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RESIDUE CURRENTS CONSTRUCTED FROM RESOLUTIONS OF MONOMIAL IDEALS

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Abstract. Given a free resolution of an ideal J of holomorphic functions, one can construct a vector-valued residue current R, whose annihilator is precisely J. In this paper we compute R in case J is a monomial ideal and the resolution is a cellular resolution in the sense of Bayer and Sturmfels. A description of R is given in terms of the underlying polyhedral cell complex and it is related to irreducible decompositions of J.

1. Introduction

The duality principle for residue currents, due to Dickenstein and Sessa, [11], and Passare, [18], asserts that a complete intersection ideal of holomorphic functions can be represented as the annihilator ideal of a so-called Coleff-Herrera current, [10]. It has been widely used, for example to obtain effective solutions to division problems, [8], and explicit versions of the Ehrenpreis-Palamodov fundamental principle, [9], see also [7]. In [2] we generalized the duality principle to general ideals of holomorphic functions by constructing, from a free resolution of an ideal J, a vector-valued residue current R, whose annihilator ideal is precisely J. This was used to extend several results previously known for complete intersections.

The degree of explicitness of the current R of course directly depends on the degree of explicitness of the free resolution. In case J is a complete intersection the Koszul complex is exact and the corresponding current is the classical Coleff-Herrera current, compare to [19] and [1]. In general, though, explicit resolutions are hard to find. In this paper we will focus on monomial ideals, for which there has recently been a lot of work done, see for example the book [16] and the references mentioned therein. We compute residue currents associated with so-called cellular resolutions, which were introduced by Bayer and Sturmfels in [5], and which can be nicely encoded into polyhedral cell complexes. Our main result, Theorem 5.3, is a complete description of the residue current of a so-called generic monomial ideal.

Because of their simplicity and nice combinatorial description monomial ideals serve as a good toy model for illustrating general ideas and

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results in commutative algebra and algebraic geometry, see [22] for examples, which make them a natural first example to consider. In [24] residue currents of Bochner-Martinelli type, in the sense of [19], were computed for monomial ideals, and in [2] and [25], there are presented some explicit computations of residue currents of certain simple monomial ideals that are not complete intersections. Also, many results for general ideals can be proved by specializing to monomial ideals. In fact, recall that the existence of Bochner-Martinelli as well as the residue currents in [2] is proved by reducing to a monomial situation by resolving singularities.

We start by considering Artinian, that is, zero-dimensional, monomial ideals in Section 3. Residue currents associated with general monomial ideals are computed essentially by reducing to this simpler case. A priori, the residue current R associated with a cellular resolution of an Artinian monomial ideal has one entry R_F for each $(n-1)$-dimensional face F of the underlying polyhedral cell complex. The main technical result in this paper, Proposition 3.1, asserts that each R_F is a certain nice Coleff-Herrera current:

$$c\,\bar\partial\Big[\frac{1}{z_1^{\alpha_1}}\Big]\wedge\ldots\wedge\bar\partial\Big[\frac{1}{z_n^{\alpha_n}}\Big],$$

where $\alpha=(\alpha_1,\ldots,\alpha_n)$ can be read off from the cell complex and c is a constant. In particular, if $c\neq 0$ the ideal of functions annihilating R_F, $\operatorname{ann} R_F$, is $(z_1^{\alpha_1},\ldots,z_n^{\alpha_n})$. A monomial ideal of this form, where the generators are powers of variables, is called *irreducible*. One can show that every monomial ideal can be written as a finite intersection of irreducible ideals; this is called an *irreducible decomposition* of the ideal. Note that an irreducible ideal is primary so an irreducible decomposition of a monomial ideal is a refinement of a primary decomposition. Since one has to annihilate all entries R_F to annihilate R, $\bigcap \operatorname{ann} R_F$ yields an irreducible decomposition of the ideal $\operatorname{ann} R$, which by the duality principle equals J, and so the (nonvanishing) entries of R can be seen to correspond to components in an irreducible decomposition. In particular, the number of nonvanishing entries are bounded from below by the minimal number of components in an irreducible decomposition.

In general, we can not extract enough information from our computations to determine which entries R_F that are nonvanishing. Still, for "most" monomial ideals we can; if the monomial ideal J is generic, which means that the exponents in the set of minimal generators fulfill a certain genericity condition (see Section 2 for a precise definition), then Theorem 3.3 states that R_F is nonvanishing precisely when F is a facet of the Scarf complex introduced by Bayer, Peeva and Sturmfels, [6]. In particular, if the underlying cell complex is the Scarf complex, then all entries of R are nonvanishing. The cellular resolution so obtained is in fact a minimal resolution of the generic ideal J. Theorem 3.5 asserts that whenever the cellular resolution is minimal, the corresponding residue current has only nonvanishing entries. Also, the number of entries is equal to the minimal number of components in an irreducible decomposition.

In Section 5 we extend the results for Artinian monomial ideals to general monomial ideals. The basic idea is to decompose the residue current into simpler parts, which can be computed essentially as in the Artinian case. In [3] it was shown that the residue current R constructed from a free resolution of the ideal J can be naturally decomposed with respect to the set of associated prime ideals of J, $\mathrm{A}ssJ$;

$$R = \sum_{\mathfrak{p} \in \mathrm{A}ssJ} R^{\mathfrak{p}}, \tag{1.1}$$

where $R^{\mathfrak{p}}$ has support on the variety $V(\mathfrak{p})$ of $\mathfrak{p}$ and has the so-called *standard extension property (SEP)* with respect to $V(\mathfrak{p})$, which basically means that it is determined by what it is generically on $V(\mathfrak{p})$. Moreover, each $\operatorname{ann} R^{\mathfrak{p}}$ is $\mathfrak{p}$-primary, and it turns out that to annihilate R one has to annihilate all the currents $R^{\mathfrak{p}}$ and so

$$J = \operatorname{ann} R = \bigcap_{\mathfrak{p} \in \mathrm{A}ssJ} \operatorname{ann} R^{\mathfrak{p}} \tag{1.2}$$

gives a minimal primary decomposition of J. For a reference on primary decompositions we refer to [4]. Now, the simpler currents $R^{\mathfrak{p}}$ associated with a monomial ideal M can be computed by reducing to the Artinian case, using ideas from [24]. The result is a vector of certain simple currents that in particular have irreducible annihilator ideals and that correspond to the $\mathfrak{p}$-primary components in an irreducible decomposition of M. Our main result, Theorem 5.3 is a complete description of the residue current associated with a generic monomial ideal M, generalizing Theorem 3.3. In particular, we get a decomposition of R, which is a refinement of (1.1) and which corresponds to the irredundant irreducible decomposition of M.

The technical core of this paper is the proof of Proposition 3.1, which occupies Section 4. It is inspired by [24], where similar results were obtained for currents of Bochner-Martinelli type corresponding to the Koszul complex. When considering general cellular resolutions the computations get more involved though; in particular, they involve finding inverses of all mappings in the resolution. As in [24], the proof amounts to computing currents in a certain toric variety constructed from the generators of the ideal, using ideas originally due to Khovanskii [14] and Varchenko [23].

2. Preliminaries and background

Let us start by briefly recalling the construction of residue currents in [2]. Consider an arbitrary complex of Hermitian holomorphic vector bundles over a complex manifold Ω,

$$0 \to E_N \xrightarrow{f_N} \ldots \xrightarrow{f_3} E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0, \tag{2.1}$$

that is exact outside an analytic variety Z of positive codimension, and suppose that the rank of E_0 is 1. In $\Omega \setminus Z$, let σ_k be the minimal inverse of f_k, with respect to some Hermitian metric, let $\sigma = \sigma_0 + \ldots + \sigma_N$, $u = \sigma(I - \bar\partial\sigma)^{-1} = \sigma + \sigma(\bar\partial\sigma) + \sigma(\bar\partial\sigma)^2 + \ldots$, and let R be the analytic continuation of $\bar\partial|F|^{2\lambda}\wedge u$ to $\lambda = 0$, where F is any tuple of holomorphic functions that vanishes on Z. It turns out that R is a well defined current taking values in $\text{End}(E)$, where $E = \oplus_k E_k$, which has support on Z, and which in a certain way measures the lack of exactness of the associated complex of locally free sheaves of $\mathcal{O}$-modules $\mathcal{O}(E_k)$ of holomorphic sections of E_k,

$$0 \to \mathcal{O}(E_N) \xrightarrow{f_N} \cdots \xrightarrow{f_2} \mathcal{O}(E_1) \xrightarrow{f_1} \mathcal{O}(E_0). \tag{2.2}$$

In particular, if $\mathcal{J}$ is the ideal sheaf $\text{Im}\,(\mathcal{O}(E_1) \to \mathcal{O}(E_0))$ and $\varphi \in \mathcal{O}(E_0)$ fulfills that the (E-valued) current $R\varphi = 0$, then locally $\varphi \in \mathcal{J}$.

