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ASYMPTOTIC ERROR EXPANSIONS FOR THE FINITE ELEMENT METHOD FOR SECOND ORDER ELLIPTIC PROBLEMS IN R^N, $N \geq 2$. I: LOCAL INTERIOR EXPANSIONS

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ABSTRACT. Our aim here is to give sufficient conditions on the finite element spaces near a point so that the error in the finite element method for the function and its derivatives at the point have exact asymptotic expansions in terms of the mesh parameter h, valid for h sufficiently small. Such expansions are obtained from the so-called asymptotic expansion inequalities valid in $\mathbb{R}^N$ for $N \geq 2$, studies by Schatz in [22] and [24].

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

The aim here is to give sufficient conditions on the finite element spaces near a point so that the error in the finite element method for the function and its derivatives at the point have asymptotic expansions in terms of the mesh size parameter h, valid for h sufficiently small. So-called asymptotic error expansion inequalities at a point for the finite element method were derived in Schatz [24]. These results which will be stated in a moment, will be our main tools in deriving asymptotic error expansions. In the paper [1] Richardson extrapolation was justified using asymptotic error expansion inequalities. In this paper we shall use asymptotic expansion inequalities to derive asymptotic expansions that are extended ways to justify Richardson extrapolations.

Let us first describe the problem more precisely and then state the results. Let Ω be a domain in $\mathbb{R}^N$, $N \geq 2$, and for $d > 0$ let $B_d(x_0)$ be a ball of radius d centered at x_0. We shall assume that at the point x_0, that we are interested in, there exists a $d > 0$ such that $B_d = B_d(x_0) \subset\subset \Omega$.

Consider the second order elliptic equation of the form

$$Lu(x) = \sum_{i,j=1}^{N} a_{ij} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} = f(x) \qquad \text{in } B_d(x_0). \tag{1.1}$$

The local weak formulation of (1.1) is

$$A(u,v) = \int_{B_d(x_0)} f v dx \qquad \text{for all } v \in \mathring{W}_2^1(B_d(x_0)), \tag{1.2}$$

where

$$A(u,v) = \int_{B_d(x_0)} \Big(\sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \Big) dx. \tag{1.3}$$

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Consider the finite element approximations u_h of u. For this purpose for each $0 < h < 1$, let $S_r^h(B_d) \subset W^1_\infty(B_d)$ be finite element space satisfying the conditions in [24]. For our purpose here, think of them as any variety of spaces of continuous functions on $B_d(x_0)$, which on each set τ_h of a quasi-uniform partition (of roughly size h) which covers $B_d(x_0)$, contains all polynomials of degree $\le r-1$ where $r \ge 2$. For any set $G \subset S_r^h(B_d)$, $S_r^h(G)$ is the restriction of $S_r^h(B_d)$ to G and $\mathring{S}_r^h(G)$ are the subspaces of $S_r^h(G)$ of functions supported in G.

Now the finite element approximation $u_h \in S_r^h(B_d)$ of u is assumed to satisfy

$$A(u - u_h, \varphi) = 0 \quad \text{for all} \quad \varphi \in \mathring{S}_r^h(B_d(x_0)). \tag{1.4}$$

Let us begin with discussing asymptotic expansions for the values of functions at the point x_0.

Definition 1.1. A γ term asymptotic expansion for the function values at a point x_0 is an expression of the form

$$u(x_0) = u_h(x_0) + \sum_{r \le |\alpha| \le \gamma - 1} h^{r+|\alpha|} E_{|\alpha|}(x_0, u) + \mathcal{R}_\gamma(h; x_0, u), \tag{1.5}$$

where the error coefficients $E_{|\alpha|}$ are independent of h and the remainder term $\mathcal{R}_\gamma$ is $\mathcal{O}(h^\gamma)$.

Our main result will be concerned with deriving $2r-2$ term asymptotic expansions for equations of the form (1.1), for $r \ge 3$. The general case of variable coefficients and lower order terms:

$$\mathcal{L}u = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\Big(a_{ij}(x)\frac{\partial u}{\partial x_i}\Big) + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x) \text{ in } B_d(x_0), \tag{1.6}$$

can also be treated by the methods given in [23] and [1]. However, in this case, which is not treated here, we are only able to prove a *one term* asymptotic expansion.

In order to find such expansions we shall impose so-called *self-similarity conditions*, on the finite element spaces in a neighborhood of the point x_0.

Expansions of this type have been known for some time for some finite difference methods (c.f., e.g. Böhmer [6]). The main contributions to the finite element literature are on plane domains and are due to Q. Lin and coworkers. In the case of finite elements with $r = 2$ (in particular piecewise linear or bilinear elements), such expansions were first derived at points x which are the vertices of a uniform triangulation of the plane in Lin and Zhu [16], Lin and Lu [12] and Lin and Wang [13]. Improvements and extensions of these expansions were then given in the paper by Blum, Lin and Rannacher [4] which contains an excellent presentation of the derivation of the exact asymptotic expansion. Further results, of this kind, can be found in Lin and Wang [14], Lin and Xie [15], Blum [3], Rannacher [18], Chen and Lin [7], Wang [29], Lin [11], Ding and Lin [9], and Chen and Rannacher [8] , where other references can be found. For an asymptotic expansion near a corner see Blum and Rannacher [5]. For a related study of the extrapolation of the energy functional see Rüde [20] and [21]. Corner domains are addressed in Huang and coworkers [10], where they assume sufficiently smooth initial data and rectangular domains. Lin [11] considers an extrapolation technique for the eigenvalues on non-convex domains. We recommend the survey articles by Rannacher [19] and Blum [2].

An outline of the remaining part of this paper is as follows: In Section 1.1 we give assumptions on the finite element spaces, in particular the self-similarity property and state the main results of the paper: Theorems 1.1 and 1.2 which are concerned the accuracy of pointwise error estimates for the function and its first order partial derivatives, i.e., $(u - u_h)(x_0)$ and $\frac{\partial}{\partial x_i}(u - u_h)(x_0)$, $i = 1, \dots, N$, respectively, at certain (similarity) points x_0 of the grid. Section 2 contains preliminaries for the proofs of Theorems 1.1 and 1.2. Section 3 is devoted to the proofs of the main results. Finally in our concluding in Section 4 (Appendix I) we recall the usual finite element assumptions used throughout the paper. Below we denote by C a general constant independent of the parameters involved in the estimates unless otherwise explicitly stated or clear from the context.

1.1. **Some assumptions on the subspaces and statement of the main theorems.** The requirements we shall make on the subspaces, near a point x_0, are motivated by looking at the meshes which are systematically refined in a neighborhood of x_0. For example as in so-called nested spaces constructed to be used in the multigird methods. In order to do so it will be more convenient to work with a sequence of subspaces $S_r^{h_j}(B_1(x_0))$ where for some sufficiently small h and fixed $K > 1$

$$h_j = \frac{h}{K^j}, \qquad \text{(for example } K = 2, 3, 4.) \tag{1.7}$$

Now we state the most important assumption **A.5.** on the finite element spaces (the usual properties **A.1.-A.4.** of the finite element spaces are listed in Appendix I).

