\nabla^2 G_i(\mathbf{x})\big)_{1\le i\le r} < r,\Big\}.$$

For $s=-1,0,1,\dots,n-1$, let $T_s = \Big\{\mathbf{y} \in \mathbb{P}^{n-1}_{\mathbb{F}_q};\ \dim S_{\mathbf{y}} \ge s\Big\}$. Then T_s is Zariski closed and $\dim T_s \le n-s-2$.

(iii) For each s, let $T_s^{(1)}, T_s^{(2)}, \dots$ be the irreducible components of T_s. Then

$$\sum_j \deg(T_s^{(j)}) = O_{n,r,d_1,\dots,d_r}(1).$$

To prove Lemma 3 we shall need the following lemma.

Lemma 4. *Let k be a field, and let V be a closed subscheme of $\mathbb{P}^n_k \times \mathbb{P}^n_k$. Let $\Delta \subseteq \mathbb{P}^n \times \mathbb{P}^n$ be the diagonal, $\Delta = \{(\mathbf{x},\mathbf{x});\ \mathbf{x}\in\mathbb{P}^n_k\}$. If $\dim V \ge n$, then $V \cap \Delta \ne \emptyset$.*

Proof. Consider the rational map

$$f : \mathbb{P}^{2n+1} \dashrightarrow \mathbb{P}^n \times \mathbb{P}^n$$

given by

$$(X_0 : \ldots : X_{2n+1}) \mapsto ((X_0:\ldots:X_n),(X_{n+1}:\ldots:X_{2n+1})).$$

Its domain of definition is the Zariski open set $U := \mathbb{P}^{2n+1} \setminus (L \cup M)$, where $L = \{X_0 = \ldots = X_n = 0\}$ and $M = \{X_{n+1} = \ldots = X_{2n+1} = 0\}$. Moreover, let $\hat{\Delta}$ be the variety in $\mathbb{P}^{2n+1}$ defined by $X_0 = X_{n+1}, \dots, X_n = X_{2n+1}$. Then f is an isomorphism between $\hat{\Delta}$ and Δ. Let $\hat{V}$ be the Zariski closure in $\mathbb{P}^{2n+1}$ of $f^{-1}(V)$. Then

$$\dim \hat{V} = \dim V + 1 \ge n+1,$$

so that

$$\operatorname{codim}\hat{\Delta} + \operatorname{codim}\hat{V} \le 2n+1.$$

Thus, by the Projective Dimension Theorem [15, Ex. 3.3.4], $\hat{\Delta}\cap\hat{V}$ is nonempty. But a point P in this intersection automatically lies in U, since $\hat{\Delta}\cap(L\cup M)$ is empty, and we get a point $f(P)$ in $\Delta\cap V$. □

Proof of Lemma 3. (i) Assume that $\dim S \ge n-1$. According to Lemma 4, we then must have $S\cap\Delta \ne \emptyset$. Thus, suppose $(\mathbf{x},\mathbf{x}) \in S\cap\Delta$. By the definition of S, we then have

$$\begin{cases} \mathbf{x}\cdot\nabla G_i(\mathbf{x}) = 0,\ i=1,\dots,r \\ \operatorname{rank}\big(\mathbf{x}\cdot\nabla^2 G_i(\mathbf{x})\big)_{1\le i\le r} < r. \end{cases}$$

But $\mathbf{x}\cdot\nabla^2 G_i(\mathbf{x}) = \nabla(\mathbf{x}\cdot\nabla G_i(\mathbf{x}))$, so by Euler's identity we have (since q does not divide any of the degrees of the G_i)

$$\begin{cases} G_i(\mathbf{x}) = 0,\ i = 1,\dots,r \\ \operatorname{rank}\left(\nabla G_i(\mathbf{x})\right)_{1\le i\le r} < r. \end{cases}$$

Therefore, by the Jacobian Criterion, $\mathbf{x}$ is a singular point of V, in contradiction with the hypothesis.

(ii) Let $\pi : S \to \mathbb{P}^{n-1}$ be the projection onto the second coordinate, $(\mathbf{x},\mathbf{y}) \mapsto \mathbf{y}$. Then $S_{\mathbf{y}} = \pi^{-1}(\mathbf{y}) \times \{\mathbf{y}\}$. The fact that T_s is closed follows from Chevalley's semicontinuity theorem [7, Cor 13.1.5]. Now let $S_s = S \cap \left(\mathbb{P}^{n-1}\times T_s\right)$ for each $s = -1,\dots,n-1$. Since S_s is the disjoint union of fibres

$$S_s = \bigcup_{\mathbf{y}\in T_s} \pi^{-1}(\mathbf{y}),$$

we have, by (i)

$$\dim T_s + s \le \dim S_s \le \dim S \le n-2,$$

whence $\dim T_s \le n-s-2$.

(iii) As in Lemma 2, we shall let $\mathbb{P}_i$ be the projective spaces parametrizing hypersurfaces of degree d_i in $\mathbb{P}^n_{\mathbb{Z}}$, and put $\mathbf{P} = \mathbb{P}_1\times\ldots\times\mathbb{P}_r$. Now, let

$$\mathcal{S} = \Big\{(G_1,\dots,G_r,\mathbf{x},\mathbf{y}) \in \mathbf{P}\times\mathbb{P}^{n-1}_{\mathbb{Z}}\times\mathbb{P}^{n-1}_{\mathbb{Z}};\ \mathbf{y}\cdot\nabla G_i(\mathbf{x}) = 0,\ i=1,\dots,r, \\ \operatorname{rank}\left(\mathbf{y}\cdot\nabla^2 G_i(\mathbf{x})\right)_{1\le i\le r} < r, \Big\}.$$

Let $\tilde\pi : \mathcal{S} \to \mathbf{P}\times\mathbb{P}^{n-1}_{\mathbb{Z}}$ be the projection $(G_1,\dots,G_r,\mathbf{x},\mathbf{y}) \mapsto (G_1,\dots,G_r,\mathbf{y})$, and define for each s

$$\mathcal{T}_s = \left\{\mathcal{P} = (G_1,\dots,G_r,\mathbf{y}); \dim\tilde\pi^{-1}(\mathcal{P}) \ge s\right\}.$$

Then $\mathcal{T}_s$ is closed by Chevalley's theorem, so it is defined in $\mathbf{P}\times\mathbb{P}^{n-1}_{\mathbb{Z}}$ by multihomogeneous polynomials $H_1,\dots,H_t$ where $t = O_{n,r,d_1,\dots,d_r}(1)$. Now we fix polynomials $G_1,\dots,G_r$ and a prime q. The set T_s is then defined in $\mathbb{P}^{n-1}_{\mathbb{F}_q}$by $H_1|_{G_1,\dots,G_r},\dots,H_t|_{G_1,\dots,G_r}$. Now by Bézout's Theorem [6, Ex. 8.4.6] we have

$$\sum_j \deg(T_s^{(j)}) \le \prod_i \deg(H_i) \ll_{n,r,d_1,\dots,d_r} 1.$$

□

3. Points on Complete Intersections over $\mathbb{F}_q$

The following result is well-known and trivial, but we include a proof for the sake of completeness.

Lemma 5. *Let $X = \operatorname{Spec}\mathbb{F}_q[X_1,\dots,X_n]/(f_1,\dots,f_\rho)$ be a closed subscheme of $\mathbb{A}^n_{\mathbb{F}_q}$, and let $d = \max_i(\deg f_i)$. Let $B \ge 1$. Then, for any box $\mathsf{B} = [a_1-b_1,a_1+b_1]\times\ldots\times[a_n-b_n,a_n+b_n]$, with $|b_i|\le B$, containing at most one representative of each congruence class modulo q, we have*

$$N(X,\mathsf{B},q) \ll_{n,\rho,d} B^{\dim X}.$$

Proof. We identify $\mathbb{A}^n_{\mathbb{F}_q}$ with the open subset $\{X_0 \neq 0\}$ of $\mathbb{P}^n_{\mathbb{F}_q}$ and consider the scheme-theoretic closure Y of X in $\mathbb{P}^n_{\mathbb{F}_q}$ defined by the homogenizations $F_1, \ldots, F_\rho$ of $f_1, \ldots, f_\rho$. Then the sum D_X of the degrees of the irreducible components of Y is at most d^ρ by Bézout's Theorem [6, Ex. 8.4.6]. Thus it suffices to show that $N(X, \mathsf{B}, q) \ll_{n,D_X} B^{\dim X}$ for every closed subscheme X. We prove this by induction over $\nu = \dim X$. If $\nu = 0$, then $\#X(\mathbb{F}_q) \leq D_X$, so we are done. Thus, suppose that $\nu \geq 1$. Since X has at most D_X irreducible components, it is enough to prove that $N(X', \mathsf{B}, q) \ll_{n,D_X} B^\nu$ for an arbitrary irreducible component X' of X. For some $i \in \{1, \ldots, n\}$, all the hyperplanes $H_a : x_i = a$, where a ranges over $\mathbb{F}_q$, intersect X' properly. Since $D_{X \cap H_a} \leq D_X$, the induction hypothesis yields that $N(X' \cap H_a, \mathsf{B}, q) \ll_{n,D_X} B^{\nu-1}$ for each $a \in \mathbb{F}_q$. Since we only need to consider at most $2B$ values of a, we get

$$N(X', \mathsf{B}, q) = \sum_a N(X' \cap H_a, \mathsf{B}, q) \leq 2B \cdot O_{n,D_X}(B^{\nu-1}) \ll_{n,D_X} B^\nu,$$

as desired. □

Delignes work on the Weil Conjectures [3] yields a sharp asymptotic formula for the number of $\mathbb{F}_q$-points on a non-singular projective complete intersection. In the paper by Hooley [12] (with an appendix by Katz) an extension to the singular case is proven. The following lemma is an affine reformulation of Hooley's result.

Lemma 6. *Let Y be a closed subscheme of $\mathbb{P}^n_{\mathbb{F}_q}$ that is a complete intersection of codimension $r \leq n$ and multidegree $(d_1, \ldots, d_r)$. Let $Z = Y \cap \{x_0 = 0\}$ and suppose that $\dim Z = \dim Y - 1$. Put $X = Y \setminus Z$ and $s = \dim \mathrm{Sing} Z$. Then we have*

$$\#X(\mathbb{F}_q) = q^{n-r} + O_{n,d_1,\ldots,d_r}(q^{(n-r+2+s)/2}).$$

Proof. In case $n = r$ the lemma is a trivial consequence of Bézout's Theorem. We may thus assume that $n > r$. By [12, Appendix, Thm. 1] we have

$$\#Z(\mathbb{F}_q) = 1 + q + \ldots + q^{n-r-1} + O(q^{(n-r+s)/2}).$$

However, $s \geq \dim \mathrm{Sing} Y - 1$ by Remark 6, so by the same theorem we get

$$\#Y(\mathbb{F}_q) = 1 + q + \ldots + q^{n-r} + O(q^{(n-r+2+s)/2}).$$

Subtracting these two equations, we get

$$\#X(\mathbb{F}_q) = q^{n-r} + O(q^{(n-r+2+s)/2}),$$

as stated. □

The following result is a generalization of Theorem 3 in [10]. However, even in the case of a hypersurface we get a slightly sharper estimate. The reason for this is the use of estimates by Katz [14] for "singular" exponential sums. A similar application of those results are found in a paper by Luo [16].

Notation. For an element $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{Z}^n$ we let

$$\mathbf{x}_q = (x_1 + q\mathbb{Z}, \ldots, x_n + q\mathbb{Z}) \in \mathbb{F}_q^n.$$

Theorem 3. *Let $W : \mathbb{R}^n \to \mathbb{R}$ be an infinitely differentiable function, supported in a cube of side $2L$. Let q be a prime and B a real number with $1 \le B \ll_L q$. Let*

$$X = \text{Spec } \mathbb{Z}[X_1, \dots, X_n]/(f_1, \dots, f_r),$$

where the leading forms $F_1, \dots, F_r$ of $f_1, \dots, f_r$ are of degree at least 2, and let

$$Z_q = \text{Proj } \mathbb{Z}[X_1, \dots, X_n]/(q, F_1, \dots, F_r).$$

Assume that $\dim Z_q = n - 1 - r$. Let $s = \dim \mathrm{Sing} Z_q$ and $d = \max_i(\deg F_i)$. Define a weighted counting function

$$N_W(X, B, q) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ \mathbf{x}_q \in X_q}} W\left(\frac{1}{B}\mathbf{x}\right).$$

Then we have

$$\begin{aligned} N_W(X, B, q) &= q^{-r} N_W(\mathbb{A}^n, B, q) \\ &\quad + O_{n,d,L}\left(D_{n+1} B^{s+1} q^{(n-r-s-2)/2}(B + q^{1/2})\right), \end{aligned} \tag{5}$$

where, for each natural number k, D_k is the maximum over $\mathbb{R}^n$ of all partial derivatives of W of order k.

Proof. We begin with some preparatory considerations, to justify the use of Lemma 6 later in the proof. Let

$$Y_q = \text{Proj } \mathbb{Z}[X_0, \dots, X_n]/(q, G_1, \dots, G_r),$$

where $G_i(X_0, \dots, X_n) = X_0^{d_i} f_i(X_1/X_0, \dots, X_n/X_0)$ for $i = 1, \dots, n$. Then $Z_q = Y_q \cap \{X_0 = 0\}$ and $X_q = Y_q \setminus Z_q$. Moreover, since $\dim Z_q = n - 1 - r$ we must have $\dim Y_q = n - r$.

We shall follow the approach of Heath-Brown [10] and use induction with respect to s, starting with the case when Z_q is non-singular, that is, when $s = -1$. In case $n - r \ge 2$ we shall use Katz' results. We begin, however, with two trivial cases. Suppose firstly that $n - r = 1$. Then

$$N_W(X, B, q) \ll_{n,L} D_0 N(X, B, q) \ll_{n,d} D_0 B$$

by Lemma 5, and

$$q^{-r} N_W(\mathbb{A}^n, B, q) \ll_{n,L} D_0 q^{-n+1} B^n \ll_{n,L} D_0 B,$$

so

$$N_W(X, B, q) - q^{-r} N_W(\mathbb{A}^n, B, q) \ll_{n,d,L} D_{n+1}(B + q^{1/2})$$

as required for (5). Next, suppose that $n - r = 0$. Also in this case the formula (5) holds, since $N_W(X, B, q) \ll_{n,d,L} D_0$ and $q^{-r} N_W(\mathbb{A}^n, B, q) \ll_{n,L} D_0 q^{-n} B^n \ll_{n,L} D_0$, whereas the error term required for (5) is $D_{n+1}(Bq^{-1/2} + 1)$.

From now on, we assume that $n-r \geq 2$. By the Poisson Summation Formula we have

$$\begin{aligned} N_W(X,B,q) &= \sum_{\mathbf{z}\in X_q}\sum_{\mathbf{u}\in\mathbb{Z}^n} W\left(\frac{1}{B}(\mathbf{z}+q\mathbf{u})\right) \\ &= \sum_{\mathbf{z}\in X_q}\left(\frac{B}{q}\right)^n \sum_{\mathbf{a}\in\mathbb{Z}^n} e_q(\mathbf{a}\cdot\mathbf{z})\hat{W}\left(\frac{B}{q}\mathbf{a}\right) \\ &= \left(\frac{B}{q}\right)^n \sum_{\mathbf{a}\in\mathbb{Z}^n} \hat{W}\left(\frac{B}{q}\mathbf{a}\right)\Sigma_q(\mathbf{a}), \end{aligned}$$

where

$$\Sigma_q(\mathbf{a}) = \sum_{\mathbf{z}\in X_q} e_q(\mathbf{a}\cdot\mathbf{z}),$$

a sum which we shall now investigate. In case $\mathbf{a}\equiv\mathbf{0} \pmod q$, we can use Lemma 6 to conclude that we have

$$\Sigma_q(\mathbf{a}) = \#X_q(\mathbb{F}_q) = q^{n-r} + O_{n,d}(q^{(n-r+1)/2}).$$

Next we consider $\Sigma_q(\mathbf{a})$ for $\mathbf{a}\not\equiv\mathbf{0} \pmod q$. Since Z_q is a projective complete intersection of dimension at least 1, it is geometrically connected. Being non-singular, it is thus geometrically integral. The hypothesis that $\deg F_i \geq 2$ for all i now implies that for each $\mathbf{a}\in\mathbb{F}_q^n\setminus\{\mathbf{0}\}$ we have $\dim(Z_q\cap H_{\mathbf{a}}) = n-r-2$, where $H_{\mathbf{a}}$ is the hyperplane defined by $\mathbf{a}\cdot\mathbf{x}=0$. Then, by Theorems 23 and 24 in [14], we have

$$\Sigma_q(\mathbf{a}) \ll q^{(n-r+1+\delta(\mathbf{a}))/2},$$

where $\delta(\mathbf{a}) = \dim \operatorname{Sing}(Z_q\cap H_{\mathbf{a}})$. Thus we get

$$\begin{aligned} N_W(X,B,q) &= \left(\frac{B}{q}\right)^n \left(\sum_{q\mid\mathbf{a}} \hat{W}\left(\frac{B}{q}\mathbf{a}\right)\left(q^{n-r} + O_{n,d}\left(q^{(n-r+1)/2}\right)\right)\right) \\ &\quad + O\left(\left(\frac{B}{q}\right)^n \sum_{\mathbf{a}\in\mathbb{Z}^n}\left|\hat{W}\left(\frac{B}{q}\mathbf{a}\right)\right| q^{(n-r+1+\delta(\mathbf{a}))/2}\right). \end{aligned} \tag{6}$$