Moreover, letting R^ℓ_k denote the component of R that takes values in $\text{Hom}\,(E_\ell, E_k)$ and $R^\ell = \sum_k R^\ell_k$, it turns out that $R^\ell = 0$ for $\ell \geq 1$ is equivalent to that (2.2) is exact, in other words that it is a resolution of $\mathcal{O}(E_0)/\mathcal{J}$, see Theorem 3.1 in [2]. We then write $R_k = R^0_k$ without any risk of confusion. In this case, $R\varphi = 0$ precisely when $\varphi \in \mathcal{J}$.

Let us continue with the construction of cellular complexes from [5]. Let S be the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$ and let $\deg m$ denote the multidegree of a monomial m in S. When nothing else is mentioned we will assume that monomials and ideals are in S.

Next, a polyhedral cell complex X is a finite collection of convex polytopes (in a real vector space $\mathbb{R}^d$ for some d), the *faces* of X, that fulfills that if $F \in X$ and G is a face of F (for the definition of a face of a polytope, see for example [26]), then $G \in X$, and moreover if F and G are in X, then $F \cap G$ is a face of both F and G. The dimension of a face F, $\dim F$, is defined as the dimension of its affine hull (in $\mathbb{R}^d$) and the dimension of X, $\dim X$, is defined as $\max_{F \in X} \dim F$. Let X_k denote the set of faces of X of dimension $(k-1)$ (X_0 should be interpreted as $\{\emptyset\}$). Faces of dimension 0 are called *vertices*. We will frequently identify $F \in X$ with its set of vertices. Maximal faces (with respect to inclusion) are called *facets*. A face F is a *simplex* if the number of vertices, $|F|$, is equal to $\dim F + 1$. If all faces of X are simplices, we say that X is a *simplicial complex*. A polyhedral cell complex $X' \subset X$ is said to be a *subcomplex* of X. Moreover, we say that X is *labeled* if there is monomial m_i in S associated to each vertex i. An arbitrary face F of X is then labeled by the least common multiple of the labels of the vertices of F, that is $m_F = \mathrm{lcm}\{m_i | i \in F\}$. Let $\mathbb{N}^n \ni \alpha_F = \deg(m_F)$. By $\mathbb{N}$ we mean $0, 1, 2, \ldots$. We will sometimes be sloppy and not differ between the faces of labeled complex and their labels.

Now, let M be a monomial ideal in S with minimal generators $\{m_1, \ldots, m_r\}$ (recall that the set of minimal generators of a monomial ideal is unique). Throughout this paper M will be supposed to be of this form if nothing else is mentioned. Moreover, let X be a polyhedral cell complex with vertices $\{1, \ldots, r\}$ endowed with some orientation and labeled by $\{m_i\}$. We will associate with X a graded complex of free S-modules: for $k = 0, \ldots, \dim X + 1$, let A_k be the free S-module with basis $\{e_F\}_{F \in X_k}$ and let the differential $f_k : A_k \to A_{k-1}$ be defined by

$$f_k : e_F \mapsto \sum_{\text{facets } G \subset F} \operatorname{sgn}(G, F) \frac{m_F}{m_G} e_G, \tag{2.3}$$

where the sign $\operatorname{sgn}(G, F)$ $(= \pm 1)$ comes from the orientation on X. Note that m_F/m_G is a monomial. The complex

$$\mathbb{F}_X : 0 \longrightarrow A_{\dim X - 1} \xrightarrow{f_{\dim X - 1}} \cdots \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0$$

is the *cellular complex supported on* X, which was introduced in [5]. It is exact if the labeled complex X fulfills a certain acyclicity condition. More precisely, for $\beta \in \mathbb{N}^n$ let $X_{\preceq \beta}$ denote the subcomplex of X consisting of all faces F for which $\alpha_F \leq \beta$ with respect to the usual ordering in $\mathbb{Z}^n$. Then $\mathbb{F}_X$ is exact if and only if $X_{\preceq \beta}$ is acyclic, which means that it is empty or has zero reduced homology, for all $\beta \in \mathbb{N}^n$, see Proposition 4.5 in [16]. We then say that $\mathbb{F}_X$ is a *cellular resolution* of S/M.

In particular, if X is the $(r-1)$-simplex this condition is fulfilled and we obtain the classical *Taylor resolution*, introduced by Diana Taylor, [21]. Note that if M is a complete intersection, then the Taylor resolution coincides with the Koszul complex. If X is an arbitrary simplicial complex, $\mathbb{F}_X$ is the more general *Taylor complex*, introduced in [6]. Observe that if X is simplicial the orientation comes implicitly from the ordering on the vertices.

Recall that a graded free resolution $\cdots \longrightarrow A_k \xrightarrow{f_k} A_{k-1} \longrightarrow \cdots$ is *minimal* if and only if for each k, f_k maps a basis of A_k to a minimal set of generators of $\operatorname{Im} f_k$, see for example Corollary 1.5 in [13]. The Taylor complex $\mathbb{F}_X$ is a minimal resolution if and only if it is exact and for all $F \in X$, the monomials m_F and $m_{F \setminus i}$ are different, see Lemma 6.4 in [16].

Now, to put the cellular resolutions into the context of [2], let us consider the vector bundle complex of the form (2.1), where E_k for $k = 0, \ldots, N = \dim X + 1$ is a trivial bundle over $\mathbb{C}^n$ of rank $|X_k|$, endowed with the trivial metric, and with a global frame $\{e_F\}_{F \in X_k}$, and where the differential is given by (2.3). Alternatively, we can regard f_k

as a section of $E_k^* \otimes E_{k-1}$, that is,

$$f_k = \sum_{F \in X_k} \sum_{\text{facets } G \subset F} \text{sgn } (G, F) \, \frac{m_F}{m_G} \, e_F^* \otimes e_G.$$

We will say that the corresponding residue current R is associated with X, and we will use R_F to denote the coefficient of $e_F \otimes e_\emptyset^*$. It is well known that the induced sheaf complex (2.2) is exact if and only if $\mathbb{F}_X$ is. For example it can be seen from the Buchsbaum-Eisenbud theorem, Theorem 20.9 in [12], and residue calculus - the proof of Theorem 3.1 in [2].

Observe that the elements in S (holomorphic polynomials) can be regarded as holomorphic sections of E_0. In this paper, by the annihilator ideal of a current T, $\operatorname{ann} T$, we will mean the ideal in S which consists of the elements $\varphi \in S$ for which $R\varphi = 0$.

For $b = (b_1, \ldots, b_n) \in \mathbb{N}^n$ we will use the notation $\mathfrak{m}^b$ for the irreducible ideal $(z_1^{b_1}, \ldots, z_n^{b_n})$. If $M = \cap_{i=1}^q \mathfrak{m}^{b^i}$, for some $b^i \in \mathbb{N}^n$, is an irreducible decomposition of the monomial ideal M, such that no intersectand can be omitted the decomposition is said to be *irredundant*, and the ideals $\mathfrak{m}^{b^i}$ are then called the *irreducible components* of M. One can prove that each monomial ideal M in S has a unique irredundant irreducible decomposition. Giving the irreducible components is in a way dual to giving the generators of the ideal (see Chapter 5 on Alexander duality in [16]), and the uniqueness of the irredundant irreducible decomposition corresponds to the uniqueness of the set of minimal generators of a monomial ideal. This duality will be illustrated in Example 1.

We will be particularly interested in so called generic monomial ideals. A monomial $m' \in S$ *strictly divides* another monomial m if m' divides m/z_i for all variables z_i dividing m. We say that a monomial ideal M is *generic* if whenever two distinct minimal generators m_i and m_j have the same positive degree in some variable, then there exists a third generator m_k that strictly divides the least common multiple of m_i and m_j. In particular M is generic if no two generators have the same positive degree in any variable. Almost all monomial ideals are generic in the sense that those which fail to be generic lie on finitely many hyperplanes in the matrix space of exponents, see [6].

We will use the notation $\bar\partial[1/f]$ for the analytic continuation of $\bar\partial|f|^{2\lambda}/f$ to $\lambda = 0$, and analogously by $[1/f]$ we will mean $|f|^{2\lambda}/f|_{\lambda=0}$, that is, just the principal value of $1/f$. By iterated integration by parts we have that

$$\int_z \bar\partial\left[\frac{1}{z^p}\right] \wedge \varphi dz = \frac{2\pi i}{(p-1)!} \frac{\partial^{p-1}}{\partial z^{p-1}} \varphi(0). \tag{2.4}$$

In particular, the annihilator of $\bar\partial[1/z^p]$ is (z^p).

3. Artinian monomial ideals

We are now ready to present our results concerning residue currents R associated with cellular complexes of Artinian monomial ideals. We are interested in the component R^0, which takes values in $\mathrm{Hom}\,(E_0, E)$. In fact, when (2.2) is exact $R = R^0$. From Proposition 2.2 in [2] we know that if M is Artinian, then $R^0 = R^0_n$, where R^0_n is a $\mathrm{Hom}\,(E_0, E_n)$-valued current. Thus, a priori we know that R^0 consists of one entry $R_F\ e_F \otimes e^*_\emptyset$ for each $F \in X_n$. We will suppress the factor $e^*_\emptyset$ in the sequel.