Definition 1.2. Two subspaces $S_r^{h_j}(B_{h_j}(x_0))$ and $S_r^{h_i}(B_{h_i}(x_0))$ are said to be similar near x_0 if the mapping (a scaling about x_0)

$$\Big(T\varphi\Big)(x) = \varphi\Big(x_0 + \frac{h_j}{h_i}(x - x_0)\Big) \tag{1.8}$$

is a one to one mapping of $S_r^{h_j}(B_{h_j}(x_0))$ onto $S_r^{h_i}(B_{h_i}(x_0))$.

Examples of subspaces satisfying this definition are given in [1]

Assumption A.5. on the mesh. We shall now assume that given x_0 there exists a $d > 0$ and an integer k_0 such that for any pair of integers j and k with $j > k > k_0$, (i.e., $h_j < h_k < h_{k_0}$), the scaling $S_r^{h_k}(B_{\frac{h_k}{h_j}d}(x_0))$ is similar to $S_r^{h_j}(B_{h_j}(x_0))$. This just says that from some mesh size h_{k_0} on, the mesh on a disk of radius d is constantly uniformly refined about x_0 resulting in self-similar subspaces about x_0.

In the sequel we shall use the following notation: For $m \geq 0$ an integer, $1 \leq p \leq \infty$ and $G \subseteq \Omega$, $W_p^m(G)$ denotes the usual Sobolev space of functions with distributional derivatives of order $\leq m$ which are in $L_p(G)$. Define the seminorms

$$|u|_{W_p^j(G)} = \begin{cases} \Big(\sum\limits_{|\alpha|=j} \|D^\alpha u\|^p_{L_p(G)}\Big)^{1/p} & \text{if } \ 1 \leq p < \infty, \\ \sum\limits_{|\alpha|=j} \|D^\alpha u\|_{L_\infty(G)} & \text{if } \ p = \infty, \end{cases}$$

and the norms

$$\|u\|_{W_p^m(G)} = \begin{cases} \Big(\sum_{j=1}^{m} |u|^p_{W_p^j(G)}\Big)^{1/p} & \text{if} \quad 1 \le p < \infty, \\ \sum_{j=1}^{m} |u|_{W_\infty^j(G)} & \text{if} \quad p = \infty. \end{cases}$$

If $m \ge 0$, $W_p^{-m}(G)$ is the completion of $C_0^\infty(G)$ under the norm

$$\|u\|_{W_p^{-m}(G)} = \sup_{\substack{\psi \in C_0^\infty(G) \\ \|\psi\|_{W_q^m(G)} = 1}} \int_G u\psi dx, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Our first result is ad follow:

Theorem 1.1. *Let $r \ge 3$, $r+1 \le \gamma \le 2r-2$, and suppose that* **A.5.** *is satisfied in a ball of radius $d \ge Ch$ centered at x_0 and suppose $u - u_h$ satisfies* (1.4). *Further suppose that $u \in W_\infty^\gamma(B_d(x_0))$. Then*

$$u(x_0) = u_h(x_0) + \Big(\sum_{r \le |\alpha| \le \gamma - 1} C_\alpha D^\alpha u(x_0) h^{|\alpha|}\Big) + \mathcal{R}_\gamma, \tag{1.9}$$

where

$$\mathcal{R}_\gamma \le C\Big(h^\gamma \|u\|_{W_\infty^\gamma(B_d(x_0))} + d^{-t-N/p}\|u - u_h\|_{W_p^{-t}(B_d(x_0))}\Big). \tag{1.10}$$

We now give an asymptotic expansion for first derivatives. Note however, if x_0 is a point of the mesh where $\frac{\partial u_h}{\partial x_i}$ is discontinuous, then we define

$$\frac{\partial \widetilde{u}_h}{\partial x_i}(x_0, \beta) = \lim_{s \to 0} \frac{\partial u_h}{\partial x_i}(x_0 + s\beta), \tag{1.11}$$

where $\beta = (\beta_1, \dots, \beta_N)$ is any unit vector chosen so that for s sufficiently small, say $0 < s \le s_0$, $\dfrac{\partial u_h}{\partial x_i}$ exists and has a limit as $s \to 0$. There may be many possible choices of $\dfrac{\partial u_h}{\partial x_i}(x_0, \beta)$. Obviously, $\dfrac{\partial \widetilde{u}_h}{\partial x_i}(x_0, \beta) = \dfrac{\partial u_h}{\partial x_i}(x_0)$ at points x_0 where $\dfrac{\partial u_h}{\partial x_i}$ is continuous.

Theorem 1.2. *Suppose the assumptions of theorem 1.1 are hold, except that now we assume $r \ge 2$, then for $i = 1, 2, \dots, N$, and $u \in W_\infty^{\gamma+1}(B_d)$,*

$$\frac{\partial u(x_0)}{\partial x_i} = \frac{\partial \widetilde{u}_h(x_0, \beta)}{\partial x_i} + \Big(\sum_{r \le |\alpha| \le \gamma - 1} C_\alpha D^\alpha u(x_0) h^{|\alpha|-1}\Big) + \mathcal{R}'_\gamma \tag{1.12}$$

where $h = h_j$, $j = k, k+1, \dots,$ and

$$\mathcal{R}'_\gamma \le C\Big(h^\gamma \, \|u\|_{W_\infty^{\gamma+1}(B_d)} + d^{-t-1-N/p} \, \|u - u_h\|_{W_p^{-t}(B_d)}\Big). \tag{1.13}$$

Remark 1.1. Such expansions, at a point, even for any $r < \gamma$ would justify the usual procedure of verifying the rate of convergence of the method at x_0.

Remark 1.2. It is unreasonable to expect that an estimate like (1.9) holds at a point if the "shape of the mesh" changes at the point as h changes. This is suggested by the fact that the interpolation error at a point depends heavily on the shape of the domain in which x_0 is located. Hence our need for the assumption **A.5.** of the similarity near the point x_0.

2. Preliminaries

Our starting point in proving Theorems 1.1 and 1.2 will be results from Schatz in [24], so-called asymptotic expansion inequalities which we state for equations of the form (1.4).

Lemma 2.1. *For $r \geq 3$, $r+1 \leq \gamma \leq 2r-2$, γ integer, let $u \in W^{\gamma}_{\infty}(B_d(x_0))$ and suppose $d \geq \hat{C}h$ for some $\hat{C}$ chosen sufficiently large. Then the following "asymptotic expansion inequality holds":*

$$
\begin{aligned}
|(u-u_h)(x_0)| \leq C\Big(\ln\frac{d}{h}\Big)^{\overline{\gamma}}\Big[h^r \sum_{|\alpha|=r} |D^\alpha u(x_0)| + \cdots \\
+ h^{\gamma-1}\sum_{|\alpha|=\gamma-1} |D^\alpha u(x_0)| + h^\gamma \|u\|_{W^{\gamma}_{\infty}(B_d(x_0))}\Big] \\
+ d^{-t-N/p}\|u-u_h\|_{W^{-t}_p(B_d(x_0))}.
\end{aligned}
\tag{2.1}
$$

Here, $\overline{\gamma}=1$ if $\gamma=2r-2$ and $\overline{\gamma}=0$ otherwise.

