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# UNIQUENESS AND FACTORIZATION OF COLEFF-HERRERA CURRENTS

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Abstract. We prove a uniqueness result for Coleff-Herrera currents which in particular means that if $f = (f_1, \ldots, f_m)$ defines a complete intersection, then the classical Coleff-Herrera product associated to $f$ is the unique Coleff-Herrera current that is cohomologous to 1 with respect to the operator $\delta_f - \bar\partial$, where $\delta_f$ is interior multiplication with $f$. From the uniqueness result we deduce that any Coleff-Herrera current on a variety $Z$ is a finite sum of products of residue currents with support on $Z$ and holomorphic forms.

## 1. Introduction

Let $X$ be an $n$-dimensional complex manifold and let $Z$ be an analytic variety of pure codimension $p$. The sheaf of Coleff-Herrera currents (or currents of *residual type*) $\mathcal{CH}_Z$ consists of all $\bar\partial$-closed $(*, p)$-currents $\mu$ with support on $Z$ such that $\bar\psi\mu = 0$ for each $\psi$ vanishing on $Z$, and which in addition fulfills the so-called standard extension property, SEP, see below. Locally, any $\mu \in \mathcal{CH}_Z$ can be realized as a meromorphic differential operator acting on the current of integration $[Z]$ (combined with contractions with holomorphic vector fields), see, e.g., [3] and [4].

The model case of a Coleff-Herrera current is the Coleff-Herrera product associated to a complete intersection $f = (f_1, \ldots, f_p)$,

$$\mu^f = \big[\bar\partial\frac{1}{f_1}\wedge\ldots\wedge\bar\partial\frac{1}{f_p}\big], \tag{1.1}$$

introduced by Coleff and Herrera in [5]. Equivalent definitions are given in [7] and [8]; see also [10]. It was proved in [6] and [7] that the annihilator of $\mu^f$ is equal to the ideal $\mathcal{J}(f)$ generated by $f$. Notice that formally (1.1) is just the pullback under $f$ of the product $\mu^w = \bar\partial(1/w_1)\wedge\ldots\wedge\bar\partial(1/w_p)$. One can also express $\mu^w$ as $\bar\partial$ of the Bochner-Martinelli form

$$B(w) = \sum_j (-1)^j \bar w_j d\bar w_1\wedge\ldots\wedge d\bar w_{j-1}\wedge d\bar w_{j+1}\wedge\ldots\wedge d\bar w_p/|w|^{2p}.$$

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In [9], $f^*B$ is defined as a principal value current, and it is proved that $\mu^f_{BM} = \bar\partial f^*B$ is indeed equal to $\mu^f$. However the proof is quite involved. An alternative but still quite technical proof appeared in [1]. In this paper we prove a uniqueness result which states that any Coleff-Herrera current that is cohomologous to 1 with respect to the operator $\delta_f - \bar\partial$ (see Section 3 for definitions) must be equal to $\mu^f$. In particular this implies that $\mu^f = \mu^f_{BM}$.

It is well-known that any Coleff-Herrera current can be written $\alpha\wedge\mu^f$, where $\alpha$ is a holomorphic $(*,0)$-form and $\mu^f$ is a Coleff-Herrera product for a complete intersection $f$. However, unless $Z$ is a complete intersection itself the support of $\mu^f$ is larger than $Z$. Using the uniqueness result we can prove

**Theorem 1.1.** *For any $\mu \in \mathcal{CH}_Z$ (locally) there are residue currents $R_I$ with support on $Z$ and holomorphic $(*,0)$-forms $\alpha_I$ such that*

$$\mu = \sum_{|I|=p}' R_I\wedge\alpha_I. \tag{1.2}$$

Here $R_I$ are currents of Bochner-Martinelli type from [9] associated with a not necessarily complete intersection. In particular, it follows that the Lelong current $[Z]$ admits a factorization (1.2).