Proposition 3.1. *Let $M = (m_1, \ldots, m_r)$ be an Artinian monomial ideal, and let R be the residue current associated with the polyhedral cell complex X with vertices $\{m_1, \ldots, m_r\}$. Then*

$$R^0 = \sum_{F \in X_n} R_F\ e_F, \tag{3.1}$$

where

$$R_F = c_F\ \bar\partial\Big[\frac{1}{z_1^{\alpha_1}}\Big] \wedge \ldots \wedge \bar\partial\Big[\frac{1}{z_n^{\alpha_n}}\Big]. \tag{3.2}$$

Here c_F is a constant and $(\alpha_1, \ldots, \alpha_n) = \alpha_F$. If any of the entries of α_F is 0, (3.2) should be interpreted as 0.

The proof of Proposition 3.1 is given in Section 4.

Observe that the proposition gives a complete description of R^0 except for the constants c_F. We are particularly interested in whether the c_F are zero or not. Indeed, note that

$$\mathrm{ann}\,\bar\partial\Big[\frac{1}{z_1^{\alpha_1}}\Big] \wedge \ldots \wedge \bar\partial\Big[\frac{1}{z_n^{\alpha_n}}\Big] = \mathfrak{m}^{\alpha_F},$$

so that $\mathrm{ann}\,R_F = \mathfrak{m}^{\alpha_F}$ if $c_F \neq 0$. Note in particular that $\mathrm{ann}\,R_F$ depends only on c_F and m_F and not on the particular vertices of F nor the remaining faces in X. Furthermore, to annihilate R^0 one has to annihilate each entry R_F and therefore

$$\mathrm{ann}\,R^0 = \bigcap_{F \in X;\ c_F \neq 0} \mathfrak{m}^{\alpha_F}.$$

Now, suppose that the cellular complex $\mathbb{F}_X$ is exact. Then, $R = R^0$, and from Theorems 3.1 and 7.2 in [2] we know that

$$\mathrm{ann}\,R = M.$$

Thus a necessary condition for c_F to be nonvanishing is that $M \subset \mathfrak{m}^{\alpha_F}$. In general though, Proposition 3.1 does not give enough information to give a sufficient condition, as will be illustrated in Example 2. Below we will discuss two situations, however, in which we can determine exactly which c_F that are nonzero.

First we will consider generic monomial ideals. For this purpose, let us introduce the *Scarf complex* Δ_M of M, which is the collection of subsets $I \subset \{1, \ldots, r\}$ whose corresponding least common multiple m_I is unique, that is,

$$\Delta_M = \{I \subset \{1, \ldots, r\} | m_I = m_{I'} \Rightarrow I = I'\}.$$

One can prove that the Scarf complex is a simplicial complex, and that its dimension is a most $n-1$. In fact, when M is Artinian, Δ_M is a regular triangulation of $(n-1)$-simplex. For details, see for example [16]. In [6] it was proved that if M is generic, then the cellular complex supported on Δ_M gives a resolution of S/M, which is moreover minimal. Furthermore, if M in addition is Artinian, then

$$M = \bigcap_{F \text{ facet of } \Delta_M} \mathfrak{m}^{\alpha_F}, \tag{3.3}$$

yields the unique irredundant irreducible decomposition of M. To be precise, originally in [6], a less inclusive definition of generic ideals was used, but the results above were extended in [17] to the more general definition of generic ideals we use.

We can now deduce the following.

Proposition 3.2. *Let $M \subset S$ be an Artinian generic monomial ideal and let R be the residue current associated with the polyhedral cell complex X. Suppose that $\mathbb{F}_X$ is a resolution of S/M. Then c_F in* (3.2) *is non-zero if and only if $F \in X_n$ is a facet of the Scarf complex Δ_M.*

Proof. Suppose that $F \in X_n$ is not a facet of Δ_M. We show that $M \not\subset \mathfrak{m}^{\alpha_F}$, which forces c_F to be zero.

Let J be the largest subset of $\{1, \ldots, r\}$ such that $m_J = m_F$. Then for some $j \in J$ it holds that $m_{J \setminus j} = m_F$, as follows from the definition of Δ_M. If m_j strictly divides m_F then clearly $m_j \notin \mathfrak{m}^{\alpha_F}$ and we are done. Otherwise, it must hold for some $k \in J \setminus j$ that m_k and m_j have the same positive degree in one of the variables. Then, since M is generic, there is a generator m_ℓ that strictly divides the least common multiple of m_j and m_k and consequently also strictly divides m_F. Hence $m_\ell \notin \mathfrak{m}^{\alpha_F}$.

On the other hand, since (3.3) is irredundant, c_F has to be nonzero whenever F is a facet of Δ_M. □

Thus, to sum up, Propositions 3.1 and 3.2 yield the following description of the residue current of a generic monomial ideal.

Theorem 3.3. *Let $M \subset S$ be an Artinian generic monomial ideal and let R be the residue current associated with the polyhedral cell complex X. Suppose that $\mathbb{F}_X$ is a resolution of S/M. Then*

$$R = \sum_{F \text{ facet of } \Delta_M} R_F \; e_F,$$

where Δ_M is the Scarf complex of M, R_F is given by (3.2), *and the constant c_F there is nonvanishing.*

In particular if we choose X as the Scarf complex Δ_M we get that all coefficients c_F are nonzero.

Remark 1. Observe that it follows from Theorem 3.3 that X must contain the Scarf complex as a subcomplex. Compare to Proposition 6.12 in [16]. □

An immediate consequence is the following.

Corollary 3.4. *Let $M \subset S$ be an Artinian generic monomial ideal and let R be the residue current associated with the polyhedral cell complex X. Suppose that $\mathbb{F}_X$ is a resolution of S/M. Then*

$$M = \bigcap_{F \in X} \mathit{ann}\, R_F$$

is the irredundant irreducible decomposition of M.

Another situation in which we can determine the set of nonvanishing constants c_F is when $\mathbb{F}_X$ is a minimal resolution of S/M. Indeed, in [15] (Theorem 5.12, see also Theorem 5.42 in [16]) was proved a generalization of (3.3); if M is Artinian and $\mathbb{F}_X$ is a minimal resolution of S/M, then the irredundant irreducible decomposition is given by

$$M = \bigcap_{F \text{ facet of } X} \mathfrak{m}^{\alpha_F}. \tag{3.4}$$

Hence, from (3.4) and Proposition 3.1 we conclude that in this case all c_F are nonvanishing.

Theorem 3.5. *Let $M \subset S$ be an Artinian generic monomial ideal and let R be the residue current associated with the polyhedral cell complex X. Suppose that $\mathbb{F}_X$ is a minimal resolution of S/M. Then*

$$R = \sum_{F \textit{ facet of } X} R_F \; e_F,$$

where R_F is given by (3.2) *and the constant c_F there is nonvanishing.*

Finally, we should remark, that even though we can not determine the set of non-vanishing entries of the residue current associated with an arbitrary cell complex, we can still estimate the number of nonvanishing entries from below by the number or irreducible components of the corresponding ideal.

Let us now illustrate our results by some examples. First observe that the ideal $(z^A) = (z^a = z_1^{a_1} \cdots z_n^{a_n} | a \in A \subset \mathbb{N}^n)$ in S is precisely the set of functions that have support in $\bigcup_{a \in A}(a + \mathbb{R}_+^n)$, where

$$\operatorname{supp} \sum_{a \in \mathbb{Z}^n} c_a z^a = \{a \in \mathbb{Z}^n | c_a \neq 0\},$$

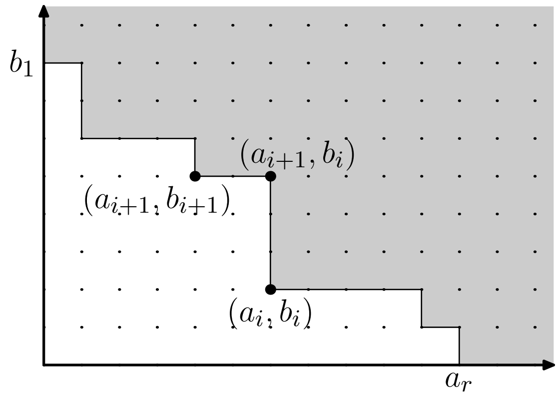

FIGURE 1. The staircase diagram of M in Example 1.

and thus we can represent the ideal by this set, see Figure 1. Such pictures of monomial ideals are usually referred to as *staircase diagrams*. The generators $\{z^a\}$ should be identified as the "inner corners of" the staircase, whereas the "outer corners" correspond to the exponents in the irredundant irreducible decomposition.