Remark 2.1. The estimate (2.1) is valid on irregular meshes.

Remark 2.2. The case $r=2$ is excluded from (2.1).

Let us state the corresponding result for the first derivatives which includes also $r=2$ and is as follows:

Lemma 2.2. *Suppose that $r \geq 2$ and Assumptions A.1–A.4 (given in the Appendix I) are satisfied. Let t be a non negative integer, $1 \leq p \leq \infty$, and γ an integer, $r < \gamma \leq 2r-2$. Let $x \in \Omega_0$ and $d \geq kh$ for some k sufficiently large, and $u \in W^{\gamma+1}_{\infty}(B_d(x_0))$ and $u_h \in S^h_r(B_d(x_0))$ satisfy (1.4). Then*

$$
\begin{aligned}
\|u-u_h\|_{W^1_\infty(B_h(x_0))} \leq C\Big(\ln\frac{d}{h}\Big)^{\overline{\overline{\gamma}}}\Big(h^{r-1} \sum_{|\alpha|=r} |D^\alpha u(x_0)| + \cdots \\
+ h^{\gamma-1}\sum_{|\alpha|=\gamma} |D^\alpha u(x_0)| + h^\gamma \|u\|_{W^{\gamma+1}_{\infty}(B_d(x_0))}\Big) \\
+ C\Big(d^{-1-t-N/p}\|u-u_h\|_{W^{-t}_p(B_d(x_0))}\Big).
\end{aligned}
\tag{2.2}
$$

Here $\overline{\overline{\gamma}}=1$ if $\gamma=2r-2$, and $\overline{\overline{\gamma}}=0$ if $r \leq \gamma < 2r-2$.

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1.

Step I. Let us perform a simple step which is still valid at *any point of an irregular mesh*. We will reduce the problem to one which more clearly shows what has to be proved in order to obtain an asymptotic expansion at a point.

Without loss of generality assume for the moment that $x_0 = \{0\}$ and $d = 1$. The case of arbitrary $d > 0$ will be considered later. Assume further that $A(\cdot,\cdot)$ is coercive over $W_2^1(B_1(0))$, i.e., there is constant C such that

$$C\,\|u\|^2_{W_2^1(B_1(0))} \le A(u,u) \quad \text{for all} \quad u \in W_2^1(B_1(0)). \tag{3.1}$$

Note that A corresponds to the operator L given in (1.1) which, in general, does not satisfy (3.1). However, it is easy to construct a form $\widetilde{A}$ by adding a sufficiently large lower order term which vanishes for $|x - x_0| \le \frac{1}{2}$ and satisfies

$$C\|u\|^2_{W_2^1(B_1)} \le \widetilde{A}(u,u) \qquad \text{for all } u \in W_2^1(B_1(x_0))$$

and

$$\widetilde{A}(u - u_h, \varphi) = A(u - u_h, \varphi) \qquad \forall \varphi \in \mathring{S}_r^h\Big(B_{1/2}(x_0)\Big).$$

In order to keep the notation to a minimum we shall assume this has been done at the beginning and set $\widetilde{A} = A$, keeping in mind that $u - u_h$ satisfies the original equation in $B_{1/2}(x_0)$ but the w_h^α are determined relative to $\widetilde{A}$ on B_1.

Now for each monomial $x^\alpha \equiv w^\alpha(x)$ let $w_h^\alpha(x) \in S_r^h(B_1)$ be the finite element approximation defined by

$$A(w^\alpha - w_h^\alpha, \varphi) = 0 \qquad \forall \varphi \in S_r^h(B_1). \tag{3.2}$$

Then the functions

$$\psi = u(x) - \sum_{r \le |\alpha| \le \gamma - 1} \frac{D^\alpha u(0)}{\alpha!} w^\alpha(x)$$

and $\psi_h \in S_r^h(B_1)$ defined by

$$\psi_h = u_h(x) - \sum_{r \le |\alpha| \le \gamma - 1} \frac{D^\alpha u(0)}{\alpha!} w_h^\alpha(x)$$

satisfy the local equations

$$A(\psi - \psi_h, \varphi) = 0 \qquad \forall \varphi \in \mathring{S}_r^h(B_1). \tag{3.3}$$

Hence the estimate (2.1) can be applied at $x_0 = \{0\}$. Noticing that $\psi(0) = u(0)$ and the fact that $(D^\alpha\psi)(0) = 0$ for all $r \le |\alpha| \le \gamma - 1$ we obtain

$$\begin{aligned} u(0) &= u_h(0) - \sum_{r \le |\alpha| \le \gamma - 1} \frac{D^\alpha u(0)}{\alpha!} w_h^\alpha(0) + \mathcal{R}_\gamma \\ &= u_h(0) - \sum_{r \le |\alpha| \le \gamma - 1} \frac{D^\alpha u(0)}{\alpha!} \cdot \frac{w_h^\alpha(0)}{h^{|\alpha|}} h^{|\alpha|} + \mathcal{R}_\gamma \end{aligned} \tag{3.4}$$

where

$$\mathcal{R}_\gamma \le C\Big(h^\gamma \|u\|_{W_\infty^\gamma(B_1(0))} + \|u - u_h\|_{W_p^{-t}(B_1(0))}\Big)$$

where we have used (2.1).

Note that (3.4) is valid on an irregular mesh without assumption of similarity at the point 0. However, (3.3) is not! an asymptotic expansion because the coefficients $\frac{w_h^\alpha(0)}{h^{|\alpha|}}$ are functions of h. In order to obtain an asymptotic expansion we shall restrict ourselves to a sequence of meshes with mesh size h_j as defined previously and require similarity at the point 0. Then we shall show

Step II. We claim that for $r \le |\alpha| \le 2r-3$

$$\lim_{h\to 0} \frac{w^\alpha_{h_j}(0)}{h_j^{|\alpha|}} = C_\alpha \quad \text{exists and} \quad \Big|\frac{w^\alpha_{h_j}(0)}{h_j^{|\alpha|}} - C_\alpha\Big| \le C h_j^{2r-2-|\alpha|}. \tag{3.5}$$

Towards this end we shall show that there exists a k_0 such that for each $j \ge k_0$

$$\Big|\frac{w^\alpha_{h_j}(0)}{h_j^{|\alpha|}} - \frac{w^\alpha_{h_k}(0)}{h_k^{|\alpha|}}\Big| \le C h_j^{2r-2-|\alpha|}, \qquad \text{for all } k \ge j. \tag{3.6}$$

Then (3.6) would imply that, for $|\alpha| < 2r-2$, $\{\frac{w^\alpha_{h_j}(0)}{h_j^{|\alpha|}}\}$ is a Cauchy sequence and therefore (3.5) holds and hence (1.9) is proved for $d = 1$.