The first term here equals

$$\begin{aligned} &\left(\frac{B}{q}\right)^n q^{n-r}\sum_{\mathbf{v}\in\mathbb{Z}^n}\hat{W}(B\mathbf{v}) + O_{n,d}\left(\left(\frac{B}{q}\right)^n q^{(n-r+1)/2}\sum_{\mathbf{v}\in\mathbb{Z}^n}\hat{W}(B\mathbf{v})\right) \\ &\quad = q^{-r}N_W(\mathbb{A}^n,B,q) + O_{n,d,L}\left(B^n q^{-(n+r-1)/2}\right), \end{aligned} \tag{7}$$

by the Poisson formula in the reverse direction and since $N_W(\mathbb{A}^n,B,q) = O_{n,d,L}(B^n)$. Now we shall try to estimate the sum

$$\sum_{\mathbf{a}\in\mathbb{Z}^n}\left|\hat{W}\left(\frac{B}{q}\mathbf{a}\right)\right| q^{(n-r+1+\delta(\mathbf{a}))/2}. \tag{8}$$

It follows from a result of Zak (see [12, Appendix, Thm. 2]) that $\delta(\mathbf{a}) = -1$ or 0 for all $\mathbf{a}$. By Lemma 2, all $\mathbf{a}$ for which $\delta(\mathbf{a}) = 0$ satisfy $\Phi(\mathbf{a}) \equiv 0$

(mod q) for a non-zero polynomial $\Phi(\xi_1, \ldots, \xi_n)$ with integer coefficients, whose degree is $O_{n,d}(1)$. Thus, let us split (8) into two sums

$$\Sigma_1 + \Sigma_2 = \sum_{\substack{\mathbf{a}\in\mathbb{Z}^n \\ \Phi(\mathbf{a})\equiv 0(q)}} \left|\hat{W}\left(\frac{B}{q}\mathbf{a}\right)\right| q^{(n-r+1)/2} + \sum_{\substack{\mathbf{a}\in\mathbb{Z}^n \\ \Phi(\mathbf{a})\not\equiv 0(q)}} \left|\hat{W}\left(\frac{B}{q}\mathbf{a}\right)\right| q^{(n-r)/2}.$$

We observe that, since the infinitely differentiable function W has compact support, we have an estimate $\left|\hat{W}(\mathbf{t})\right| \ll_{n,L} D_k |\mathbf{t}|^{-k}$ for $|\mathbf{t}| \geq 1$ and any $k \geq 0$, and moreover $D_k \ll_{n,L} D_{k+1}$ for every k. In particular, for any $\mathbf{t} \in \mathbb{R}^n$ we have the estimate

$$\left|\hat{W}(\mathbf{t})\right| \ll_{n,L} D_k \min(1, |\mathbf{t}|^{-k}),\ k \geq 0 \tag{9}$$

In order to estimate Σ_2, we calculate

$$\begin{aligned}\sum_{\Phi(\mathbf{a})\not\equiv 0(q)} \left|\hat{W}\left(\frac{B}{q}\mathbf{a}\right)\right| &\ll_{n,L} D_{n+1} \sum_{\mathbf{a}\in\mathbb{Z}^n} \min\left(1, \left|\frac{B}{q}\mathbf{a}\right|^{-n-1}\right) \\ &\ll_{n,L} D_{n+1} \sum_{|\mathbf{a}|\leq q/B} 1 + D_{n+1}\left(\frac{q}{B}\right)^{n+1} \sum_{|\mathbf{a}|>q/B} |\mathbf{a}|^{-n-1} \\ &\ll D_{n+1}\left(\frac{q}{B}\right)^n,\end{aligned}$$

which yields

$$\Sigma_2 \ll_{n,L} D_{n+1}\left(\frac{q}{B}\right)^n q^{(n-r)/2}.$$

For Σ_1 we write

$$\begin{aligned}\sum_{\Phi(\mathbf{a})\equiv 0(q)} \left|\hat{W}\left(\frac{B}{q}\mathbf{a}\right)\right| &\ll_{n,L} D_{n+1} \sum_{\Phi(\mathbf{a})\equiv 0(q)} \min\left(1, \left|\frac{B}{q}\mathbf{a}\right|^{-n-1}\right) \\ &\ll_{n,L} D_{n+1} \sum_{\substack{|\mathbf{a}|\leq q/B \\ \Phi(\mathbf{a})\equiv 0(q)}} 1 + D_{n+1}\left(\frac{q}{B}\right)^{n+1} \sum_{\substack{|\mathbf{a}|>q/B \\ \Phi(\mathbf{a})\equiv 0(q)}} |\mathbf{a}|^{-n-1}.\end{aligned}$$

The first term is $O_{n,d}(D_{n+1}(q/B)^{n-1})$ by Lemma 5. Moreover we claim that

$$\sum_{\substack{|\mathbf{a}|>q/B \\ \Phi(\mathbf{a})\equiv 0(q)}} |\mathbf{a}|^{-n-1} \ll_{n,d} \left(\frac{q}{B}\right)^{-2}.$$

To see this, we note that the contribution to the sum from an interval $A/2 \leq |(a)| < A$, where $A > 0$, is $O_{n,d}(A^{-2})$. Indeed, by Lemma 5 the number of terms is $O_{n,d}(A^{n-1})$ and the size of each term is $O_{n,d}(A^{-n-1})$. Putting $A = 2^k$ and summing over k for which $2^k \geq q/B$, we get

$$\sum_{\substack{|\mathbf{a}|>q/B \\ \Phi(\mathbf{a})\equiv 0(q)}} |\mathbf{a}|^{-n-1} \ll_{n,d} \sum_{2^k\geq q/B} 2^{-2k} \ll \left(\frac{q}{B}\right)^{-2}.$$

Thus we conclude that

$$\Sigma_1 \ll_{n,d,L} D_{n+1}\left(\frac{q}{B}\right)^{n-1} q^{(n-r+1)/2},$$

so we arrive at the estimate

$$\Sigma_1 + \Sigma_2 \ll_{n,d,L} D_{n+1} \left(\frac{q}{B}\right)^n q^{(n-r-1)/2}(B + q^{1/2}). \tag{10}$$

Inserting (7) and (10) into the formula (6) yields

$$N_W(X, B, q) = q^{-r} N_W(\mathbb{A}^n, B, q) + O_{n,d,L}\left(D_{n+1} q^{(n-r-1)/2}(B + q^{1/2})\right),$$

as required for the case $s = -1$.

Suppose now that Z_q is singular, so that $s \geq 0$. Following Heath-Brown [10] we will count points on hyperplane sections. We begin with remarking that it is enough to prove the theorem for q greater than some constant $q_0 = q_0(n, d)$. Indeed, if $q \ll_{n,d} 1$, then $B \ll_{n,d,L} 1$, so that trivially we have $N_W(X, B, q) - q^{-r} N_W(\mathbb{A}^n, B, q) \ll_{n,d,L} 1$. Thus, using Lemma 2, we can assume that it is possible to find a primitive integer vector $\mathbf{b}$, with $\mathbf{b} \ll_{n,d} 1$, such that $\dim(Z_q \cap H_{\mathbf{b}}) = n - r - 2$ and $\dim \operatorname{Sing}((Z_q \cap H_{\mathbf{b}})_q) = s - 1$, where $H_{\mathbf{b}}$ is the hyperplane in $\mathbb{P}^{n-1}$ defined by $\mathbf{b} \cdot \mathbf{x} = 0$. We can find a unimodular integer matrix M, all of whose entries are $O_{n,d}(1)$ such that the automorphism of $\mathbb{P}^{n-1}_{\mathbb{Z}}$ induced by M maps $H_{\mathbf{b}}$ onto the hyperplane $X_n = 0$, which we identify with $\mathbb{P}^{n-2} = \operatorname{Proj} \mathbb{Z}[X_1, \ldots, X_{n-1}]$. Let $\tilde{Z}_q$ be the image of $Z_q \cap H_{\mathbf{b}}$. Then

$$\tilde{Z}_q = \operatorname{Proj} \mathbb{Z}[X_1, \ldots, X_{n-1}]/(q, G_1, \ldots, G_r)$$

where $G_i(X_1, \ldots, X_{n-1}) = F_i(M^{-1}(X_1, \ldots, X_{n-1}, 0))$ for $i = 1, \ldots, r$, and each G_i is of the same degree as F_i. Obviously we have $\dim \operatorname{Sing}\tilde{Z}_q = s - 1$. Moreover,

$$N_W(X, B, q) = \sum_{\mathbf{x}_q \in \mathbb{X}_q} W\left(\frac{1}{B}\mathbf{x}\right) = \sum_{\mathbf{x}_q \in \tilde{X}_q} \tilde{W}\left(\frac{1}{B}\mathbf{x}\right),$$

where $\tilde{X}$ is the image of X under the automorphism of $\mathbb{A}^n$ induced by M and where $\tilde{W}(\mathbf{t}) = W(M^{-1}\mathbf{t})$. Then $\tilde{W}$ is supported in a cube of side $L' \ll_{n,d} L$, so we can write

$$N_W(X, B, q) = \sum_{-BL' \leq c \leq BL'} \sum_{\substack{\mathbf{x}_q \in \tilde{X}_q \\ x_n = c}} \tilde{W}\left(\frac{1}{B}\mathbf{x}\right). \tag{11}$$

For each $c \in \mathbb{Z}$, the intersection of $\tilde{X}$ with the hyperplane $x_n = c$ is isomorphic to

$$\tilde{X}_c = \operatorname{Spec} \mathbb{Z}[X_1, \ldots, X_{n-1}]/(g_1^c, \ldots, g_r^c)$$

where $g_i^c(X_1, \ldots, X_{n-1}) = f_i(X_1, \ldots, X_{n-1}, c)$ for $i = 1, \ldots, r$. For each c and i, the leading form of g_i^c is G_i, so our induction assumption applies to $\tilde{X}_c, \tilde{Z}_q$ and the new weight function $\tilde{W}_c$ on $\mathbb{R}^{n-1}$ defined by $\tilde{W}_c(\mathbf{t}) = \tilde{W}(\mathbf{t}, c)$. We get

$$\sum_{\substack{\mathbf{x}_q \in \tilde{X}_q \\ x_n = c}} \tilde{W}\left(\frac{1}{B}\mathbf{x}\right) = N_{\tilde{W}_c}(\tilde{X}_c, B, q)$$

$$= q^{-r} N_{\tilde{W}_c}(\mathbb{A}^{n-1}, B, q) + O_{n,d,L}\left(D_{n+1} B^s q^{(n-r-s-2)/2}(B + q^{1/2})\right).$$

We shall now add the contributions from all c in the interval $[-BL', BL']$. Observe that

$$\begin{aligned}\sum_{-BL'\leq c\leq BL'} N_{\tilde{W}_c}(\mathbb{A}^{n-1}, B, q) &= \sum_{-BL'\leq c\leq BL'} \sum_{\mathbf{y}\in\mathbb{Z}^{n-1}} \tilde{W}\left(\frac{1}{B}(\mathbf{y}, c)\right) \\ &= \sum_{\mathbf{x}\in\mathbb{Z}^n} W\left(\frac{1}{B}M^{-1}\mathbf{x}\right) = \sum_{\mathbf{x}'\in\mathbb{Z}^n} W\left(\frac{1}{B}\mathbf{x}'\right) \\ &= N_W(\mathbb{A}^n, B, q),\end{aligned}$$

since M is unimodular. Thus, summing according to (11) we deduce that

$$\begin{aligned}N_W(X, B, q) = q^{-r} N_W(\mathbb{A}^n, B, q) \\ + O_{n,d,L}\left(D_{n+1} B^{s+1} q^{(n-r-s-2)/2}(B + q^{1/2})\right)\end{aligned}$$

and the induction step is finished. □

4. Proof of the Main Result

The aim of this section is to prove Theorems 1 and 2.

Proof of Theorem 2. Throughout the proof, any implicit constant is allowed to depend only on n and d, and we will omit the subscripts n, d from the O- and $\ll$-notation.

Note. It will suffice to prove the theorem under the somewhat weaker hypothesis that $p < 2B+1 < q$, but with the additional assumption that $2B+1$ is a multiple of p. We will now prove that the general case follows from this case. If p and q are given primes and B is an arbitrary real number such that $2p < 2B + 1 < q - p$, then there are real numbers B_1 and B_2, with $B \ll B_1 \leq B \leq B_2 \ll B$, such that $2B_1 + 1$ and $2B_2 + 1$ are multiples of p and $p < 2B_i + 1 < q$ for $i = 1, 2$. Indeed, we can take $B_1 = B - \rho/2$ and $B_2 = B + (p - \rho)/2$, where $2B + 1 \equiv \rho \pmod p$ and $0 \leq \rho < p$. We have

$$\begin{aligned}N(X, B, pq) - \frac{(2B+1)^n}{p^r q^r} &\leq N(X, B_2, pq) - \frac{(2B+1)^n}{p^r q^r} \\ &= N(X, B_2, pq) - \frac{(2B_2+1)^n}{p^r q^r} + O(B^{n-1} p^{-r+1} q^{-r}),\end{aligned}$$

and similarly

$$N(X, B, pq) - \frac{(2B+1)^n}{p^r q^r} \geq N(X, B_1, pq) - \frac{(2B_1+1)^n}{p^r q^r} + O(B^{n-1} p^{-r+1} q^{-r}).$$

Thus, if we assume Theorem 2 to be true for B_1 and B_2, then we see that it must also hold for B, since $B_1, B_2 \asymp B$.

From now on we assume that $2B + 1$ is a multiple of p between p and q. To facilitate the notation we introduce the characteristic function of the box $\mathsf{B} = [-B, B]^n$,

$$\chi_{\mathsf{B}}(\mathbf{x}) = \begin{cases} 1 & \text{if } \max |x_i| \leq B, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$N := N(X,B,pq) = \sum_{\substack{\mathbf{x}\in\mathbb{Z}^n \\ pq|f_i(\mathbf{x})}} \chi_{\mathsf{B}}(\mathbf{x}) = \sum_{\substack{\mathbf{w}\in\mathbb{F}_p^n \\ p|f_i(\mathbf{w})}} \sum_{\substack{\mathbf{x}\equiv\mathbf{w}(p) \\ q|f_i(\mathbf{x})}} \chi_{\mathsf{B}}(\mathbf{x}).$$

The "expected value" of the inner sum is

$$K := p^{-n}q^{-r}(2B+1)^n,$$

so let us write

$$N = \sum_{\substack{\mathbf{w}\in\mathbb{F}_p^n \\ p|f_i(\mathbf{w})}} \left(\sum_{\substack{\mathbf{x}\equiv\mathbf{w}(p) \\ q|f_i(\mathbf{x})}} \chi_{\mathsf{B}}(\mathbf{x}) - K \right) + K \sum_{\substack{\mathbf{w}\in\mathbb{F}_p^n \\ p|f_i(\mathbf{w})}} 1.$$

If we denote the first of these two sums by S, then, using Lemma 6, we get

$$\begin{aligned} N &= S + K\#X(\mathbb{F}_p) = S + K\left(p^{n-r} + O(p^{(n-r+1)/2})\right) \\ &= \frac{(2B+1)^n}{p^r q^r} + S + O(B^n p^{-(n+r-1)/2} q^{-r}). \end{aligned} \tag{12}$$

Now we turn our attention to S. By Cauchy's inequality

$$S^2 \le \left(\sum_{\substack{\mathbf{w}\in\mathbb{F}_p^n \\ p|f_i(\mathbf{w})}} 1 \right) \left(\sum_{\substack{\mathbf{w}\in\mathbb{F}_p^n \\ p|f_i(\mathbf{w})}} \left(\sum_{\substack{\mathbf{x}\equiv\mathbf{w}(p) \\ q|f_i(\mathbf{x})}} \chi_{\mathsf{B}}(\mathbf{x}) - K \right)^2 \right),$$

so that, if we denote the expression in the rightmost parentheses by Σ, and apply Lemma 5, we get

$$S \ll p^{(n-r)/2}\Sigma^{1/2}. \tag{13}$$

We estimate Σ by adding some extra (positive) terms:

$$\begin{aligned} \Sigma &\le \sum_{\mathbf{w}\in\mathbb{F}_p^n} \sum_{\mathbf{a}\in\mathbb{F}_q^r} \left(\sum_{\substack{\mathbf{x}\equiv\mathbf{w}(p) \\ f_i(\mathbf{x})\equiv a_i(q)}} \chi_{\mathsf{B}}(\mathbf{x}) - K \right)^2 \\ &= \sum_{\mathbf{w}\in\mathbb{F}_p^n} \sum_{\mathbf{a}\in\mathbb{F}_q^r} \left(\sum_{\substack{\mathbf{x}\equiv\mathbf{w}(p) \\ f_i(\mathbf{x})\equiv a_i(q)}} \chi_{\mathsf{B}}(\mathbf{x}) \right)^2 - 2K \sum_{\mathbf{x}\in\mathbb{Z}^n} \chi_{\mathsf{B}}(\mathbf{x}) + p^n q^r K^2. \end{aligned}$$

The middle term here is just $-2p^n q^r K^2$, so, denoting the first sum by $\mathcal{Z}$ we get

$$\Sigma \le \mathcal{Z} - p^n q^r K^2. \tag{14}$$

To analyze $\mathcal{Z}$, we write

$$\mathcal{Z} = \sum_{\mathbf{x}\in\mathbb{Z}^n} \chi_{\mathsf{B}}(\mathbf{x}) \sum_{\substack{\mathbf{x}'\in\mathbb{Z}^n \\ \mathbf{x}'\equiv\mathbf{x}(p) \\ f_i(\mathbf{x}')\equiv f_i(\mathbf{x})(q)}} \chi_{\mathsf{B}}(\mathbf{x}').$$

We make the variable change $\mathbf{x}' = \mathbf{x} + p\mathbf{y}$ in the second sum, introducing the "differenced" polynomials

$$f_i^{\mathbf{y}}(\mathbf{x}) = f_i(\mathbf{x} + p\mathbf{y}) - f_i(\mathbf{x}).$$