Example 1. Let us consider the case when $n = 2$. Note that then all monomial ideals are generic. If M is an Artinian monomial ideal, we can write

$$M = (w^{b_1}, z^{a_2}w^{b_2}, \ldots, z^{a_{r-1}}w^{a_{r-1}}, z^{a_r}),$$

for some integers $a_2 < \ldots < a_r$ and $b_1 > \ldots > b_{r-1}$. Now Δ_M is one-dimensional and its facets are the pairs of adjacent generators in the staircase. Moreover $m_{\{i,i+1\}} = z^{a_{i+1}}w^{b_i}$, which corresponds precisely to the ith outer corner of the staircase. Thus, according to Theorem 3.3 the residue current R associated with a cellular resolution of M is of the form

$$R = \sum_{i=1}^{n-1} c_i \ \bar{\partial} \left[\frac{1}{z^{a_{i+1}}}\right] \wedge \bar{\partial} \left[\frac{1}{w^{b_i}}\right] e_{\{i,i+1\}},$$

for some nonvanishing constants c_i. The annihilator of the ith entry is the irreducible component $(z^{a_{i+1}}, z^{b_i})$.

Figure 1 illustrates the two dual ways of thinking of M, either as a staircase with inner corners (a_i, b_i), corresponding to the generators, or as a staircase with outer corners (a_{i+1}, b_i), corresponding to the irreducible components or equivalently the annihilators of the entries of R. □

Let us also give an example that illustrates how we in general fail to determine the set of nonzero c_F when the ideal is not generic.

Example 2. Consider the non-generic ideal $M = (x^2, xy, y^2, yz, z^2) =: (m_1, \ldots, m_5)$. The Scarf complex Δ_M, depicted in Figure 2, consists of the 2-simplex $\{2, 3, 4\}$ together with the one-dimensional "handle"

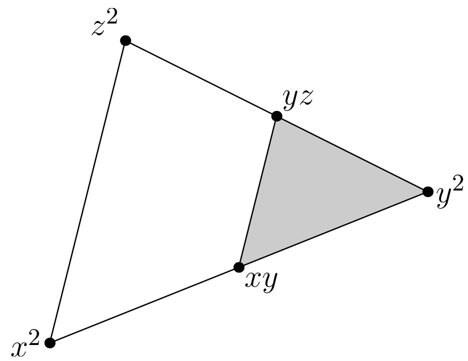

FIGURE 2. The Scarf complex Δ_M of the ideal M in Example 2.

made up from the edges $\{1,2\}$, $\{1,5\}$ and $\{4,5\}$. Moreover the irredundant irreducible decomposition is given by $M = (x, y^2, z) \cap (x^2, y, z^2)$.

Let X be the full 4-simplex with vertices $\{1, \dots, 5\}$ corresponding to the Taylor resolution. It is then easily checked that for the associated residue current, $c_{\{2,3,4\}}$ and at least one of $c_{\{1,2,5\}}$ and $c_{\{1,4,5\}}$ have to be zero, whereas $c_{\{1,2,4\}}$ and $c_{\{2,4,5\}}$ can be either zero or nonzero. The remaining c_F has to be zero since for them $M \not\subset \mathfrak{m}^{\alpha_F}$. Thus, in general Proposition 3.1 does not provide enough information to determine which of the coefficients c_F that vanish.

However, let instead X' be the polyhedral cell complex consisting of the two facets $\{2,3,4\}$ and $\{1,2,4,5\}$, that is the triangle and the quadrilateral in Figure 2. The resolution obtained from X', which is in fact the so called Hull resolution introduced in [5], is minimal. Thus, according to Theorem 3.5 the two entries of the associated residue current, which correspond to the two facets of X' are both nonvanishing, with annihilators (x, y^2, z) and (x^2, y, z^2) respectively. This could of course be seen directly since we already knew the irredundant irreducible decomposition of M. □

4. PROOF OF PROPOSITION 3.1

The proof of Proposition 3.1 is inspired by the proof of Theorem 3.1 in [24]. We will compute R^0 as a push-forward of corresponding currents on a certain toric variety. To do this we will have use for the following simple lemma which is proved essentially by integration by parts.

Lemma 4.1. *Let v be a strictly positive smooth function in $\mathbb{C}$, φ a test function in $\mathbb{C}$, and p a positive integer. Then*

$$\lambda \mapsto \int v^\lambda |z|^{2\lambda} \varphi(z) \frac{dz \wedge d\bar{z}}{z^p}$$

and

$$\lambda \mapsto \int \bar\partial (v^\lambda |z|^{2\lambda}) \wedge \varphi(z) \frac{dz}{z^p}$$

both have meromorphic continuations to the entire plane with poles at integer points on the negative real axis. At $\lambda = 0$ they are both independent of v and equal to $[1/z^p]$ and $\bar\partial[1/z^p]$ respectively (acting on suitable test forms). Moreover, if $\varphi(z) = \bar z \psi(z)$ or $\varphi = d\bar z \wedge \psi$, then the value of the second integral at $\lambda = 0$ is zero.

Before presenting the proof of Proposition 3.1, let us just give a very brief overview of it. First, we will give a description of the current R^0 in terms of the cell complex X. After that we will introduce the toric variety mentioned above and show that R^0 equals the push-forward of certain currents on this variety. Finally, we will compute these currents.

Let us start by recalling from Section 2 in [2] that R^0_n is the analytic continuation to $\lambda = 0$ of $\bar\partial |F|^{2\lambda} \wedge u^0_n$, where F is a holomorphic function that vanishes at the origin and

$$u^0_n = (\bar\partial \sigma_n)(\bar\partial \sigma_{n-1}) \cdots (\bar\partial \sigma_2)\sigma_1.$$

Here

$$\sigma_k = \frac{\delta_{f_k}^{q_k - 1} S_k}{|F_k|^2}, \tag{4.1}$$

where q_k is the rank of f_k, δ_{f_k} is contraction with f_k, $F_k = (f_k)^{q_k}/q_k!$ and $S_k = (s_k)^{q_k}/q_k!$ is the dual section of F_k. For details, we refer to Section 2 in [2], see also [25]. Furthermore, s_k is the section of $E_k \otimes E^*_{k-1}$ that is dual to f_k with respect to the trivial metric, that is,

$$s_k = \sum_{G \in X_k} \sum_{\text{facets } H \subset G} \text{sgn}\,(H, G)\, \frac{\overline{m_G}}{m_H}\, e_G \otimes e^*_H.$$

Here $\overline{m_G}$ just denotes the conjugate of m_G. Notice that, since $\sigma_k \sigma_{k-1} = 0$, as follows by definition, it holds that only the terms obtained when the $\bar\partial$ fall in the numerator survive, and so

$$u^0_n = \frac{\bar\partial(\delta_{f_n}^{q_n - 1} S_n) \cdots \bar\partial(\delta_{f_2}^{q_2 - 1} S_2) \delta_{f_1}^{q_1 - 1} S_1}{|F_n|^2 \cdots |F_1|^2}.$$

Observe furthermore that the numerator of the right hand side of (4.1) is a sum of terms of the form

$$v_k = \pm |\omega_k|^2 \frac{\overline{m_G}}{m_H}\, e_G \otimes e^*_H, \tag{4.2}$$

where $G \in X_k$ and $H \in X_k$ is a facet of G and

$$\omega_k = \frac{m_{G_1} \cdots m_{G_{q_k - 1}}}{m_{H_1} \cdots m_{H_{q_k - 1}}},$$

where for $1 \leq \ell \leq q_k - 1$, $G_\ell \in X_k$ and $H_\ell \in X_{k-1}$ is a facet of G_ℓ. The $\pm$ in front of $|\omega_k|$ depends on the orientation on X. Note that the coefficients are monomials. It follows that u_n^0 is a sum of terms of the form

$$u_v = u_{\{v_1,\ldots,v_n\}} = \frac{(\bar\partial v_n)\cdots(\bar\partial v_2)v_1}{|F_n|^2\cdots|F_1|^2},$$

where each v_k is of the form (4.2), and where

$$v_n\cdots v_1 = \pm|\omega_n\cdots\omega_1|^2\ \overline{m_F}\ e_F\otimes e_\emptyset^* \tag{4.3}$$

for some $F\in X_n$.

Observe that each F_k has monomial entries. By ideas originally from [14] and [23], one can show that there exists a toric variety $\mathcal{X}$ and a proper map $\widetilde\Pi:\mathcal{X}\to\mathbb{C}^n$ that is biholomorphic from $\mathcal{X}\setminus\widetilde\Pi^{-1}(\{z_1\cdots z_n=0\})$ to $\mathbb{C}^n\setminus\{z_1\cdots z_n=0\}$, such that locally, in a coordinate chart $\mathcal{U}$ of $\mathcal{X}$, it holds for all k that the pullback of one of the entries of F_k divides the pullbacks of all entries of F_k. In other words we can write $\widetilde\Pi^*F_k=F_k^0F_k'$, where F_k^0 is a monomial and F_k' is nonvanishing, and analogously we have $\widetilde\Pi^*F=F^0F'$. The construction is based on the so called Newton polyhedra associated with F_k and we refer to [7] and the references therein for details. The mapping $\widetilde\Pi$ is locally in the chart $\mathcal{U}$ given by

$$\begin{array}{rcl}\Pi:\mathcal{U} & \to & \mathbb{C}^n\\ t & \mapsto & t^P,\end{array}$$

where $P=(\rho_{ij})$ is a matrix with determinant ±1 and t^P is a shorthand notation for $(t_1^{\rho_{11}}\cdots t_n^{\rho_{n1}},\ldots,t_1^{\rho_{1n}}\cdots t_n^{\rho_{nn}})$. Hence, the pullback Π^* transforms the exponent of monomials by the linear mapping P;

$$\Pi^*z^a=\Pi^*z_1^{a_1}\cdots z_n^{a_n}=t_1^{\rho_1\cdot a}\cdots t_n^{\rho_n\cdot a}=t^{Pa},$$

where ρ_i denotes the ith row of P, so that the pullback of a monomial is itself a monomial.