The proof of (3.6) is somewhat involved. We start with a fixed h_j for j sufficiently large so that the estimate (2.1) holds (h_j sufficiently small). By assumption **A.5.** each $S_r^{h_k}\Big(B_{\frac{h_k}{h_j}}(0)\Big)$ is similar to $S_r^{h_j}\Big(B_1(0)\Big)$ for all $k \ge j$. Set $\frac{h_k}{h_j} = \lambda$ and note that $\frac{1}{\lambda^{|\alpha|}} w^\alpha_{h_k}(\lambda x) \in S_r^{h_j}\Big(B_1(0)\Big)$, ($\lambda < 1$ is scaling factor).

We shall now show that

$$A\Big(x^\alpha - \frac{1}{\lambda^{|\alpha|}} w^\alpha_{h_k}(\lambda x), \varphi(x)\Big) = 0, \qquad \text{for all } \varphi \in \mathring{S}_r^{h_j}\Big(B_{1/2}(0)\Big). \tag{3.7}$$

Since the a_{ij} are constants, we have by the change of variable $x = \frac{y}{\lambda}$ that the left side of (3.7) is equal to

$$\begin{aligned}
&\int_{B_{1/2}(0)} \sum_{i,j=1}^N a_{ij} \frac{\partial}{\partial x_i}\Big(x^\alpha - \frac{1}{\lambda^{|\alpha|}} w^\alpha_{h_k}(\lambda x)\Big) \cdot \frac{\partial \varphi(x)}{\partial x_j}\, dx \\
&\qquad = \lambda^{1-N-|\alpha|} \int_{B_{1/2}(0)} \sum_{i,j=1}^N \frac{\partial}{\partial y_i}\Big(y^\alpha - w^\alpha_{h_k}(y)\Big) \cdot \frac{\partial}{\partial y_j} \varphi(\frac{y}{\lambda})\, dy = 0,
\end{aligned}$$

which proves (3.7).

Because of (3.7) and (3.2), recalling that $x^\alpha \equiv w^\alpha$, it follows that $w^\alpha_{h_j}(x) - \frac{1}{\lambda^{|\alpha|}} w^\alpha_{h_k}(\lambda x) \in S_r^{h_j}(B_1(0))$ and satisfies

$$A\Big(w^\alpha_{h_j}(x) - \frac{1}{\lambda^{|\alpha|}} w^\alpha_{h_k}(\lambda x), \varphi(x)\Big) = 0, \qquad \text{for all } \varphi \in S_r^{h_j}\Big(B_{1/2}(0)\Big). \tag{3.8}$$

Thus the difference in (3.8) is a discrete A harmonic function in $B_{1/2}(0)$ and it follows from (2.1): first proved by Schatz and Wahlbin [26], with $u = 0$ and $u_h = w^\alpha_{h_j}(x) - \frac{1}{\lambda^{|\alpha|}} w^\alpha_{h_k}(\lambda x)$, that

$$\begin{aligned}
\Big|(w^\alpha_{h_j}(0) - \frac{1}{\lambda^{|\alpha|}} w^\alpha_{h_k}(0)\Big| &\le \Big\| w^\alpha_{h_j} - \frac{1}{\lambda^{|\alpha|}} w^\alpha_{h_k} \Big\|_{W_2^{2-r}(B_{1/2}(0))} \\
&\le \Big\| x^\alpha - w^\alpha_{h_j} \Big\|_{W_2^{2-r}(B_{1/2}(0))} + \Big\| x^\alpha - \frac{1}{\lambda^{|\alpha|}} w^\alpha_{h_k} \Big\|_{W_2^{2-r}(B_{1/2}(0))} \\
&:= J_1 + J_2
\end{aligned} \tag{3.9}$$

We shall now estimate J_1 and J_2.

Setting $x^\alpha - w^\alpha_{h_j} = E^\alpha_j$ a standard duality argument leads to

$$J_1 = \left\|E^\alpha_j\right\|_{W_2^{2-r}(B_{1/2}(0))} = \sup_{\substack{\psi \in \mathring{W}_2^{r-2}(B_{1/2}(0)) \\ \|\psi\|_{W_2^{r-2}(B_{1/2}(0))} = 1}} (E^\alpha_j, \psi). \tag{3.10}$$

Now let

$$(v, \psi) = A(v, Z) \qquad \text{for all } v \in \mathring{W}_2^1(B_{1/2}(0)),$$

then for any $\chi \in S_r^h(B_{1/2}(0))$,

$$|(E^\alpha_j, \psi)| = A(E^\alpha_j\,,\, Z - \chi) \leq C h_j^{2r-2}\, \|\psi\|_{W_2^{r-2}(B_{1/2}(0))}\,.$$

Thus

$$J_1 \leq C h_j^{2r-2} \tag{3.11}$$

The estimate of J_2 is rather lengthy. We begin by using a duality argument where setting $\widetilde{E}^\alpha_k := x^\alpha - \frac{1}{\lambda^{|\alpha|}} w^\alpha_{h_k}(\lambda x)$ we have

$$J_2 = \sup_{\substack{\psi \in \mathring{W}_2^{r-2}(B_{1/2}(0)) \\ \|\psi\|_{W_2^{r-2}(B_{1/2}(0))} = 1}} (\widetilde{E}^\alpha_k, \psi), \tag{3.12}$$

where $|\psi|_{W_2^{r-2}(B_{1/2}(0))}$ is the seminorm and we have used the obvious fact that $|\psi|_{W_2^{r-2}(B_{1/2}(0))} \leq \|\psi\|_{W_2^{r-2}(B_{1/2}(0))}$. We again make the change of variable $x = \frac{y}{\lambda}$ and obtain

$$\begin{aligned}
(\widetilde{E}^\alpha_k, \psi) &= \int_{B_{1/2}(0)} \widetilde{E}^\alpha_k(x)\psi(x)\,dx = \int_{B_{\lambda/2}(0)} \widetilde{E}^\alpha_k(\frac{y}{\lambda})\psi(\frac{y}{\lambda})\,\frac{dy}{\lambda^N} \\
&= \frac{1}{\lambda^{N+|\alpha|}} \int_{B_{\lambda/2}(0)} \Big(y^\alpha - w^\alpha_{h_k}(y)\Big)\widetilde{\psi}(y)\,dy,
\end{aligned}$$

where $\widetilde{\psi}(y) = \psi(\frac{y}{\lambda}) \in \mathring{W}_2^{r-2}(B_{\lambda/2}(0))$ and

$$\begin{aligned}
|\psi|_{W_2^{r-2}(B_{1/2}(0))} &= \Big(\sum_{|\beta|=r-2} \int \left|D^\beta_x \psi(x)\right|^2\,dx\Big)^{1/2} \\
&= \lambda^{r-2-N/2}\Big(\sum_{|\beta|=r-2} \int \left|D^\beta_y \widetilde{\psi}(y)\right|^2\,dy\Big)^{1/2}.
\end{aligned}$$