If $\mathsf{B}_{\mathbf{y}}$ denotes the new box $\mathsf{B} \cap (\mathsf{B} - p\mathbf{y}) = \{\mathbf{x} \in \mathbb{Z}^n; \mathbf{x} \in \mathsf{B}, \mathbf{x} + p\mathbf{y} \in \mathsf{B}\}$, we get

$$\begin{aligned} \mathcal{Z} &= \sum_{\mathbf{x} \in \mathbb{Z}^n} \chi_{\mathsf{B}}(\mathbf{x}) \sum_{\substack{\mathbf{y} \in \mathbb{Z}^n \\ f_i^{\mathbf{y}}(\mathbf{x}) \equiv 0 (q)}} \chi_{\mathsf{B}}(\mathbf{x} + p\mathbf{y}) \\ &= \sum_{\mathbf{y} \in \mathbb{Z}^n} \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ f_i^{\mathbf{y}}(\mathbf{x}) \equiv 0 (q)}} \chi_{\mathsf{B}_{\mathbf{y}}}(\mathbf{x}). \end{aligned}$$

Let us define

$$\Delta(\mathbf{y}) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ f_i^{\mathbf{y}}(\mathbf{x}) \equiv 0 (q)}} \chi_{\mathsf{B}_{\mathbf{y}}}(\mathbf{x}) - q^{-r} \sum_{\mathbf{x} \in \mathbb{Z}^n} \chi_{\mathsf{B}_{\mathbf{y}}}(\mathbf{x}),$$

and write

$$\mathcal{Z} = \sum_{\mathbf{y} \in \mathbb{Z}^n} \Delta(\mathbf{y}) + q^{-r} \sum_{\mathbf{y} \in \mathbb{Z}^n} \sum_{\mathbf{x} \in \mathbb{Z}^n} \chi_{\mathsf{B}_{\mathbf{y}}}(\mathbf{x}).$$

Now one sees that, since we are assuming $p \mid (2B+1)$,

$$\begin{aligned} \sum_{\mathbf{y} \in \mathbb{Z}^n} \sum_{\mathbf{x} \in \mathbb{Z}^n} \chi_{\mathsf{B}_{\mathbf{y}}}(\mathbf{x}) &= \prod_{i=1}^{n} \left(\sum_{y_i \in \mathbb{Z}} \sum_{x_i \in \mathbb{Z}} \chi_{[-B,B]}(x_i) \chi_{[-B-py_i, B-py_i]}(x_i) \right) \\ &= \left(\frac{(2B+1)^2}{p} \right)^n = p^n q^{2r} K^2. \end{aligned}$$

In other words, $\mathcal{Z} = \sum \Delta(\mathbf{y}) + p^n q^r K^2$, so we get by (14)

$$\Sigma \le \sum_{\mathbf{y} \in \mathbb{Z}^n} \Delta(\mathbf{y}). \tag{15}$$

Our task is now to estimate $\sum \Delta(\mathbf{y})$. To this end, denote the leading forms of $f_1^{\mathbf{y}}, \dots, f_r^{\mathbf{y}}$ by $F_1^{\mathbf{y}}, \dots F_r^{\mathbf{y}}$ and let

$$\begin{aligned} X_{\mathbf{y}} &= \mathrm{Spec}\ \mathbb{F}_q[x_1, \dots, x_n]/(f_1^{\mathbf{y}}, \dots, f_r^{\mathbf{y}}), \\ Z_{\mathbf{y}} &= \mathrm{Proj}\ \mathbb{F}_q[x_1, \dots, x_n]/(F_1^{\mathbf{y}}, \dots, F_r^{\mathbf{y}}). \end{aligned}$$

Observe that for each $i = 1, \dots, r$ we have

$$F_i^{\mathbf{y}} = p\mathbf{y} \cdot \nabla F_i,$$

unless the right hand side vanishes identically (mod q) in $\mathbf{x}$. Due to the non-singularity of Z, this happens only if $\mathbf{y} \equiv 0 \pmod q$. Indeed, if $\mathbf{y} \cdot \nabla F_i$ is identically zero for some i, then, in the notation of Lemma 3, $S_{\mathbf{y}} = \mathbb{P}_{\mathbb{F}_q}^{n-1}$. Thus $\mathbf{y}$ is a point on the affine cone over $T_{n-1} = \emptyset$.

Lemma 7.

$$\begin{aligned} \sum_{\mathbf{y} \in \mathbb{Z}^n} \Delta(\mathbf{y}) \ll\ & B^{n+1} p^{-n} q^{(n-r-1)/2} (\log q)^n + B^{n+1} p^{-r} q^{-1/2} (\log q)^n \\ & + B^n p^{-n} q^{(n-r)/2} (\log q)^n + B^n (\log q)^n. \end{aligned}$$

Proof. First, we note that $\Delta(\mathbf{y}) = 0$ for all $\mathbf{y}$ with $|\mathbf{y}| \geq (2B+1)/p$. Thus, we only need to sum over the set

$$\mathcal{B} = \{\mathbf{y} \in \mathbb{Z}^n; |\mathbf{y}| < (2B+1)/p\}.$$

Let us decompose this set into subsets: $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_r$, where

$$\mathcal{B}_\sigma = \{\mathbf{y} \in \mathcal{B}; \mathrm{codim} Z_{\mathbf{y}} = \sigma\}, \quad \sigma = 0, \ldots, r.$$

For $\mathbf{y} \in \mathcal{B}_r$, we can use Theorem 1 of the Appendix [17] to get

$$\Delta(\mathbf{y}) \ll_{n,d} B^{s(\mathbf{y})+1} q^{(n-r-s(\mathbf{y})-2)/2}(B + q^{1/2})(\log q)^n,$$

where $s(\mathbf{y}) = \dim \mathrm{Sing}(Z_{\mathbf{y}})$. Next we need to find out how often each value of $s(\mathbf{y})$ arises. We consult Lemma 3. Since $Z_{\mathbf{y}}$ is a complete intersection of codimension r, the Jacobian Criterion implies that $\mathrm{Sing}(Z_{\mathbf{y}}) = S_{\mathbf{y}}$. Thus, the set of all $\mathbf{y}$ such that $s(\mathbf{y}) = s$ is contained in the affine cone over the set T_s. By part (ii) of Lemma 3, T_s has projective dimension $n - s - 2$, so by part (iii) and Lemma 5, we get

$$\#\{\mathbf{y} \in \mathcal{B}_r; s(\mathbf{y}) = s\} \ll_{n,d} \left(\frac{B}{p}\right)^{n-s-1}.$$

Summing, we get

$$\begin{aligned}\sum_{\mathbf{y} \in \mathcal{B}_r} \Delta(\mathbf{y}) &\ll \sum_{s=-1}^{n-r-1} \left(\frac{B}{p}\right)^{n-s-1} B^{s+1} q^{(n-r-s-2)/2}(B + q^{1/2})(\log q)^n \\ &\ll B^n (\log q)^n \left(Bp^{-n} q^{(n-r-1)/2} + p^{-n} q^{(n-r)/2} + Bp^{-r} q^{-1/2} + p^{-r}\right).\end{aligned}$$

It remains to consider the contribution from $\mathbf{y} \in \mathcal{B}_\sigma$, $\sigma < r$. We make a simple observation about the varieties $Z_{\mathbf{y}}$ originating from these values of $\mathbf{y}$: now the set $S_{\mathbf{y}}$ is very large.

Lemma 8. *Let $G_1, \ldots, G_r$ be forms in the variables $X_1, \ldots, X_n$. Let*

$$V = \{G_1 = \ldots = G_r = 0\} \subseteq \mathbb{P}^{n-1}$$

and let

$$W = \left\{G_1 = \ldots = G_r = 0, \ \mathrm{rank}\left(\frac{\partial G_i}{\partial X_j}\right) < r\right\}.$$

Suppose that $\mathrm{codim}(V) = \sigma < r$. *Then W contains all irreducible components of V of dimension $n - 1 - \sigma$. In particular,* $\dim W = n - 1 - \sigma$.

Proof. Let V' be an irreducible component of V with $\dim V' = n - 1 - \sigma$. Assume that there were a point $P \in V'$ such that $\mathrm{rank}\left(\frac{\partial G_i}{\partial X_j}\right)(P) = r$. Then we would have

$$\dim T_P V' = n - 1 - r < n - 1 - \sigma = \dim V',$$

a contradiction. Thus $V' \subseteq W$. □

We see that if $\mathbf{y} \in \mathcal{B}_\sigma$, then, by Lemma 8, $\dim S_{\mathbf{y}} = n - 1 - \sigma$. Recalling that, in the notation of Lemma 3, $T_{n-1-\sigma}$ has dimension less than or equal to $\sigma - 1$, we must have

$$|\mathcal{B}_\sigma| \ll \left(\frac{B}{p}\right)^\sigma.$$

Using Lemma 5 to get the trivial estimate $\Delta(\mathbf{y}) \ll B^{n-\sigma}$ for $\mathbf{y} \in \mathcal{B}_\sigma$, we compute the contribution from the $\mathcal{B}_\sigma$, $\sigma < r$:

$$\sum_{\sigma=0}^{r-1} \sum_{\mathbf{y}\in\mathcal{B}_\sigma} \Delta(\mathbf{y}) = \sum_{\sigma=0}^{r-1} \left(\frac{B}{p}\right)^{\sigma} B^{n-\sigma} = B^n \sum_{\sigma=0}^{r-1} p^{-\sigma} \ll B^n.$$

In sum, then,

$$\begin{aligned}\sum_{\mathbf{y}\in\mathcal{B}} \Delta(\mathbf{y}) &= \sum_{\sigma=0}^{r} \sum_{\mathbf{y}\in\mathcal{B}_\sigma} \Delta(\mathbf{y}) \\ &\ll B^{n+1}p^{-n}q^{(n-r-1)/2}(\log q)^n + B^{n+1}p^{-r}q^{-1/2}(\log q)^n \\ &\quad + B^n p^{-n} q^{(n-r)/2}(\log q)^n + B^n(\log q)^n,\end{aligned}$$

and Lemma 7 follows. □

Working our way back through the estimates (15), (13) and (12), we now arrive at

$$\begin{aligned} N = \frac{(2B+1)^n}{p^r q^r} &+ O\Big(B^{(n+1)/2}p^{-r/2}q^{(n-r-1)/4}(\log q)^{n/2} \\ +B^{(n+1)/2}p^{(n-2r)/2}q^{-1/4}(\log q)^{n/2} &+ B^{n/2}p^{-r/2}q^{(n-r)/4}(\log q)^{n/2} \\ +B^{n/2}p^{(n-r)/2}(\log q)^{n/2} &+ B^n p^{-(n+r-1)/2}q^{-r}\Big). \end{aligned} \tag{16}$$

This completes the proof of Theorem 2. □

We shall now prove Theorem 1, where the modest dependence upon $\|F_i\|$ is due to the following lemma.

Lemma 9. *Let X and Z be defined as in Theorem 2, and assume that $Z_{\mathbb{Q}}$ is non-singular of dimension $n-1-r$. If $P \geq \left(\sum_{i=1}^r \log\|F_i\|\right)^{1+\delta}$, then there is a prime $p \asymp_\delta P$ such that Z_p is non-singular of dimension $n-1-r$.*

Proof. As in the proof of Lemma 2, let $\mathbf{P} = \mathbb{P}_1 \times \ldots \times \mathbb{P}_r$, where $\mathbb{P}_i$ is the projective space parametrizing all hypersurfaces of degree d_i in $\mathbb{P}^{n-1}_{\mathbb{Z}}$. By a semicontinuity argument analogous to that in the proof of Lemma 2, the subset $U \subseteq \mathbf{P}$ defined by

$$(G_1,\ldots,G_r) \in U \Leftrightarrow V(G_1,\ldots,G_r) \text{ is non-singular of codimension } r,$$

is Zariski open, its complement thus being defined by multihomogeneous polynomials $H_1,\ldots,H_t$ in the coefficients of $G_1,\ldots,G_r$. Now by the hypotheses, for some j we must have $H_j(F_1,\ldots,F_r) \neq 0$. We observe firstly that

$$\log|H_j(F_1,\ldots,F_r)| \ll_{n,d} \sum_{i=1}^r \log\|F_i\|.$$

Secondly, for an arbitrary positive number A we have

$$\#\{p > AP; p \mid H_j(F_1,\ldots,F_r)\} \ll \frac{\log|H_j(F_1,\ldots,F_r)|}{\log AP}.$$

Thus, if we choose A large enough, there are fewer than

$$a := \left\lceil \sum_{i=1}^{r} \log \|F_i\| \right\rceil$$

such primes. Hence among the a first prime numbers greater than AP, there must be one prime p such that $p \nmid H_j(F_1, \dots, F_r)$. By Chebyshev's Theorem it is possible to find an interval $[AP, c_\delta AP]$ that contains more than $P^{1/(1+\delta)}$ primes. Since $P \geq a^{1+\delta}$, this interval must contain p. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Theorem 2 yields in particular that

$$\begin{aligned} N(X, B, pq) \ll_{n,d} \Bigg[& \frac{B^n}{p^r q^r} + B^{(n+1)/2} p^{-r/2} q^{(n-r-1)/4} \\ & + B^{(n+1)/2} p^{(n-2r)/2} q^{-1/4} + B^{n/2} p^{-r/2} q^{(n-r)/4} + B^{n/2} p^{(n-r)/2} \\ & + B^n p^{-(n+r-1)/2} q^{-r} + B^{n-1} p^{-r+1} q^{-r} \Bigg] (\log q)^{n/2} \end{aligned}$$

Thus we want to optimize the expression

$$\begin{aligned} & \frac{B^n}{p^r q^r} + B^{(n+1)/2} p^{-r/2} q^{(n-r-1)/4} + B^{(n+1)/2} p^{(n-2r)/2} q^{-1/4} \\ & + B^{n/2} p^{-r/2} q^{(n-r)/4} + B^{n/2} p^{(n-r)/2} + B^n p^{-(n+r-1)/2} q^{-r} + B^{n-1} p^{-r+1} q^{-r} \end{aligned}$$

by choosing appropriate p and q. It turns out that

$$p \asymp B^{1 - \frac{5nr - r^2 - 5r}{n^2 + 4nr - n - r^2 - r}}, \qquad q \asymp B^{2 - \frac{2(4nr - r^2)}{n^2 + 4nr - n - r^2 - r}}. \tag{17}$$

would be an optimal choice. (Note that the last two terms in the expression are dominated by the first term, so the optimization consists of trying to get the first five terms to be of approximately equal order of magnitude.) The restriction $n \geq 4r + 2$ ensures that (17) is compatible with the requirement that $2p < 2B + 1 < q - p$. The trouble is now to make sure that the intervals specified in (17) contain "good" primes, that is, primes such that both Z_p and Z_q are non-singular of dimension $n - 1 - r$.

For B large enough, (17) is a valid choice. Indeed, if

$$B \geq \left(\sum_{i=1}^{r} \log \|F_i\| \right)^{e_1}, \text{ where}$$

$$e_1 = \left(1 - \frac{5nr - r^2 - 5r}{n^2 + 4nr - n - r^2 - r} \right)^{-1} \left(1 + \frac{1}{2r} \right),$$

then by Lemma 9 (with $\delta = (2r)^{-1}$) we can choose p and q, satisfying (17), such that Theorem 2 holds. For these B, and with p and q subject to (17), Theorem 2 implies that

$$N(X, B) \leq N(X, B, pq) \ll_{n,d} B^{n - 3r + r^2 \frac{13n - 3r - 5}{n^2 + 4nr - n - r^2 - r}} (\log B)^{n/2}.$$

For $B < (\sum_{i=1}^{r} \log \|F_i\|)^{e_1}$, we use the trivial estimate

$$N(X, B) \ll_{n,d} B^{n-r}$$

obtained by Lemma 5 to get

$$N(X,B) \ll_{n,d} B^{n-3r+r^2\frac{13n-3r-5}{n^2+4nr-n-r^2-r}} \left(\sum_{i=1}^{r} \log \|F_i\|\right)^{e_2}, \text{ where}$$

$$e_2 = e_1 \left(2r - r^2 \frac{13n-3r-5}{n^2+4nr-n-r^2-r}\right) \leq 2r+1.$$

This proves the theorem. □

Remark. If we are content with just an upper bound for $N(X,B,pq)$ in Theorem 2, we can get rid of the factor $(\log q)^{n/2}$ and thus prove a slightly sharpened version of Theorem 1, without the factor $(\log B)^{n/2}$. This can be achieved by introducing an infinitely differentiable weight function into the proof of Theorem 2, as in [10], and using Theorem 3 in the place of [17, Thm. 1] . More precisely, if instead of $N(X,B,pq)$ we consider the weighted counting function

$$N_W(X,B,pq) = \sum_{\substack{\mathbf{x}\in\mathbb{Z}^n \\ \mathbf{x}_p\in X_p \\ \mathbf{x}_q \in X_q}} W\left(\frac{1}{2B}\mathbf{x}\right),$$

where W is a non-negative, infinitely differentiable weight function on $\mathbb{R}^n$ supported in $[-1,1]^n$, we can prove an asymptotic formula for $N_W(X,B,pq)$ where the main term is

$$p^{-r}q^{-r}\sum_{\mathbf{x}\in\mathbb{Z}^n} W\left(\frac{1}{2B}\mathbf{x}\right).$$

The error term would then consist of the first four error terms of Theorem 2 with the factor $(\log q)^{n/2}$ removed, the fifth error term unchanged, and an additional term which is $o\left(p^{-r}q^{-r}B^n\right)$ and thus negligible for the application of Theorem 1. To prove this asymptotic formula one imitates the proof of Theorem 2, with $\chi_{\mathsf{B}}(\mathbf{x})$ replaced by $W\left(\frac{1}{2B}\mathbf{x}\right)$ and K by

$$K_W = p^{-n}q^{-r}\sum_{\mathbf{x}\in\mathbb{Z}^n} W\left(\frac{1}{2B}\mathbf{x}\right).$$

One is then led to estimate expressions

$$\Delta_W(\mathbf{y}) = \sum_{\substack{\mathbf{x}\in\mathbb{Z}^n \\ f_i^{\mathbf{y}}(\mathbf{x})\equiv 0 (q)}} W_{\mathbf{y}}(\mathbf{x}) - q^{-r}\sum_{\mathbf{x}\in\mathbb{Z}^n} W_{\mathbf{y}}(\mathbf{x}),$$

where $W_{\mathbf{y}}(\mathbf{x}) = W\left(\frac{1}{2B}\mathbf{x}\right) W\left(\frac{1}{2B}(\mathbf{x}+p\mathbf{y})\right)$. At this point we invoke Theorem 3. Here the error term, in contrast to the unweighted formula of Theorem 1 in the Appendix, contains no factor $(\log q)^n$, whence the promised improvement of the upper bound. The only main divergence from the proof of Theorem 2 lies in the calculation of the sum $\sum_{\mathbf{y}\in\mathbb{Z}^n}\sum_{\mathbf{x}\in\mathbb{Z}^n} W_{\mathbf{y}}(\mathbf{x})$. This can be done by means of Poisson summation (see [10, p. 20]) and gives rise to the additional error term mentioned above.