By the the uniqueness result we also obtain simple proofs of the equivalence of various definitions of the SEP, as well as the equivalence of various conditions for the vanishing of a Coleff-Herrera current.

We will adopt the following definition of SEP: *Given any holomorphic $h$ that does not vanish identically on any irreducible component of $Z$, the function $|h|^{2\lambda}\mu$, a priori defined only for $\mathrm{Re}\,\lambda >> 0$, has a current-valued analytic extension to $\mathrm{Re}\,\lambda > -\epsilon$, and the value at $\lambda = 0$ coincides with $\mu$.* The reason for this choice is merely practical; for the equivalence to the classical definition, see Section 5. Now, if $\mu \in \mathcal{CH}_Z$ has support on $Z \cap \{h = 0\}$, then $|h|^{2\lambda}\mu$ must vanish if $\mathrm{Re}\,\lambda$ is large enough, and by the uniqueness of analytic continuation thus $\mu = 0$. In particular, $\mu = 0$ identically if $\mu = 0$ on $Z_{reg}$.

## 2. The Coleff-Herrera product

Let $f_1, \dots, f_p$ define a complete intersection in $X$, i.e., $\mathrm{codim}\, Z^f = p$, where $Z^f = \{f = 0\}$. Notice that (1.1) is elementarily defined if each $f_j$ is a power of a coordinate function. The general definition relies on the possibility to resolve singularities: By Hironaka's theorem we can locally find a resolution $\pi\colon \tilde{\mathcal{U}} \to \mathcal{U}$ such that locally in $\tilde{\mathcal{U}}$, each $\pi^* f_j$ is a monomial times a non-vanishing factor. It turns out that locally $\mu^f$ is a sum of terms

$$\sum_\ell \pi_*\tau_\ell \tag{2.1}$$

where each $\tau_\ell$ is of the form

$$\tau_\ell = \bar\partial \frac{1}{t_1^{a_1}} \wedge \ldots \wedge \bar\partial \frac{1}{t_p^{a_p}} \wedge \frac{\alpha}{t_{p+1}^{a_{p+1}} \cdots t_r^{a_r}},$$

$t$ is a suitable local coordinate system in $\tilde{\mathcal{U}}$, and $\alpha$ is a smooth function with compact support. It is well-known that $\mu^f$ is in $\mathcal{CH}_{Z^f}$ but for further reference we sketch a proof: It follows immediately from the definition that $\mu^f$ is a $\bar\partial$-closed $(0,p)$-current with support on $Z^f$. Given any holomorphic function $\psi$ we may choose the resolution so that also $\pi^*\psi$ is a monomial. Notice that each $|\pi^*\psi|^{2\lambda}\tau_\ell$ has an analytic continuation to $\lambda = 0$ and that the value at 0 is equal to $\tau_\ell$ if none of $t_1, \ldots, t_p$ is a factor in $\pi^*\psi$ and zero otherwise. According to this let us subdivide the set of $\tau_\ell$ into two groups $\tau'_\ell$ and $\tau''_\ell$. Notice that $|\psi|^{2\lambda}\mu^f = \sum_\ell \pi_*(|\pi^*\psi|^{2\lambda}\tau_\ell)$ admits an analytic continutation and that the value at $\lambda = 0$ is $\sum \pi_*\tau''_\ell$. If $\psi = 0$ on $Z^f$, then $0 = |\psi|^{2\lambda}\mu^f$, and hence $\mu^f = \sum_\ell \pi_*\tau'_\ell$; it now follows that $\bar\psi\mu^f = d\bar\psi\wedge\mu^f = 0$. If $h$ is holomorphic and the zero set of $h$ intersects $Z^f$ properly, then $T = \mu^f - |h|^{2\lambda}\mu^f|_{\lambda=0}$ is a current of the type (2.1) with support on $Y = Z^f \cap \{h = 0\}$ that has codimension $p+1$. For the same reason as above, $d\bar\psi\wedge T = 0$ for each holomorphic $\psi$ that vanishes on $Y$ and by a standard argument it now follows that $T = 0$ for degree reasons. Thus $\mu^f$ has the SEP and so $\mu^f \in \mathcal{CH}_{Z^f}$. This proof is inspired by a forthcoming joint paper, [2], with Elizabeth Wulcan.