Now, from Lemma 2.1 in [2] we know that $F_k^0\Pi^*\sigma_k$ is smooth in $\mathcal{U}$. However,

$$F_k^0\Pi^*\sigma_k=\sum_j\frac{\Pi^*v_k^j}{\bar F_k^0|F_k'|^2}=\sum_{\alpha\in\mathbb{N}^n}\ \sum_{\deg\Pi^*v_k^j=\alpha}\frac{\Pi^*v_k^j}{\bar F_k^0|F_k'|^2},$$

where v_k^j are just the different terms v_k that appear in the numerator of σ_k. Therefore clearly for each $\alpha\in\mathbb{N}^n$ the sum

$$\sum_{\deg\Pi^*v_k^j=\alpha}\frac{\Pi^*v_k^j}{\bar F_k^0|F_k'|^2},$$

which is just equal to $Ct^\alpha/(\bar F_k^0|F_k'|^2)$ for some constant C, has to be smooth and consequently $t^\alpha/(\bar F_k^0|F_k'|^2)$ is smooth or $C=0$. Hence, to compute R^0 we only need to consider terms u_v, where $v=(v_1,\ldots,v_n)$

is such that $\widetilde{\Pi}^* v_k/(\bar{F}_k^0 |F_k'|^2)$ is smooth on $\mathcal{X}$ for all k. For such a v let us define

$$R_v^0 := \bar{\partial}|F|^{2\lambda} \wedge u_v|_{\lambda=0} \quad \text{and} \quad \widetilde{R}_v^0 := \widetilde{\Pi}^*(\bar{\partial}|F|^{2\lambda} \wedge u_v)|_{\lambda=0}. \tag{4.4}$$

From below it follows that R_v^0 and $\widetilde{R}_v^0$ are well defined (globally defined) currents and moreover that $\widetilde{\Pi}_* \widetilde{R}_v^0 = R_v^0$. Furthermore, it is clear that $R^0 = \sum R_v^0$, where the sum is taken over all v. Next, observe that, in view of (4.2), the frame element of u_v is $e_F \otimes e_\emptyset^*$, where $F \in X_n$ is determined by v_n. Hence $R_F\, e_F$ in (3.1) will be the sum of currents R_v^0, where v is such that v_n contains the frame element e_F. Thus, to prove the proposition it suffices to show that R_v^0 is of the desired form.

Let us therefore consider $\widetilde{R}_v^0$ in $\mathcal{U}$. Observe that

$$\widetilde{R}_v^0 = \bar{\partial}|F^0 F'|^{2\lambda} \wedge \left. \frac{\Pi^*((\bar{\partial} v_n) \cdots (\bar{\partial} v_2) v_1)}{|F_n^0 \cdots F_1^0|^2 \nu(t)} \right|_{\lambda=0}, \tag{4.5}$$

where $\nu(t) := (|F_n'| \cdots |F_1'|)^2$ is nonvanishing. For further reference, note that $\nu(t)$ only depends on $|t_1|, \ldots, |t_n|$. Moreover, let us denote $\deg(F_n^0 \cdots F_1^0)$ by $\mathbb{N}^n \ni \gamma = (\gamma_1, \ldots, \gamma_n)$ and $\deg(\omega_n \cdots \omega_1)$ by β, and recall that $\deg m_F = \alpha_F$. By Leibniz' rule and Lemma 4.1, recalling (4.3), we see that (4.5) is equal to a sum of terms of the form a constant times

(4.6)

$$\bar{\partial}\left[\frac{1}{t_i^{\gamma_i - \rho_i \cdot \beta}}\right] \otimes \Big[\prod_{j \neq i} |t_j|^{2(\rho_j \cdot \beta - \gamma_j)}\Big] \wedge \frac{\bar{t}_i^{\rho_i \cdot (\alpha_F + \beta) - \gamma_i} \prod_{j \neq i} \bar{t}_j^{\rho_j \cdot \alpha_F - 1}}{\nu(t)} \widehat{d\bar{t}_i}\; e_F \otimes e_\emptyset^*,$$

where t_i is one of the variables which fulfills that t_i divides the monomials F^0 and $F_n^0 \cdots F_1^0$, whereas $t_1 \cdots t_{i-1} t_{i+1} \cdots t_n$ divides $\Pi^* m_F$. In fact, it is not hard to check that, unless the latter requirement is fulfilled, the corresponding contribution will vanish for symmetry reasons. Moreover $\widehat{d\bar{t}_i}$ is just shorthand for $d\bar{t}_1 \wedge \ldots \wedge d\bar{t}_{i-1} \wedge d\bar{t}_{1+1} \wedge \ldots \wedge d\bar{t}_n$. Note that since $\Pi^* v_k/(F_k^0 |F_k'|^2)$ is smooth there will be no occurrences of any of the coordinate functions $\bar{t}_j$ in the denominator, except for them in $\nu(t)$, and in particular it follows that $\gamma_j - \rho_j \cdot \beta \geq 0$ when $j \neq i$. Moreover, due to Lemma 4.1, (4.6) vanishes whenever there is an occurrence of $\bar{t}_i$ in the numerator. Hence a necessary condition for (4.6) not to vanish is that

$$\rho_i \cdot (\alpha_F + \beta) - \gamma_i = 0.$$

We will now compute the action of $\widetilde{R}_v^0$ on the pullback of a test form $\phi = \varphi(z) dz$ of bidegree $(n, 0)$. Here $dz = dz_1 \wedge \ldots \wedge dz_n$. Let $\{\mathcal{U}_\tau\}$ be the cover of $\mathcal{X}$ that naturally comes from the construction of $\mathcal{X}$ as described in the proof of Theorem 3.1 in [24], and let $\{\chi_\tau\}$ be a partition of unity on $\mathcal{X}$ subordinate $\{\mathcal{U}_\tau\}$. It is not hard to see that we can choose the partition in such a way that the χ_τ are circled, that is, they only depend on $|t_1|, \ldots, |t_n|$. Now $\widetilde{R}_v^0 = \sum_\tau \chi_\tau \widetilde{R}_v^0$. We will start by computing the contribution from our fixed chart $\mathcal{U}$ (with corresponding cutoff function χ), where $\widetilde{R}^0_v$ is realized as a sum of terms (4.6).

Recall that R has support at the origin; hence it only depends on finitely many derivatives of φ at the origin. Moreover we know that $\bar{h}$ annihilates R if h is a holomorphic function which vanishes on Z, see Proposition 2.2 in [2]. For that reason, to determine R^0_v it is enough to consider the case when φ is a holomorphic polynomial. We can write φ as a finite Taylor expansion,

$$\varphi = \sum_a \frac{\varphi_a(0)}{a!} z^a,$$

where $a = (a_1, \dots, a_n)$, $\varphi_a = \frac{\partial^{a_1}}{\partial z_1^{a_1}} \cdots \frac{\partial^{a_n}}{\partial z_n^{a_n}} \varphi$ and $a! = a_1! \cdots a_n!$, with pullback to $\mathcal{U}$ given by

$$\Pi^* \varphi = \sum_a \frac{\varphi_a(0)}{a!} t^{Pa} = \sum_a \frac{\varphi_a(0)}{a!} t_1^{\rho_1 \cdot a} \cdots t_n^{\rho_n \cdot a}.$$

Moreover a computation similar to the proof of Lemma 4.2 in [24] yields

$$\Pi^* dz = \det P \ t^{(P-I)\mathbf{1}} \ dt,$$

where $\mathbf{1} = (1, 1, \dots, 1)$.

