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Interacting particle systems in a randomly evolving environment

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ABSTRACT

This thesis concerns interacting particle systems in a randomly evolving environment. In the first paper, we consider the so called contact process in a randomly evolving environment (CPREE), introduced by Erik Broman. This process is a generalization of the contact process where the recovery rate can vary between two values. The rate which it chooses is determined by a background process, which evolves independently at different sites. As for the contact process, we can similarly define a critical value in terms of survival for this process. We prove that this definition is independent of how we start the background process, that finite and infinite survival (meaning nontriviality of the upper invariant measure) are equivalent and finally that the process dies out at criticality.

In the second paper, we consider spin systems on $\mathbb{Z}$ (i.e. interacting particle systems on $\mathbb{Z}$ in which each coordinate has only two possible values and only one coordinate changes in each transition) whose rates are determined by a background process, which is more general than in the first paper. We prove a generalization of a result by Liggett, that under certain conditions on the rates there are only two extremal invariant distributions.

Keywords: Interacting particle systems, contact process, randomly evolving environment, spin systems.

Preface

This thesis consists of the following papers:

- ▷ Jeffrey E. Steif and Marcus Warfheimer "The critical contact process in a randomly evolving environment dies out", (Submitted).
- ▷ Marcus Warfheimer, "Attractive nearest-neighbor spin systems on the integers in a randomly evolving environment".

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Marcus Warfheimer
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Part I

INTRODUCTION

1
Introduction

The field of interacting particle systems is a branch of probability theory. However, the motivation often comes from physical or biological systems. In loose terms, one tries to formulate a mathematical model for objects (particles, people, cars, etc) which interact with each other in a certain way. One way to construct such a model is to place each object at a site in a graph structure and declare that each one of them can be in a finite number of different states. (A graph is just a finite or countable set of vertices equipped with a relation that determines which vertices are neighbors.) One then assigns some initial configuration and lets the system evolve according to some probabilistic rules. It is at this point where the interactions come into play. Each object is changing its state depending on the states of the other (usually neighboring) objects as well as itself. Another common situation to model is particle motion. In that case one places particles at the sites of a graph and lets them move according to some probabilstic rule. In this situation the state of a site reflects whether there is a particle present at the site or not.

From a more mathematical point of view, interacting particle systems are a special class of so called Markov processes. Markov processes have the property that given the present, the future is independent of the past. Denote the set of sites and possible states by S and A respectively. The state space, or configuration space, for our Markov process is then A^S. The most common situation is when A consists of only two elements and that only one coordinate of the process is allowed to change at a time. Such processes are called spin systems. In this situation, the evolution is described by a rate function, $c(x, \eta)$, $x \in S$ and $\eta \in A^S$, which gives the rate at which the coordinate at x flips when the system is in state η. Having something occur "at a rate λ" means informally that the time for this to occur has an exponential distribution with mean $1/\lambda$.

In this generality not much can be said. Therefore one concentrates upon specific types of models of which I will name a few.

The contact process on the d-dimensional lattice $\mathbb{Z}^d$. This process was introduced by Harris (1974) and is a model for spread of an infection. The model is such that infected people recover at rate 1 and healthy people are infected with a rate proportional to the number of infected neighbors. The state of the system is described by a configuration $\eta \in \{0,1\}^{\mathbb{Z}^d}$, where $\eta(x) = 0$ represents that the individual at x is healthy and $\eta(x) = 1$ represents it is infected. Also, the dynamics are specified by the following rate function

$$c(x, \eta) = \begin{cases} 1 & \text{if } \eta(x) = 1 \\ \lambda \sum\limits_{y \sim x} \eta(y) & \text{if } \eta(x) = 0, \end{cases}$$

where $\mathbf{y} \sim x$ means that x and y are neighbors and λ is a positive parameter called the infection rate. To simplify notation, we will identify $\{0,1\}^{\mathbb{Z}^d}$ with subsets of $\mathbb{Z}^d$ by letting $\eta \in \{0,1\}^{\mathbb{Z}^d}$ correspond to

$$\{\, x \in \mathbb{Z}^d : \ \eta(x) = 1 \,\}.$$

Let P_λ^A denote the distribution of the process with parameter λ and initial configuration $A \subseteq \mathbb{Z}^d$. We say that the process *survives at* λ if

$$P_\lambda^{\{0\}}[\, \eta_t \neq \emptyset \text{ for all } t \geq 0 \,] > 0;$$

otherwise it is said to *die out at* λ. One can rather easily show that

$$P_\lambda^{\{0\}}[\, \eta_t \neq \emptyset \text{ for all } t \geq 0 \,] = 0$$

for small values of λ and

$$P_\lambda^{\{0\}}[\, \eta_t \neq \emptyset \text{ for all } t \geq 0 \,] > 0$$

for large values of λ. In words, when we start the process with one site infected, the infection will almost surely eventually disappear for small values of λ and will last forever with positive probability for large values of λ. From this, it is natural to define the critical value:

$$\lambda_c := \inf\{\, \lambda : \, P_\lambda^{\{0\}}[\, \eta_t \neq \emptyset \text{ for all } t \geq 0 \,] > 0 \,\}$$

and the previous statement just means that $0 < \lambda_c < \infty$. A much harder question, and one which had been open for approximately 15 years, is whether the contact process survives or dies out *at* the critical value λ_c. A celebrated theorem by Bezuidenhout and Grimmett, [1], gives us the answer.

THEOREM 1 (BEZUIDENHOUT AND GRIMMETT) *The critical contact process dies out.*

For a proof of this, see [1] or [5].

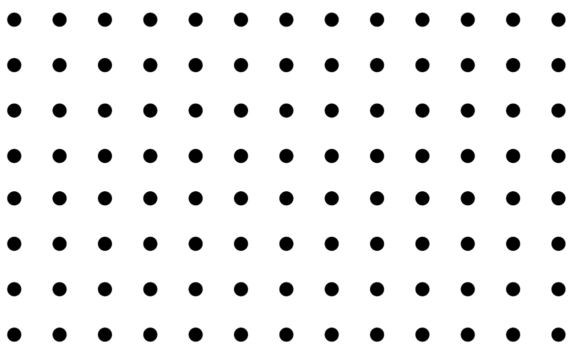

Figure 1.1: *A small portion of the lattice* $\mathbb{Z}^2$.

Remark: We can parameterize the contact process in an equivalent way as follows: Let the recovery rate be $\delta > 0$ and the infection rate be equal to the number of infected neighbors. In other words, we change λ to 1 and let δ be the recovery rate, which of course just corresponds to a time scaling. We will denote the corresponding critical value by δ_c.

The voter model on the d-dimensional lattice $\mathbb{Z}^d$. This process was introduced independently by Clifford and Sudbury (1973) and by Holley and Liggett (1975). Here, the state of the system is described by a configuration

$\eta \in \{0,1\}^{\mathbb{Z}^d}$ where $d \geq 1$ and the transition mechanism is described by saying that $\eta(x)$ flips to $1-\eta(x)$ at rate

$$\frac{1}{2d}\sum_{y\sim x} 1_{\{\eta(y)\neq\eta(x)\}}.$$

One interpretation, made by Holley and Liggett, is to think of the sites in $\mathbb{Z}^d$ as representing voters who can hold either of two political positions, which are denoted by 0 and 1. In this representation the dynamics of the model can be described as follows: A voter waits an exponentially distributed time with mean 1 and then takes the opinion of a neighbor choosen at random. Clearly, if we start the process with all voters in state 0 or all voters in state 1, nothing happens. In mathematical terms the point masses on these two configurations are so called *invariant distributions*, meaning that if we start the process with such a distribution μ, the distribution of the process at any time later is still μ. (Of course, in this case they are also absorbing states.) At this point, one may ask if there are any other invariant distributions. To answer that question it turns out that the dimension d plays a prominent role. Namely, when $d \leq 2$ there are no other than those two above (plus their convex combinations), but when $d \geq 3$ there are in fact other ones. (To people with a little background in probability theory, this result is intimately related to the fact that simple random walk is recurrent when $d \leq 2$ and transient when $d \geq 3$.)

The stochastic Ising model on $\mathbb{Z}^d$. This is a model for magnetism introduced by Glauber (1963). The state space of this model is $\{-1,1\}^{\mathbb{Z}^d}$. Imagine that atoms are laid out on all of $\mathbb{Z}^d$ and that each of them either can have a spin (state) of $+1$ or -1. The resulting configuration describing the system is an element of $\{-1,1\}^{\mathbb{Z}^d}$. The dynamics of the evolution is described by declaring a spin $\eta(x)$ at a site x to flip to $-\eta(x)$ at rate

$$\exp\left(-\beta\sum_{y\sim x}\eta(x)\eta(y)\right)$$

where β is a nonnegative parameter called the inverse temperature. Note that the flip rate is higher when the spin at x differs from most of its neighbors than it is when it agrees with most of them. In other words, the system prefers configurations in which the spins tend to be aligned with one another. When $\beta = 0$ it is easy to see that there is only one invariant distribution, namely the product measure μ on $\{-1,1\}^{\mathbb{Z}^d}$ with density $\frac{1}{2}$ and in addition, for any initial distribution, the distribution at time t converges weakly as $t \to \infty$ to μ. Such a process is called *ergodic*. In the non-ergodic case, i.e. when we have more than one invariant distribution, one says that a phase transition has occurred and each of the invariant distributions corresponds to different "phases" of the system. The problem is to determine for which choices of β and d the process is ergodic. If $d = 1$, then it turns out that the process is ergodic for all β but when $d \geq 2$ there is a critical value $0 < \beta_d < \infty$ such that the process is ergodic if $\beta < \beta_d$ and non-ergodic if $\beta > \beta_d$.