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Appendix

PER SALBERGER[1,2]

The aim of this note is to count $\mathbb{F}_q$-points in boxes on affine varieties. If $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ and q is a prime, then we set $\mathbf{x}_q = (x_1 + q\mathbb{Z}, \ldots, x_n + q\mathbb{Z}) \in \mathbb{F}_q^n$. If B is a box in $\mathbb{R}^n$ and W a closed subscheme of $\mathbb{A}^n_{\mathbb{Z}}$, then we let

$$N(W, \mathsf{B}, q) = \#\left\{\mathbf{x} = (x_1, \ldots, x_n) \in \mathsf{B} \cap \mathbb{Z}^n : \mathbf{x}_q \in W(\mathbb{F}_q)\right\}.$$

Lemma 1. *Let q be a prime and B be a box in $\mathbb{R}^n$ such that each side has length at most $2B < q$. Let $f_1, \ldots f_r, l_1, \ldots, l_{s+1}$ be polynomials in $\mathbb{Z}[x_1, \ldots, x_n]$, $r+s+1 \leq n$ such that the leading forms $F_1, \ldots, F_r$ of $f_1, \ldots f_r$ are of degree ≥ 2 and the leading forms $L_1, \ldots, L_{s+1}$ of $l_1, \ldots, l_{s+1}$ are of degree 1. Let*

$$X = \text{Spec } \mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_r, l_1 \ldots, l_{s+1}),$$
$$\Lambda = \text{Spec } \mathbb{Z}[x_1, \ldots, x_n]/(l_1 \ldots, l_{s+1}) \text{ and}$$
$$Z = \text{Proj } \mathbb{Z}[x_1, \ldots, x_n]/(F_1, \ldots, F_r, L_1, \ldots, L_{s+1}).$$

Suppose that $Z_q = Z_{\mathbb{F}_q}$ is non-singular of codimension $r+s+1$ in $\mathbb{P}^{n-1}_{\mathbb{F}_q}$. Then

$$N(X, \mathsf{B}, q) = q^{-r} N(\Lambda, \mathsf{B}, q) + O_{n,d}(q^{(n-r-s-2)/2}(B + q^{1/2})(\log q)^n),$$

where $d = \max_i \deg F_i$.

Proof. If $r+s+1 = n$, then $\#X(\mathbb{F}_q) \leq d^n$ by the theorem of Bezout and hence $N(X, \mathsf{B}, q) - q^{-r} N(\Lambda, \mathsf{B}, q) \ll_{n,d} 1 \leq q^{(n-r-s-2)/2}(B + q^{1/2})$. If $r+s+1 = n-1$, then $N(X, \mathsf{B}, q) = O_{n,d}(B)$ by Lemma 5 in the main paper, so that $N(X, \mathsf{B}, q) - q^{-r} N(\Lambda, \mathsf{B}, q) \ll_{n,d} B \leq q^{(n-r-s-2)/2}(B + q^{1/2})$. We may thus assume that $r+s+1 \leq n-2$. Then, Z_q is geometrically connected since it is a complete intersection of dimension ≥ 1 (see [1, Ex. II.8.4(c)]). It is thus geometrically integral since it is non-singular. Therefore, by the homogeneous Nullstellensatz we obtain that a linear form $\mathbf{a} \cdot \mathbf{x} = a_1 x_1 + \ldots + a_n x_n$, $(a_1, \ldots, a_n) \in \mathbb{F}_q^n$ vanishes on Z_q if and only if $\mathbf{a} \cdot \mathbf{x}$ belongs to the linear $\mathbb{F}_q$-space V of linear forms in $(x_1, \ldots, x_n)$ generated by the reductions of $L_1, \ldots, L_{s+1}$ (mod q). We now follow the approach of [3]. Let $S_1(\mathbf{a}) = \sum_{\mathbf{b} \in \mathsf{B} \cap \mathbb{Z}^n} e_q(-\mathbf{a} \cdot \mathbf{b})$ and $S_2(\mathbf{a}) = \sum_{\mathbf{x} \in X(\mathbb{F}_q)} e_q(\mathbf{a} \cdot \mathbf{x})$ for $\mathbf{a} \in \mathbb{F}_q$. Then,

$$N(X, \mathsf{B}, q) = q^{-n} \sum_{\mathbf{a} \in \mathbb{F}_q^n} S_1(\mathbf{a}) S_2(\mathbf{a}).$$

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Let $\Pi_{\mathbf{a}} = \text{Proj } \mathbb{F}_q[x_1, \dots, x_n]/(a_1x_1 + \dots + a_nx_n)$ for $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{F}_q^n$. Then,

$$q^{-(s+1)} \sum_{\mathbf{a}\in V} S_1(\mathbf{a})S_2(\mathbf{a}) = q^{-(s+1)} \sum_{\mathbf{a}\in V} \sum_{\mathbf{x}\in X(\mathbb{F}_q)} \sum_{\mathbf{b}\in \mathsf{B}\cap\mathbb{Z}^n} e_q(\mathbf{a}\cdot(\mathbf{x}-\mathbf{b}))$$

$$= \sum_{\mathbf{x}\in X(\mathbb{F}_q)} \sum_{\mathbf{b}\in \mathsf{B}\cap\mathbb{Z}^n} \prod_{i=1}^{s+1} \left(\frac{1}{q} \sum_{a\in\mathbb{F}_q} e_q(aL_i(\mathbf{x}-\mathbf{b})) \right)$$

$$= \#\left\{(\mathbf{x},\mathbf{b}) \in X(\mathbb{F}_q) \times (\mathsf{B}\cap\mathbb{Z}^n) : L_i(\mathbf{x}-\mathbf{b}) \equiv 0 \pmod q, 1 \le i \le s+1\right\}$$

$$= \#\left\{(\mathbf{x},\mathbf{b}) \in X(\mathbb{F}_q) \times (\mathsf{B}\cap\mathbb{Z}^n) : l_i(\mathbf{b}) \equiv 0 \pmod q, 1 \le i \le s+1\right\}$$

$$= \#X(\mathbb{F}_q)N(\Lambda, \mathsf{B}, q).$$

Here $\#X(\mathbb{F}_q) = q^{n-r-s-1} + O_{n,d}(q^{(n-r-s)/2})$ by Lemma 6 in the main paper. There is also a set of $n-s-1$ indices $i(1), \dots, i(n-s-1) \in \{1, \dots, n\}$ such that any $\mathbf{b} = (b_1, \dots, b_n) \in \mathsf{B}\cap\mathbb{Z}^n$ with $\mathbf{b}_q \in \Lambda(\mathbb{F}_q)$ is uniquely determined by $(b_{i(1)}, \dots, b_{i(n-s-1)})$. Hence, $\#N(\Lambda, \mathsf{B}, q) \ll_n B^{n-s-1}$. We have thus shown that

$$q^{-n} \sum_{\mathbf{a}\in V} S_1(\mathbf{a})S_2(\mathbf{a}) = q^{-(n-s-1)} \#X(\mathbb{F}_q)N(\Lambda, \mathsf{B}, q)$$

$$= q^{-r}N(\Lambda, \mathsf{B}, q) + O_{n,d}(q^{-(n-s-1)+(n-r-s)/2}B^{n-s-1}).$$

As $q^{-(n-s-1)+(n-r-s)/2}B^{n-s-1} < q^{(n-r-s-2)/2}B$, we conclude that

$$q^{-n} \sum_{\mathbf{a}\in V} S_1(\mathbf{a})S_2(\mathbf{a}) = q^{-r}N(\Lambda, \mathsf{B}, q) + O_{n,d}(q^{(n-r-s-2)/2}B).$$

We now estimate $q^{-n}\sum_{\mathbf{a}\in\mathbb{F}_q^n\setminus V} S_1(\mathbf{a})S_2(\mathbf{a})$. Since $\dim Z_q \cap \Pi_{\mathbf{a}} < \dim Z_q$ for $\mathbf{a} \notin V$, we obtain from the theorem of Katz (cf. [3]) that

$$S_2(\mathbf{a}) \ll_{n,d} q^{(n-r-s+\delta)/2}$$

where $\delta = \dim \text{Sing}(Z_q \cap \Pi_{\mathbf{a}}) < \dim Z_q \in \{-1, 0\}$. As

$$\sum_{\mathbf{a}\in\mathbb{F}_q^n} |S_1(\mathbf{a})| \ll_{n,d} q^n (\log q)^n$$

(see [3]), we get that the total contribution to $q^{-n}\sum_{\mathbf{a}\in\mathbb{F}_q^n\setminus V} S_1(\mathbf{a})S_2(\mathbf{a})$ from all $\mathbf{a} \in \mathbb{F}_q^n \setminus V$ where $Z_q \cap \Pi_{\mathbf{a}}$ is non-singular is $O_{n,d}(q^{(n-r-s-1)/2}(\log q)^n)$.

To estimate the contribution from the remaining $\mathbf{a} \in \mathbb{F}_q^n$, we use that there exists a form $\Phi \in \mathbb{Z}[y_1, \dots, y_n]$ of degree $O_{n,d}(1)$ in the dual coordinates $(y_1, \dots, y_n)$ of $(x_1, \dots, x_n)$ such that $\Phi(\mathbf{a}) = 0$ in $\mathbb{Z}/q\mathbb{Z}$ for all n-tuples $\mathbf{a}$ where $Z_q \cap \Pi_{\mathbf{a}}$ is singular (cf. Lemma 2 in the main paper). Hence,

$$\sum_{\substack{\mathbf{a}\in\mathbb{F}_q^n \\ \text{Sing}(Z_q\cap\Pi_{\mathbf{a}})\neq\emptyset}} |S_1(\mathbf{a})| \le \sum_{\substack{\mathbf{a}\in\mathbb{F}_q^n \\ \Phi(\mathbf{a})=0}} |S_1(\mathbf{a})| \ll_{n,d} q^{n-1}B(\log q)^{n-1},$$

where the last inequality comes from an argument in [3]. The n-tuples $\mathbf{a}$ where $Z_q \cap \Pi_{\mathbf{a}}$ is singular will therefore contribute with

$$O_{n,d}(q^{(n-r-s-2)/2}B(\log q)^{n-1})$$

to $q^{-n}\sum_{\mathbf{a}\in\mathbb{F}_q^n} S_1(\mathbf{a})S_2(\mathbf{a})$. This completes the proof of the lemma. □

For a linear form $L = a_1x_1 + \ldots + a_nx_n \in \mathbb{Z}[x_1, \ldots, x_n]$, we will write $\|L\| = \sup(|a_1|, \ldots, |a_n|)$.

Theorem 1. *Let q be a prime and B be a box in $\mathbb{R}^n$ such that each side has length at most $2B < q$. Let $f_1, \ldots f_r$ be polynomials in $\mathbb{Z}[x_1, \ldots, x_n]$, $r < n$ with leading forms $F_1, \ldots, F_r$ of degree ≥ 2. Let*

$$X = \text{Spec } \mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_r) \text{ and}$$
$$Z = \text{Proj } \mathbb{Z}[x_1, \ldots, x_n]/(F_1, \ldots, F_r)$$

Suppose that $Z_q = Z_{\mathbb{F}_q}$ is a closed subscheme of $\mathbb{P}^{n-1}_{\mathbb{F}_q}$ of codimension r with singular locus of dimension s. Then,

$$N(X, \mathsf{B}, q) = q^{-r}N(\mathbb{A}^n_{\mathbb{Z}}, \mathsf{B}, q) + O_{n,d}(B^{s+1}q^{(n-r-s-2)/2}(B + q^{1/2})(\log q)^n),$$

where $d = \max_i \deg F_i$.

Proof. It is enough to prove the statement for q greater than some constant q_0 depending only on n and d, since for $q \ll_{n,d} 1$ we have $B \ll_{n,d} 1$ and thus, trivially, $N(X, \mathsf{B}, q) - q^{-r}N(\mathbb{A}^n_{\mathbb{Z}}, \mathsf{B}, q) \ll_{n,d} 1$. Thus, assuming that q is large enough, we choose $s + 1$ linear forms $L_1, \ldots, L_{s+1} \in \mathbb{Z}[x_1, \ldots, x_n]$ such that $\|L_i\| = O_{d,n}(1)$ and such that

$$Z^i_q = \text{Proj } \mathbb{Z}[x_1, \ldots, x_n]/(q, F_1, \ldots, F_r, L_1, \ldots, L_i)$$

is a closed subscheme of codimension $r + i$ in $\mathbb{P}^{n-1}_{\mathbb{F}_q}$ with singular locus of dimension $s - i$ for $i = 1, \ldots, s + 1$. Such forms were used already in [2] and one gets a proof of their existence from Lemma 2 in the main paper.

Let $I = L(\mathsf{B} \cap \mathbb{Z}^n)$ for the map $L : \mathbb{Z}^n \to \mathbb{Z}^{s+1}$ which sends $\mathbf{b} = (b_1, \ldots, b_n)$ to $(L_1(\mathbf{b}), \ldots, L_{s+1}(\mathbf{b}))$. Then $\#I = O_{n,d}(B^{s+1})$. Moreover, if $\mathbf{c} = (c_1, \ldots, c_{s+1}) \in \mathbb{Z}^{s+1}$, then we may apply Lemma 1 to the affine subscheme $X_{\mathbf{c}}$ of $\mathbb{A}^n_{\mathbb{Z}}$ defined by $(f_1, \ldots, f_r, L_1 - c_1, \ldots, L_{s+1} - c_{s+1})$ and conclude that

$$N(X_{\mathbf{c}}, \mathsf{B}, q) = q^{-r}N(\Lambda_{\mathbf{c}}, \mathsf{B}, q) + O_{n,d}(q^{(n-r-s-2)/2}(B + q^{1/2})(\log q)^n)$$

for $\Lambda_{\mathbf{c}} = \text{Spec } \mathbb{Z}[x_1, \ldots, x_n]/(L_1 - c_1, \ldots, L_{s+1} - c_{s+1})$. If we sum over all $\mathbf{c} = (c_1, \ldots, c_{s+1}) \in I$, then we get the desired asymptotic formula for $N(X, \mathsf{B}, q)$. This finishes the proof. □

Remark. Note that $q^{-r}N(\mathbb{A}^n_{\mathbb{Z}}, \mathsf{B}, q) = q^{-r}\#(\mathsf{B} \cap \mathbb{Z}^n)$, since different elements in $\mathsf{B} \cap \mathbb{Z}^n$ are non-congruent $\pmod q$ by the assumption on B.

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THE DENSITY OF INTEGRAL POINTS ON HYPERSURFACES OF DEGREE AT LEAST 4

OSCAR MARMON

1. Introduction

Given a polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$ we wish to study the solutions in $\mathbb{Z}^n$ to the Diophantine equation

$$f(x_1, \ldots, x_n) = 0. \tag{1}$$

We are interested in the density of solutions, that is, for a given positive real number B we want to estimate the number of solutions $\mathbf{x}$ to (1) satisfying $|\mathbf{x}| \leq B$, where $|\mathbf{x}| = \max_i |x_i|$. To this end we introduce the *counting function*

$$N(f, B) = \#\{\mathbf{x} \in \mathbb{Z}^n; f(\mathbf{x}) = 0, |\mathbf{x}| \leq B\}.$$

We shall use congruences as a tool to estimate $N(f, B)$. Thus, we introduce the counting functions

$$N(f, B, m) = \#\{\mathbf{x} \in \mathbb{Z}^n; f(\mathbf{x}) \equiv 0 \,(\mathrm{mod}\, m), |\mathbf{x}| \leq B\}.$$

Trivially, for any m, $N(f, B, m)$ is an upper bound for $N(f, B)$. We extend this notation to systems of equations in the obvious way:

$$N(f_1, \ldots, f_r, B) = \#\{\mathbf{x} \in \mathbb{Z}^n; f_1(\mathbf{x}) = \cdots = f_r(\mathbf{x}) = 0, |\mathbf{x}| \leq B\},$$
$$N(f_1, \ldots, f_r, B, m) = \#\{\mathbf{x} \in \mathbb{Z}^n; f_1(\mathbf{x}) \equiv \cdots \equiv f_r(\mathbf{x}) \equiv 0 \,(\mathrm{mod}\, m), |\mathbf{x}| \leq B\}.$$

By the *leading form* of the polynomial f we shall mean the homogeneous part of maximal degree. Heath-Brown [5] proved that for a polynomial $f \in \mathbb{Z}[X_1, \ldots, X_n]$ of degree at least 3 such that the leading form F is non-singular (i.e. defines a non-singular hypersurface in $\mathbb{P}^n_{\mathbb{C}}$), we have the estimate

$$N(f, B) \ll_F B^{n-3+15/(n+5)}$$

for $n \geq 5$. To prove this, Heath-Brown studied $N(f, B, pq)$ for two different primes p, q, and devised a version of van der Corput's method of exponential sums as a key step in the estimation of this counting function. By incorporating an exponential sum estimate by Katz [8] into Heath-Browns method, the author [9] sharpened this result slightly, to

$$N(f, B) \ll_F B^{n-3+(13n-8)/(n^2+3r-2)} (\log B)^{n/2}$$

for $n \geq 6$. Salberger [10] was able to sharpen the estimate further, through a new geometric argument. He proved

$$N(f, B) \ll_F B^{n-3+9/(n+2)} (\log B)^{n/2}$$

for $n \geq 4$.