Since $\det P \neq 0$, it follows that $\chi \widetilde{R}^0_v . \Pi^* \phi$ is equal to a sum of terms of the form a constant times

$$\int \bar{\partial} \left[\frac{1}{t_i^{\rho_i \cdot \alpha_F}} \right] \otimes \left[\prod_{j \neq i} |t_j|^{2(\rho_j \cdot \beta - \gamma_j)} \right] \wedge \frac{\prod_{j \neq i} \bar{t}_j^{\rho_j \cdot \alpha_F - 1}}{\nu(t)} \widehat{d\bar{t}_i} \ e_F \otimes e_\emptyset^* \wedge$$
$$\chi(t) \sum_a \frac{\varphi_a(0)}{a!} t^{Pa} t^{(P-I)\mathbf{1}} dt = \sum_a I_a \ \wedge \frac{\varphi_a(0)}{a!} \ e_F \otimes e_\emptyset^*,$$

where

$$I_a = \int \bar{\partial} \left[\frac{1}{t_i^{\rho_i \cdot (\alpha_F - a - \mathbf{1}) + 1}} \right] \otimes [\mu_a] \wedge \frac{\chi(t)}{\nu(t)} \ \widehat{d\bar{t}_i} \wedge dt. \tag{4.7}$$

Here μ_a is the Laurent monomial

$$\mu_a = \prod_{j \neq i} t_j^{\rho_j \cdot (\beta + a + \mathbf{1}) - \gamma_j - 1} \ \bar{t}_j^{\rho_j \cdot (\beta + \alpha_F) - \gamma_j - 1}.$$

Invoking (2.4) we evaluate the t_i-integral. Since ν and χ depend on $|t_1|, \dots, |t_n|$ it follows that $\frac{\partial^\ell}{\partial t_i^\ell} \frac{\chi}{\nu(t)} |_{t_i = 0} = 0$ for $\ell \geq 1$ and thus (4.7) is equal to

$$2\pi i \int_{\widehat{t_i}} \frac{\chi(t)|_{t_i=0} [\mu_a]}{\nu(t)|_{t_i=0}} \ \widehat{d\bar{t}_i} \wedge \widehat{dt_i}, \tag{4.8}$$

if

$$\rho_i \cdot (\alpha_F - a - \mathbf{1}) + 1 = 1, \tag{4.9}$$

and zero otherwise. Moreover, for symmetry reasons, (4.8) vanishes unless

$$\rho_j \cdot (\alpha_F - a - \mathbf{1}) = 0 \tag{4.10}$$

for $j \neq i$, that is, unless μ_a is real.

Thus, since P is invertible, the system of equations (4.9) and (4.10) has the unique solution $a = \alpha_F - \mathbf{1}$ if $\alpha_F \geq \mathbf{1}$. Otherwise there is no solution, since a has to be larger than $(0, \ldots, 0)$. With this value of a the Laurent monomial μ_a is nonsingular and so the integrand of (4.8),

$$\frac{\chi(t)|_{t_i=0} \prod_{j\neq i} |t_j|^{2(\rho_j\cdot(\beta+\alpha_F)-\gamma_j-1)}}{\nu(t)|_{t_i=0}},$$

becomes integrable. Hence I_a is equal to some finite constant if $a = \alpha_F - \mathbf{1}$ and zero otherwise.

Now, recall that the chart $\mathcal{U}$ was arbitrarily chosen. Thus adding contributions from all charts reveals that R_v^0 and thus R_F is of the desired form (3.2), and so Proposition 3.1 follows.

Remark 2. We should compare Proposition 3.1 to Theorem 3.1 in [24]. It states that the residue current of Bochner-Martinelli type of an Artinian monomial ideal is a vector with entries of the form (3.2), but it also tells precisely which of these entries that are non-vanishing. If we had not cared about whether a certain entry was zero or not we could have used the proof of Proposition 3.1 above. Indeed, the Koszul complex, which gives rise to residue currents of Bochner-Martinelli type, can be seen as the cellular complex supported on the full $(r-1)$-dimensional simplex with labels $m_F = \{\prod_{i\in F} m_i\}$. It is not hard to see that the proof above goes through also with this non-conventional labeling. □

5. General monomial ideals

If the monomial ideal $M \subset S$ is of positive dimension, the computation of the residue current R associated with a cellular resolution of S/M gets more involved. In general $R = R_p + \ldots + R_\mu$, where R_k has bidegree $(0, k)$ and takes values in $\mathrm{Hom}\,(E_0, E_k)$, $p = \mathrm{codim}\, M$ and $\mu = \min(n, r)$. Our strategy is to decompose R into the simpler currents $R^{\mathfrak{p}}$, compare to (1.1), which can be computed essentially as $R = R_n$ in the Artinian case following the proof of Theorem 5.2 in [24].

The currents $R^{\mathfrak{p}}$ in (1.1) are obtained as certain "restrictions" of R to subsets of its support. As a basic tool we introduce in [25] a class of currents that we call hypermeromorphic and that admits restrictions to varieties and more generally constructible sets. A current on a complex manifold X is hypermeromorphic if it can be written as a locally finite sum $\sum \Pi_* \tau_\ell$, where τ_ℓ is a current of the form

$$\bar\partial\Big[\frac{1}{\sigma_1^{a_1}}\Big]\wedge\ldots\wedge\bar\partial\Big[\frac{1}{\sigma_q^{a_q}}\Big]\wedge\Big[\frac{1}{\sigma_{q+1}^{a_{q+1}}}\Big]\cdots\Big[\frac{1}{\sigma_\nu^{a_\nu}}\Big]\wedge\alpha,$$

on some manifold $\tilde X_s$. Here α is a smooth form with compact support, and $\Pi = \Pi_1\circ\cdots\circ\Pi_s$ is a corresponding composition of resolution of singularities $\Pi_1 : \widetilde X_1 \to X_1 \subset X, \ldots, \Pi_s : \widetilde X_s \to X_s \subset \widetilde X_{s-1}$. By the arguments in Section 2 in [2], the residue currents constructed from generically exact complexes (2.1) as well as its components R^ℓ_k, compare to Section 2, are hypermeromorphic.

Let T be a hypermeromorphic current on a complex manifold X and let $U \subset X$ be a Zariski-open set. Then the restriction of T to U has a natural standard extension to X, which we denote by $T|_U$ and which is hypermeromorphic. It is defined as the analytic continuation to $\lambda = 0$ of $|h|^{2\lambda}T$, where h is a tuple of holomorphic functions that vanish on $V = U^C$. Proposition 2.2 in [3] asserts that $T|_U$ is independent of the particular choice of h. The current $T - T|_U$, which has support on V, is a kind of residue; we call this the *restriction of* T *to* V and denote it by $T|_V$. With this notation, a hypermeromorphic current T has the standard extension property (SEP) with respect to the variety V if $T|_{V'} = 0$ for all subvarieties $V' \subset V$ of strictly smaller dimension. Moreover, the notion of restrictions can be extended in a unique way to any constructible set $W \subset X$, so that $T|_{W^C} = T - T|_W$ and $T|_{W\cap W'} = T|_W|_{W'}$ if $W' \subset X$ is another constructible set. In particular it follows that $\operatorname{supp} T|_W \subset \operatorname{supp} T \cap \overline{W}$. The current $R^\mathfrak{p}$ is now defined as $R|_{V(\mathfrak{p})\setminus\bigcup_{\mathfrak{q}\in \mathrm{Ass} J, \supset \mathfrak{p}} V(\mathfrak{q})}$.

We are now ready to present our result; as in the Artinian case, we start by a technical proposition. Since $R^\mathfrak{p}$ has the SEP, it behaves essentially like its component of lowest degree, that is, $R^\mathfrak{p}_\ell$ if $\operatorname{codim}\mathfrak{p} = \ell$. Indeed, outside the set $Z_{\ell+1}\cup\cdots\cup Z_N$, where Z_j denotes the set where the mapping f_j in (2.1) does not have optimal rank, $R^\mathfrak{p} = \beta R^\mathfrak{p}_\ell$, where

$$\beta := \sum_{j\geq 0} (\bar\partial\sigma)^j \tag{5.1}$$

with $(\bar\partial\sigma)^0$ interpreted as the identity map on E, is smooth. By arguments similar to the proof of Proposition 2.2 in [3] one can show that βT has a standard extension over $Z_{\ell+1}\cup\cdots\cup Z_N$ for any E_ℓ-valued hypermeromorphic current T. By the Buchsbaum-Eisenbud theorem, see [12], $\operatorname{codim}(Z_{\ell+1}\cup\cdots\cup Z_N) \geq \ell+1$ when (2.2) is exact, and so, since $R^\mathfrak{p}$ has the SEP with respect to $V(\mathfrak{p})$ of codimension ℓ, $R^\mathfrak{p}$ is equal to the standard extension of $\beta R^\mathfrak{p}_\ell$.

Recall that each associated prime ideal of a monomial ideal is monomial and therefore generated by a subset of the variables, see for example [20]. For $K \subset \{1,\ldots,n\}$ let $\mathfrak{p}_K$ denote the prime ideal $(z_i)_{i\in K}$.

If $\mathfrak{p}_K$ of codimension ℓ is associated with M we have a priori that $R_\ell^{\mathfrak{p}_K}$ consists of one entry $R_{\ell,F}^{\mathfrak{p}_K} e_F \otimes e_\emptyset^*$ for each $F \in X_\ell$. It follows from above that $R^{\mathfrak{p}_K} = \sum_{F\in X_\ell} R_{(K,F)}\ e_F \otimes e_\emptyset^*$, where $R_{(K,F)}$ is the standard extension of $\beta R_{\ell,F}^{\mathfrak{p}_K}$. We will suppress factor $e_\emptyset^*$ in the sequel.