## 3. The uniqueness result

Let $f = (f_1, \ldots, f_m)$ be a holomorphic tuple on some complex manifold $X$. It is practical to introduce a (trivial) vector bundle $E \to X$ with global frame $e_1, \ldots, e_m$ and consider $f = \sum f_j e_j^*$ as a section of the dual bundle $E^*$, where $e_j^*$ is the dual frame. Then $f$ induces a mapping $\delta_f$, interior multiplication with $f$, on the exterior algebra $\Lambda E$. Let $\mathcal{C}_{0,k}(\Lambda^\ell E)$ be the sheaf of $(0,k)$-currents with values in $\Lambda^\ell E$, considered as as sections of the bundle $\Lambda(E \oplus T^*(X))$; thus a section of $\mathcal{C}_{0,k}(\Lambda^\ell E)$ is given by an expression $v = \sum'_{|I|=\ell} f_I \wedge e_I$ where $f_I$ are $(0,k)$-currents and $d\bar z_j \wedge e_k = -e_k \wedge d\bar z_j$ etc. Notice that both $\bar\partial$ and $\delta_f$ act as antiderivations on these spaces, i.e., $\bar\partial(f\wedge g) = \bar\partial f \wedge g + (-1)^{\deg f} f \wedge \bar\partial g$, if at least one of $f$ and $g$ is smooth, and similarly for $\delta_f$. It is straight forward to check that $\delta_f\bar\partial = -\bar\partial\delta_f$. Therefore, if $\mathcal{L}^k = \oplus_j \mathcal{C}_{0,j-k}(\Lambda^j E)$ and $\nabla_f = \delta_f - \bar\partial$, then $\nabla_f \colon \mathcal{L}^k \to \mathcal{L}^{k+1}$, and $\nabla_f^2 = 0$. For example, $v \in \mathcal{L}^{-1}$ is of the form $v = v_1 + \cdots + v_m$, where $v_k$ is a $(0, k-1)$-current with values in $\Lambda^k E$. Also for a general current the subscript will denote degree in $\Lambda E$.

*Example* 1 (The Coleff-Herrera product). Let $f = (f_1, \ldots, f_m)$ be a complete intersection in $X$. The current

$$V = \Big[\frac{1}{f_1}\Big]e_1 + \Big[\frac{1}{f_2}\bar\partial\frac{1}{f_1}\Big] \wedge e_1 \wedge e_2 + \Big[\frac{1}{f_3}\bar\partial\frac{1}{f_2} \wedge \bar\partial\frac{1}{f_1}\Big] \wedge e_1 \wedge e_2 \wedge e_3 + \cdots \tag{3.1}$$

is in $\mathcal{L}^{-1}$ and solves $\nabla_f V = 1 - \mu^f \wedge e$, where $\mu^f$ is the Coleff-Herrera product and $e = e_1 \wedge \ldots \wedge e_m$. For definition of the coefficients of $V$ and the computational rules used here, see [7]; one can obtain a simple proof of these rules by arguing as in Section 2, see [2]. □

*Example* 2 (Residues of Bochner-Martinelli type). Introduce a Hermitian metric on $E$ and let $\sigma$ be the section of $E$ over $X \setminus Z^f$ with minimal pointwise norm such that $\delta_f\sigma = f \cdot \sigma = 1$. Then

$$u = \frac{\sigma}{\nabla_f \sigma} = \sigma + \sigma\wedge\bar\partial\sigma + \sigma\wedge(\bar\partial\sigma)^2 + \cdots$$