Inserting these into (3.12) yields

$$J_2 \leq \lambda^{2-r-|\alpha|-N/2} \sup_{\widetilde{\psi} \in \mathring{W}_2^{r-2}(B_{\lambda/2}(0))} \frac{\Big(y^\alpha - w^\alpha_{h_k}(y), \widetilde{\psi}(y)\Big)}{\left|\widetilde{\psi}\right|_{W_2^{r-2}(B_{\lambda/2}(0))}}.$$

Using a duality argument let

$$A(\eta, v) = (\eta, \widetilde{\psi}), \tag{3.13}$$

and let v_I be the interpolant of v, then it follows that for each $\widetilde{\psi}$,

$$\begin{aligned}
\lambda^{2-r-|\alpha|-N/2}\Big(y^\alpha - w^\alpha_{h_k}, \widetilde{\psi}\Big) &= \lambda^{2-r-|\alpha|-N/2} A(y^\alpha - w^\alpha_{h_k}, v - v_I) \\
&\le \lambda^{2-r-|\alpha|-N/2}\Big(\Big\| (|y|+\lambda)^{r-|\alpha|}\nabla E^\alpha_k(y)\Big\|_{L_\infty(B_{1/2}(0))} \times \\
&\quad \times \Big\| (|y|+\lambda)^{|\alpha|-r}\nabla(v-v_I)(y)\Big\|_{L_1(B_{1/2})(0)} \\
&\quad + \|E^\alpha_k\|_{W^1_2(B_1\setminus B_{1/2})}\, \|v-v_I\|_{W^1_2(B_1\setminus B_{1/2})}\Big) \\
&\equiv \lambda^{2-r-|\alpha|-N/2}\Big(J_{2a}J_{2b} + J_{2c}J_{2d}\Big).
\end{aligned} \tag{3.14}$$

Notice that $\frac{\lambda}{2} < \frac{1}{4}$, if $\lambda < \frac{1}{2}$, also the return from $\widetilde{E}^\alpha_k$ to E^α_k in (3.14). One can always decompose the unit disk into two parts as an inner and an outer region so that v satisfies a homogeneous differential equation on the outer region. Estimating the above J_2 terms in the reverse order we obtain for J_{2d},

$$J_{2d} = \|v - v_I\|_{W^1_2(B_1\setminus B_{1/2})} \le C\lambda^{r-1}h_j^{r-1}\, \|v\|_{W^r_2(B_1\setminus B_{7/16})}\,.$$

By well known a priori error estimates, see Lions and Magenes [17]

$$\|v\|_{W^r_2(B_1\setminus B_{7/16})} \le \|v\|_{W^{2-\delta'}_2(B_1)}\,, \qquad \forall \delta' > \frac{N}{2}+1.$$

Thus

$$J_{2d} \le \lambda^{r-1}h_j^{r-1}\Big\|\widetilde{\psi}\Big\|_{W^{-\delta'}_2(B_1)} \le \lambda^{2r-3+N/2}h_j^{r-1}\Big|\widetilde{\psi}\Big|_{W^{r-2}_2(B_1)}\,, \tag{3.15}$$

where we have used the fact that the measure of $B_{\lambda/2}$ is proportional to λ^N, since,

$$\begin{aligned}
\Big\|\widetilde{\psi}\Big\|_{W^{-\delta'}_2(B_1)} &= \sup_{\|\varphi\|_{W^{\delta'}_2(B_1)}=1} (\widetilde{\psi},\varphi) \le \sup_{\|\varphi\|_{W^{\delta'}_2(B_1)}=1} \Big\|\widetilde{\psi}\Big\|_{L_1(B_{\lambda/2})}\|\varphi\|_{L_\infty} \\
&\le \sup_{\|\varphi\|_{W^{\delta'}_2(B_1)}=1} \lambda^{N/2}\Big|\widetilde{\psi}\Big|_{L_2(B_{\lambda/2})}\|\varphi\|_{L_\infty} \le C\lambda^{r-2+N/2}\Big|\widetilde{\psi}\Big|_{W^{r-2}_2(B_1)}\,,
\end{aligned}$$

where we have used the Sobolev inequality to obtain the bound: $\|\varphi\|_{L_\infty} \le C$. As for J_{2c} we have

$$J_{2c} = \|E^\alpha_k\|_{W^1_2(B_1\setminus B_{1/2})} \le \|E^\alpha_k\|_{W^1_2(B_1)} \le \lambda^{r-1}h_j^{r-1}. \tag{3.16}$$

Taken together (3.15) and (3.16) yield

$$\begin{aligned}
\lambda^{r-2-|\alpha|-N/2}J_{2c}J_{2d} &\le C\lambda^{r-2-|\alpha|-N/2}h_j^{r-1}\lambda^{3r-4+N/2}h_j^{r-1}\Big|\widetilde{\psi}\Big|_{W^{r-2}_2(B_1)} \\
&\le C\lambda^{2r-2-|\alpha|}h_j^{2r-2}\Big|\widetilde{\psi}\Big|_{W^{r-2}_2(B_1)}\,.
\end{aligned} \tag{3.17}$$

This is half of the estimate for J_2. We shall now estimate J_{2b} by writing

$$\begin{aligned}
J_{2b} &\le C\Big\|(|y|+\lambda)^{|\alpha|-r}\nabla(v-v_I)\Big\|_{L_1(B_{4\lambda})} \\
&\quad + \sum_{4\lambda\le\rho_j\le\frac{1}{2}}\Big\|(|y|+\lambda)^{|\alpha|-r}\nabla(v-v_I)\Big\|_{L_1(\Omega_\ell)}\,,
\end{aligned} \tag{3.18}$$

where Ω_ℓ are the annuli

$$\Omega_\ell = \{x \in B_1 :\ \rho_{\ell+1} \le |x| \le \rho_\ell\}, \quad \rho_\ell = 2^{-\ell}, \quad \ell = 1,2,\ldots.$$

Then by approximation theory and the Cauchy Schwarz inequality, from (3.18) we get

$$(3.19)\quad J_{2b} \le C\lambda^{|\alpha|-1+N/2} h_j^{r-1} \, |v|_{W_2^r(B_{5\lambda})} + \sum_{1\le \ell \le \bar\ell} \rho_\ell^{|\alpha|-r+N/2} \lambda^{r-1} h_j^{r-1} \, |v|_{W_2^r(\Omega'_\ell)},$$

where $\bar\ell = [(\ln \frac{1}{4\lambda})/\ln 2]$, with $[\tau]$ denoting the integer part of τ, and $\Omega'_\ell = \cup_{m=\ell-1}^{\ell+1}\Omega_m$ is the union of Ω_ℓ and its closest adjacent neighbors. Similarly, using local estimates for the continuous problem, with $\Omega''_\ell = \cup_{m=\ell-1}^{\ell+1}\Omega'_m$, and the fact that $3 \le 2r - |\alpha| \le r$,

$$(3.20)\qquad |v|_{W_2^r(\Omega'_\ell)} \le C\rho_\ell^{r-|\alpha|-N/2} \, |v|_{W_1^{2r-|\alpha|}(\Omega''_\ell)}.$$