Proposition 5.1. *Let $M \subset S$ be a monomial ideal and let R be the residue current associated with the polyhedral cell complex X. Suppose that $\mathbb{F}_X$ is a resolution of S/M. Then*

$$R = \sum_{\mathfrak{p}_K \in AssM} R^{\mathfrak{p}_K},$$

where

$$R^{\mathfrak{p}_K} = \sum_{F\in X_\ell} R_{(K,F)}\ e_F$$

if $\mathfrak{p}_K = (z_{k_1}, \dots, z_{k_\ell})$ is of codimension ℓ. Here $R_{(K,F)}$ is the standard extension of

$$\beta\ C_{(K,F)}(\eta) \otimes \bar\partial\Big[\frac{1}{z_{k_1}^{\alpha_{k_1}}}\Big] \wedge \dots \wedge \bar\partial\Big[\frac{1}{z_{k_\ell}^{\alpha_{k_\ell}}}\Big], \tag{5.2}$$

where β is given by (5.1)*, η denotes the variables $z_i, i \notin K$, $C_{(K,F)}(\eta)$ is a smooth function outside a set of codimension $> \ell$, and $(\alpha_1, \dots, \alpha_n) = \alpha_F$. If any of the entries of $\alpha_{k_1}, \dots, \alpha_{k_\ell}$ is* 0*,* (5.2) *should be interpreted as* 0.

Proof. Throughout this proof we will use the notation from the proof of Proposition 3.1. We will start by computing $R_n|_{\{0\}} = R_n^0|_{\{0\}}$; note that if the maximal ideal $\mathfrak{m} = (z_1, \dots, z_n)$ is associated with M, then this is precisely $R_n^{\mathfrak{m}} = R^{\mathfrak{m}}$. Following the proof of Proposition 3.1, we have that $R_n^0 = \sum R_v^0$, where R_v^0 is given by (4.4). Furthermore $R_v^0 = \widetilde\Pi_* \widetilde R_v^0$, where $\widetilde R_v^0$ is locally a finite sum of currents τ_ℓ of the form (4.6). Now, according the proof of Proposition 2.2 in [3], $R_v^0|_{\{0\}} = \sum \widetilde\Pi_* \tau_{\ell'}$ where the sum is taken over ℓ' such that t_i in $\tau_{\ell'}$ divides $\Pi^{-1}(\{0\})$. Moreover, $R_v^0|_{\{0\}}$ is a hypermeromorphic current that has support at the origin; in particular it follows that it is annihilated by $\bar h$ for all holomorphic functions h that vanish at the origin, see Proposition 2.3 in [3]. Hence the rest of the proof of Proposition 3.1 goes through and we get that $R_n|_{\{0\}}$ is of the form (3.1), and so $R^{\mathfrak{m}}$ is of the desired form.

Now, consider an associated prime $\mathfrak{p}_K$ of codimension ℓ. We want to compute $R_\ell^{\mathfrak{p}_K}$, which is equal to $R_\ell|_{V_K}$, where $V_K = V(\mathfrak{p}_K) = \{z_{k_1} = \dots = z_{k_\ell} = 0\}$, since by Corollary 2.4 in [3] a hypermeromorphic current of bidegree (p,q) that has support on a variety of codimension $> q$ vanishes. Let W_K denote the Zariski-open set $\{z_i \neq 0\}_{i\notin K}$. Note that codim $(V_k \setminus W_k) = \ell + 1$. Thus, since R_ℓ is a hypermeromorphic current of bidegree $(0,\ell)$ we have that $R_\ell|_{V_K} = R_\ell|_{V_K}|_{W_K}$.

The current $R_\ell|_{V_K}|_{W_K}$ can be computed analogously to $R_n|_{\{0\}}$ above. As in the proof of Proposition 3.1 we get that u_ℓ^0 is a sum of terms

$$u_v = \frac{(\bar\partial v_\ell)\cdots(\bar\partial v_2)v_1}{|F_\ell|^2\cdots|F_1|^2},$$

where v_k is given by (4.2) and $F_k = (f_k)^{q_k}/q_k!$. Denote the $z_i, i \in K$ by ζ and the $z_i, i \notin K$ by η and let ϕ be a test form of bidegree $(n, n-\ell)$ with compact support in W_K. Then R_ℓ acting on ϕ is the analytic continuation to $\lambda = 0$ of a sum of terms

$$\int_\eta \int_\zeta \bar\partial |F|^{2\lambda} \wedge u_v \wedge \varphi(\zeta,\eta) d\zeta \wedge d\bar\eta \wedge d\eta, \tag{5.3}$$

where v fulfills that $\Pi^* v_k/(\bar F_k^0 |F_k'|^2)$ is smooth, compare to Section 4. It is easily checked that (5.3) vanishes unless ϕ is of the form $\varphi(\zeta,\eta)d\zeta \wedge d\bar\eta \wedge d\eta$. We can now compute the inner integral of (5.3) as we computed $R_n|_{\{0\}}$ above. Indeed, V_K corresponds to the origin in the ζ-plane and since η is nonvanishing in W_K we can regard the coefficients of v_k as monomials in ζ times monomials in the parameters η. Thus we get that the inner integral is of the form

$$C(\eta) \otimes \bar\partial\Big[\frac{1}{z_{k_1}^{\alpha_{k_1}}}\Big] \wedge \ldots \wedge \bar\partial\Big[\frac{1}{z_{k_\ell}^{\alpha_\ell}}\Big] \wedge e_F \wedge \varphi(\zeta,\eta) d\zeta,$$

where C is smooth. Hence summing over all u_v gives that $R_\ell|_{V_K}|_{W_K}$, and consequently also $R^{\mathfrak{p}_K}$, is of the desired form. Note that $C(\eta)$ extends as a distribution over W_K^C. □

Observe that Proposition 5.1 gives a complete description of R except for the functions $C_{(K,F)}$. As in the Artinian case we are particularly interested in whether the $C_{(K,F)}$ are zero or not. Indeed, if we denote by $M_{(K,F)}$ the ideal generated by $\{z_i^{\alpha_i};\ i \in K,\ \alpha_F = (\alpha_1,\ldots,\alpha_n)\}$, then

$$\operatorname{ann} \bar\partial\Big[\frac{1}{z_{k_1}^{\alpha_{k_1}}}\Big] \wedge \ldots \wedge \bar\partial\Big[\frac{1}{z_{k_\ell}^{\alpha_{k_\ell}}}\Big] = M_{(K,F)},$$

and so $\operatorname{ann} R_{(K,F)} = M_{(K,F)}$ if $C_{(K,F)} \not\equiv 0$. Note that β does not affect the annihilator. Furthermore, to annihilate $R^{\mathfrak{p}_K}$ one has to annihilate each entry $R_{(K,F)}$ and therefore

$$\operatorname{ann} R^{\mathfrak{p}_K} = \bigcap_{(K,F);\ C_{(K,F)}\not\equiv 0} M_{(K,F)}.$$

Thus, in light of (1.2), a necessary condition for $C_{(K,F)}$ to be not identically zero is that $M \subset M_{(K,F)}$ and as in the Artinian case it turns out that we can determine precisely for which pairs (K,F) this happens if M is generic.

To this end we need to recall from [6] how one can find the irredundant irreducible decomposition of a generic monomial ideal M

from the Scarf complex of a certain associated ideal. When M is Artinian all irreducible components in the decomposition are of course $\mathfrak{m}$-primary and we saw above, in Section 3, that they correspond to facets of the Scarf complex Δ_M. The idea is now that adding to M a set of "ghost generators" $\{z_i^D\}$, where D is some integer larger than the degree of any generator m_i, enables us to identify a $\mathfrak{p}_K$-primary component $(z_k^{\alpha_k})_{k\in K}$ in the irreducible decomposition of M with the $\mathfrak{m}$-primary irreducible ideal generated by $\{z_k^{\alpha_k}\}_{k\in K} \cup \{z_i^D\}_{i\notin K}$ which in turn corresponds to a facet of the Scarf complex of the Artinian ideal $M^* := (m_1, \dots, m_r, z_1^D, \dots, z_n^D)$. More precisely, for each subset J of the underlying vertex set $\{1, \dots, r, 1_{\text{ghost}}, \dots, n_{\text{ghost}}\}$ with labels $\{m_1, \dots, m_r, z_1^D, \dots, z_n^D\}$ let M_J be the irreducible ideal generated by $\{z_i^{\alpha_i}, \alpha_i = \deg_{z_i}(m_J), \alpha_i < D\}$. In other words, M_J has a generator $z_i^{\alpha_i}$ precisely when J does not contain the ith ghost vertex; in particular if J does not have any ghost vertices, then the extra condition is empty and $M_J = \mathfrak{m}^{\alpha_J}$. Note also that M_J is independent on the particular choice of D. Bayer, Peeva and Sturmfels proved that a generic monomial ideal M is the intersection of the irreducible ideals M_J, where J runs over all facets of Δ_{M^*} and moreover that this intersection is irredundant; this is Theorem 3.7 in [6]. To be able to use this result let us observe that $J \subset \{1, \dots, r, 1_{\text{ghost}}, \dots, n_{\text{ghost}}\}$ can be identified with a pair (K, F) above, by letting K be determined by the set of ghost vertices and F by the remaining vertices. Indeed, given J, let $K = \{i \in \{1, \dots, n\}; i_{\text{ghost}} \notin J\}$ and $F = \{i \in \{1, \dots, r\}; i \in J\}$. With this identification the ideals M_J and $M_{(K,F)}$ coincide.