For polynomials of degree at least 4, one can try to iterate the Weyl (or van der Corput) differencing step in [5] twice to get even sharper estimates, and that is the approach we will take in this paper. The aim is to prove the following result:

Theorem 1.1. *Let f be a polynomial in $\mathbb{Z}[x_1, \dots, x_n]$ of degree $d \geq 4$ with leading form F. Let $Z = \operatorname{Proj} \mathbb{Z}[x_1, \dots, x_n]/(F)$, and suppose that $Z_{\mathbb{Q}}$ is a non-singular subscheme of $\mathbb{P}^{n-1}_{\mathbb{Q}}$. Then, provided $n \geq 10$, we have the estimate*

$$N(f, B) \ll_F B^{n-4+36/(n+8)}.$$

It will be convenient to seek to estimate a weighted counting function rather than the original one. More precisely, let $W : \mathbb{R}^n \to [0, 1]$ be an infinitely differentiable function, supported on $[-2, 2]^n$. Then we define weighted counting functions

$$N_W(f, B, m) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ f_m(\mathbf{x})=0}} W\left(\frac{1}{B}\mathbf{x}\right).$$

Here f_m denotes the image of f in $(\mathbb{Z}/m\mathbb{Z})[x_1, \dots, x_n]$ under the homomorphism induced by the canonical epimorphism $\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$. In the proof of Theorem 1.1 we shall take W to be the function defined by

$$(2) \qquad W(\mathbf{t}) = \prod_{i=1}^{n} w(t_i/2), \text{ where } w(t) = \begin{cases} \exp(-1/(1-t^2)), & |t| < 1, \\ 0, & |t| \geq 1. \end{cases}$$

It is then clear that $N(f, B, m) \ll N_W(f, B, m)$. Approximating the characteristic function of the cube $[-B, B]^n$ with a smooth function in this way allows us to sharpen some of the estimates involved.

The proof of Theorem 1.1 is carried out in Section 3. We will use a modulus which is a product of three distinct primes $m = \pi p q$, where the primes π, p can be viewed as parameters connected to the two consecutive differencing steps. The two differencings put us in the position to apply results on the density of $\mathbb{F}_q$-rational points on a family of new varieties over $\mathbb{F}_q$, parametrized by integral n-tuples $\mathbf{y}, \mathbf{z}$. These results, behind which lie Deligne's bounds for exponential sums over non-singular varieties, become weaker as the dimensions of the singular loci of the varieties increase, and thus we need to control these dimensions. Section 2 is devoted to this problem.

2. Preliminary geometric results

The geometric arguments in this section extend those of Salberger [10]. A priori, some of our results are valid in characteristic zero only. But by standard arguments from elimination theory and Chevalley's theorem on upper semicontinuity of fibre dimension, we obtain conditions on primes p ensuring the truth of the statements in characteristic p.

Let $F(x_1, \dots, x_n)$ be a homogeneous polynomial of degree $d \geq 3$ with coefficients in a perfect field K of characteristic p, where $p > d$ or $p = 0$, and suppose that the variety $Z \subset \mathbb{P}^{n-1}_K$ defined by the vanishing of F is non-singular. Furthermore, for each $\mathbf{y} \in K^n$, define

$$F^{\mathbf{y}}(\mathbf{x}) = \mathbf{y} \cdot \nabla F(\mathbf{x}) = y_1 \frac{\partial F}{\partial x_1} + \cdots + y_n \frac{\partial F}{\partial x_n},$$

and let

$$Z_{\mathbf{y}} = \operatorname{Proj} K[x_1, \ldots, x_n]/(F, F^{\mathbf{y}}), \quad \tilde{Z}_{\mathbf{y}} = \operatorname{Proj} K[x_1, \ldots, x_n]/(F^{\mathbf{y}}).$$

Furthermore still, for each pair $\mathbf{y}, \mathbf{z}$ of n-tuples of elements of K, we define

$$F^{\mathbf{y},\mathbf{z}}(\mathbf{x}) = (\operatorname{Hess}(F))\mathbf{y} \cdot \mathbf{z} = \sum_{1 \le i,j \le n} \frac{\partial^2 F}{\partial x_i \partial x_j} y_i z_j,$$

and the corresponding projective subscheme

$$Z_{\mathbf{y},\mathbf{z}} = \operatorname{Proj} K[x_1, \ldots, x_n]/(F, F^{\mathbf{y}}, F^{\mathbf{y},\mathbf{z}}).$$

When x (or any other letter) is used to denote a K-point of $\mathbb{P}_K^{n-1}$, we will use the corresponding bold letter $\mathbf{x}$ to denote an element of K^n representing x, and vice versa, given $\mathbf{x} \in K^n \setminus \{\mathbf{0}\}$, we denote its homothety class by x.

Definition. In general, if $V \subset \mathbb{P}_K^{n-1}$ is a non-singular hypersurface defined by a homogeneous polynomial $G(x_1, \ldots, x_n)$, then the *Gauss morphism* $\mathcal{G} : V \to \mathbb{P}_K^{n-1}$ is defined by $x \mapsto [\nabla G(\mathbf{x})]$, where $\nabla G(\mathbf{x}) = (\frac{\partial G}{\partial x_1}, \ldots, \frac{\partial G}{\partial x_n})$. It can be extended to the whole of $\mathbb{P}_K^{n-1}$, since if $\nabla G(\mathbf{x}) = \mathbf{0}$ then $dG(\mathbf{x}) = \mathbf{x} \cdot \nabla G(\mathbf{x}) = 0$, so $G(\mathbf{x}) = 0$ by the assumption on the characteristic. Thus $\mathcal{G}$ is well-defined outside V.

Remark 2.1. It is easy to prove that the fibres of $\mathcal{G}$ are finite. In particular, this implies that the polynomial $G^{\mathbf{y}}$, as defined above, cannot vanish identically for $\mathbf{y} \neq \mathbf{0}$, since then the image of $\mathbb{P}_K^{n-1}$ under the Gauss map would be contained in a hyperplane.

Note that, by the remark above, $Z_{\mathbf{y}}$ is a complete intersection of codimension 2 and multidegree $(d, d-1)$, unless $\mathbf{y} = \mathbf{0}$.

For fixed $\mathbf{y}$, we wish to characterize the choices of $\mathbf{z}$ giving rise to particular values of the dimension of the singular locus of $Z_{\mathbf{y},\mathbf{z}}$. Thus, for each $s = -1, 0, \ldots, n-2$, define $T_s(Z_{\mathbf{y}})$ to be the closed subset of $z = [\mathbf{z}] \in \mathbb{P}_K^{n-1}$ such that $\dim \operatorname{Sing} Z_{\mathbf{y},\mathbf{z}} \ge s$. Also define the closed subset $T_{\deg}(Z_{\mathbf{y}}) = \{z;\ \dim Z_{\mathbf{y},\mathbf{z}} = \dim Z_{\mathbf{y}}\}$.

Lemma 2.1. *Let $n \ge 4$. Suppose that $\mathbf{y} \neq \mathbf{0}$ and that $\tilde{Z}_{\mathbf{y}}$ and $Z_{\mathbf{y}}$ are non-singular varieties. Then*

(i) $T_{\deg}(Z_{\mathbf{y}}) = \varnothing$.

(ii) *Suppose furthermore that $p = 0$. Then for each $s = -1, 0, \ldots, n-4$, we have*

$$\dim T_s(Z_{\mathbf{y}}) \le n - 2 - s.$$

Proof. To prove the first assertion of the lemma, we need only note that by the remark above, since $F^{\mathbf{y}}$ is non-singular, $F^{\mathbf{y},\mathbf{z}}$ does not vanish identically for $\mathbf{z} \neq \mathbf{0}$. Thus it has degree $d-2$. Moreover, since $Z_{\mathbf{y}}$ is a non-singular complete intersection of dimension at least 1, it is geometrically integral. If $Z_{\mathbf{y}}$ and $Z_{\mathbf{y},\mathbf{z}}$ were to have equal dimension, then $F^{\mathbf{y},\mathbf{z}}$ would have to vanish on the whole of $Z_{\mathbf{y}}$, implying, by the homogeneous Nullstellensatz, that $F^{\mathbf{y},\mathbf{z}} \in \operatorname{Rad}(F, F^{\mathbf{y}})$. However, the ideal $(F, F^{\mathbf{y}})$ is prime, hence radical, so we would have $F^{\mathbf{y},\mathbf{z}} \in (F, F^{\mathbf{y}})$, which is impossible for degree reasons. This proves that $T_{\deg}(Z_{\mathbf{y}}) = \varnothing$.

We shall now prove the second assertion. Since $\tilde{Z}_{\mathbf{y}}$ is non-singular, we can define the Gauss morphism

$$\mathcal{G} : \mathbb{P}_K^{n-1} \to \mathbb{P}_K^{n-1}, \mathbf{x} \mapsto (\xi_1, \ldots, \xi_n) = \left(\frac{\partial F^{\mathbf{y}}}{\partial x_1}, \ldots, \frac{\partial F^{\mathbf{y}}}{\partial x_n} \right).$$

Note that, using the notation $H_{\mathbf{z}}$ for the hyperplane $\mathbf{z} \cdot \boldsymbol{\xi} = 0$, we have $Z_{\mathbf{y},\mathbf{z}} = Z_{\mathbf{y}} \cap \mathcal{G}^{-1}(H_{\mathbf{z}})$. We shall recursively find a sequence of linear subspaces $\Pi_{-1}, \Pi_0, \ldots, \Pi_{n-4}$ of $\mathbb{P}_K^{n-1}$ such that $Z_{\mathbf{y}} \cap \mathcal{G}^{-1}(\Pi_s)$ is non-singular of dimension $n - 4 - s$ for $s = -1, 0, \ldots, n-4$. Let $\Pi_{-1} = \mathbb{P}_K^{n-1}$. Then $Z_{\mathbf{y}} \cap \mathcal{G}^{-1}(\Pi_{-1}) = Z_{\mathbf{y}}$ is non-singular by assumption. Suppose next that we have already found a linear subspace Π_s, $s \in \{-1, 0, \ldots, n-5\}$ of dimension $n - 4 - s$ such that $Z_{\mathbf{y},s} := Z_{\mathbf{y}} \cap \mathcal{G}^{-1}(\Pi_s)$ is non-singular , and let $\mathcal{G}_s : Z_{\mathbf{y},s} \to \Pi_s$ be the restriction of $\mathcal{G}$ to $Z_{\mathbf{y},s}$. Then, by Bertini's theorem [7, Cor 6.11(2)], we may find a hyperplane $\Pi_{s+1} \subset \Pi_s$ such that $\mathcal{G}_s^{-1}(\Pi_{s+1}) = Z_{\mathbf{y}} \cap \mathcal{G}^{-1}(\Pi_{s+1})$ is non-singular of dimension $n - 3 - s$. Here we use the fact that K has characteristic zero.

Now, for each $s = -1, 0, \ldots, n-4$, let Λ_s be the s-dimensional linear subspace of $\mathbb{P}_K^{n-1} = \operatorname{Proj} K[z_1, \ldots, z_n]$ parametrizing hyperplanes $H_{\mathbf{z}}$ such that $H_{\mathbf{z}} \supseteq \Pi_s$. We shall now prove that $T_s(Z_{\mathbf{y}}) \cap \Lambda_s = \varnothing$, and the statement (ii) will then follow from the projective intersection theorem. Therefore, suppose that $z = [\mathbf{z}] \in \Lambda_s$. Since then $H_{\mathbf{z}} \supseteq \Pi_s$, there is a linear subvariety $\Gamma_z \subseteq \mathbb{P}_K^{n-1}$ of codimension s such that $\Pi_s = H_{\mathbf{z}} \cap \Gamma_z$. By the above, however,

$$Z_{\mathbf{y}} \cap \mathcal{G}^{-1}(H_{\mathbf{z}}) \cap \mathcal{G}^{-1}(\Gamma_z) = Z_{\mathbf{y}} \cap \mathcal{G}^{-1}(\Pi_s)$$

is non-singular, so we must have

$$(\operatorname{Sing} Z_{\mathbf{y},\mathbf{z}}) \cap \mathcal{G}^{-1}(\Gamma_z) = \varnothing. \tag{3}$$

By Remark 2.1 it follows that

$$\dim \mathcal{G}^{-1}(\Gamma_z) = \dim \Gamma_z = n - 1 - s.$$

Now (3), along with the projective dimension theorem, implies that $\dim \operatorname{Sing} Z_{\mathbf{y},\mathbf{z}} \leq s - 1$. Thus we have $z \notin T_s(Z_{\mathbf{y}})$, as promised.

□

Next, let us consider the (possibly) singular case.

Lemma 2.2. *There is a constant Q, depending only on d and n, with the following property. Put*

$$\sigma = \max\{\dim \operatorname{Sing} Z_{\mathbf{y}}, \dim \operatorname{Sing} \tilde{Z}_{\mathbf{y}}\}.$$

Assume that $\sigma \leq n - 5$. Then the following holds:

(i) *Suppose that either $p = 0$ or $p \geq Q$. Then*

$$\dim T_{\mathrm{deg}}(Z_{\mathbf{y}}) \leq \sigma.$$

(ii) *Suppose that $p = 0$. Then, for each $s = 0, \ldots, n - 2 - \sigma$ we have*

$$\dim T_{\sigma+s+1}(Z_{\mathbf{y}}) \leq n - 2 - s.$$

Proof. For $\sigma = -1$, this is precisely the statement of the previous lemma, so we assume that $\sigma \geq 0$. It is then a consequence of Bertini's theorem that there exists a linear subspace $L \subseteq \mathbb{P}_K^{n-1}$ of codimension $\sigma+1$ such that $Z_{\mathbf{y}} \cap L$ and $\tilde{Z}_{\mathbf{y}} \cap L$ are non-singular. This follows from repeated application of [9, Lemma 1]. (As stated there, the lemma assumes that K be algebraically closed, but using an argument by Ballico [1] this hypothesis can be replaced by the requirement that the cardinality of K be greater than a constant depending only on d and n.)

Without loss of generality, assume that L is given by $x_n = x_{n-1} = \cdots = x_{n-\sigma} = 0$. Then $Y = Z_{\mathbf{y}} \cap L$ and $\tilde{Y} = \tilde{Z}_{\mathbf{y}} \cap L$ are non-singular subschemes of $\mathbb{P}_K^{n-\sigma-2} = \operatorname{Proj} K[x_1, \dots, x_{n-\sigma-1}]$. For every $\mathbf{z} = (z_1, \dots, z_{n-\sigma-1}, 0, \dots, 0) \in L$, we have $Z_{\mathbf{y},\mathbf{z}} \cap L = Y_{\tilde{\mathbf{z}}}$, where $\tilde{\mathbf{z}} = (z_1, \dots, z_{n-\sigma-1})$.

It is easy to prove (see [8, Lemma 3]) that for any complete intersection V in projective space, and any hyperplane H, we have

$$\dim \operatorname{Sing}(V \cap H) \geq \dim \operatorname{Sing} V - 1.$$

By repeated application of this fact, we get that

$$T_{\sigma+s+1}(Z_{\mathbf{y}}) \cap L \subseteq T_s(Y).$$

Now we can use Lemma 2.1 (note that $n - (\sigma + 1) \geq 4$, so the lemma applies) to conclude that

$$\dim T_s(Y) \leq n - (\sigma + 1) - 2 - s,$$

and hence $\dim T_{\sigma+s+1}(Z_{\mathbf{y}}) \leq n - 2 - s$ by subadditivity of codimension. This proves (ii).

Similarly, we note that $T_{\deg}(Z_{\mathbf{y}}) \cap L = \varnothing$ by Lemma 2.1, which yields the assertion (i). □

In what follows, let $F \in \mathbb{Z}[x_1, \dots, x_n]$ be a homogeneous polynomial of degree $d \geq 3$. Let

$$Z = \operatorname{Proj} \mathbb{Z}[x_1, \dots, x_n]/(F).$$

For any prime q, put

$$\begin{aligned} Z_q &= \operatorname{Proj} \mathbb{Z}[x_1, \dots, x_n]/(q, F) \\ &= \operatorname{Proj} \mathbb{F}_q[x_1, \dots, x_n]/(F_q), \end{aligned}$$

where we use the notation F_q to denote the image of a polynomial under the canonical epimorphism $\mathbb{Z} \to \mathbb{Z}/q\mathbb{Z}$. This notation will be used also for composite moduli. Next, for any $\mathbf{y} = (y_1, \dots, y_n)$ in $\mathbb{Z}^n$ we let

$$F^{\mathbf{y}}(x_1, \dots, x_n) = \mathbf{y} \cdot \nabla F(x_1, \dots, x_n) = y_1 \frac{\partial F}{\partial x_1} + \ldots + y_n \frac{\partial F}{\partial x_n},$$

and define

$$Z_{q,\mathbf{y}} = \operatorname{Proj} \mathbb{Z}[x_1, \dots, x_n]/(q, F, F^{\mathbf{y}}), \quad \tilde{Z}_{q,\mathbf{y}} = \operatorname{Proj} \mathbb{Z}[x_1, \dots, x_n]/(q, F^{\mathbf{y}}).$$

We also introduce the abbreviated notations $s_q(\mathbf{y}) = \dim \operatorname{Sing} Z_{q,\mathbf{y}}$, $\tilde{s}_q(\mathbf{y}) = \dim \operatorname{Sing} \tilde{Z}_{q,\mathbf{y}}$ and $\sigma_q(\mathbf{y}) = \max(s_q(\mathbf{y}), \tilde{s}_q(\mathbf{y}))$. Furthermore, let us define

$$T_s(Z_q) = \{y \in \mathbb{P}_{\mathbb{F}_q}^{n-1}; s_q(\mathbf{y}) \geq s\}.$$

For any pair of n-tuples $\mathbf{y}, \mathbf{z}$ we put

$$F^{\mathbf{y},\mathbf{z}}(x_1, \dots, x_n) = (\operatorname{Hess}(F))\mathbf{y} \cdot \mathbf{z} = \sum_{i,j=1}^{n} y_i z_j \frac{\partial^2 F}{\partial x_i \partial x_j}.$$

We define

$$Z_{q,\mathbf{y},\mathbf{z}} = \operatorname{Proj} \mathbb{Z}[x_1, \dots, x_n]/(q, F, F^{\mathbf{y}}, F^{\mathbf{y},\mathbf{z}})$$

and $s_q(\mathbf{y},\mathbf{z}) = \dim \operatorname{Sing} Z_{q,\mathbf{y},\mathbf{z}}$. Furthermore, as above, let

$$T_s(Z_{q,\mathbf{y}}) = \{z \in \mathbb{P}^{n-1}_{\mathbb{F}_q}; s_q(\mathbf{y},\mathbf{z}) \geq s\}.$$

The first part of the following uniformity statement is proven in [9, Lemma 2.9 (iii)].