For further reading about interacting particle systems, there are three standard reference books, namely Liggett [4], [5] and Durrett [3]. The first one deals with the construction of interacting particle systems from given rates as well as more or less all results in the field until 1985. The second one concentrates upon three models; the contact process, the voter model and the so called exclusion process, a model of particle motion, and covers many of the results concerning these models until 1999. The third book considers, among other things, the contact process, the voter model and some variants thereof.

1.1 Paper I

In this paper we consider the so called *contact process in a randomly evolving environment* (CPREE), introduced by Erik Broman [2]. This process is a generalization of the contact process, where the recovery rate is allowed to vary between two values, δ_0 and δ_1. (Recall the equivalent parameterization of the contact process.) The rate which is chosen is determined by a background process, which evolves independently at different sites. To be precise, we consider the Markov process $\{(B_t, C_t)\}_{t\geq 0}$ with state space $\{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d}$ which performs transitions according to the following rates at a site $x \in \mathbb{Z}^d$:

transition	rate
$(0,0) \to (0,1)$	$\sum_{y \sim x} C(y)$
$(1,0) \to (1,1)$	$\sum_{y \sim x} C(y)$
$(0,1) \to (0,0)$	δ_0
$(1,1) \to (1,0)$	δ_1
$(0,0) \to (1,0)$	γp
$(0,1) \to (1,1)$	γp
$(1,0) \to (0,0)$	$\gamma(1-p)$
$(1,1) \to (0,1)$	$\gamma(1-p)$

where $d \geq 1, \gamma, \delta_0, \delta_1 > 0$ with $\delta_1 \leq \delta_0$ and $p \in [0,1]$. In other words, at each site x independently, $\{B_t(x)\}_{t \geq 0}$ is a 2-state Markov chain with infinitesimal matrix

$$\begin{pmatrix} -\gamma p & \gamma p \\ \gamma(1-p) & -\gamma(1-p) \end{pmatrix}$$

which in turn determines the recovery rate of $\{C_t(x)\}_{t \geq 0}$ in the following way. For each x and t, the recovery rate at time t and site x is δ_0 or δ_1 depending on whether $B_t(x) = 0$ or $B_t(x) = 1$. Also, the infection rate is always the number of infected neighbors. (Actually Broman did this on a more general graph, but here we will only consider $\mathbb{Z}^d$.) Broman referred to $\{B_t\}_{t \geq 0}$ as *the background process* and the whole process $\{(B_t, C_t)\}_{t \geq 0}$ as *the contact process in a randomly evolving environment* (CPREE). Let $\{C_t^\rho\}_{t \geq 0}$ denote the right marginal where the initial distribution of the whole process is ρ. In the case where $\rho = \mu \times \nu$, we write $\{C_t^{\mu,\nu}\}_{t \geq 0}$. Further, let $\mathbf{P}_p$ denote the measure governing the process for the parameters p, γ, δ_0 and δ_1, where γ, δ_0 and δ_1 are considered fixed. Also, denote the product measure with density $q \in [0,1]$ by π_q. Broman defined the critical value

$$p_c := \inf \left\{ p : \mathbf{P}_p[\, C_t^{\pi_p, \{0\}} \neq \emptyset \, \forall t > 0 \,] > 0 \right\}$$

(p_c is taken to be 1 if no p satisfies this) and proved that if $\delta_1 < \delta_c < \delta_0$ and $\gamma > \max(2d, \delta_c - \delta_1)$, then $p_c \in (0,1)$. (Recall the definition of δ_c from the remark after Theorem 1.) At the end of his paper he asked whether the critical value is affected if we vary the initial distribution of the background process. Our first result answers this question. Given $\gamma, \delta_0, \delta_1 > 0$ with $\delta_1 \leq \delta_0$, $q \in [0,1]$ and $A \subseteq \mathbb{Z}^d$ with $|A| < \infty$, define

$$p_c(q,A) := \inf\left\{p : \mathbf{P}_p[\, C_t^{\pi_q, A} \neq \emptyset \,\forall t > 0\,] > 0\right\}.$$

THEOREM 2 *Given $A, A' \subseteq \mathbb{Z}^d$ with $|A|, |A'| < \infty$ and p, q, $q' \in [0,1]$,*

$$\mathbf{P}_p[\, C_t^{\pi_q, A} \neq \emptyset \,\forall t > 0\,] > 0 \quad \Longleftrightarrow \quad \mathbf{P}_p[\, C_t^{\pi_{q'}, A'} \neq \emptyset \,\forall t > 0\,] > 0.$$

In particular, $p_c(q,A)$ is independent of both q and A.

We will let p_c denote this common value. (Recall, p_c of course depends on γ, δ_0 and δ_1.) Also, if $\mathbf{P}_p[\, C_t^{\pi_q, A} \neq \emptyset \,\forall t > 0\,] > 0$ holds (which we now know is independent of q and A), we say that $\{C_t\}$ *survives at* p; otherwise it is said to *die out at* p.

Standard arguments yield that the limiting distribution starting from all 1's exists and we will denote the limit by $\bar{\nu}_p$. Also, we will refer to this measure as the *upper invariant measure*. This measure gives us another natural way to define a critical value:

$$p'_c := \inf\{\, p : \bar{\nu}_p \neq \pi_p \times \delta_\emptyset \,\}.$$

For general attractive interacting particle systems it might or might not be the case that these two critical values coincide. However, for the ordinary contact process this is the case (due to its self-duality) and our next result shows that this is also true in our situation.

THEOREM 3 *$\{C_t\}$ survives at p if and only if $\bar{\nu}_p \neq \pi_p \times \delta_\emptyset$. In particular $p_c = p'_c$.*

Our final result is a generalization of Theorem 1.

THEOREM 4 *If $\{C_t\}$ survives at $p > 0$, then there exists $\delta > 0$ so that it survives at $p - \delta$. In particular, if $p_c \in (0,1]$, then the critical contact process in a randomly evolving environment dies out.*

1.2 Paper II

Recall that spin systems are interacting particle systems where each coordinate has two possible states and only one coordinate changes in each transition. In this paper we consider spin systems on $\mathbb{Z}$ in a randomly evolving environment, where the environment is more general than in the previous paper. To decribe the process we are dealing with in mathematical terms, let $c_0(x,\eta)$, $c_1(x,\eta)$ and $b(x,\eta)$ be given rate functions and define a Markov process $\{(\beta_t,\eta_t)\}_{t\geq 0}$ on $\{0,1\}^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}}$ with the dynamics at a site x specified in the following way:

transition	rate
$(\beta,\eta) \to (\beta,\eta_x)$	$c_0(x,\eta)$ if $\beta(x)=0$
$(\beta,\eta) \to (\beta,\eta_x)$	$c_1(x,\eta)$ if $\beta(x)=1$
$(\beta,\eta) \to (\beta_x,\eta)$	$b(x,\beta)$

Here, for given $\eta \in \{0,1\}^{\mathbb{Z}^d}$ and $x \in \mathbb{Z}$, η_x is the element in $\{0,1\}^{\mathbb{Z}}$ defined by

$$\eta_x(y) = \begin{cases} \eta(y) & \text{if } y \neq x \\ 1-\eta(x) & \text{if } y = x. \end{cases}$$

As before, $\{\beta_t\}_{t\geq 0}$ will be referred to as the background process. Of course, to be able to say anything about the process we need some conditions on the rate functions above. We will assume that the dynamics are translation invariant, that

$$\begin{aligned} c_0(x,\eta) &\leq c_1(x,\eta) \quad \text{if} \quad \eta(x)=0, \\ c_1(x,\eta) &\leq c_0(x,\eta) \quad \text{if} \quad \eta(x)=1, \end{aligned} \tag{1}$$

that $c_0(x,\eta)$ and $c_1(x,\eta)$ only depend on η through $\eta(x-1)$, $\eta(x)$ and $\eta(x+1)$ and that c_0, c_1 and b satisfy the following attractivity condition:

DEFINITION 1 *A spin system on $\mathbb{Z}$, with rate function $c(x,\eta)$ is said to be attractive if whenever $\eta \leq \eta'$,*

$$\begin{aligned} c(x,\eta) &\leq c(x,\eta') \quad \textit{if} \quad \eta(x)=\eta'(x)=0, \quad \textit{and} \\ c(x,\eta) &\geq c(x,\eta') \quad \textit{if} \quad \eta(x)=\eta'(x)=1. \end{aligned} \tag{2}$$

(Here, $\leq$ refers to the usual partial ordering on $\{0,1\}^{\mathbb{Z}}$, i.e., $\eta \leq \eta'$ if and only if $\eta(x) \leq \eta'(x)$ for all $x \in \mathbb{Z}$.) This condition is exactly what is needed to be able to couple two copies, with initial configurations stochastically ordered, such that the two copies continue to be ordered for all times. Furthermore, note that we can equivalently view our process on $\{0,1\}^{\mathbb{Z}\times\{0,1\}}$ and that the conditions (1) and (2) just means that the whole process is attractive on that space. (Definition 1 can of course be generalizad to $\{0,1\}^S$ where S is countable.) The attractivity can be used to show (via monotonicity) the existence of two extremal stationary distrubutions ν_0 and ν_1 defined by

$$\nu_0 = \lim_{t\to\infty} \delta_0 S(t) \qquad\qquad \nu_1 = \lim_{t\to\infty} \delta_1 S(t),$$

where δ_0 and δ_1 denote the point masses corresponding to the elements $\eta \equiv 0$ and $\eta \equiv 1$ in $\{0,1\}^{\mathbb{Z}\times\{0,1\}}$ and $\{S(t)\}_{t\geq 0}$ denotes the semigroup associated to $\{(\beta_t, \eta_t)\}_{t\geq 0}$.