is smooth outside $Z^f$ and $\nabla_f u = 1$ there. It turns out, see [1], that $U$ has a natural current extension $U$ across $Z^f$, and if $p = \operatorname{codim} Z^f$, then $\nabla_f U = 1 - R^f$, where $R^f = R^f_p + \cdots + R^f_m$. Moreover, these currents have representations like (2.1) so if $\xi \in \mathcal{O}(\Lambda^{m-p}E)$ and $\xi\wedge R^f_p$ is $\bar\partial$-closed, then it is in $\mathcal{CH}^f_Z$ by the arguments given in Section 2. Notice that

$$R^f_k = \sum_{|I|=k}{}' R^f_I \wedge e_{I_1} \wedge \ldots \wedge e_{I_k}. \tag{3.2}$$

If we choose the trivial metric, the coefficients $R^f_I$ are precisely the currents introduced in [9]. In particular, if $f$ is a complete intersection, i.e., $m = p$, then $R^f_{1,\ldots,p} = \mu^f_{BM}\wedge e$. □

**Theorem 3.1** (Uniqueness for Coleff-Herrera currents)**.** *Assume that $Z^f$ has pure codimension $p$. If $\tau \in \mathcal{CH}_{Z^f}$ and there is a solution $V \in \mathcal{L}^{p-m-1}$ to $\nabla_f V = \tau\wedge e$, then $\tau = 0$.*

*Remark* 1. If $Z^f$ does not have pure codimension, the theorem still holds (with the same proof) with $\mathcal{CH}_{Z^f}$ replaced by $\mathcal{CH}_{Z'}$, where $Z'$ is the irreducible components of $Z^f$ of maximal dimension. □

In view of Examples 1 and 2 we get

**Corollary 3.2.** *Assume that $f$ is a complete intersection. If $\mu \in \mathcal{CH}_{Z^f}$ and there is a current $U \in \mathcal{L}^{-1}$ such that $\nabla_f U = 1 - \mu\wedge e$, then $\mu$ is equal to the Coleff-Herrera product $\mu^f$. In particular, $\mu^f_{BM} = \mu^f$.*

The proof of Theorem 3.1 relies on the following lemma, which is probably known. However, for the reader's convenience we include an outline of a proof.

**Lemma 3.3.** *If $\mu$ is in $\mathcal{CH}_Z$ and for each neighborhood $\omega$ of $Z$ there is a current $V$ with support in $\omega$ such that $\bar\partial V = \mu$, then $\mu = 0$.*

*Proof.* Locally on $Z_{reg}$ we can choose coordinates $(z, w)$ such that $Z = \{w = 0\}$. Since $\bar w_j \mu = 0$ and $\bar\partial \mu = 0$ it follows that $d\bar w_j \wedge \mu = 0$, $j = 1, \dots, p$, and hence $\mu = \mu_0 d\bar w_1 \wedge \dots \wedge d\bar w_p$. From a Taylor expansion in $w$ we get that

$$\mu = \sum_{|\alpha| \le M-p} a_\alpha(z) \bar\partial \frac{1}{w_1^{\alpha_1+1}} \wedge \dots \wedge \bar\partial \frac{1}{w_p^{\alpha_p+1}}, \tag{3.3}$$

where $a_\alpha$ are the push-forwards of $\mu \wedge w^\alpha dw/(2\pi i)^p$ under the projection $(z, w) \mapsto z$. Since $\mu$ is $\bar\partial$-closed it follows that $a_\alpha$ are holomorphic. Notice that

$$\bar\partial \frac{1}{w_1^{\beta_1}} \wedge \dots \wedge \bar\partial \frac{1}{w_p^{\beta_p}} \wedge dw_1^{\beta_1} \wedge \dots \wedge dw_p^{\beta_p}/(2\pi i)^p = \beta_1 \cdots \beta_p [w = 0],$$

where $[w = 0]$ denote the current of integration over $Z_{reg}$. Now assume that $\bar\partial \gamma = \mu$ and $\gamma$ has support close to $Z$. We have, for $|\beta| = M$, that

$$\bar\partial(\gamma \wedge dw^\beta) = (2\pi i)^p a_{\beta-1}(z) \beta_1 \cdots \beta_p [w = 0].$$