Lemma 2.3. *The sum of the degrees of the irreducible components of $T_s(Z_q)$ is bounded in terms of n and d only. The same holds for $T_s(Z_{q,\mathbf{y}})$ for any $\mathbf{y}$.*

Definition. Let q be a prime number. We say that F satisfies the property

($\mathrm{R}_0(q)$) if Z_q is a non-singular variety.

($\mathrm{R}_1(q)$) if for every $s = -1, 0, \dots, n-3$,

$$\dim T_s(Z_q) \leq n - 2 - s.$$

($\mathrm{R}_2(q)$) if for every $\mathbf{y} \in \mathbb{F}_q^n$ and every $s = -1, 0, \dots, n-4$,

$$\dim T_{s+\sigma_q(\mathbf{y})+1}(Z_{q,\mathbf{y}}) \leq n - 2 - s.$$

We shall now derive a sufficient condition for F to satisfy the condition ($\mathrm{R}_2(q)$). Let H be the Hilbert scheme of all degree d hypersurfaces in $\mathbb{P}^{n-1}_{\mathbb{Z}}$. Homogeneous coordinates for H are given by $\mathbf{t} = (t_I)$, where I runs over all n-tuples $(i_1, \dots, i_n)$ of non-negative integers such that $i_1 + \cdots + i_n = d$. If $\mathbf{x} = (x_1, \dots, x_n)$ are homogeneous coordinates for $\mathbb{P}^{n-1}_{\mathbb{Z}}$, then $\mathbf{x}^I$ denotes the monomial $x_1^{i_1} \cdots x_n^{i_n}$. Consider the multiprojective space

$$\mathcal{P} = H \times \mathbb{P}^{n-1}_{\mathbb{Z}} \times \mathbb{P}^{n-1}_{\mathbb{Z}} \times \mathbb{P}^{n-1}_{\mathbb{Z}}$$

with multihomogeneous coordinates $(\mathbf{t},\mathbf{y},\mathbf{z},\mathbf{x})$.

Lemma 2.4. *There exists a finite number of forms $\Phi_1(\mathbf{t}), \dots, \Phi_k(\mathbf{t})$ with integer coefficients, such that a sufficient condition for the hypersurface defined by $\sum \mathbf{a}_I \mathbf{x}^I$ to satisfy ($\mathrm{R}_2(p)$) at a prime p is that p does not divide all of the integers $\Phi_i(\mathbf{a})$. Moreover, if $\sum \mathbf{a}_I \mathbf{x}^I$ defines a non-singular hypersurface over $\mathbb{Q}$, then the $\Phi_i(\mathbf{a})$ do not all vanish.*

Proof of Lemma 2.4 and Lemma 2.3. Consider the following polynomials:

$$\Phi(\mathbf{a},\mathbf{x}) = \sum a_I \mathbf{x}^I,$$

$$\Psi(\mathbf{a},\mathbf{y},\mathbf{x}) = \sum y_i \frac{\partial \Phi}{\partial x_i},$$

$$\Theta(\mathbf{a},\mathbf{y},\mathbf{z},\mathbf{x}) = \sum z_j \frac{\partial \Psi}{\partial x_j} = \sum_{i,j} y_i \frac{\partial^2 \Phi}{\partial x_i \partial x_j} z_j.$$

Let $\mathcal{M}$ be the closed subscheme of $\mathcal{P}$ defined by Φ, Ψ, Θ and all 3×3-minors of the matrix

$$\begin{bmatrix} \partial\Phi/\partial x_1 & \cdots & \partial\Phi/\partial x_n \\ \partial\Psi/\partial x_1 & \cdots & \partial\Psi/\partial x_n \\ \partial\Theta/\partial x_1 & \cdots & \partial\Theta/\partial x_n \end{bmatrix}.$$

Let $\mathcal{N}$ be the closed subscheme of $\mathcal{P}$ defined by Φ, Ψ and all 2×2-minors of the matrix

$$\begin{bmatrix} \partial\Phi/\partial x_1 & \cdots & \partial\Phi/\partial x_n \\ \partial\Psi/\partial x_1 & \cdots & \partial\Psi/\partial x_n \end{bmatrix},$$

and $\tilde{\mathcal{N}}$ the closed subscheme of $\mathcal{P}$ defined by Ψ and $\partial\Psi/\partial x_1, \dots, \partial\Psi/\partial x_n$.

Let $\pi : \mathcal{M} \to H \times \mathbb{P}^{n-1}_{\mathbb{Z}} \times \mathbb{P}^{n-1}_{\mathbb{Z}}$ be the projection onto the first three factors. Let $\mathcal{I}_s$, for $s = -1, 0, 1, \dots$, be the subset of points $P \in H \times \mathbb{P}^{n-1}_{\mathbb{Z}} \times \mathbb{P}^{n-1}_{\mathbb{Z}}$ such that $\dim \pi^{-1}(P) \geq s$. By semicontinuity of fiber dimension, this is a closed set.

To see the relevance of $\mathcal{I}_s$, let $\pi_s : \mathcal{I}_s \to H$ be the projection onto the first factor, and consider a point $h \in H(k)$ representing the hypersurface $Z \subseteq \mathbb{P}^{n-1}_k$. Then the Jacobian criterion implies that

$$\pi_s^{-1}(h) = \{h\} \times \{(\mathbf{y}, \mathbf{z}) \in \mathbb{P}^{n-1}_k \times \mathbb{P}^{n-1}_k; \dim \operatorname{Sing} Z_{\mathbf{y},\mathbf{z}} \geq s\}.$$

Lemma 2.3 follows from the closedness of $\mathcal{I}_s$, since the product of the $\mathbf{z}$-degrees of the multihomogeneous polynomials defining $\mathcal{I}_s$ gives a bound for the sum of the degrees of the irreducible components of $T_s(Z_{q,\mathbf{y}})$.

Similarly, let $p : \mathcal{N} \to H \times \mathbb{P}^{n-1}_{\mathbb{Z}}$ and $\tilde{p} : \tilde{\mathcal{N}} \to H \times \mathbb{P}^{n-1}_{\mathbb{Z}}$ be the projections onto the first two factors, and let $\mathcal{J}_r$ be the closed subset of points $P \in H \times \mathbb{P}^{n-1}_{\mathbb{Z}}$ such that $\min\{\dim p^{-1}(P), \dim \tilde{p}^{-1}(P)\} \geq r$. Suppose that $\mathcal{J}_r$ is defined by forms $U_1^{(r)}(\mathbf{t}, \mathbf{y}), \dots, U_{m_r}^{(r)}(\mathbf{t}, \mathbf{y})$. Furthermore, let $\mathcal{U}_r$ be the complement of $\mathcal{J}_r$ in $H \times \mathbb{P}^{n-1}_{\mathbb{Z}}$.

Next, let $\varpi_s : \mathcal{I}_s \to H \times \mathbb{P}^{n-1}_{\mathbb{Z}}$ be the projection onto the first two factors. Then define $\mathcal{Q}_{s,u}$, $u = -1, 0, 1, \dots$, to be the closed subset of points $P \in H \times \mathbb{P}^{n-1}_{\mathbb{Z}}$ such that $\dim \varpi_s^{-1}(P) \geq u$.

It is known that the projection $p_1 : H \times \mathbb{P}^{n-1}_{\mathbb{Z}} \to H$ onto the first factor is proper. Hence $p_1(\mathcal{Q}_{s,u})$ is a closed subset of H for all s and u, defined by some homogeneous polynomials, say $Q_1^{(s,u)}(\mathbf{t}), \dots, Q_{m_{s,u}}^{(s,u)}(\mathbf{t})$. Suppose now that the tuple $\mathbf{a} = (a_I)$ of integers represents a hypersurface $Z \subseteq \mathbb{P}^{n-1}_{\mathbb{Z}}$, and that p is a prime number at which Z does not satisfy $(\mathrm{R}_2(p))$. Then for some s and σ and some primitive n-tuple $\mathbf{y} \in \mathbb{Z}^n$, we have $(\mathbf{a}_p, \mathbf{y}_p) \in \mathcal{Q}_{\sigma+s+1,n-1-s} \cap \mathcal{U}_{r+1}$. In particular,

$$Q_i^{(\sigma+s+1,n-1-s)}(\mathbf{a}) \equiv 0 \pmod{p}, i = 1, \dots, m_{\sigma+s+1,n-1-s}.$$

This shows that the forms $Q_i^{(\sigma+s+1,n-1-s)}(\mathbf{t})$ suffice for the first assertion of the lemma.

Assume now that $Z_{\mathbb{Q}}$ is non-singular. The fact that $(\mathbf{a}_p, \mathbf{y}_p) \in \mathcal{U}_{r+1}$ implies that $p \nmid U_i^{(r+1)}(\mathbf{a}_p, \mathbf{y}_p)$ for some i, so that in particular $U_i^{(r+1)}(\mathbf{a}_p, \mathbf{y}_p) \neq 0$, implying that $(\mathbf{a}, \mathbf{y}) \in \mathcal{U}_{r+1}$. But by Lemma 2.2, $\mathcal{Q}_{\sigma+s+1,n-1-s} \cap \mathcal{U}_{r+1}$ contains no $\mathbb{Q}$-points, so for some j we must have $Q_j^{(\sigma+s+1,n-1-s)}(\mathbf{a}) \neq 0$. This proves Lemma 2.4. □

The corresponding result with $(\mathrm{R}_2(p))$ replaced by $(\mathrm{R}_1(p))$ is shown in [10, §1]. With $(\mathrm{R}_2(p))$ replaced by $(\mathrm{R}_0(p))$ it is a standard result. Thus we get the following

Corollary 2.1. *For each polynomial $F \in \mathbb{Z}[x_1, \dots, x_n]$ of degree at least 3 defining a non-singular hypersurface in $\mathbb{P}^{n-1}_{\mathbb{Q}}$, there is a finite set of primes $\mathcal{P}(F)$ such that F satisfies both $(\mathrm{R}_0(p))$, $(\mathrm{R}_1(p))$ and $(\mathrm{R}_2(p))$ for every $p \notin \mathcal{P}(F)$.*

3. Proof of the main result

We begin with some remarks on the results from the author's paper [9] that we will use.

Remark 3.1. The error term

$$D_{n+1}B^{s+1}q^{(n-r-s-2)/2}(B+q^{1/2})$$

in [9, Theorem 3.3] can be given by the simpler expression

$$D_{n+1}B^{s+2}q^{(n-r-s-2)/2}.$$

Indeed, if $q^{1/2} \gg B$, then one would have $B^{s+2}q^{(n-r-s-2)/2} \gg B^{n-r}$, so that the theorem would be true by means of a trivial estimate, such as [9, Lemma 3.1].

We shall in the proof use the weighted asymptotic formula mentioned in [9, Remark 4.4]. Let us therefore state this result. Appealing to Remark 3.1, we may simplify the error term somewhat.

Theorem 3.1. *Let $W : \mathbb{R}^n \to [0,1]$ be an infinitely differentiable function supported on $[-2,2]^n$. Let $f_1,\dots,f_r$ be polynomials in $\mathbb{Z}[x_1,\dots,x_n]$ of degree at least 3, with leading forms $F_1,\dots,F_r$. Let*

$$Z = \operatorname{Proj}\mathbb{Z}[x_1,\dots,x_n]/(F_1,\dots,F_r)$$

and suppose that p and q are primes, with $p \le B \le q$, such that both Z_p and Z_q are non-singular subvarieties of $\mathbb{P}^{n-1}_{\mathbb{F}_q}$ of dimension $n-1-r$. Then we have

$$\begin{aligned} N_W(f_1,\dots,f_r,B,pq) &- p^{-r}q^{-r}N_W(0,B,pq) \\ &\ll_{W,n,d,C} B^{(n+1)/2}p^{-r/2}q^{(n-r-1)/4} + B^{(n+1)/2}p^{(n-2r)/2}q^{-1/4} \\ &\qquad + B^n p^{-(n+r-1)/2}q^{-r} + B^{n-C/2}p^{(C-r)/2}q^{-r/2} \end{aligned}$$

for any $C > 0$, where $d = \max_i(\deg f_i)$.

The following result is standard [9, Lemma 3.1].

Proposition 3.1. *Let $X = \operatorname{Spec}\mathbb{F}_q[x_1,\dots,x_n]/(f_1,\dots,f_\rho)$ be a closed subscheme of $\mathbb{A}^n_{\mathbb{F}_q}$, and let $d = \max_i(\deg f_i)$. For any box $\mathsf{B} = [a_1-b_1, a_1+b_1]\times \dots \times [a_n-b_n, a_n+b_n]$, with $|b_i| \le B$, containing at most one representative of each congruence class modulo q, let B_q be its image in $(\mathbb{Z}/q\mathbb{Z})^n$. Then we have*

$$\#(\mathsf{B}_q \cap X(\mathbb{F}_q)) \ll_{n,\rho,d} B^{\dim X}.$$

The following asymptotic formula for the number of rational points on a complete intersection, due to Hooley [6], is a consequence of the Weil conjectures [2]. The version below is proven in [9, Lemma 3.2].

Proposition 3.2. *Let $f_1,\dots,f_r$ be polynomials in $\mathbb{F}_q[x_1,\dots,x_n]$ with leading forms $F_1,\dots,F_r$, respectively. Let*

$$X = \operatorname{Spec}\mathbb{F}_q[x_1,\dots,x_n]/(f_1,\dots,f_r),$$
$$Z = \operatorname{Proj}\mathbb{F}_q[x_1,\dots,x_n]/(F_1,\dots,F_r).$$

Suppose that $\dim Z = n-1-r$ *and let* $s = \dim \operatorname{Sing} Z$. *Then*

$$\# X(\mathbb{F}_q) = q^{n-r} + O_{n,d}\left(q^{(n-r+2+s)/2}\right),$$

where $d = \max_i(\deg F_i)$.

The following result is a simple exercise in Poisson summation. The argument appears in [5].

Lemma 3.1. *Let* $\phi : \mathbb{R}^n \to \mathbb{R}$ *be an infinitely differentiable function supported in the box* $[-M, M]^n$, *and let* D_k, *for* $k = 0, 1, 2, 3, \ldots$, *be the maximum over* $\mathbb{R}^n$ *of all partial derivatives of* ϕ *of order* k. *Let* a *and* B *be real numbers such that* $B \geq 1$ *and* $1 \leq a \leq B$. *Then we have*

$$\sum_{\mathbf{x}\in\mathbb{Z}^n} \phi\left(\frac{1}{B}\mathbf{x}\right) \sum_{\mathbf{y}\in\mathbb{Z}^n} \phi\left(\frac{1}{B}(\mathbf{x}+a\mathbf{y})\right) = a^{-n}\left(\sum_{\mathbf{x}\in\mathbb{Z}^n} \phi\left(\frac{1}{B}\mathbf{x}\right)\right)^2 + O_{n,M,k}\left(D_k B^{2n-k} a^{-n+k}\right).$$

for any $k \in \mathbb{Z}_{\geq 0}$.

Proof. Since ϕ is infinitely differentiable and compactly supported, we have for the Fourier transform $\hat{\phi}$ the estimate

$$\hat{\phi}(\boldsymbol{\xi}) \ll_{n,M,k} D_k |\boldsymbol{\xi}|^{-k}, \tag{4}$$

The function $\Phi(\mathbf{x}) = \phi((1/B)\mathbf{x})$ has the Fourier transform $\hat{\Phi}(\boldsymbol{\xi}) = B^n \hat{\phi}(B\boldsymbol{\xi})$. Thus, by Poisson's summation formula and (4), we get

$$\begin{aligned} \sum_{\mathbf{x}\in\mathbb{Z}^n} \phi\left(\frac{1}{B}\mathbf{x}\right) &= B^n \sum_{\boldsymbol{\xi}\in\mathbb{Z}^n} \hat{\phi}(B\boldsymbol{\xi}) \\ &= B^n \hat{\phi}(\mathbf{0}) + O_{n,M,k}(D_k B^{n-k}). \end{aligned} \tag{5}$$

For fixed $\mathbf{x}$, put $\psi(\mathbf{y}) = \phi((1/B)(\mathbf{x}+a\mathbf{y}))$. Then

$$\hat{\psi}(\boldsymbol{\eta}) = \left(\frac{B}{a}\right)^n \exp(-2\pi i a^{-1}\mathbf{x}\cdot\boldsymbol{\eta}) \hat{\phi}\left(\frac{B}{a}\boldsymbol{\eta}\right).$$

By Poisson's summation formula and (5) we calculate

$$\begin{aligned} \sum_{\mathbf{y}\in\mathbb{Z}^n} \phi\left(\frac{1}{B}(\mathbf{x}+a\mathbf{y})\right) &= \left(\frac{B}{a}\right)^n \exp(-2\pi i a^{-1}\mathbf{x}\cdot\boldsymbol{\eta}) \sum_{\boldsymbol{\eta}\in\mathbb{Z}^n} \hat{\phi}\left(\frac{B}{a}\boldsymbol{\eta}\right) \\ &= a^{-n}\left(B^n + O_{n,M,k}(D_k B^{n-k} a^k)\right) \\ &= a^{-n} \sum_{\mathbf{x}\in\mathbb{Z}^n} \phi\left(\frac{1}{B}\mathbf{x}\right) + O_{n,M,k}(D_k B^{n-k} a^{-n+k}), \end{aligned} \tag{6}$$

which yields the desired formula. □

The density of solutions to $f(\mathbf{x}) \equiv 0 \pmod{\pi p q}$**.** This subsection constitutes the technical heart of the proof of Theorem 1.1. In what follows, let f be a polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$ of degree $d \geq 4$, with leading form F. Let W be the infinitely differentiable weight function in (2). We shall assume the existence of three different prime numbers π, p, q, with $\pi, p \leq B \leq q$, such that F satisfies ($\mathrm{R}_0(\pi)$), ($\mathrm{R}_0(p)$), ($\mathrm{R}_1(p)$), ($\mathrm{R}_0(q)$), ($\mathrm{R}_1(q)$) and ($\mathrm{R}_2(q)$).