The assumptions on c_0 and c_1 imply that they together only can attain at most 16 different values. To describe the values we will use the following notation: $c_i(001) = c_i(x,\eta)$ when $\eta(x-1) = 0$, $\eta(x) = 0$ and $\eta(x+1) = 1$ etc. For technical reasons, define

$$\begin{aligned} C_1 = \{\, & c_i(100) + c_j(110),\ c_i(001) + c_j(011), \\ & c_i(011) + c_j(110),\ c_i(100) + c_j(001),\ i = 0,1,\ j = 0,1 \,\} \end{aligned}$$

and let

$$C = \min(C_1)\,.$$

Before we state our main result, we want to emphasize that the case when $c_0 = c_1$, i.e. no background process, has been studied before by Liggett. Let $\mathcal{I}$ denote the set of invariant distributions for $\{(\beta_t, \eta_t)\}_{t\geq 0}$ and let $\mathcal{I}_e$ denote its extreme points.

THEOREM 5 (LIGGETT) *Suppose $c_0 = c_1 := c$ and let c satisfies*

$$c(x,\eta) + c(x,\eta_x) > 0 \quad \textit{whenever} \quad \eta(x-1) \neq \eta(x+1). \tag{3}$$

Then $\mathcal{I}_e = \{\nu_0, \nu_1\}$.

For a proof, see Theorem 3.13, page 152 in [4].

Finally, we state our main result whose proof follows the ideas of the proof of Theorem 5.

THEOREM 6 *Suppose that the background process is ergodic and* $C > 0$. *Then* $\mathcal{I}_e = \{\nu_0, \nu_1\}$.

Remark: In the case when $c_0 = c_1$ the condition of Theorem 6 is stronger than the condition of Theorem 5. However, if (3) holds and $C = 0$, then it can be seen more directly that $\mathcal{I}_e = \{\nu_0, \nu_1\}$. (See the beginning of Theorem 3.13, page 152 in [4].)

References

[1] C. Bezuidenhout and G. Grimmett, *The critical contact process dies out*, Ann. Probab. **18** (1990), 1462–1482.

[2] E. I. Broman, *Stochastic domination for a hidden markov chain with applications to the contact process in a randomly evolving environment*, Ann. Probab. **35** (2007), 2263–2293.

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Part II

PAPERS

PAPER I

The critical contact process in a randomly evolving environment dies out

Jeffrey E. Steif and Marcus Warfheimer

2

PAPER I

ABSTRACT
Bezuidenhout and Grimmett proved that the critical contact process dies out. Here, we generalize the result to the so called contact process in a random evolving environment (CPREE), introduced by Erik Broman. This process is a generalization of the contact process where the recovery rate can vary between two values. The rate which it chooses is determined by a background process, which evolves independently at different sites. As for the contact process, we can similarly define a critical value in terms of survival for this process. In this paper we prove that this definition is independent of how we start the background process, that finite and infinite survival (meaning nontriviality of the upper invariant measure) are equivalent and finally that the process dies out at criticality.

Key words and phrases: contact process, varying environment

Subject classification : 60K35.

2.1 Introduction and main results

The contact process, introduced by Harris [5], is a simple model for the spread of an infection on a lattice. The state at a certain time is described by a configuration, $\eta \in \{0,1\}^{\mathbb{Z}^d}$, where $\eta(x) = 0$ means that the individual at location x is healthy and $\eta(x) = 1$ means it is infected. The model is such that infected people recover at rate 1 and healthy people are infected with a rate proportional to the number of infected neighbors. In more mathematical language, the contact process is a Markov process, $\{\eta_t\}_{t\geq 0}$, with state space $\{0,1\}^{\mathbb{Z}^d}$ where the configuration changes its state at site $x \in \mathbb{Z}^d$ as follows:

$$\begin{array}{llll} \eta \to \eta_x & \text{with rate} & 1 & \text{if} \quad \eta(x) = 1 \\ \eta \to \eta_x & \text{with rate} & \lambda \sum_{y\sim x} \eta(y) & \text{if} \quad \eta(x) = 0, \end{array}$$

where $y \sim x$ means that x and y are neighbors,

$$\eta_x(y) = \begin{cases} \eta(y) & \text{if } y \neq x \\ 1 - \eta(x) & \text{if } y = x \end{cases}$$

and λ is a positive parameter called the infection rate. See the standard references Liggett [7] and Durrett [4] for how these informal rates determine a Markov process and for much on the contact process as well as other interacting particle systems. Denote the distribution of this process when it starts with the configuration η by P_λ^η. We say that the process *dies out at* λ if

$$P_\lambda^{\{0\}}[\,\eta_t = \emptyset \text{ some } t \geq 0\,] = 1;$$

otherwise it is said to *survive at* λ. Here, the initial configuration $\{0\}$ means there is a single infection at the origin and the configuration $\emptyset$ means the element in $\{0,1\}^{\mathbb{Z}^d}$ consisting of all zeros. (As usual, we identify $\{0,1\}^{\mathbb{Z}^d}$ with subsets of $\mathbb{Z}^d$.) Using an easy monotonicity in λ, it is natural to define the critical value

$$\lambda_c := \inf\{\,\lambda : \; P_\lambda^{\{0\}}[\,\eta_t \neq \emptyset \text{ for all } t \geq 0\,] > 0\,\}.$$

A fundamental first question concerning this model is whether it survives when λ is large and whether it dies out for small values of λ, in other words whether $0 < \lambda_c < \infty$, and it is not very hard to show that this indeed is the case.

Furthermore, since the contact process is attractive (see Liggett [7] for this definition), we can define

$$\lambda_c' := \inf\{\, \lambda : \bar{\nu}_\lambda \neq \delta_\emptyset \,\},$$

where $\bar{\nu}_\lambda$ is the so called upper invariant measure, defined to be the limiting distribution starting from all 1's. A self-duality equation (see [4] or [7]) easily leads to $\lambda_c = \lambda_c'$. A much harder question, and one which had been open for approximately 15 years, is whether the contact process survives or dies out *at* the critical value. A celebrated theorem by Bezuidenhout and Grimmett, [1], gives us the answer.

THEOREM 1 (BEZUIDENHOUT AND GRIMMETT) *The critical contact process dies out.*

For a proof of this, see [1] or [9].

Note that changing λ to 1 and the recovery rate to δ corresponds to a trivial time scaling and so the process could have instead been defined in this way. We will denote the corresponding critical value by δ_c. This should be kept in mind in what follows.

In 1991, Bramson, Durrett and Schonmann [2] introduced the contact process in a random environment, in which the recovery rates are taken to be independently and identically distributed random variables and then fixed in time. For further results concerning this model see for example, Liggett [8], Klein [6] and Newman and Volchan [10]. Recently, Broman [3] introduced another variant where the environment changes in time in a simple Markovian way. More precisely, he considered the Markov process, $\{(B_t, C_t)\}_{t\geq 0}$ on

$\{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d}$ described by the following rates at a site x:

transition	rate
$(0,0) \to (0,1)$	$\sum_{y\sim x} C(y)$
$(1,0) \to (1,1)$	$\sum_{y\sim x} C(y)$
$(0,1) \to (0,0)$	δ_0
$(1,1) \to (1,0)$	δ_1
$(0,0) \to (1,0)$	γp
$(0,1) \to (1,1)$	γp
$(1,0) \to (0,0)$	$\gamma(1-p)$
$(1,1) \to (0,1)$	$\gamma(1-p)$

where $d \geq 1, \gamma, \delta_0, \delta_1 > 0$ with $\delta_1 \leq \delta_0$ and $p \in [0,1]$. In other words, at each site x independently, $\{B_t(x)\}_{t\geq 0}$ is a 2-state Markov chain with infinitesimal matrix

$$\begin{pmatrix} -\gamma p & \gamma p \\ \gamma(1-p) & -\gamma(1-p) \end{pmatrix}$$

which in turn determines the recovery rate of $\{C_t(x)\}_{t\geq 0}$ in the following way. For each t, the recovery rate at location x is δ_0 or δ_1 depending on whether $B_t(x) = 0$ or $B_t(x) = 1$. In addition, the infection rate is always taken to be the number of infected neighbors. (Actually, Broman did this on a more general graph, but here we will only consider $\mathbb{Z}^d$.) Broman referred to $\{B_t\}_{t\geq 0}$ as *the background process* and the whole process $\{(B_t, C_t)\}_{t\geq 0}$ as *the contact process in a randomly evolving environment* (CPREE). Let $\{C_t^\rho\}_{t\geq 0}$ denote the right marginal where the initial distribution of the whole process is ρ. In the case where $\rho = \mu \times \nu$ we write $\{C_t^{\mu,\nu}\}_{t\geq 0}$. Furthermore, let $\mathbf{P}_p$ denote the measure governing the process for the parameters p, γ, δ_0 and δ_1, where γ, δ_0 and δ_1 are considered fixed. Also, denote the product measure with density $q \in [0,1]$ by π_q. Broman defined the critical value

$$p_c := \inf\left\{p : \mathbf{P}_p[\,C_t^{\pi_p,\{0\}} \neq \emptyset\,\forall t > 0\,] > 0\right\}$$

(p_c is taken to be 1 if no p satisfies this) and proved that if $\delta_1 < \delta_c < \delta_0$ and $\gamma > \max(2d, \delta_c - \delta_1)$, then $p_c \in (0,1)$. At the end of his paper he asked whether the critical value is affected if we vary the initial distribution of the background process. Our first result answers this question. Given $\gamma, \delta_0, \delta_1 > 0$ with $\delta_1 \leq \delta_0$, $q \in [0,1]$ and $A \subseteq \mathbb{Z}^d$ with $|A| < \infty$, define

$$p_c(q, A) := \inf\left\{p : \mathbf{P}_p[\, C_t^{\pi_q, A} \neq \emptyset\, \forall t > 0\,] > 0\right\}.$$

THEOREM 2 *Given $A, A' \subseteq \mathbb{Z}^d$ with $|A|$, $|A'| < \infty$ and p, q, $q' \in [0,1]$,*

$$(1) \quad \mathbf{P}_p[\, C_t^{\pi_q, A} \neq \emptyset\, \forall t > 0\,] > 0 \quad \Longleftrightarrow \quad \mathbf{P}_p[\, C_t^{\pi_{q'}, A'} \neq \emptyset\, \forall t > 0\,] > 0.$$

In particular, $p_c(q, A)$ is independent of both q and A.