If $\nu$ is the component of $\gamma \wedge dw^\beta$ of bidegree $(p, p-1)$ in $w$, thus

$$d_w \nu = \bar\partial_w \nu = (2\pi i)^p a_{\beta-1} \beta_1 \cdots \beta_p [w = 0].$$

Integrating with respect to $w$ we get that $a_{\beta-1}(z) = 0$. By finite induction we can conclude that $\mu = 0$. Thus $\mu$ vanishes on $Z_{reg}$ and by the SEP it follows that $\mu = 0$. □

*Proof of Theorem 3.1.* Let $\omega$ be any neighborhood of $Z$ and take a cutoff function $\chi$ that is 1 in a neighborhood of $Z$ and with support in $\omega$. Let $u$ be any smooth solution to $\nabla_f u = 1$ in $X \setminus Z^f$, cf., Example 2. Then $g = \chi - \bar\partial \chi \wedge u$ is a smooth form in $\mathcal{L}^0(\omega)$ and $\nabla_f g = 0$. Moreover, the scalar term $g_0$ is 1 in a neighborhood of $Z^f$. Therefore,

$$\nabla_f [g \wedge V] = g \wedge \tau \wedge e = g_0 \tau \wedge e = \tau \wedge e,$$

and hence the current coefficient $W$ of the top degree component of $g \wedge V$ is a solution to $\bar\partial W = \tau$ with support in $\omega$. In view of Lemma 3.3 we have that $\tau = 0$. □

## 4. The factorization

The double sheaf complex $\mathcal{C}_{0,k}(\Lambda^\ell E)$ is exact in the $k$ direction except at $k = 0$, where we have the cohomology $\mathcal{O}(\Lambda^\ell E)$. By a standard argument there are natural isomorphisms

$$\operatorname{Ker}_{\delta_f} \mathcal{O}(\Lambda^\ell E)/\delta_f \mathcal{O}(\Lambda^{\ell+1}) \simeq \operatorname{Ker}_{\nabla_f} \mathcal{L}^{-\ell}/\nabla_f \mathcal{L}^{-\ell-1}. \tag{4.1}$$

When $\ell = 0$ the left hand side is $\mathcal{O}/\mathcal{J}(f)$, where $\mathcal{J}(f)$ is the ideal sheaf generated by $f$. We have the following factorization result.

**Theorem 4.1.** *Assume that $Z^f$ has pure codimension $p$ and let $\mu \in \mathcal{CH}_{Z^f}$ be $(0,p)$ and such that $\mathcal{J}(f)\mu = 0$. Then there is locally $\xi \in \mathcal{O}(\Lambda^{m-p}E)$ such that*

$$\mu\wedge e = R^f_p\wedge\xi. \tag{4.2}$$

*Proof.* Since $\nabla_f(\mu\wedge e) = 0$, by (4.1) there is $\xi \in \mathcal{O}(\Lambda^{m-p}E)$ such that $\nabla_f V = \xi - \mu\wedge e$. On the other hand, if $U$ is the current from Example 2, then $\nabla_f(U\wedge\xi) = \xi - R^f\wedge\xi = \xi - R^f_p\wedge\xi$. Now (4.2) follows from Theorem 3.1. □

*Proof of Theorem 1.1.* With no loss of generality we may assume that $\mu$ has bidegree $(0,p)$. Let $g = (g_1, \ldots, g_m)$ be a tuple such that $Z^g = Z$. If $f_j = g_j^M$ and $M$ is large enough, then $\mathcal{J}(f)\mu = 0$ and hence by Theorem 4.1 there is a form

$$\xi = \sum_{|J|=m-p}' \xi_J\wedge e_J$$

such that (4.2) holds. Then, cf., (3.2), (1.2) holds if $\alpha_I = \pm\xi_{I^c}$, where $I^c = \{1, \ldots, m\} \setminus I$. □