Hence the last term on the right hand side of (3.19) can be estimated as

$$(3.21)\qquad \sum_{1\le \ell\le\bar\ell} \rho_\ell^{|\alpha|-r+N/2} \lambda^{r-1} h_j^{r-1} \, |v|_{W_2^r(\Omega'_\ell)} \le C\lambda^{r-1} h_j^{r-1} \, |v|_{W_1^{2r-|\alpha|}(B_1\setminus B_{2\lambda})}.$$

The last term will be estimated by Green's function. For any multi-index β, with $|\beta| = 2r - |\alpha|$ we have, with $\widetilde\psi$ and v satisfying (3.13),

$$(3.22)\qquad \begin{aligned} \int_{|x|\ge 2\lambda} \left|D^\beta v(x)\right| \, dx &= \int_{|x|\ge 2\lambda} \Big(\int_{|y|\le 2\lambda} \left|\widetilde\psi(y)\right| \left|D_x^\beta G(x,y)\right| \, dy \Big) \, dx \\ &\le \int_{|y|\le 2\lambda} \left|\widetilde\psi(y)\right| \Big(\int_{|x|\ge 2\lambda} \frac{1}{|x-y|^{N-2+2r-|\alpha|}} \, dx \Big) dy. \end{aligned}$$

For each y let $R = |x-y|$, then in spherical coordinates

$$(3.23)\qquad \begin{aligned} \int_{|x|\ge 2\lambda} \frac{1}{|x-y|^{N-2+2r-|\alpha|}} \, dx &\le C \int_{\lambda\le R\le 1} \frac{R^{N-1}}{R^{N-2+2r-|\alpha|}} \, dR \\ &= C \int_{\lambda\le R\le 1} R^{|\alpha|+1-2r} \, dR \le C\lambda^{|\alpha|+2-2r}, \end{aligned}$$

where we use the inequality $-r+1 \le |\alpha|+1-2r \le -2$. Therefore using (3.23) in (3.22) and the Poincare inequality on $\widetilde\psi$ we get for $\widetilde\psi \in \mathring{W}_2^{r-2}(B_{\lambda/2})$,

$$\int_{|x|\ge 2\lambda} \left|D^\beta v(x)\right| \, dx \le C\lambda^{|\alpha|+2-2r} \left\|\widetilde\psi\right\|_{L_1(B_{\lambda/2})} \le C\lambda^{|\alpha|-r+N/2} \left|\widetilde\psi\right|_{W_2^{r-2}(B_{\lambda/2})},$$

and from using (3.22) in (3.20)

$$(3.24)\qquad J_{2b} \le C\lambda^{|\alpha|-1+N/2} h_j^{r-1} \left|\widetilde\psi\right|_{W_2^{r-2}}.$$

Finally, it remains, to estimate J_{2a}. In order to do this we shall need a simple variant of a result proved in Schatz [24] for

$$(3.25)\qquad J_{2a} = \left\| \Big(\frac{1}{|y|+\lambda} \Big)^{|\alpha|-r} \nabla E_k^\alpha(y) \right\|_{L_\infty(B_{1/2})}.$$

The result in [24] is as follows: For any $0 \le s \le r-1$,

(3.26)

$$|\nabla(u-u_{h_k})(y)| \le C \left\| \Big(\frac{h_k}{|y-z|+h_k} \Big)^s \nabla(u-\chi)(z) \right\|_{L_\infty(B_{1/4})} + \|e\|_{W_2^{2-r}(B_{1/4})}.$$

Applying (3.26) to $u = x^\alpha$, $u_h = w^\alpha_{h_k}$ we obtain for any $z \in B_{1/2}(0)$,

$$\left|\Big(\frac{1}{|y|+\lambda}\Big)^{|\alpha|-r}\nabla E^\alpha_{h_k}(y)\right| \le \tag{3.27}$$
$$\le \left\|\Big(\frac{1}{|y|+\lambda}\Big)^{|\alpha|-r}\Big(\frac{h_k}{|y-z|+h_k}\Big)^{|\alpha|-r}\nabla(u-\chi)(z)\right\|_{L_\infty(B_{3/4})}$$
$$+\Big(\frac{1}{|y|+\lambda}\Big)^{|\alpha|-r}\left\|\nabla E^\alpha_{h_k}\right\|_{W_2^{2-r}(B_1)}.$$

Now, since $\lambda = h_k/h_j < 1$, thus $h_k/(|y|+\lambda) \approx h_j < 1$, and hence

$$\Big(\frac{1}{|y|+\lambda}\cdot\frac{h_k}{|y-z|+h_k}\Big)\Big(\frac{|z|+\lambda}{|z|+\lambda}\Big) \le \frac{1}{|z|+\lambda}\Big(\frac{|y-z|+|y|+\lambda}{(|y-z|+h_l)(|y|+\lambda)}\Big)\times h_k$$
$$\le \frac{1}{|z|+\lambda}\Big(\frac{h_k}{|y|+\lambda}+\frac{h_k}{|y-z|+h_k}\Big).$$

Therefore from the above inequality and (3.27), taking the supremum over all $y \in B_{1/2}$ we get

$$J_{2a} \le C\left\|\Big(\frac{2}{|z|+\lambda}\Big)^{|\alpha|-r}\nabla(x^\alpha - x^\alpha_I)(z)\right\|_{L_\infty(B_{3/4})} + C\lambda^{r-|\alpha|}\left\|\nabla E^\alpha_k\right\|_{W_2^{2-r}(B_1)}. \tag{3.28}$$

Now

$$\lambda^{r-|\alpha|}\left\|\nabla E^\alpha_k\right\|_{W_2^{2-r}(B_1)} \le \lambda^{r-|\alpha|}\lambda^{2r-2}h_j^{2r-2} \le \lambda^{3r-|\alpha|-2}h_j^{2r-2}. \tag{3.29}$$

To estimate the first term on the right hand side of (3.28) we use the same diadic decomposition as before and write $B_{3/4} = B_{\lambda h_j}\bigcup_\ell B_{\Omega_\ell}$, where $\lambda h_j \le \rho_1$, Then on $B_{\lambda h_j}$,

$$\Big(\frac{1}{|z|+\lambda}\Big)^{|\alpha|-r} \le \lambda^{r-|\alpha|},$$

and

$$|\nabla(x^\alpha - x^\alpha_I)| \le \lambda^{r-1}h_j^{r-1}\left\|x^\alpha\right\|_{W^r_\infty} \le C\lambda^{r-1}h_j^{r-1}\left|\lambda h_j\right|^{|\alpha|-r}.$$