We will let p_c denote this common value. (Recall, p_c of course depends on γ, δ_0 and δ_1.) Also, if $\mathbf{P}_p[\, C_t^{\pi_q, A} \neq \emptyset\, \forall t > 0\,] > 0$ holds (which we now know is independent of q and A), we say that $\{C_t\}$ *survives at* p; otherwise it is said to *die out at* p.

Later on, we will see that the process is attractive. (See Proposition 5.) This yields that the limiting distribution starting from all 1's exists and we will denote the limit by $\bar{\nu}_p$. Also, we will refer to this measure as the *upper invariant measure*. This measure gives us another natural way to define a critical value:

$$p_c' := \inf\{\, p : \bar{\nu}_p \neq \pi_p \times \delta_\emptyset \,\}.$$

For general attractive systems it might or might not be the case that these definitions coincide. However, for the ordinary contact process, this is the case (due to its self-duality) and our next result shows that this is also true in our situation.

THEOREM 3 *$\{C_t\}$ survives at p if and only if $\bar{\nu}_p \neq \pi_p \times \delta_\emptyset$. In particular $p_c = p_c'$.*

Our final result is a generalization of Theorem 1.

THEOREM 4 *If $\{C_t\}$ survives at $p > 0$, then there exists $\delta > 0$ so that it survives at $p - \delta$. In particular, if $p_c \in (0, 1]$, then the critical contact process in a randomly evolving environment dies out.*

The rest of the paper is organized as follows. In Section 2, we provide some preliminaries, in Section 3, we prove Theorems 2 and 3 and in Section 4, we prove Theorem 4.

2.2 Some preliminaries

In this section we will present the basic construction of the CPREE via a graphical representation that is suitable for our situation. We will also prove the elementary fact that the CPREE is an attractive process. However, we will start off with some notation and basic definitions. When the initial distribution of the process is ρ, we will denote the distribution at time t by $\rho S_p(t)$, supressing γ, δ_0 and δ_1 in the notation. (Of course, ρ is a probability measure on $\{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d}$.) When ρ is a product measure, $\rho = \mu \times \nu$, we will denote the process by $\{(B_t^{\mu}, C_t^{\mu,\nu})\}_{t\geq 0}$. In the case where, $\mu = \delta_\beta$ and $\nu = \delta_\eta$ for some $\beta, \eta \in \{0,1\}^{\mathbb{Z}^d}$ we write $\{(B_t^{\beta}, C_t^{\beta,\eta})\}_{t\geq 0}$. To simplify notation, we freely interchange between talking about elements in $\{0,1\}^{\mathbb{Z}^d}$ and subsets of $\mathbb{Z}^d$. For $\eta, \eta' \in \{0,1\}^{\mathbb{Z}^d}$ we write $\eta \leq \eta'$ if $\eta(x) \leq \eta'(x)$ $\forall x \in \mathbb{Z}^d$. Furthermore, for $(\beta,\eta), (\beta',\eta') \in \{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d}$ we write $(\beta,\eta) \leq (\beta',\eta')$ if both $\beta \leq \beta'$ and $\eta \leq \eta'$. These relations induce the concept of increasing function in the usual way.

DEFINITION 1 *We say that a function f on $\{0,1\}^{\mathbb{Z}^d}$ (or $\{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d}$) is increasing if $f(\eta) \leq f(\eta')$ ($f(\beta,\eta) \leq f(\beta',\eta')$) whenever $\eta \leq \eta'$ ($(\beta,\eta) \leq (\beta',\eta')$).*

In our analysis we make extensive use of the concept of *stochastic domination.*

DEFINITION 2 *For, μ_1, μ_2, probability measure on $\{0,1\}^{\mathbb{Z}^d}$ we say that μ_1 is stochastically dominated by μ_2 if $\mu_1(f) \leq \mu_2(f)$ $\forall$ increasing continuous functions f and we denote this by $\mu_1 \leq \mu_2$. If μ_i is the distribution of X_i, $i = 1,2$, we also write $X_1 \leq_D X_2$.*

It is well known (see for example [7]) that this is equivalent to the existence of random variables X_1, X_2 on a common probability space such that $X_1 \sim \mu_1$, $X_2 \sim \mu_2$ and $X_1 \leq X_2$ a.s. (The $\sim$ here means distributed according to.) Also, since we can identify $\{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d}$ with $\{0,1\}^{\mathbb{Z}^d \times \{0,1\}}$ we have a similar result for measures on $\{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d}$. (Of course, stochastic domination makes sense on any space of the form $\{0,1\}^S$ where S is countable.)

Now, we turn to the graphical representation from which our process will be defined. Let $\gamma, \delta_0, \delta_1 > 0$ with $\delta_1 \leq \delta_0$ and $p \in [0,1]$ be given parameters.

Let $\{e_j\}_{j=1}^d$ denote the standard basis on $\mathbb{Z}^d$, i.e, for $i, j \in \{1, \dots, d\}$

$$e_j(i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Define the following stochastic elements on a common probability space such that they are independent:

- $M^{b,0\to 1} = \{M_t^{b,0\to 1}\}_{t\geq 0}$, a process with state space $\mathbb{N}^{\mathbb{Z}^d}$ where each marginal independently evolves as a Poisson process with intensity γp. (This process will correspond to the 0 to 1 flips in the background process, see below.)

- $M^{b,1\to 0} = \{M_t^{b,1\to 0}\}_{t\geq 0}$, a process with state space $\mathbb{N}^{\mathbb{Z}^d}$ where each marginal independently evolves as a Poisson process with intensity $\gamma(1-p)$. (This process will correspond to the 1 to 0 flips in the background process, see below.)

- $N^{\delta_1} = \{N_t^{\delta_1}\}_{t\geq 0}$, a process with state space $\mathbb{N}^{\mathbb{Z}^d}$ where each marginal independently evolves as a Poisson process with intensity δ_1.

- $N^{\delta_0-\delta_1} = \{N_t^{\delta_0-\delta_1}\}_{t\geq 0}$, a process with state space $\mathbb{N}^{\mathbb{Z}^d}$ where each marginal independently evolves as a Poisson process with intensity $\delta_0 - \delta_1$.

- $\vec{N}^j = \{\vec{N}_t^j\}_{t\geq 0}$, $j \in \{\pm e_1, \dots, \pm e_d\}$, independent processes with state space $\mathbb{N}^{\mathbb{Z}^d}$ where each marginal independently evolves as a Poisson process with intensity 1. (We think of the points in $\vec{N}^j(x)$ as being arrows from x to $x + e_j$ and will correspond to the potential spread of infection from x to $x + e_j$.)

For $s \geq 0$ and $\beta \in \{0,1\}^{\mathbb{Z}^d}$, we will begin to define a process $B^{\beta,s} = \{B_t^{\beta,s}\}_{t\geq s}$ where for each $x \in \mathbb{Z}^d$, $B^{\beta,s}(x)$ is a function of the arrivals of $M^{b,0\to 1}(x)$ and $M^{b,1\to 0}(x)$ in $[s,\infty)$. Assume for example that $\beta(x) = 0$; the case when $\beta(x) = 1$ can be handled in a similar fashion. We then define

$$\begin{aligned}
B_t^{\beta,s}(x) &= 0, & s \leq t < T_1 \\
B_t^{\beta,s}(x) &= 1, & T_1 \leq t < T_2 \\
B_t^{\beta,s}(x) &= 0, & T_2 \leq t < T_3 \\
B_t^{\beta,s}(x) &= 1, & T_3 \leq t < T_4 \\
&\vdots
\end{aligned}$$

where T_1 is the first arrival time of $M^{b,0\to 1}(x)$ after s, T_2 is the first arrival time of $M^{b,1\to 0}(x)$ after T_1, T_3 is the first arrival time of $M^{b,0\to 1}(x)$ after T_2, T_4 is the first arrival time of $M^{b,1\to 0}(x)$ after T_3 and so forth. In words, the points in $M^{b,0\to 1}$ are the times at which the background process switches to 1 (had it been in state 0) and similarily for $M^{b,1\to 0}$. Note importantly, we have all the processes $B^{\beta,s}$, as β and s vary, defined on the same probability space.

Given $B^{\beta,s}$, N^{δ_1} and $N^{\delta_0-\delta_1}$, define $X^{\beta,s} = \{X_t^{\beta,s}\}_{t\geq s}$, a point process on $\mathbb{Z}^d \times [s,\infty)$, in the following way:

$$X^{\beta,s} = \{\, (x,t) \in \mathbb{Z}^d \times [s,\infty) : \; (x,t) \in N^{\delta_1} \text{ or } \\ (x,t) \in N^{\delta_0-\delta_1} \text{ and } B_t^{\beta,s}(x) = 0 \,\}$$

In words, for each site x, we choose points in $[s,\infty)$ from $N^{\delta_1}(x)$ when the background process is in state 1 and from the union of $N^{\delta_1}(x)$ and $N^{\delta_0-\delta_1}(x)$ when the background process is in state 0.