*Example* 3. Let $[Z]$ be any variety of pure codimension and choose $f$ such that $Z = Z^f$. It is not hard to prove that (each term of) the Lelong current $[Z]$ is in $\mathcal{CH}_Z$, and hence there is a holomorphic form $\xi$ such that $R^f_p\wedge\xi = [Z]\wedge e$. (In fact, one can notice that the proof of Lemma 3.3 works for $\mu = [Z]$ just as well, and then one can obtain (4.2) for $[Z]$ in the same way as for $\mu \in \mathcal{CH}_Z$. A posteriori it follows that indeed $[Z]$ is in $\mathcal{CH}_Z$.) There are natural ways to regularize the current $R^f_p$, see, e.g., [10], and thus we get natural regularizations of $[Z]$. □

Next we recall the duality principle, [6], [7]: If $f$ is a complete intersection, then

$$\operatorname{ann}\mu^f = \mathcal{J}(f). \tag{4.3}$$

In fact, if $\phi \in \operatorname{ann}\mu$, then $\nabla_f U\phi = \phi - \phi\mu\wedge e = \phi$ and hence $\phi \in \mathcal{J}(f)$ by (4.1). Conversely, if $\phi \in \mathcal{J}(f)$, then there is a holomorphic $\psi$ such that $\phi = \delta_f\psi = \nabla_f\psi$ and hence $\phi\mu = \nabla_f\psi\wedge\mu = \nabla(\psi\wedge\mu) = 0$.

Notice that $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{J}(f), \mathcal{CH}_{Z^f}(\Lambda^p E))$ is the sheaf of currents $\mu\wedge e$ with $\mu \in \mathcal{CH}_{Z^f}$ that are annihilated by $\mathcal{J}(f)$. From (4.3) and Theorem 4.1 we now get

**Theorem 4.2.** *If $f$ is a complete intersection, then the sheaf mapping*

$$\mathcal{O}/\mathcal{J}(f) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{J}(f), \mathcal{CH}_Z(\Lambda^p E)), \quad \phi \mapsto \phi\mu^f\wedge e, \tag{4.4}$$

*is an isomorphism.*

## 5. The standard extension property

Given the other conditions in the definition of $\mathcal{CH}_Z$ the SEP is automatically fulfilled on $Z_{reg}$; this is easily seen, e.g., as in the proof of Lemma 3.3, so the interesting case is when the zero set $Y$ of $h$ contains the singular locus of $Z$. Classically the SEP is expressed as

$$\lim_{\epsilon\to 0}\chi(|h|/\epsilon)\mu=\mu, \tag{5.1}$$

where $Y\supset Z_{sing}$ and $h$ is not vanishing identically on any irreducible component of $Z$. Here $\chi(t)$ can be either the characteristic function for the interval $[1,\infty)$ or some smooth approximand.

**Proposition 5.1.** *Let $\chi$ be a fixed function as above. The class of $\bar\partial$-closed $(0,p)$-currents $\mu$ with support on $Z$ that are annihilated by $\bar I_Z$ and satisfy* (5.1) *coincides with our class $\mathcal{CH}_Z$.*

If $\chi$ is not smooth the existence of the currents $\chi(|h|/\epsilon)\mu$ in a reasonable sense for small $\epsilon>0$ is part of the statement.

*Sketch of proof.* Let $f$ be a tuple such that $Z=Z^f$. We first show that $R^f_p$ satisfies (5.1). From the arguments in Section 2, cf., Example 2, we know that $R^f_p$ has a representation (2.1) such that $\pi^*h$ is a pure monomial (since the possible nonvanishing factor can be incorporated in one of the coordinates) and none of the factors in $\pi^*h$ occurs among the residue factors in $\tau_\ell$. Therefore, the existence of the product in (5.1) and the equality follow from the simple observation that