We have the following generalization of Proposition 3.2.

Proposition 5.2. *Let $M \subset S$ be a monomial ideal and let R be the residue current associated with the polyhedral cell complex X. Suppose that $\mathbb{F}_X$ is a resolution of S/M. Then $C_{(K,F)} \not\equiv 0$ if and only if (K, F) is a facet of the Scarf complex Δ_{M^*}.*

Proof. Observe that M^* is generic. Thus, if (K, F) is not a facet of Δ_{M^*}, then there is a (minimal) generator m' of M^* such that m' strictly divides $m_{(K,F)}$, so that $m' \notin \mathfrak{m}^{\alpha(K,F)}$, as we showed in the proof of Proposition 3.2. In particular, $m' \notin M_{(K,F)}$, since clearly $M_{(K,F)} \subset \mathfrak{m}^{\alpha(K,F)}$. Moreover m' has to be in M. Indeed, it is easy to see that m' can not possibly be any of the generators z_i^D of M^*. Since D is chosen to be larger than the degree of any minimal generator of M, the degree of any of the variables in $m_{(K,F)}$ can not exceed D, and so m' that strictly divides $m_{(K,F)}$ can not have degree D in any variable. Hence, if (K, F) is not a facet of Δ_{M^*}, then $M \not\subseteq M_{(K,F)}$ and so $C_{(K,F)}$ has to be 0.

On the other hand, since (3.3) is irredundant, $C_{(K,F)}$ has to be not identically equal to zero whenever (K, F) is a facet of Δ_{M^*}. □

To conclude, Propositions 5.1 and 5.2 give the following description of the residue current associated with a cellular resolution of a generic ideal, generalizing Theorem 3.3.

Theorem 5.3. *Let $M \subset S$ be a generic monomial ideal and let R be the residue current associated with the polyhedral cell complex X. Suppose that $\mathbb{F}_X$ is a resolution of S/M. Then*

$$R = \sum_{(K,F) \text{ facet of } \Delta_{M^*}} R_{(K,F)}\, e_F, \tag{5.4}$$

where $R_{(K,F)}$ is given by (5.2) *and $C_{(K,F)}$ there is not identically equal to zero.*

Note that (5.4) is a refinement of the decomposition (1.1), corresponding to that irreducible decompositions of monomial ideals are refinements of primary decompositions.

An immediate consequence is the following generalization of Corollary 3.4.

Corollary 5.4. *Let $M \subset S$ be a monomial ideal and let R be the residue current associated with the polyhedral cell complex X. Suppose that $\mathbb{F}_X$ is a resolution of S/M. Then*

$$M = \bigcap_{(K,F) \text{ facet of } \Delta_{M^*}} ann\, R_{(K,F)} \tag{5.5}$$

is the irredundant irreducible decomposition of M.

Note that (5.4) is a refinement of the decomposition (1.1), corresponding to that the irreducible decomposition (5.5) is a refinement of the primary decomposition (1.2).

For a monomial ideal M there exists a unique "maximal" monomial primary decomposition. Indeed, for each prime $\mathfrak{p}$ associated with M, there is a unique monomial $\mathfrak{p}$-primary component $\mathfrak{q}$ that is maximal among all monomial $\mathfrak{p}$-primary components that could appear in a monomial primary decomposition of M. Moreover in any primary decomposition the $\mathfrak{p}$-primary component may be replaced by $\mathfrak{q}$, see for example [20] or Exercise 3.11 in [12]. It is not hard to check that $\mathfrak{q}$ equals the intersection of all $\mathfrak{p}$-primary components in the irredundant irreducible decomposition of M. Hence we immediately get the following.

Corollary 5.5. *Let $M \subset S$ be a monomial ideal and let R be the residue current associated with the polyhedral cell complex X. Suppose that $\mathbb{F}_X$ is a resolution of S/M. Then*

$$M = \bigcap_{\mathfrak{p}_K \in Ass M} ann\, R^{\mathfrak{p}_K}$$

is the maximal monomial primary decomposition of M.

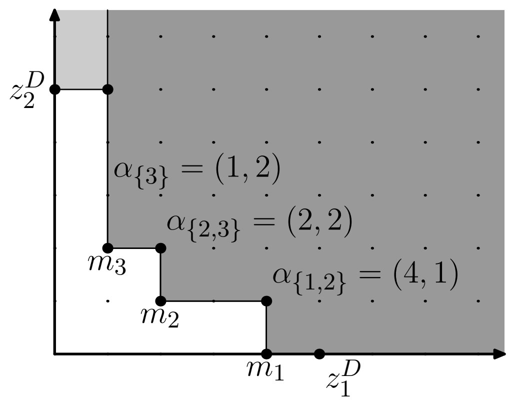

FIGURE 3. The ideals M (dark grey) and M^* (light grey) in Example 3.

As in the Artinian case, we should remark, that even though we can not determine the set of non-vanishing entries of the residue current associated with an arbitrary cellular resolution, we can still estimate the number of them from below by the number or irreducible components of the corresponding ideal.

Finally, let us illustrate our results by an example.

Example 3. Let M be the (generic) ideal

$$M = (z_1^4, z_1^2 z_2, z_1 z_2^2) =: (m_1, m_2, m_3),$$

and let M^* be the corresponding Artinian ideal obtained by adding the ghost vertices z_1^D and z_2^D (in fact, only z_2^D comes into account), see Figure 3.

As in Example 1 the Scarf complex Δ_{M^*} is one-dimensional and the facets are the pairs of adjacent generators, that is $F_1 = \{z_1^4, z_1^2 z_2\}$, $F_2 = \{z_1^2 z_2, z_1 z_2^2\}$ and $F_3 = \{z_1 z_2^2, z_2^D\}$. Note that the associated prime ideals of M are $\mathfrak{p}_{\{1\}} = (z_1)$ and $\mathfrak{p}_{\{1,2\}} = (z_1, z_2)$, corresponding to $K = \{1\}$ and $\{1,2\}$ respectively.

Let R be the residue current obtained from a cellular resolution. By Theorem 5.3 the current $R^{\mathfrak{p}_{\{1\}}}$ is equal to a sum of terms $R_{(\{1\},F)}\, e_F$, where the sum is taken over facets $(\{1\}, F)$ of Δ_{M^*}. Recall that (K, F) should be interpreted as the facet of Δ_{M^*} that has the vertex set $\{z_j^D\}_{j\notin K} \cup \{m_i\}_{i\in F}$, and thus we are looking for facets containing the ghost vertex z_2^D. However, there is only one such facet, namely $F_3 = (\{1\}, \{3\})$. Moreover, $\alpha_{\{3\}} = (1,2)$, and so

$$R_{F_3} = (\bar\partial \sigma_2) C(z_2) \otimes \bar\partial \left[\frac{1}{z_1}\right],$$

with annihilator equal to (z_1), which is precisely M_{F_3}.

Next, $R^{\mathfrak{p}_{\{1,2\}}} = \sum R_{(\{1,2\},F)}\, e_F$, where the sum now is taken over facets of Δ_{M^*} that contain no ghost vertices. There are two such facets,

$F_1 = (\{1,2\},\{1,2\})$ and $F_2 = (\{1,2\},\{2,3\})$, with corresponding currents

$$R_{F_1} = \bar\partial\left[\frac{1}{z_1^4}\right]\wedge\bar\partial\left[\frac{1}{z_2}\right] \text{ and } R_{F_2} = \bar\partial\left[\frac{1}{z_1^2}\right]\wedge\bar\partial\left[\frac{1}{z_2^2}\right].$$

The annihilators are $(z_1^4, z_2) = M_{F_1}$ and $(z_1^2, z_2^2) = M_{F_2}$, respectively, and so $\operatorname{ann} R^{\mathfrak{p}_{\{1,2\}}} = (z_1^4, z_2)\cap(z_1^2,z_2^2) = (z_1^4, z_1^2 z_2, z_2^2)$.

To conclude, the maximal monomial primary decomposition of M is given by

$$M = (z_1)\cap(z_1^4, z_1^2 z_2, z_2^2),$$

whereas the irredundant irreducible decomposition is given by

$$M = (z_1)\cap(z_1^4, z_2)\cap(z_1^2, z_2^2),$$

as follows from Corollaries 5.5 and 5.4. Of course, these two decompositions can easily be derived directly from M. □

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