DEFINITION 3 *Given space-time points (x,s) and (y,t) with $t > s$ and $\beta \in \{0,1\}^{\mathbb{Z}^d}$, we say that there is a β-active path from (x,s) to (y,t) if there is a sequence of times $s = s_0 < s_1 < \ldots < s_m < s_{m+1} = t$ and space points $x = x_0, x_1, \ldots, x_m = y$ so that for $i = 1,\ldots,m$, there is an arrow from x_{i-1} to x_i at time s_i and there are no points in $X^{\beta,s}$ on the vertical segments $\{x_i\} \times (s_i, s_{i+1})$, $i = 0,\ldots,m$.*

Remark: Note importantly, that both $B^{\beta,s}$ and the existence of a β-active path from (x,s) to (y,t) are measurable with respect to the Poisson processes after time s and hence are independent of everything in the Poisson processes up to that time. The reason that these objects are introduced for $s > 0$ is that they are useful objects to which the original process can be usefully compared as will be done in the proof of Theorem 4.

To define the process $\{(B_t^{\beta}, C_t^{\beta,\eta})\}_{t\geq 0}$ for a given initial configuration $(\beta,\eta) \in \{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d}$, we let $B_t^{\beta} = B_t^{\beta,0}$ and

$$C_t^{\beta,\eta} = \{\, y \in \mathbb{Z}^d : \text{ for some } x \in \mathbb{Z}^d \text{ with } \eta(x) = 1, \\ \text{there is a } \beta\text{-active path from } (x,0) \text{ to } (y,t) \,\}.$$

This is our formal definition of the CPREE. Note as β and η vary, we have all the processes $\{(B_t^{\beta}, C_t^{\beta,\eta})\}_{t\geq 0}$ defined on the same probability space.

Having defined $\{(B_t, C_t)\}_{t\geq 0}$ with initial configuration (β, η), it is a simple matter to extend the definition to an arbitrary initial distribution ρ. Just add to our probability space, independently of all the random variables already defined, two random variables on $\{0,1\}^{\mathbb{Z}^d}$ with joint distribution ρ. We will denote the probability measure governing all these variables by $\mathbf{P}_p$, suppressing γ, δ_0 and δ_1 in the notation.

The first easy fact about the CPREE we will show is that it is an attractive process.

PROPOSITION 5 *(B_t, C_t) satisfies the attractivity condition:*

$$\rho \leq \sigma \quad \Longrightarrow \quad \rho S_p(t) \leq \sigma S_p(t) \quad \forall t > 0. \tag{2}$$

Proof. It is standard that (2) is equivalent to $(\delta_\beta \times \delta_\eta) S_p(t)$ being stochastically increasing in (β, η) for all $t \geq 0$. However, it is immediate from the construction that if $\beta_1 \leq \beta_2$ and $\eta_1 \leq \eta_2$, then for all $t \geq 0$

$$B_t^{\beta_1} \leq B_t^{\beta_2}$$

and

$$C_t^{\beta_1,\eta_1} \leq C_t^{\beta_2,\eta_2}.$$

This gives the stochastic domination (with an explicit coupling). □

2.3 Proofs of Theorems 2 and 3

Recall, given $\gamma, \delta_0, \delta_1 > 0$ with $\delta_1 \leq \delta_0$ and $q \in [0,1]$ we have defined

$$p_c(q, A) := \inf\left\{p : \mathbf{P}_p[\, C_t^{\pi_q, A} \neq \emptyset \, \forall t > 0\,] > 0\right\}$$

where $A \subseteq \mathbb{Z}^d$, $|A| < \infty$, and π_q denotes product measure with density q.

Proof of Theorem 2. We will prove the statements:

- For all $A \subseteq \mathbb{Z}^d$ with $|A| < \infty$ and $p, q \in [0,1]$,

$$\mathbf{P}_p[\, C_t^{\pi_q, A} \neq \emptyset \, \forall t > 0\,] > 0 \iff \mathbf{P}_p[\, C_t^{\pi_q, \{0\}} \neq \emptyset \, \forall t > 0\,] > 0. \tag{3}$$

- For all $p \in [0,1]$,

(4) $\mathbf{P}_p[\, C_t^{\emptyset,\{0\}} \neq \emptyset \, \forall t > 0\,] > 0 \iff \mathbf{P}_p[\, C_t^{\mathbb{Z}^d,\{0\}} \neq \emptyset \, \forall t > 0\,] > 0.$

Combining these two will yield the statement in Theorem 2. For (3), the left implication follows from translation invariance and the right implication follows easily from the additivity property of the process meaning

$$C_t^{\beta,A\cup B} = C_t^{\beta,A} \cup C_t^{\beta,B} \quad \forall A,B \subseteq \mathbb{Z}^d,\ \forall \beta \in \{0,1\}^{\mathbb{Z}^d}.$$

To prove (4), observe that the right implication is immediate from Proposition 5 and so we assume $\mathbf{P}_p[\, C_t^{\mathbb{Z}^d,\{0\}} \neq \emptyset \, \forall t > 0\,] > 0$. Define

$$\varphi_t(x) = 1_{\{B_t^{\emptyset}(x) = B_t^{\mathbb{Z}^d}(x)\}} \quad x \in \mathbb{Z}^d,\ t \geq 0.$$

(Recall this is well defined since $\{B_t^{\emptyset}\}_{t\geq 0}$ and $\{B_t^{\mathbb{Z}^d}\}_{t\geq 0}$ are defined on the same probability space.) Note that φ_t has the property that for each site independently, after an exponential distributed time with mean $\frac{1}{\gamma}$, the process flips to one and stays there. Therefore we have $\mathbf{P}_p[\, \varphi_t(x) = 1\,] = 1 - e^{-\gamma t}$. For $A \subseteq \mathbb{Z}^d$, define $\{\tilde{C}_t^A\}_{t\geq 0}$ from the graphical representation in the same way as $\{C_t^{\cdot,A}\}_{t\geq 0}$ except that all recoveries are ignored. Observe that this is the so called Richardson model.

LEMMA 6 $\mathbf{P}_p[\, \tilde{C}_t^{\{0\}} \subseteq \varphi_t\,, \forall t \geq n\,] \to 1$ *as* $n \to \infty$.

Proof. Let $I_n = \{-n^2, \ldots, n^2\}^d$ and for $x \in \mathbb{Z}^d$ define

$$t(x) = \inf\{\, t : \, x \in \tilde{C}_t^{\{0\}}\,\}.$$

From page 16 of [4], we get that there are constants $c_1, c_2, c_3 \in (0,\infty)$ such that

$$\mathbf{P}_p[\, t(x) < c_1 |x|_\infty\,] \leq c_2 e^{-c_3 |x|_\infty},$$

where $|\cdot|_\infty$ is the L^∞ norm. This easily gives us the estimate

$$\mathbf{P}_p[\, \tilde{C}_{c_1(n+1)}^{\{0\}} \not\subseteq I_n\,] \leq P(n) e^{-c_3 n},$$

where $P(n)$ is a polynomial in n, and from the Borel Cantelli lemma we can conclude

(5) $$\mathbf{P}_p[\, \exists N \geq 1 \text{ such that } \tilde{C}_{c_1(n+1)}^{\{0\}} \subseteq I_n\,, \forall n \geq N\,] = 1.$$

Furthermore, independence gives

$$\mathbf{P}_p[\, I_n \subseteq \varphi_{c_1 n}\,] = (1 - e^{-\gamma c_1 n})^{(2n^2+1)^d}.$$

and since

$$\sum_{n=1}^{\infty} 1 - (1 - e^{-\gamma c_1 n})^{(2n^2+1)^d} < \infty,$$

the Borel Cantelli lemma again yields

$$\mathbf{P}_p[\, \exists N \geq 1 \text{ such that } \mathrm{I_n} \subseteq \varphi_{\mathrm{c_1 n}}\,, \forall \mathrm{n} \geq \mathrm{N}\,] = 1. \tag{6}$$

Combining (5) and (6), we obtain

$$\mathbf{P}_p[\, \exists N \geq 1 \text{ such that } \tilde{\mathrm{C}}_{\mathrm{t}}^{\{0\}} \subseteq \varphi_{\mathrm{t}}\,, \forall \mathrm{t} \geq \mathrm{N}\,] = 1,$$

as desired. □

Since $C_t^{\mathbb{Z}^d,\{0\}} \subseteq \tilde{C}_t^{\{0\}}$ $\forall t \geq 0$, the claim tells us that, with probability one, after some time and thereafter, the two background processes influence $C_t^{\emptyset,\{0\}}$ and $C_t^{\mathbb{Z}^d,\{0\}}$ in exactly the same way. Next, countable additivity gives us that for some $n \geq 1$ we have

$$\mathbf{P}_p[\, \tilde{C}_t^{\{0\}} \subseteq \varphi_t \; \forall t \geq n\,,\; C_t^{\mathbb{Z}^d,\{0\}} \neq \emptyset \; \forall t > 0\,] > 0$$

and then that for some m (depending on n)

$$\mathbf{P}_p[\, \tilde{C}_t^{\{0\}} \subseteq \varphi_t \forall t \geq n\,,\; \tilde{C}_t^{\{0\}} \subseteq [-m,m]^d \forall t \in [0,n],\; C_t^{\mathbb{Z}^d,\{0\}} \neq \emptyset \forall t > 0\,] > 0.$$

Denote the previous event by A and define the random set

$$U = \{\, (x,t) \in [-m,m]^d \times [0,n] :\; B_t^{\mathbb{Z}^d}(x) = 1\,\}$$

and let

$$B = \{\, \text{no arrivals in } N^{\delta_0 - \delta_1} \text{ during } U\,\}.$$

It is clear that

$$A \cap B \subseteq \{\, C_t^{\emptyset,\{0\}} \neq \emptyset \; \forall t > 0\,\}$$

and so it remains to show that

$$\mathbf{P}_p[\, A \cap B\,] > 0.$$

However, if we condition on A and U, then we will not yield any information about the $N^{\delta_0-\delta_1}$ process on $[-m,m]^d \times [0,n]$ and so

$$\mathbf{P}_p[\,B\,|A,U\,] = e^{-(\delta_0-\delta_1)\mathcal{L}(U)}$$

where $\mathcal{L}(U)$ is the "length" of U. This easily gives

$$\mathbf{P}_p[\,B|\,A\,] > 0$$

and the proof is complete. □

Remark: The same argument shows that strong survival does not depend on the initial distribution of the background process in the sense that

$$\mathbf{P}_p[\,0 \in C_t^{\emptyset,\{0\}} \text{ i.o.}\,] > 0 \quad \Longleftrightarrow \quad \mathbf{P}_p[\,0 \in C_t^{\mathbb{Z}^d,\{0\}} \text{ i.o.}\,] > 0.$$

This answers another question in [3].