$$\int_{s_1,\dots,s_\mu}\chi(|s_1^{c_1}\cdots s_\mu^{c_\mu}|/\epsilon)\frac{\psi(s)}{s_1^{\gamma_1}\cdots s_\mu^{\gamma_\mu}}\to\int_{s_1,\dots,s_\mu}\frac{\psi(s)}{s_1^{\gamma_1}\cdots s_\mu^{\gamma_\mu}} \tag{5.2}$$

for test forms $\psi$, where the right hand side is a principal value integral. Let temporarily $\mathcal{CH}_Z^{cl}$ denote the class of currents defined in the proposition. Since each $\mu\in\mathcal{CH}_Z$ admits the representation (4.2) it follows that $\mu\in\mathcal{CH}_Z^{cl}$. On the other hand, Lemma 3.3 and therefore Theorem 3.1 and (4.2) hold for $\mathcal{CH}_Z^{cl}$ as well (with the same proofs), and thus we get the other inclusion. □

## 6. Vanishing of Coleff-Herrera currents

We conclude with some equivalent condition for the vanishing of a Coleff-Herrera current. This result is proved by the ideas above, it should be well-known, but we have not seen it in this way in the literature.

**Theorem 6.1.** *Assume that $X$ is Stein and that the subvariety $Z\subset X$ has pure codimension $p$. If $\mu\in\mathcal{CH}_Z(X)$ and $\bar\partial V=\mu$ in $X$, then the following are equivalent:*
*(i) $\mu=0$.*

*(ii) For all $\psi \in \mathcal{D}_{n,n-p}(X)$ such that $\bar\partial\psi = 0$ in some neighborhood of $Z$ we have that*

$$\int V\wedge\bar\partial\psi = 0.$$

*(iii) There is a solution to $\bar\partial w = V$ in $X \setminus Z$.*
*(iv) For each neighborhood $\omega$ of $Z$ there is a solution to $\bar\partial w = V$ in $X \setminus \bar\omega$.*

*Proof.* It is easy to check that (i) implies all the other conditions. Assume that (ii) holds. Locally on $Z_{reg} = \{w = 0\}$ we have (3.3), and by choosing $\xi(z,w) = \psi(z)\chi(w)dw^\beta\wedge dz\wedge d\bar z$ for a suitable cutoff function $\chi$ and test functions $\psi$, we can conclude from (ii) that $a_\beta = 0$ if $|\beta| = M$. By finite induction it follows that $\mu = 0$ there. Hence $\mu = 0$ globally by the SEP. Clearly (iii) implies (iv). Finally, assume that (iv) holds. Given $\omega \supset Z$ choose $\omega' \subset\subset \omega$ and a solution to $\bar\partial w = V$ in $X \setminus \overline{\omega'}$. If we extend $w$ arbitrarily across $\omega'$ the form $U = V - \bar\partial w$ is a solution to $\bar\partial U = \mu$ with support in $\omega$. In view of Lemma 3.3 thus $\mu = 0$. □

Notice that $V$ defines a Dolbeault cohomology class $\omega^\mu$ in $X \setminus Z$ that only depends on $\mu$, and that conditions (ii)-(iv) are statements about this class. For an interesting application, fix a current $\mu \in \mathcal{CH}_Z$. Then the theorem gives several equivalent ways to express that a given $\phi \in \mathcal{O}$ belongs to the annihilator ideal of $\mu$. In the case when $\mu = \mu^f$ for a complete intersection $f$, one gets back the equivalent formulations of the duality theorem from [6] and [7].

*Remark* 2. If $\mu$ is an arbitrary $(0,p)$-current with support on $Z$ and $\bar\partial V = \mu$ we get an analogous theorem if condition (i) is replaced by: *$\mu = \bar\partial\gamma$ for some $\gamma$ with support on $Z$.* This follows from the Dickenstein-Sessa decomposition $\mu = \mu_{CH} + \bar\partial\gamma$, where $\mu_{CH}$ is in $\mathcal{CH}_Z$. See [6] for the case $Z$ is a complete intersection and [3] for the general case. □

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