Similarly, on Ω_ℓ,

$$\sup_\ell\left\|\Big(\frac{1}{|z|+\lambda}\Big)^{|\alpha|-r}\nabla(x^\alpha - x^\alpha_I)(z)\right\|_{L_\infty(\Omega_\ell)} \le \Big(\frac{1}{\rho_\ell}\Big)^{|\alpha|-r}\lambda^{r-1}h_j^{r-1}\rho_\ell^{|\alpha|-r}.$$

Summing up we get

$$\left\|\Big(\frac{1}{|z|+\lambda}\Big)^{|\alpha|-r}\nabla(x^\alpha - x^\alpha_I)(z)\right\|_{L_\infty(B_{3/4})} \tag{3.30}$$
$$\le C\Big(\lambda^{r-|\alpha|}\lambda^{r-1}h_j^{r-1}\lambda^{|\alpha|-r} + \sup_{1\le\ell\le\bar\ell}\rho_\ell^{r-|\alpha|}\rho_\ell^{|\alpha|-r}\lambda^{r-1}h_j^{r-1}\Big)$$
$$\le C\Big(\lambda^{r-1}h_j^{r-1}\Big)$$

Taken together these last two inequalities (3.29) and (3.30) yield

$$J_{2a} \le C\Big(\lambda^{r-1}h_j^{r-1} + \lambda^{3r-|\alpha|-2}h_j^{2r-2}\Big).$$

Now since $r-1 < r+1 = 3r-2r+3-2 \le 3r-|\alpha|-2 \le 2r-2$. Thus

$$J_{2a} \le C\Big(\lambda^{r-1}h_j^{r-1}\Big). \tag{3.31}$$

Combining (3.31), (3.24) and (3.17) we have in view of (3.14) and finally (3.10) and (3.9) that

$$\begin{aligned}\left|w^\alpha_{h_j}(0) - \frac{1}{\lambda^{|\alpha|}} w^\alpha_{h_k}(0)\right| &\le C h_j^{2r-2} + C\lambda^{2r-2-|\alpha|} h_j^{2r-2} \\ &+ \lambda^{2-r-|\alpha|-N/2}\Big(\lambda^{|\alpha|-1+N/2}\lambda^{r-1}h_j^{2r-2}\Big) \\ &\le C\Big(h_j^{2r-2}\Big)\Big(1+\lambda^{2r-2-|\alpha|}+1\Big) \\ &\le C h_j^{2r-2} \qquad \text{for all} \quad h_k.\end{aligned}$$

Consequently we have the final answer for the case $\rho = 1$, viz,

$$\left|\frac{w^\alpha_{h_j}(0)}{h_j^{|\alpha|}} - \frac{w^\alpha_{h_k}(0)}{h_k^{|\alpha|}}\right| \le C h_j^{2r-2-|\alpha|} \qquad \text{for all} \quad h_k < h_j \ \ (k > j). \tag{3.32}$$

Therefore for each α, $r \le |\alpha| \le 2r-3$,

$$\lim_{j\to\infty} \frac{w^\alpha_{h_j}(0)}{h_j^{|\alpha|}} = C_\alpha$$

exists and is independent of h_j and u, and we have the asymptotic expansion

$$u(0) = u_h(0) - \sum_{r\le|\alpha|\le\gamma-1} C_\alpha \frac{D^\alpha u(0)}{\alpha!} h^{|\alpha|} + h^\gamma \|u\|_{W^\gamma_\infty(B_1)} + \|u-u_h\|_{W_p^{-t}(B_1)}\,. \tag{3.33}$$

Step III. $Ch_j < \rho < 1$, ρ fixed.
Let $A(u-u_h,\varphi) = 0 \quad \forall\varphi \in \mathring{S}_r^{h_j}(B_\rho)$ then under the change of variable $y = \frac{x}{\rho}$ $\widetilde{u} := u(\rho y)$ and $\widetilde{u}_h = u_h(\rho y)$ satisfy

$$A(\widetilde{u}-\widetilde{u}_h,\psi) = 0 \quad \forall\psi \in \mathring{S}_r^{h_j/\rho}(B_1)$$

and therefore we may apply the asymptotic expansion to obtain on unit size domain

$$\widetilde{u}(0) - \widetilde{u}_h(0) = -\sum_{r\le|\alpha|\le\gamma-1}\Big(\frac{C_\alpha}{\alpha!}\Big) D^\alpha_y \widetilde{u}(0)\cdot\Big(\frac{h}{\rho}\Big)^{|\alpha|} + \widetilde{\mathcal{R}}_\gamma \tag{3.34}$$

where

$$\left|\widetilde{\mathcal{R}}_\gamma\right| \le C\Big(\frac{h_j}{\rho}\Big)^\gamma \|\widetilde{u}\|_{W^\gamma_\infty(B_{1/2})} + C\,\|\widetilde{u}-\widetilde{u}_h\|_{W_p^{-t}(B_1)}\,. \tag{3.35}$$

Changing variables back again we get the main result for step III:

$$u(0) - u_h(0) = -\sum_{r\le|\alpha|\le\gamma-1}\Big(\frac{C_\alpha}{\alpha!}\Big) D^\alpha_x u(0) h^{|\alpha|} + \mathcal{R}_\gamma \tag{3.36}$$

where

$$\mathcal{R}_{(\gamma,\rho)} \le C\Big(h^\gamma \|u\|_{W^\gamma_\infty(B_\rho)} + \rho^{-t-N/p}\|u-u_h\|_{W_p^{-t}(B_\rho)}\Big). \quad \square \tag{3.37}$$

Proof of Theorem 1.2.

Step I. $\rho = 1$.

We shall give a sketch of the proof of Theorem 1.2, mentioning only the differences between the two proofs for Theorems 1.1 and 1.2. For $r \geq 2$, i.e. piecewise linear as above, consider

$$u(0) - u_h(0) - \sum_{r\leq|\alpha|\leq\gamma-1} \Big(\frac{D^\alpha u(0)}{\alpha!} x^{|\alpha|} - C_\alpha D^\alpha u(0) w_h^\alpha(x)\Big). \tag{3.38}$$

The result we shall need from Schatz and Wahlbin [26] is the Lemma 2.2. Applying Lemma 2.2 to calculate the error at 0 for the difference $\frac{\partial u(0)}{\partial x_i} - \frac{\partial u_h(0)}{\partial x_i}$ we have for $i = 1, 2, \dots, N$, on unit size domain

$$\frac{\partial u(0)}{\partial x_i} - \frac{\partial u_h(0)}{\partial x_i} - \sum_{|\alpha|} \frac{D^\alpha u(0)}{\alpha!} \cdot \frac{\partial w_{h_j}^\alpha(0)}{\partial x_i} = R'_\gamma,$$

where $r \leq \gamma \leq 2r - 1$ and

$$R'_\gamma \leq C\Big(h_j^\gamma \,\|u\|_{W_\infty^{\gamma+1}(B_1(0))} + \|u - u_h\|_{W_p^{-t}(B_1(0))}\Big),$$