Recall the definition of p_c' from the introduction:

$$p_c' := \inf\{\,p : \,\bar{\nu}_p \neq \pi_p \times \delta_\emptyset\,\}.$$

Here $\bar{\nu}_p = \lim_{t\to\infty}(\delta_{\mathbb{Z}^d} \times \delta_{\mathbb{Z}^d})S_p(t)$. (The limit exists due to Proposition 5.) To prove Theorem 3 we will use the next Lemma.

LEMMA 7 *Given* $p, q \in (0,1)$ *with* $q \geq p$ *we have*

$$\lim_{t\to\infty}(\pi_q \times \delta_{\mathbb{Z}^d})S_p(t) = \bar{\nu}_p.$$

Proof. By simple stochastic comparison, it is enough to consider the case when $q = p$. We begin to establish the existence of that limit. Since π_p is the stationary distribution for the background process and the right marginal always occupies less than or equal to the whole $\{0,1\}^{\mathbb{Z}^d}$, we have

$$(\pi_p \times \delta_{\mathbb{Z}^d})S_p(t) \leq \pi_p \times \delta_{\mathbb{Z}^d} \quad \forall t > 0.$$

Using attractiveness and the Markov property yields

$$(\pi_p \times \delta_{\mathbb{Z}^d})S_p(s+t) \leq (\pi_p \times \delta_{\mathbb{Z}^d})S_p(s) \quad \forall s,t > 0,$$

and so the existence of the limit is clear from monotonicity. Denote this limit by ν'_p and observe it is necessarily stationary. It is clear that $\nu'_p \leq \bar{\nu}_p$ so we are done if $\bar{\nu}_p \leq \nu'_p$. For this, note that attractiveness again gives that the map

$$\mu \mapsto E^{\mu}[f(\delta_t, \eta_t)]$$

is increasing whenever f is continuous and increasing. Using this, and the fact that any stationary distribution necessarily has as first marginal π_p, we can do the following calculation for any stationary distribution μ of (B_t, C_t) and $f : \{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R}$ continuous and increasing:

$$\int f d\mu = E^{\mu}[f(\delta_t, \eta_t)] \leq E^{\pi_p \times \delta_{\mathbb{Z}^d}}[f(\delta_t, \eta_t)] \to \int f d\nu'_p \quad \text{as } t \to \infty.$$

Hence, $\mu \leq \nu'_p$ and we are done. □

Proof of Theorem 3. When the initial distribution of the background process is π_p, it is easy to see from the graphical representation that C_t is self-dual in the sense that

$$\text{(7)} \qquad \mathbf{P}_p[\, C_t^{\pi_p, A} \cap B \neq \emptyset \,] = \mathbf{P}_p[\, C_t^{\pi_p, B} \cap A \neq \emptyset \,] \quad \forall t > 0,\ A, B \subseteq \mathbb{Z}^d.$$

If we take $A = \{0\}$, $B = \mathbb{Z}^d$ in this equation and let $t \to \infty$ using the previous lemma, we can easily conclude that

$$\mathbf{P}_p[\, C_t^{\pi_p, \{0\}} \neq \emptyset \, \forall t > 0 \,] > 0 \qquad \Longleftrightarrow \qquad \bar{\nu}_p \neq \pi_p \times \delta_{\emptyset}$$

and we are done. □

Remark: There is a weaker duality equation when the initial distribution of the background process differs from π_p, but this is less natural and seems less useful.

2.4 Proof of Theorem 4

We now turn to the proof of Theorem 4, that the critical CPREE dies out. Once Lemma 8 below is established, the rest follows similar lines as in the proofs of Theorem 1 carried out in [1] and [9]. Our main goal is to prove that if $\{C_t\}$ survives at $p > 0$, then there is a number $\delta > 0$ and integers n, a such that

$$\text{(8)} \qquad \mathbf{P}_{p-\delta}[\, C_t^{\emptyset, [-n,n]^d} \text{ survives in } \mathbb{Z} \times [-5a, 5a]^{d-1} \times [0, \infty) \,] > 0.$$

If $p_c \in (0,1]$, this will immediately imply

$$\mathbf{P}_{p_c}[\,C_t^{\emptyset,\{0\}} \neq \emptyset\,\forall t \geq 0\,] = 0.$$

To achieve (8), we begin by showing that if the CPREE survives, then it is very likely to have survival if the initial configuration is sufficiently large even if we start with all zeros in the background process.

LEMMA 8 *If $\{C_t\}$ survives at $p > 0$ then*

$$\lim_{n\to\infty} \mathbf{P}_p[\,C_t^{\emptyset,[-n,n]^d} \neq \emptyset\,\forall t > 0\,] = 1.$$

For the proof of this we use the following result.

LEMMA 9 *For all $n \geq 1$, we have*

$$\lim_{\epsilon\to 0} \mathbf{P}_p[\,C_t^{\pi_{p-\epsilon},[-n,n]^d} \neq \emptyset\,\forall t > 0\,] = \mathbf{P}_p[\,C_t^{\pi_p,[-n,n]^d} \neq \emptyset\,\forall t > 0\,].$$

Proof. Fix $n \geq 1$. The probability on the left increases when ϵ decreases and so the limit exists and is clearly at most the right hand side. For the other inequality let $\delta > 0$ and define

$$\varphi_t^\epsilon(x) = 1_{\{B_t^{\pi_{p-\epsilon}}(x) = B_t^{\pi_p}(x)\}} \quad x \in \mathbb{Z}^d,\ t \geq 0,$$

where $\pi_{p-\epsilon}$ and π_p are coupled in the usual monotone way. Recall the definition of φ_t from the proof of Theorem 2 and observe that

$$\varphi_t \subseteq \varphi_t^\epsilon \quad \forall t > 0,\ \forall \epsilon > 0.$$

Also, an easy modifictaion of the proof of Lemma 6 yields

$$\lim_{T\to\infty} \mathbf{P}_p[\,\tilde{C}_t^{[-n,n]^d} \subseteq \varphi_t\,, \forall t \geq T\,] = 1.$$

(Recall that $\tilde{C}_t^A$ is the CPREE starting from the configuration A but with no recoveries.) This allows us to choose $T > 0$ such that

$$\mathbf{P}_p[\,C_t^{\pi_p,[-n,n]^d} \neq \emptyset\,\forall t > 0\,] \leq$$
$$\leq \mathbf{P}_p[\,\tilde{C}_t^{[-n,n]^d} \subseteq \varphi_t\,, \forall t \geq T,\ C_t^{\pi_p,[-n,n]^d} \neq \emptyset\,\forall t > 0\,] + \delta.$$

Given this T, choose $m \geq 1$ such that

$$\mathbf{P}_p[\, \tilde{C}_t^{[-n,n]^d} \subseteq [-m,m]^d \,\forall\, 0 \leq t \leq T\,] > 1-\delta$$

and for that m choose $\epsilon_0 > 0$ such that

$$\mathbf{P}_p[\, B_0^{\pi_{p-\epsilon}} = B_0^{\pi_p} \text{ on } [-m,m]^d\,] > 1-\delta, \quad \forall\, 0 < \epsilon \leq \epsilon_0.$$

Now since

$$\begin{aligned}&\{\, \tilde{C}_t^{[-n,n]^d} \subseteq \varphi_t\,, \forall t \geq T,\, \tilde{C}_t^{[-n,n]^d} \subseteq [-m,m]^d \,\forall\, 0 \leq t \leq T,\\ &B_0^{\pi_{p-\epsilon}} = B_0^{\pi_p} \text{ on } [-m,m]^d,\, C_t^{\pi_p,[-n,n]^d} \neq \emptyset \,\forall t > 0\,\} \subseteq\\ &\subseteq \{\, C_t^{\pi_{p-\epsilon},[-n,n]^d} \neq \emptyset \,\forall t > 0\,\},\end{aligned}$$

we get

$$\mathbf{P}_p[\, C_t^{\pi_p,[-n,n]^d} \neq \emptyset \,\forall t > 0\,] \leq \mathbf{P}_p[\, C_t^{\pi_{p-\epsilon},[-n,n]^d} \neq \emptyset \,\forall t > 0\,] + 3\delta,$$

whenever $0 < \epsilon \leq \epsilon_0$ and so the proof is complete. □

Proof of Lemma 8. Let $\delta > 0$. From the self-duality equation (7), Lemma 7 and the easily verified fact that the second marginal of $\bar{\nu}_p$ gives zero measure to $\emptyset$, we easily get that there is an $n \geq 1$ such that

$$\mathbf{P}_p[\, C_t^{\pi_p,[-n,n]^d} \neq \emptyset \,\forall t > 0\,] > 1-\delta.$$