Lemma 3.2. *In the notation of above, we have the following results:*

(i) *Put*

$$K = \pi^{-n} p^{-1} q^{-1} N_W(0, B, \pi pq).$$

Then

$$N_W(f, B, \pi pq) = (\pi pq)^{-1} N_W(0, B, \pi pq) + O\left(\pi^{(n-1)/2} \Sigma^{1/2}\right) + O\left(B^n \pi^{-n/2} p^{-1} q^{-1}\right), \quad \text{where} \tag{7}$$

$$\Sigma = \sum_{\mathbf{u} \in \mathbb{F}_\pi^n} \left(\sum_{\substack{\mathbf{x} \equiv \mathbf{u}\,(\pi) \\ f_{pq}(\mathbf{x}) = 0}} W\left(\frac{1}{B}\mathbf{x}\right) - K \right)^2.$$

(ii) *For any* $\mathbf{y} \in \mathbb{Z}^n$, *let*

$$f^{\mathbf{y}}(\mathbf{x}) := \frac{1}{\pi}\left(f(\mathbf{x} + \pi\mathbf{y}) - f(\mathbf{x})\right), \quad W_{\mathbf{y}}(\mathbf{x}) = W(\mathbf{x}) W(\mathbf{x} + \pi\mathbf{y}),$$

and put

$$\Delta(\mathbf{y}) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ f_{pq}(\mathbf{x}) = f_{pq}(\mathbf{x} + \pi\mathbf{y}) = 0}} W_{\mathbf{y}}\left(\frac{1}{B}\mathbf{x}\right) - p^{-2} q^{-2} \sum_{\mathbf{x} \in \mathbb{Z}^n} W_{\mathbf{y}}\left(\frac{1}{B}\mathbf{x}\right).$$

Then

$$\Sigma = \sum_{\mathbf{y} \ll B/\pi} \Delta(\mathbf{y}) + O_C\left(B^{2n-C} \pi^{-n+C} p^{-2} q^{-2}\right) + \mathcal{E}_1 \tag{8}$$

for any $C > 0$.

(iii) *Suppose that* $\mathbf{y} \neq 0$. *For any* $\mathbf{z} \in \mathbb{Z}^n$, *let*

$$\begin{aligned} f^{\mathbf{z}}(\mathbf{x}) &= \frac{1}{p}(f(\mathbf{x} + p\mathbf{z}) - f(\mathbf{x})), \\ f^{\mathbf{y},\mathbf{z}}(\mathbf{x}) &= \frac{1}{p}(f_{\mathbf{y}}(\mathbf{x} + p\mathbf{z}) - f_{\mathbf{y}}(\mathbf{x})) \\ &= \frac{1}{\pi p}\big(f(\mathbf{x} + \pi\mathbf{y} + p\mathbf{z}) - f(\mathbf{x} + \pi\mathbf{y}) - f(\mathbf{x} + p\mathbf{z}) + f(\mathbf{x})\big), \\ W_{\mathbf{y},\mathbf{z}}(\mathbf{x}) &= W_{\mathbf{y}}(\mathbf{x}) W_{\mathbf{y}}(\mathbf{x} + p\mathbf{z}) \end{aligned}$$

and put

$$\Delta(\mathbf{y}, \mathbf{z}) := \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \\ f_q(\mathbf{x}) = f_q^{\mathbf{z}}(\mathbf{x}) = \\ f_q^{\mathbf{y},\mathbf{z}}(\mathbf{x}) = 0}} W_{\mathbf{y},\mathbf{z}}\left(\frac{1}{B}\mathbf{x}\right) - q^{-3} \sum_{\mathbf{x} \in \mathbb{Z}^n} W_{\mathbf{y},\mathbf{z}}\left(\frac{1}{B}\mathbf{x}\right).$$

Then

$$\Delta(\mathbf{y}) = p^{(n-2)/2} \left(\sum_{\mathbf{z} \ll B/p} \Delta(\mathbf{y}, \mathbf{z}) \right)^{1/2} + \mathcal{E}_2(\mathbf{y}) + \mathcal{E}_4(\mathbf{y}). \tag{9}$$

(iv) *Furthermore, we have*

$$\Delta(\mathbf{0}) \ll B^n p^{-1} q^{-1} + B^{(n+1)/2} p^{-1/2} q^{(n-2)/4} + B^{(n+1)/2} p^{(n-2)/2} q^{-1/4}. \tag{10}$$

All the implied constants depend only on n and d, unless otherwise specified. The $\mathcal{E}_i$ are error terms, spelled out in more detail within the proof.

Proof. Starting from the definition of $N_W(f, B, \pi pq)$, we write

$$\begin{aligned} N_W(f, B, \pi pq) &= \sum_{\substack{\mathbf{u} \in \mathbb{F}_\pi^n \\ f_\pi(\mathbf{u})=0}} \left(\sum_{\substack{\mathbf{x} \equiv \mathbf{u}\,(\pi) \\ f_{pq}(\mathbf{x})=0}} W\left(\frac{1}{B}\mathbf{x}\right) - K \right) + K \sum_{\substack{\mathbf{u} \in \mathbb{F}_\pi^n \\ f_\pi(\mathbf{u})=0}} 1 \\ &= S + K\left(\pi^{n-1} + O(\pi^{n/2})\right), \end{aligned}$$

where

$$S = \sum_{\substack{\mathbf{u} \in \mathbb{F}_\pi^n \\ f_\pi(\mathbf{u})=0}} \left(\sum_{\substack{\mathbf{x} \equiv \mathbf{u}\,(\pi) \\ f_{pq}(\mathbf{x})=0}} W\left(\frac{1}{B}\mathbf{x}\right) - K \right).$$

Here we have used the property ($\mathrm{R}_0(\pi)$), applying Proposition 3.2 to the hypersurface defined by $f_\pi(\mathbf{u}) = 0$. By Cauchy's inequality,

$$S^2 \ll \pi^{n-1} \sum_{\mathbf{u} \in \mathbb{F}_\pi^n} \left(\sum_{\substack{\mathbf{x} \equiv \mathbf{u}\,(\pi) \\ f_{pq}(\mathbf{x})=0}} W\left(\frac{1}{B}\mathbf{x}\right) - K \right)^2,$$

so we have

$$N_W(f, B, \pi pq) = \pi^{n-1}K + O\left(\pi^{(n-1)/2}\Sigma^{1/2}\right) + O\left(B^n \pi^{-n/2} p^{-1} q^{-1}\right),$$

and (i) is proven. Now,

$$\begin{aligned} \Sigma &= \sum_{\mathbf{u} \in \mathbb{F}_\pi^n} \left(\sum_{\substack{\mathbf{x} \equiv \mathbf{u}\,(\pi) \\ f_{pq}(\mathbf{x})=0}} W\left(\frac{1}{B}\mathbf{x}\right) - K \right)^2 \\ &= \sum_{\mathbf{u} \in \mathbb{F}_\pi^n} \left(\sum_{\substack{\mathbf{x} \equiv \mathbf{u}\,(\pi) \\ f_{pq}(\mathbf{x})=0}} W\left(\frac{1}{B}\mathbf{x}\right) \right)^2 - 2K N_W(f, B, pq) + \pi^n K^2. \end{aligned}$$

By Theorem 3.1 we have

$$N_W(f, B, pq) = \pi^n K + \mathcal{E}_0,$$

where

$$\begin{aligned} \mathcal{E}_0 \ll_C B^{(n+1)/2} p^{-1/2} q^{(n-2)/4} &+ B^{(n+1)/2} p^{(n-2)/2} q^{-1/4} \\ &+ B^n p^{-n/2} q^{-1} + B^{n-C/2} p^{(C-1)/2} q^{-1/2}, \end{aligned}$$

since Z_p and Z_q are non-singular. We conclude that

$$\Sigma = \sum_{\mathbf{u}\in\mathbb{F}_\pi^n} \left(\sum_{\substack{\mathbf{x}\equiv\mathbf{u}\,(\pi) \\ f_{pq}(\mathbf{x})=0}} W\left(\frac{1}{B}\mathbf{x}\right) \right)^2 - \pi^n K^2 + \mathcal{E}_1,$$

where $\mathcal{E}_1 \ll K\mathcal{E}_0$. Introducing a new variable $\mathbf{y}$, we expand the sum of squares as a double sum

$$\sum_{\mathbf{u}\in\mathbb{F}_\pi^n} \left(\sum_{\substack{\mathbf{x}\equiv\mathbf{u}\,(\pi) \\ f_{pq}(\mathbf{x})=0}} W\left(\frac{1}{B}\mathbf{x}\right) \right)^2 = \sum_{\substack{\mathbf{x}\in\mathbb{Z}^n \\ f_{pq}(\mathbf{x})=0}} W\left(\frac{1}{B}\mathbf{x}\right) \sum_{\substack{\mathbf{y}\in\mathbb{Z}^n \\ f_{pq}(\mathbf{x}+\pi\mathbf{y})=0}} W\left(\frac{1}{B}(\mathbf{x}+\pi\mathbf{y})\right)$$

$$= \sum_{\mathbf{y}\ll B/\pi} \sum_{\substack{\mathbf{x}\in\mathbb{Z}^n \\ f_{pq}(\mathbf{x})=f_{pq}(\mathbf{x}+\pi\mathbf{y})=0}} W_{\mathbf{y}}\left(\frac{1}{B}\mathbf{x}\right).$$

Recalling the definition of $\Delta(\mathbf{y})$ above, we have

$$\Sigma = \sum_{\mathbf{y}\ll B/\pi} \Delta(\mathbf{y}) + p^{-2}q^{-2} \sum_{\mathbf{y}\in\mathbb{Z}^n} \sum_{\mathbf{x}\in\mathbb{Z}^n} W_{\mathbf{y}}\left(\frac{1}{B}\mathbf{x}\right) - \pi^n K^2 + \mathcal{E}_1.$$

By Lemma 3.1, however,

$$p^{-2}q^{-2} \sum_{\mathbf{y}\in\mathbb{Z}^n} \sum_{\mathbf{x}\in\mathbb{Z}^n} W_{\mathbf{y}}\left(\frac{1}{B}\mathbf{x}\right) - \pi^n K^2 \ll_C B^{2n-C}\pi^{-n+C}p^{-2}q^{-2},$$

so we have proven (ii).

We note that the ideal $(f(\mathbf{x}), f(\mathbf{x}+\pi\mathbf{y}))$ in $\mathbb{Z}[\mathbf{x}]$ can also be written as $(f(\mathbf{x}), f^{\mathbf{y}}(\mathbf{x}))$. By Remark 2.1, neither of $F_q^{\mathbf{y}}$ and $F_p^{\mathbf{y}}$ is identically zero. Thus $f_q^{\mathbf{y}}$ and $f_p^{\mathbf{y}}$ are polynomials of degree $d-1$ with leading form $F_q^{\mathbf{y}}$ and $F_p^{\mathbf{y}}$, respectively, and moreover

$$\dim Z_{q,\mathbf{y}} = \dim Z_{p,\mathbf{y}} = n-3.$$

Let

$$X_{\mathbf{y}} = \operatorname{Spec} \mathbb{Z}[X_1,\ldots,X_n]/(f, f^{\mathbf{y}}).$$

Now we write

$$\Delta(\mathbf{y}) = S(\mathbf{y}) + \mathcal{E}_2(\mathbf{y}), \tag{11}$$

where we have defined

$$S(\mathbf{y}) = \sum_{\substack{\mathbf{v}\in\mathbb{F}_p^n \\ f_p(\mathbf{v})=f_p^{\mathbf{y}}(\mathbf{v})=0}} \left(\sum_{\substack{\mathbf{x}\equiv\mathbf{v}\,(p) \\ f_q(\mathbf{x})=f_q^{\mathbf{y}}(\mathbf{x})=0}} W_{\mathbf{y}}\left(\frac{1}{B}\mathbf{x}\right) - K(\mathbf{y}) \right)$$

and

$$\mathcal{E}_2(\mathbf{y}) = \sum_{\substack{\mathbf{v}\in\mathbb{F}_p^n \\ f_p(\mathbf{v})=f_p^{\mathbf{y}}(\mathbf{v})=0}} K(\mathbf{y}) - p^{-2}q^{-2} \sum_{\mathbf{x}\in\mathbb{Z}^n} W_{\mathbf{y}}\left(\frac{1}{B}\mathbf{x}\right),$$

with

$$K(\mathbf{y}) := p^{-n}q^{-2}\sum_{\mathbf{x}\in\mathbb{Z}^n} W_{\mathbf{y}}\left(\frac{1}{B}\mathbf{x}\right).$$

But then

$$\mathcal{E}_2(\mathbf{y}) = K(\mathbf{y})\left(\#X_{\mathbf{y}}(\mathbb{F}_p) - p^{n-2}\right),$$

and by Proposition 3.2 we have $\#X_{\mathbf{y}}(\mathbb{F}_p) = p^{n-2} + O\left(p^{(n+s_p(\mathbf{y}))/2}\right)$, yielding

$$\mathcal{E}_2(\mathbf{y}) \ll B^n p^{-(n-s_p(\mathbf{y}))/2} q^{-2}.$$

Thus we turn now to $S(\mathbf{y})$. We again apply Cauchy's inequality, using Proposition 3.1 to estimate the number of $\mathbb{F}_p$-points on $X_{\mathbf{y}}$. Thus we get

$$S(\mathbf{y})^2 \ll p^{n-2}\Sigma(\mathbf{y}), \tag{12}$$

where

$$\Sigma(\mathbf{y}) = \sum_{\mathbf{v}\in\mathbb{F}_p^n}\left(\sum_{\substack{\mathbf{x}\equiv\mathbf{v}\,(p)\\ f_q(\mathbf{x})=f_q^{\mathbf{y}}(\mathbf{x})=0}} W_{\mathbf{y}}\left(\frac{1}{B}\mathbf{x}\right) - K(\mathbf{y})\right)^2.$$

In this second differencing step, we will complete the sum (mod p) as in Heath-Brown's original argument [5], with respect to one of the two polynomials involved. We have

$$\Sigma(\mathbf{y}) \le \sum_{\mathbf{v}\in\mathbb{F}_p^n}\sum_{a=1}^{q}\left(\sum_{\substack{\mathbf{x}\equiv\mathbf{v}\,(p)\\ f_q(\mathbf{x})=0\\ f_q^{\mathbf{y}}(\mathbf{x})=a}} W_{\mathbf{y}}\left(\frac{1}{B}\mathbf{x}\right) - K(\mathbf{y})\right)^2. \tag{13}$$

Denote by $\Sigma'(\mathbf{y})$ the right hand side of (13). Expanding the square, we get

$$\Sigma'(\mathbf{y}) = \sum_{\mathbf{v}\in\mathbb{F}_p^n}\sum_{a=1}^{q}\left(\sum_{\substack{\mathbf{x}\equiv\mathbf{v}\,(p)\\ f_q(\mathbf{x})=0\\ f_q^{\mathbf{y}}(\mathbf{x})=a}} W_{\mathbf{y}}\left(\frac{1}{B}\mathbf{x}\right)\right)^2 - 2K(\mathbf{y})N_W(f,B,q) + p^n qK(\mathbf{y})^2.$$

By Remark 3.1 we have the estimate

$$N_W(f,B,q) = p^n qK(\mathbf{y}) + O\left(Bq^{(n-2)/2}\right),$$

insertion of which yields

$$\begin{aligned}\Sigma'(\mathbf{y}) = \sum_{\mathbf{v}\in\mathbb{F}_p^n}\sum_{a=1}^{q}&\left(\sum_{\substack{\mathbf{x}\equiv\mathbf{v}\,(p)\\ f_q(\mathbf{x})=0\\ f_q^{\mathbf{y}}(\mathbf{x})=a}} W_{\mathbf{y}}\left(\frac{1}{B}\mathbf{x}\right)\right)^2 - p^n qK(\mathbf{y})^2\\ &+ O\left(B^{n+1}p^{-n}q^{(n-6)/2}\right).\end{aligned} \tag{14}$$

As before, we proceed to expand the sum of squares as a double sum, introducing a third variable $\mathbf{z}$:

$$\sum_{\mathbf{v}\in\mathbb{F}_p^n}\sum_{a=1}^{q}\left(\sum_{\substack{\mathbf{x}\equiv\mathbf{v}\,(p)\\ f_q(\mathbf{x})=0\\ f_q^{\mathbf{y}}(\mathbf{x})=a}} W_{\mathbf{y}}\left(\frac{1}{B}\mathbf{x}\right)\right)^2 = \sum_{\mathbf{z}\ll B/p}\sum_{\substack{\mathbf{x}\in\mathbb{Z}^n\\ f_q(\mathbf{x})=f_q^{\mathbf{z}}(\mathbf{x})\\ =f_q^{\mathbf{y},\mathbf{z}}(\mathbf{x})=0}} W_{\mathbf{y},\mathbf{z}}\left(\frac{1}{B}\mathbf{x}\right),$$

and then we compare the sum over $\mathbf{x}$ to its expected value $\Delta(\mathbf{y},\mathbf{z})$. Another application of Lemma 3.1 yields that

$$q^{-3}\sum_{\mathbf{z}\ll B/p}\sum_{\mathbf{x}\in\mathbb{Z}^n} W_{\mathbf{y},\mathbf{z}}\left(\frac{1}{B}\mathbf{x}\right) - p^n qK(\mathbf{y})^2 \ll_C B^{2n-C}p^{-n+C}q^{-3},$$

so it follows, in view of (12), (13) and (14), that

$$\Delta(\mathbf{y}) \ll p^{(n-2)/2}\left(\sum_{\mathbf{z}\ll B/p}\Delta(\mathbf{y},\mathbf{z})\right)^{1/2} + \mathcal{E}_4(\mathbf{y}), \tag{15}$$

where

$$\mathcal{E}_4(\mathbf{y}) \ll_C B^{(n+1)/2}p^{-1}q^{(n-6)/4} + B^{n-C}p^{-1+C}q^{-3/2}, \quad \text{for any } C>0.$$

This proves (iii). Finally, (iv) follows directly from Theorem 3.1.