Thus in this case we are led to showing that

$$M = \left| \frac{\frac{\partial w_{h_j}^\alpha(0)}{\partial x_i}}{h_j^{|\alpha|-1}} - \frac{\frac{\partial w_{h_k}^\alpha(0)}{\partial x_i}}{h_k^{|\alpha|-1}} \right| \leq C h_j^{2r-1-|\alpha|}.$$

By a result of Schatz and Wahlbin [26] and in view of the fact that $w_{h_j}^\alpha - \frac{1}{\lambda^{|\alpha|}} w_{h_l}^\alpha$ with $\lambda = \frac{h_k}{h_j}$, as proved before is discrete harmonic and in view of (3.32), satisfies

$$M \leq \frac{1}{h_j^{|\alpha|-1}} \left\| w_{h_j}^\alpha - \frac{1}{\lambda^{|\alpha|}} w_{h_j}^\alpha \right\|_{W_2^{2r-2}} \leq h_j^{2r-1-|\alpha|},$$

which proves the expansion on unit sized domain and scaling as before gives the main result for the derivatives: For $i = 1, 2, \dots, N$,

$$\frac{\partial u(0)}{\partial x_i} - \frac{\partial u_h(0)}{\partial x_i} = -\sum_{r\leq|\alpha|\leq\gamma-1} C'_\alpha D_u^\alpha(0) h^{|\alpha|-1} + \mathcal{R}'_\gamma \tag{3.39}$$

where $h = h_j,\ j = k, k+1, \dots,\ \gamma \leq 2r - 1$ and

$$\mathcal{R}'_\gamma \leq \Big(C h^{\gamma-1} \,\|u\|_{W_\infty^\gamma(B_1(0))} + \|u - u_h\|_{W_p^{-t}(B_1(0))}\Big). \tag{3.40}$$

Step II. Asymptotic expansion for $\rho < 1$.

Following the scaling argument given in the proof of the Theorem 1.1, in the corresponding case: $\rho < 1$, the proof is an exercise to the reader. The estimate for the remainder can be written as

$$\mathcal{R}'_\gamma \leq C\Big(h^{\gamma-1} \,\|u\|_{W_\infty^\gamma(B_\rho)} + \rho^{-t-1-N/p} \,\|u - u_h\|_{W_p^{-t}(B_\rho)}\Big).$$

This completes the proof of Theorem 1.2. □

4. Appendix I. Properties of the Finite Element Subspaces

Here we shall state our assumptions on the finite element subspaces used in this paper. They are basically the same as those given in Schatz and Wahlbin [27] and [26]. The precise statements here are taken from [26].

For $0 < h < 1$ a parameter and $r \geq 2$ an integer, $S_r^h(\Omega)$ will denote a family of finite dimensional subspaces of $W_\infty^1(\Omega)$. If $D \subseteq \Omega$ then $S_r^h(D)$ will denote the restriction of functions in $S_r^h(\Omega)$ to D and $\mathring{S}_r^h(D)$ is the subspace of $S_r^h(D)$ consisting of functions whose support is contained in D. In what follows $D_0 \subset\subset D_1 \subset\subset D_2$, etc. denote concentric balls which are contained in Ω. Assume that there exists a constant k such that if $\mathrm{dist}(D_0, \partial D_1) \geq kh$ and $\mathrm{dist}(D_1, \partial D_2) \geq kh$ then the following hold:

A.1 (Approximation) If $t = 0, 1$, $t \leq \ell \leq r$, $1 \leq p \leq \infty$, then for each $v \in W_p^\ell(D_2)$ there exists a $\chi \in S_r^h(D_1)$ such that

$$\|v - \chi\|_{W_p^t(D_1)} \leq C h^{\ell - t} |v|_{W_p^\ell(D_2)}.$$

Here

$$|v|_{W_p^\ell} = \begin{cases} \Big(\sum_{|\alpha| = \ell} \|D^\alpha v\|_{L_p}^p \Big)^{1/p} & \text{if } \ 1 \leq p < \infty \\ \sum_{|\alpha| = \ell} \|D^\alpha v\|_{L_\infty} & \text{if } \ p = \infty. \end{cases}$$

Furthermore, if $v \in \mathring{W}_p^\ell(D_0)$ then $\chi \in \mathring{S}_r^h(D_2)$. Here C is independent of h, v, χ and D_i, $i = 0, 1, 2$.

A.2 (Inverse Properties) If $\chi \in S_r^h(D_2)$, then for $t = 0, 1$,

$$\|\chi\|_{W_\infty^t(D_1)} \leq C h^{-N/2 - t} \|\chi\|_{L_2(D_2)},$$

and for $\ell = 0, 1$,

$$\|\chi\|_{W_2^t(D_1)} \leq C h^{\ell - t} \|\chi\|_{W_2^{-\ell}(D_2)}.$$

Here C is independent of h, χ, D_1 and D_2.

A.3 (Super-approximation) Let $\omega \in C_0^\infty(D_1)$, then for each $\chi \in S_r^h(D_2)$ there exists an $\eta \in \mathring{S}_r^h(D_2)$ such that for some integer $\gamma > 0$

$$\|\omega\chi - \eta\|_{W_2^1(D_2)} \leq C h \|\omega\|_{W_\infty^\gamma(D_1)} \|\chi\|_{W_2^1(D_3)}.$$

Furthermore, if $\omega \equiv 1$ on D_0, and $D_{-1} \subset\subset D_0$ with $\mathrm{dist}(D_{-1}, \partial D_0) \geq k$, then $\eta = \chi$ on D_{-1}, and

$$\|\omega\chi - \eta\|_{W_2^1(D_2)} \leq C h \|\omega\|_{W_\infty^\gamma(D_1)} \|\chi\|_{W_2^1(D_2 \setminus D_{-1})}.$$

Here C is independent of ω, χ, η, h, D_i, $i = -1, 0, 1, 2$.

A.4 (Scaling) Let $x_0 \in \overline{\Omega}$ and $d \geq kh$. The linear transformation $y = x_0 + (x - x_0)/d$ takes $B_d(x_0) = \{x : |x - x_0| < d\} \cap \Omega$ into a new domain $\widehat{B}_1(x_0)$ and $S_r^h(B_d(x_0))$ into a new function space $\widehat{S}_r^{h/d}(\widehat{B}_1(x_0))$. The $\widehat{S}_r^{h/d}(\widehat{B}_1(x_0))$ satisfies A.1, A.2 and A.3 with h replaced by h/d. The constants occurring in A.1, A.2 and A.3 remain unchanged, in particular independent of d.

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