The previous lemma makes it possible to now choose an $\epsilon > 0$ such that

$$\mathbf{P}_p[\, C_t^{\pi_{p-\epsilon},[-n,n]^d} \neq \emptyset \,\forall t > 0\,] > 1-\delta.$$

Denote the semigroup operator associated with the background process by $T(t)$ and note that for ϵ above there is a time s such that

$$\delta_\emptyset T(s) \geq \pi_{p-\epsilon}.$$

Now, let $B_{m,n}$ denote the box in $\mathbb{Z}^d$ with sidelength mn and write

$$B_{m,n} = \bigcup_{i=1}^{m^d} A_i,$$

where each A_i is a translation of the box with sidelength n and with the A_i's disjoint. Then, define

$$A^s_{m,n} = \{\text{ No arrivals in } N^{\delta_1} \text{ or } N^{\delta_0-\delta_1} \text{ up to time } s \text{ in some } A_i\,\}.$$

Given n and s, we can choose m so large that

$$\mathbf{P}_p[\,A^s_{m,n}\,] > 1-\delta.$$

The proof is finished by noting that monotonicity easily implies that

$$\mathbf{P}_p[\,C_t^{\emptyset,[-mn,mn]^d} \neq \emptyset\,\forall t>0\,|\,A^s_{m,n}\,] \geq \mathbf{P}_p[\,C_t^{\pi_p-\epsilon,[-n,n]^d} \neq \emptyset\,\forall t>0\,],$$

using the fact that $A^s_{m,n}$ is independent of the background process. □

We have now set up the necessary ground work for our model in order to be able to follow the steps in [9]. For $L \geq 1$ and $A \subseteq (-L,L)^d$, let ${}_LC_t^{\emptyset,A}$ be the truncated process, using only $\emptyset$-active paths (recall Definition 3) which stay in $(-L,L)^d \times [0,t]$.

LEMMA 10 *For all finite $A \subseteq \mathbb{Z}^d$ and $N \geq 1$, we have*

$$\lim_{t\to\infty}\lim_{L\to\infty}\mathbf{P}_p[\,|{}_LC_t^{\emptyset,A}| \geq N\,] = \mathbf{P}_p[\,C_t^{\emptyset,A} \neq \emptyset\,\forall t>0\,]$$

Proof. Fix A and N. Since

$$C_t^{\emptyset,A} = \bigcup_{L=1}^{\infty} {}_LC_t^{\emptyset,A},$$

we easily get that for fixed t

$$\mathbf{P}_p[\,|C_t^{\emptyset,A}| \geq N\,] = \lim_{L\to\infty}\mathbf{P}_p[\,|{}_LC_t^{\emptyset,A}| \geq N\,],$$

and so we are done if

$$\lim_{t\to\infty}\mathbf{P}_p[\,|C_t^{\emptyset,A}| \geq N\,] = \mathbf{P}_p[\,C_t^{\emptyset,A} \neq \emptyset\,\forall t>0\,].$$

For this, it is enough to check two things:

$$\lim_{t\to\infty}\mathbf{P}_p[\,|C_t^{\emptyset,A}| \geq N, C_s^{\emptyset,A} = \emptyset \text{ some } s>0\,] = 0$$
$$\lim_{t\to\infty}\mathbf{P}_p[\,|C_t^{\emptyset,A}| \geq N, C_s^{\emptyset,A} \neq \emptyset\,\forall s>0\,] = \mathbf{P}_p[\,C_t^{\emptyset,A} \neq \emptyset\,\forall t>0\,]$$

The first equality follows easily by applying Fatou's Lemma. The second one follows if

$$\lim_{t\to\infty} |C_t^{\emptyset,A}| = \infty \quad \text{a.s} \quad \text{on} \quad \{\, C_t^{\emptyset,A} \neq \emptyset \, \forall t > 0 \,\}.$$

Assuming the contrary, we get from the martingale convergence theorem and standard arguments that with positive probability the following can happen:

$$\lim_{s\to\infty} \mathbf{P}^{(\beta_s, C_s)}[\, C_t \neq \emptyset \, \forall t > 0 \,] = 1$$

$$\exists \, M > 0, \ \{\tau_i\}_{i\geq 1} \ \ni \ \tau_1 < \tau_2 < \ldots < \tau_i \to \infty \, , |C_{\tau_i}| \leq M \, \forall i.$$

However, using elementary facts about exponentially distributed variables, we get

$$\mathbf{P}^{(\beta_{\tau_i}, C_{\tau_i})}[\, C_t = \emptyset \text{ some } t > 0 \,] \geq \mathbf{P}^{(\mathbb{Z}^d, C_{\tau_i})}[\, C_t = \emptyset \text{ some } t > 0 \,]$$
$$\geq \left(\frac{\delta_1}{\delta_0 + \gamma + 2d} \right)^M \qquad \forall i,$$

which yields a contradiction and the proof is complete. □

The next step is to take care of the sides of the space-time box. Define

$$S(L,T) = \{\, (x,t) \in \mathbb{Z}^d \times [0,T] : \, |x|_\infty = L \,\}.$$

Fix $A \subseteq (-L,L)^d$ and look at all points on $S(L,T)$ that can be reached from A by an $\emptyset$-active path using vertical segments where the space coordinate is in $(-L,L)^d$ and infection arrows from $(x,\cdot)$ to $(y,\cdot)$ with $x \in (-L,L)^d$. Define $N_\emptyset^A(L,T)$ to be the maximum number of such points with the following property: If (x,t_1) and (x,t_2) are any two points with the same spatial coordinate, then $|t_1 - t_2| \geq 1$.

LEMMA 11 *Assume $L_j \nearrow \infty$ and $T_j \nearrow \infty$. Then for any $M, N \geq 1$ and finite $A \subseteq \mathbb{Z}^d$, we have*

$$\limsup_{j\to\infty} \mathbf{P}_p[\, N_\emptyset^A(L_j,T_j) \leq M \,]\, \mathbf{P}_p[\, |_{L_j} C_{T_j}^{\emptyset,A}| \leq N \,] \leq$$
$$\leq \mathbf{P}_p[\, C_t^{\emptyset,A} = \emptyset \textit{ some } t > 0 \,].$$

Proof. The proof follows the steps of Proposition 2.8 in [9] with some adjustments. Let $\mathcal{F}_{L,T}$ denote the σ-algebra generated by $M^{b,0\to1}$, $M^{b,1\to0}$, N^{δ_1}, $N^{\delta_0-\delta_1}$ and $\vec{N}^j$, $j \in \{\pm e_1, \dots, \pm e_d\}$ in $(-L,L)^d \times [0,T]$. We first argue that

$$\mathbf{P}_p[\, C_t^{\emptyset,A} = \emptyset \text{ some } t > 0 \,|\, \mathcal{F}_{L,T}\,] \ge \left(\frac{e^{-4d}\delta_1}{\delta_0+\gamma+2d}\right)^k \tag{9}$$
$$\text{a.s} \quad \text{on} \quad \{\, N_\emptyset^A(L,T) + |{}_L C_T^{\emptyset,A}| \le k \,\}$$

For $x \in {}_L C_T^{\emptyset,A}$ there is a conditional probability of at least

$$\frac{\delta_1}{\delta_0+\gamma+2d}$$

that x becomes healthy before it infects any of its neighbors. So, if $|{}_L C_T^{\emptyset,A}| = m$, then the conditional probability that no $x \in {}_L C_T^{\emptyset,A}$ contributes to survival is at least

$$\left(\frac{\delta_1}{\delta_0+\gamma+2d}\right)^m.$$

For the sides of the box, consider a time line $\{x\} \times [0,T]$, where $|x| = L$ and let

$$(x,t_1), \dots, (x,t_j)$$

be a maximal set of points that can be reached from A by an $\emptyset$-active path with the property that each pair is separated by at least distance 1. Let

$$I = \bigcup_{k=1}^{j} \{x\} \times (t_k - 1, t_k + 1)$$

and note that the probability that there are no arrows coming out from I is at least e^{-4dj}. Furthermore, for each interval of length y in the complement of I in $\{x\} \times [0,\infty)$, the probability of the event that if there is at least one arrival of the Poisson processes in the interval with the first one coming from N^{δ_1} or there is no arrivals at all is

$$\left(1 - e^{-(\delta_0+\gamma+2d)y}\right)\frac{\delta_1}{\delta_0+\gamma+2d} + e^{-(\delta_0+\gamma+2d)y} \ge \frac{\delta_1}{\delta_0+\gamma+2d}.$$

By independence, we get that the conditional probability that none of the points in the time line $\{x\} \times [0,T]$ contributes to survival is at least

$$\left(\frac{e^{-4d}\delta_1}{\delta_0+\gamma+2d}\right)^j.$$

Now, considering the contribution of different x's yields

$$\mathbf{P}_p[\, C_t^{\emptyset,A} = \emptyset \text{ some } t > 0 \,|\, \mathcal{F}_{L,T}\,] \geq$$
$$\geq \left(\frac{\delta_1}{\delta_0+\gamma+2d}\right)^{|_L C_T^{\emptyset,A}|} \left(\frac{e^{-4d}\delta_1}{\delta_0+\gamma+2d}\right)^{N^A(L,T)}$$

which implies (9). For the rest of the proof, one proceeds exactly as in the second half of Proposition 2.8 in [9], page 48-49. The needed inequality

$$\mathbf{P}_p[\, N_\emptyset^A(L,T) \leq M, |_L C_T^{\emptyset,A}| \leq N\,] \geq$$
$$\geq \mathbf{P}_p[\, N_\emptyset^A(L,T) \leq M\,]\mathbf{P}_p[\, |_L C_T^{\emptyset,A}| \leq N\,]$$

is justified by the fact that $N_\emptyset^A(L,T)$ and $|_L C_T^{\emptyset,A}|$ are increasing functions of $\vec{N}^j$, $j \in \{\pm e_1, \ldots, \pm e_d\}$ and $M^{b,0\to 1}$, and decreasing in N^{δ_1}, $N^{\delta_0-\delta_1}$ and $M^{b,1\to 0}$. This completes the proof. □

We are soon ready to state and prove the so called finite space-time condition. However, we first need two more propositions. We just state them here since the proofs are exactly the same as for Propositions 2.6 and 2.11, page 46-47 and 49 in [9]. The positive correlations needed are justified in the same way as in the end of the proof of the previous lemma.