□

Next, let us sum up the results of Lemma 3.2 in an estimate for $N_W(f,B,\pi pq)$. By (7),(8) and (9) we are led to evaluate the quantity

$$E_1 := \pi^{(n-1)/2}p^{(n-2)/4}\left(\sum_{\mathbf{y}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}}\left(\sum_{\mathbf{z}\in\mathbb{Z}^n}\Delta(\mathbf{y},\mathbf{z})\right)^{1/2}\right)^{1/2}, \tag{16}$$

which will be the strongest competitor to the main term in (7). We wish to switch the order of summation. Thus we apply Hölder's inequality [3, Theorem 65] to get

$$\sum_{\mathbf{y}\neq\mathbf{0}}\left(\sum_{\mathbf{z}\in\mathbb{Z}^n}\Delta(\mathbf{y},\mathbf{z})\right)^{1/2} \ll \left(\frac{B}{\pi}\right)^{n/2}\left(\sum_{\mathbf{y}\in\mathbb{Z}^n}\sum_{\mathbf{z}\in\mathbb{Z}^n}\Delta(\mathbf{y},\mathbf{z})\right)^{1/2}.$$

Here we have used the fact that $\Delta(\mathbf{y},\mathbf{z})$ vanishes for $\mathbf{y}\gg B/\pi$. (16) transforms into

$$E_1 = B^{n/4}\pi^{(n-2)/4}p^{(n-2)/4}\left(\sum_{\mathbf{z}\in\mathbb{Z}^n}\sum_{\mathbf{y}\in\mathbb{Z}^n}\Delta(\mathbf{y},\mathbf{z})\right)^{1/4}. \tag{17}$$

We shall now try to calculate the double sum

$$\sum_{\mathbf{z}\in\mathbb{Z}^n}\sum_{\mathbf{y}\in\mathbb{Z}^n}\Delta(\mathbf{y},\mathbf{z}). \tag{18}$$

Our domain of summation for the outer sum is then $\mathcal{C} = \mathbf{C} \cap \mathbb{Z}^n$ for some cube $\mathbf{C}$ centered at the origin and with sides of length $\ll B/p$. Denote by $\mathcal{C}''$ the exceptional set of $\mathbf{z} \in \mathcal{C}$ such that $\sigma_q(\mathbf{z}) \geq n-4$, and $\mathcal{C}' = \mathcal{C} \setminus \mathcal{C}''$.

First we take care of the case $\mathbf{z} = \mathbf{0}$. We have

$$\Delta(\mathbf{y}, \mathbf{0}) \ll N_W(f, f^{\mathbf{y}}, B, q) + B^n q^{-3} \ll B^n q^{-2}$$

for $\mathbf{y} \neq \mathbf{0}$ such that $s_q(\mathbf{y}) \leq n-5$, by Remark 3.1. If $s_q(\mathbf{y}) \geq n-4$ or $\mathbf{y} = \mathbf{0}$, we have

$$\Delta(\mathbf{y}, \mathbf{0}) \ll N_W(f, B, q) + B^n q^{-3} \ll B^n q^{-1}.$$

Recall that F satisfies $(\mathrm{R}_1(q))$, so the number of such $\mathbf{y}$ in the cube $\mathbf{y} \ll B/\pi$ is $\ll (B/\pi)^3$, by Proposition 3.1 and Lemma 2.3. This yields

$$\begin{aligned} \sum_{\mathbf{y} \in \mathbb{Z}^n} \Delta(\mathbf{y}, \mathbf{0}) &\ll \left(\frac{B}{\pi}\right)^n B^n q^{-2} + \left(\frac{B}{\pi}\right)^3 B^n q^{-1} \\ &= B^{2n} \pi^{-n} q^{-2} + B^{n+3} \pi^{-3}. \end{aligned} \tag{19}$$

Next we calculate the contribution of $\mathbf{z} \in \mathcal{C}'' \setminus \{\mathbf{0}\}$. As remarked in Section 2, $Z_{q,\mathbf{z}}$ has the expected dimension, $n-3$, for every $\mathbf{z} \neq \mathbf{0}$, so we can use the trivial estimate $\Delta(\mathbf{y}, \mathbf{z}) \ll B^{n-3}$ from Proposition 3.1. Another trivial estimate yields $\sum_{\mathbf{y} \in \mathbb{Z}^n} \Delta(\mathbf{y}, \mathbf{z}) \ll B^{2n-3} \pi^{-n}$. Again, because of the property $(\mathrm{R}_1(q))$, we have $\#\mathcal{C}'' \ll (B/p)^3$, so we get

$$\sum_{\mathbf{z} \in \mathcal{C}''} \sum_{\mathbf{y} \in \mathbb{Z}^n} \Delta(\mathbf{y}, \mathbf{z}) \ll B^{2n} \pi^{-n} p^{-3}. \tag{20}$$

It remains to calculate the contribution from $\mathbf{z} \in \mathcal{C}'$. Thus, fix $\mathbf{z} \neq \mathbf{0}$ with $\mathbf{z} \ll B/p$, and suppose that $\sigma_q(\mathbf{z}) = \sigma \leq n-5$. (This means that $Z_{q,\mathbf{z}}$ is regular in codimension one.) Our domain of summation for $\mathbf{y}$ is $\mathcal{B}_{\mathbf{z}} = \mathbf{B}_{\mathbf{z}} \cap \mathbb{Z}^n$ for a cube $\mathbf{B}_{\mathbf{z}}$ centered at the origin and with sides of length $\ll B/\pi$. Denote by $\mathcal{B}''_{\mathbf{z}}$ the exceptional set of $\mathbf{y} \in \mathcal{B}_{\mathbf{z}}$ such that $\dim Z_{q,\mathbf{z},\mathbf{y}} > n-4$, and put $\mathcal{B}'_{\mathbf{z}} = \mathcal{B}_{\mathbf{z}} \setminus \mathcal{B}''_{\mathbf{z}}$. Using [9, Theorem 3.3] and Remark 3.1 we have the estimate

$$\Delta(\mathbf{y}, \mathbf{z}) \ll B^{s_q(\mathbf{y},\mathbf{z})+2} q^{(n-5-s_q(\mathbf{y},\mathbf{z}))/2}$$

for $\mathbf{y} \in \mathcal{B}'_{\mathbf{z}}$. Furthermore, by Lemma 2.3, and since F satisfies $(\mathrm{R}_2(q))$, the number of $\mathbf{y} \in \mathcal{B}'_{\mathbf{z}}$ such that $s_q(\mathbf{y}, \mathbf{z}) = s + \sigma + 1$ is $\ll (B/\pi)^{n-1-s}$ for $-1 \leq s \leq n-5-\sigma$. Hence we calculate

$$\sum_{\mathbf{y} \in \mathcal{B}'_{\mathbf{z}}} \Delta(\mathbf{y}, \mathbf{z}) \ll \sum_{s=-1}^{n-5-\sigma} \left(\frac{B}{\pi}\right)^{n-1-s} B^{\sigma+s+3} q^{(n-6-\sigma-s)/2}. \tag{21}$$

For $\mathbf{y} \in \mathcal{B}''_{\mathbf{z}}$ we note that at least we have

$$\Delta(\mathbf{y}, \mathbf{z}) \ll N_{W_{\mathbf{y},\mathbf{z}}}(f, f^{\mathbf{z}}, B, q) + B^n q^{-3} \ll B^n q^{-2},$$

by [9, Theorem 3.3] and Remark 3.1, and since $\sigma \leq n-5$. Furthermore, provided q is greater than some constant Q independent of $\mathbf{z}$, Lemma 2.2 tells us that $\#\mathcal{B}''_{\mathbf{z}} \ll (B/\pi)^{\sigma+1}$. Thus, incorporating into (21) the contribution from $\mathbf{y} \in \mathcal{B}''_{\mathbf{z}}$ we get

$$\begin{aligned} \sum_{\mathbf{y} \in \mathbb{Z}^n} \Delta(\mathbf{y}, \mathbf{z}) &\ll B^{n+\sigma+2} \pi^{-n} q^{(n-5-\sigma)/2} + B^{n+\sigma+2} \pi^{-4-\sigma} q^{-1/2} \\ &\quad + B^{n+\sigma+1} \pi^{-\sigma-1} q^{-2}. \end{aligned} \tag{22}$$

The next step is to sum over $\mathbf{z} \in \mathcal{C}'$. The property $(\mathrm{R}_1(q))$, combined with [5, Lemma 2], yields the estimate $(B/p)^{n-1-\sigma}$ for the number of $\mathbf{z}$ such that $\sigma_q(\mathbf{z}) \geq \sigma$. Thus, summing the contributions from (22) we get

$$\sum_{\mathbf{z}\in\mathcal{C}'}\sum_{\mathbf{y}\in\mathbb{Z}^n} \Delta(\mathbf{y},\mathbf{z}) \ll \sum_{\sigma=-1}^{n-5} \left(\frac{B}{p}\right)^{n-1-\sigma} \Big(B^{n+\sigma+2}\pi^{-n}q^{(n-5-\sigma)/2} + B^{n+\sigma+2}\pi^{-4-\sigma}q^{-1/2} + B^{n+\sigma+1}\pi^{-\sigma-1}q^{-2}\Big). \tag{23}$$

Each term in this sum is dominated by either the previous term or the next. Taking this into account, we put together the three contributions (23), (20) and (19) to get the following estimate for the double sum in (18):

$$\begin{aligned}\sum_{\mathbf{z}\in\mathbb{Z}^n}\sum_{\mathbf{y}\in\mathbb{Z}^n} \Delta(\mathbf{y},\mathbf{z}) \ll{}& B^{2n+1}\pi^{-n}p^{-n}q^{(n-4)/2} + B^{2n+1}\pi^{-3}p^{-n}q^{-1/2} \\ &+ B^{2n}p^{-n}q^{-2} + B^{2n+1}\pi^{-n}p^{-4} + B^{2n+1}\pi^{-n+1}p^{-4}q^{-1/2} \\ &+ B^{2n}\pi^{-n+4}p^{-4}q^{-2} + B^{2n}\pi^{-n}p^{-3} + B^{2n}\pi^{-n}q^{-2} + B^{n+3}\pi^{-3}.\end{aligned} \tag{24}$$

We arrive at the following result:

Lemma 3.3. *Under the hypotheses of Lemma 3.2, and with E_1 defined by* (16)*, we have the estimate*

$$\begin{aligned}E_1 \ll{}& B^{(3n+1)/4}\pi^{-1/2}p^{-1/2}q^{(n-4)/8} + B^{(3n+1)/4}\pi^{(n-5)/4}p^{-1/2}q^{-1/8} \\ &+ B^{3n/4}\pi^{(n-2)/4}p^{-1/2}q^{-1/2} + B^{(3n+1)/4}\pi^{-1/2}p^{(n-6)/4} \\ &+ B^{(3n+1)/4}\pi^{-1/4}p^{(n-6)/4}q^{-1/8} + B^{3n/4}\pi^{1/2}p^{(n-6)/4}q^{-1/2} \\ &+ B^{3n/4}\pi^{-1/2}p^{(n-5)/4} + B^{3n/4}\pi^{-1/2}p^{(n-2)/4}q^{-1/2} \\ &+ B^{(2n+3)/4}\pi^{(n-5)/4}p^{(n-2)/4}.\end{aligned} \tag{25}$$

For clarity, let us recall where the different terms in (25) come from. The first three constitute the contribution from values of $\mathbf{z} \neq \mathbf{0}$ such that the 'differenced variety' $Z_{q,\mathbf{z}}$ is non-singular. The next three come from $\mathbf{z} \neq \mathbf{0}$ where $\max(\dim \operatorname{Sing} Z_{q,\mathbf{z}}, \dim \operatorname{Sing} \tilde{Z}_{q,\mathbf{z}}) = n-5$. Within these sets of three terms, the first two correspond to $\mathbf{y}$ such that $Z_{q,\mathbf{y},\mathbf{z}}$ is a complete intersection, with $\operatorname{Sing} Z_{q,\mathbf{y},\mathbf{z}}$ of minimal and maximal dimension, respectively, whereas the third term comes from $\mathbf{y}$ such that $Z_{q,\mathbf{y},\mathbf{z}}$ has higher dimension than expected. The sixth term corresponds to the case $\max(\dim \operatorname{Sing} Z_{q,\mathbf{z}}, \dim \operatorname{Sing} \tilde{Z}_{q,\mathbf{z}}) > n-5$, and the two last to $\mathbf{z} = \mathbf{0}$.

Proof of Theorem 1.1. We repeat the hypotheses of the theorem: let f be a polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$ of degree $d \geq 4$ with leading form F, let $Z = \operatorname{Proj} \mathbb{Z}[x_1, \ldots, x_n]/(F)$, and suppose that $Z_{\mathbb{Q}}$ is a non-singular subscheme of $\mathbb{P}^{n-1}_{\mathbb{Q}}$. Note that Lemma 3.2 (i) gives an asymptotic formula for $N_W(f, B, \pi pq)$. However, we shall only use it as an upper bound, and try to deduce a good upper bound for $N(f, B)$ by choosing π, p and q wisely in terms of B. It turns out that the following relations are desirable:

$$\pi \asymp p \asymp B^{(n-1)/(n-8)}, \qquad q \asymp B^{2(n-1)/(n-8)}, \tag{26}$$

since then the first, second, fourth and fifth terms in (25) will be of the same order of magnitude as the main term in (i), and all other terms involved will be smaller. To be able to use Lemma 3.2 we need to have $q \geq B$, so we cannot hope to find q satisfying (26) unless $n \geq 10$, whence this condition in the theorem. More importantly, the results of Lemma 3.2 are subject to a set of hypotheses ($\mathrm{R}_k(\cdot)$) on π, p, q. We need to show that such π, p, q exist in the specified intervals (26).

By Corollary 2.1, however, the number of primes not fulfilling these criteria is finite. Thus, provided $B \gg_F 1$, any primes π, p, q satisfying (26) will do. Bertrand's postulate [4, Theorem 418] assures that the intervals specified in (26) contain primes. We are thus allowed to insert (26) into Lemmata 3.2 and 3.3. Then we have, for the main term in (7),

$$(\pi pq)^{-1} N_W(0, B, \pi pq) \ll B^n (\pi pq)^{-1} \ll B^{n-4+36/(n+8)}.$$

The same holds for the 'main' auxiliary term - by Lemma 3.3 we have

$$E_1 \ll B^{n-4+36/(n+8)}.$$

Thus, to finish the proof of Theorem 1.1 it remains to check that the remaining terms occuring in Lemma 3.2 are small enough. The third term in (7) is obviously dominated by the main term. From the second term in (8) we get the contribution

$$B^{n-C/2} \pi^{(C-1)/2} p^{-1} q^{-1}$$

which is $\ll B^{n-4+36/(n+8)}$ provided $C \gg_n 1$. The term $\Delta(\mathbf{0})$ of (10) contributes

$$\pi^{(n-1)/2} \Delta(\mathbf{0})^{1/2} \ll B^{n-13/2+54/(n+8)}.$$

The contribution of $\mathcal{E}_1$ to (7) is

$$\pi^{(n-1)/2} \mathcal{E}_1^{1/2} \ll B^{n-19/4+171/(4(n+8))} + B^{3n/4-3/4+9/(n+8)} + B^{n-11/4+9(2C-11)/(4(n+8))}$$

for any $C > 0$. Again, choosing C large enough yields this term negligible. The error terms $\mathcal{E}_2(\mathbf{y})$ and $\mathcal{E}_4(\mathbf{y})$ have to be summed over all $\mathbf{y} \neq \mathbf{0}$. For $\mathcal{E}_4(\mathbf{y})$, this is easily done, since the expression vanishes for $\mathbf{y} \gg B/\pi$, and the estimate we have is independent of $\mathbf{y}$. The contribution to (7) is

$$\pi^{(n-1)/2} \left(\sum_{\mathbf{y} \neq \mathbf{0}} \mathcal{E}_4(\mathbf{y}) \right)^{1/2} \ll B^{n-9/2+81/(2(n+8))} + B^{n-9/4+9(4C-9)/(4(n+8))},$$

which is small enough for $C \gg_n 1$. The summation of $\mathcal{E}_2(\mathbf{y})$ requires somewhat more work. Namely, by the property ($\mathrm{R}_1(p)$), the number of $\mathbf{y}$ in the interval $\mathbf{y} \ll B/\pi$ for which $s_p(\mathbf{y}) = s$ is $O((B/\pi)^{n-1-s})$. Thus we get

$$\sum_{\mathbf{y} \neq \mathbf{0}} \mathcal{E}_2(\mathbf{y}) \ll \sum_{s=-1}^{n-3} \left(\frac{B}{\pi} \right)^{n-1-s} B^n p^{-(n-s)/2} q^{-2}$$
$$\ll B^{2n} \pi^{-n} p^{-(n+1)/2} q^{-2} + B^{n+2} \pi^{-2} p^{-3/2} q^{-2},$$

yielding the contribution

$$\pi^{(n-1)/2}\left(\sum_{\mathbf{y}\neq\mathbf{0}}\mathcal{E}_2(\mathbf{y})\right)^{1/2} \ll B^{3n/4-1/2+27/(4(n+8))} + B^{n-31/4+297/(4(n+8))}.$$

This proves the theorem.

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