PROPOSITION 12 *For every $n, N \geq 1$ and $L \geq n$, we have*

$$\mathbf{P}_p[\, |_L C_t^{\emptyset,[-n,n]^d} \cap [0,L)^d| \leq N\,] \leq \left(\mathbf{P}_p[\, |_L C_t^{\emptyset,[-n,n]^d}| \leq 2^d N\,]\right)^{2^{-d}}$$

Let

$$S_+(L,T) = \{\, (x,t) \in \mathbb{Z}^d \times [0,T] : \, x_1 = L\,, x_i \geq 0,\, 2 \leq i \leq d\}$$

and define $N_{\emptyset,+}^A(L,T)$ in a similar manner as $N_\emptyset^A(L,T)$ using $S_+(L,T)$ instead of $S(L,T)$.

PROPOSITION 13 *For any $L, M \geq 1$, $T > 0$ and $n < L$,*

$$\left(\mathbf{P}_p[\, N_{\emptyset,+}^{[-n,n]^d}(L,T) \leq M\,]\right)^{d2^d} \leq \mathbf{P}_p[\, N_\emptyset^{[-n,n]^d}(L,T) \leq Md2^d\,]$$

THEOREM 14 *If $\{C_t\}$ survives at $p > 0$, then it satisfies the following condition: For all $\epsilon > 0$ there exists $n, L \geq 1$ and $T > 0$ such that*

$$\mathbf{P}_p[\,{}_{L+n}C_{T+1}^{\emptyset,[-n,n]^d} \supseteq x + [-n,n]^d \text{ some } x \in [0,L)^d\,] > 1-\epsilon \tag{10}$$

$$\mathbf{P}_p[\,{}_{L+2n+1}C_{t+1}^{\emptyset,[-n,n]^d} \supseteq x + [-n,n]^d \text{ some } 0 \leq t < T, \text{ some } x \in \{L+n\} \times [0,L)^{d-1}\,] > 1-\epsilon \tag{11}$$

Proof. Again, we will follow the steps in [9] with some modifications. Let $0 < \delta < 1$. We will see at the end how to choose δ for a given $\epsilon > 0$. Lemma 8 gives us an n such that

$$\mathbf{P}_p[\,C_t^{\emptyset,[-n,n]^d} \neq \emptyset \,\forall t > 0\,] > 1-\delta^2. \tag{12}$$

Given n, choose N' such that

$$\left(1 - \mathbf{P}_p[\,{}_{n+1}C_1^{\emptyset,\{0\}} \supseteq [-n,n]^d\,]\right)^{N'} < \delta$$

and then choose N so large such that if $A \subseteq \mathbb{Z}^d$ with $|A| \geq N$, then there exists $B \subseteq A$ with $|B| \geq N'$ and

$$|x-y|_\infty \geq 2n+1 \quad \forall x,y \in B,\ x \neq y.$$

Let B_A be a fixed (deterministic) such choice for each A.

In a similar fashion, choose M' such that

$$(1-a)^{M'} < \delta, \tag{13}$$

where

$$a = \mathbf{P}_p[\text{There are } \emptyset\text{-active paths from the origin to every point in } [0,2n] \times [-n,n]^{d-1} \times \{1\} \text{ that stays in } [0,2n] \times [-n,n]^{d-1} \times [0,1]\,]$$

Then choose M so large such that if $A \subseteq \mathbb{Z}^d \times [0,\infty)$ is a finite set with $|A| \geq M$, where the distance in time between points with the same spatial coordinate is at least 1, then there exists $B \subseteq A$ with $|B| \geq M'$ and with the property that for each pair of points $(x,s),(y,t) \in B$ we have either

$$x = y, \quad |s-t| \geq 1 \qquad \text{or} \qquad |x-y|_\infty \geq 2n+1. \tag{14}$$

Let B_A be a fixed (deterministic) such choice for each A.

From Lemma 10, (12), the inequality $1-\delta<1-\delta^2$ and the facts that for fixed L, n and N, the map $t\mapsto \mathbf{P}_p[\,|_LC_t^{\emptyset,[-n,n]^d}|>2^dN\,]$ is continuous and that $\lim_{t\to\infty}\mathbf{P}_p[\,|_LC_t^{\emptyset,[-n,n]^d}|>2^dN\,]=0$, we can conclude that there exists $L_j\nearrow\infty$ and $T_j\nearrow\infty$ so that

$$\mathbf{P}_p[\,|_{L_j}C_{T_j}^{\emptyset,[-n,n]^d}|>2^dN\,]=1-\delta\quad\forall j\geq 1.$$

Furthermore, Lemma 11 with M and N replaced by $Md2^d$ and 2^dN respectively and with $A=[-n,n]^d$, we get that for some j

$$\mathbf{P}_p[\,N_\emptyset^{[-n,n]^d}(L_j,T_j)>Md2^d\,]>1-\delta.$$

Let $L=L_j$ and $T=T_j$ for that specific j and apply Propositions 12 and 13 to get

$$\mathbf{P}_p[\,|_LC_T^{\emptyset,[-n,n]^d}\cap[0,L)^d|>N\,]>1-\delta^{2^{-d}} \tag{15}$$

$$\mathbf{P}_p[\,N_{\emptyset,+}^{[-n,n]^d}(L,T)>M\,]>1-\delta^{2^{-d}/d}. \tag{16}$$

To obtain (10), define for $B\subseteq\mathbb{Z}^d$ and $T>0$

$$\begin{aligned}V_B^T=\{\,\exists\,(x,t)\in B\times\{T\}&\text{ such that there are }\emptyset\text{-active paths from}\\ &(x,t)\text{ to every }(y,s)\in\Big(x+[-n,n]^d\Big)\times\{T+1\}\\ &\text{that stays in }\Big(x+[-n,n]^d\Big)\times(T,T+1]\,\},\end{aligned}$$

and note that

$$\begin{aligned}&\bigcup_{A\subseteq[0,L)^d}\{\,|_LC_T^{\emptyset,[-n,n]^d}\cap[0,L)^d|>N,\ {}_LC_T^{\emptyset,[-n,n]^d}\cap[0,L)^d=A,\ V_{B_A}^T\,\}\subseteq\\ &\subseteq\{\,{}_{L+n}C_{T+1}^{\emptyset,[-n,n]^d}\supseteq\ x+[-n,n]^d\text{ some }x\in[0,L)^d\,\}.\end{aligned} \tag{17}$$

Let $\mathcal{F}_T$ be the σ-algebra generated by $M^{b,0\to1}$, $M^{b,1\to0}$, N^{δ_1}, $N^{\delta_0-\delta_1}$, and $\vec{N}^j$, $j\in\{\,\pm e_1,\ldots,\pm e_d\,\}$ up to time T and note that for given $A\subseteq[0,L)^d$ with $|A|\geq N$, $V_{B_A}^T$ is independent of $\mathcal{F}_T$ so

$$\begin{aligned}\mathbf{P}_p[\,V_{B_A}^T\,|\,\mathcal{F}_T\,]&=\mathbf{P}_p[\,V_{B_A}^T\,]\geq\\ &\geq 1-\Big(1-\mathbf{P}_p[\,{}_{n+1}C_1^{\emptyset,\{0\}}\supseteq[-n,n]^d]\Big)^{N'}>1-\delta.\end{aligned}$$

By summing up over $A \subseteq [0,L)^d$ and using (15) and (17), we get

$$\mathbf{P}_p[\,{}_{L+n}C_{T+1}^{\emptyset,[-n,n]^d} \supseteq x+[-n,n]^d \text{ some } x \in [0,L)^d\,] > (1-\delta)(1-\delta^{2^{-d}}).$$

This yields (10) when δ is chosen appropriately.

To obtain (11), define for each space-time point (x_i,t_i) we count in the variable $N_{\emptyset,+}^{[-n,n]^d}(L,T)$ a variable $\tilde{Y}_i$ which is 1 if (x_i,t_i) infects all points in

$$\left(x_i+[0,2n]\times[-n,n]^{d-1}\right)\times\{t_i+1\}$$

using $\emptyset$-active paths in

$$\left(x_i+[0,2n]\times[-n,n]^{d-1}\right)\times(t_i,t_i+1]$$

only and 0 otherwise. If $N_{\emptyset,+}^{[-n,n]^d}(L,T) > M$, we can choose M' space-time points satisfying (14). Denote the corresponding variables by Y_i, $i = 1,\ldots,M'$. Let $\mathcal{F}_{L,T}$ be as in the proof of Lemma 11 and note that conditioned on $\mathcal{F}_{L,T}$ restricted to the event $\{\,N_{\emptyset,+}^{[-n,n]^d}(L,T) > M\,\}$, the M' space-time points are specified and $Y_1,Y_2,\ldots,Y_{M'}$ are independent with the (conditional) probability of $Y_i=1$ equal to a. This implies that

$$\begin{aligned}\mathbf{P}_p[\,Y_i=1 \text{ some } i=1,\ldots,M' \mid \mathcal{F}_{L,T}\,] &= 1-(1-a)^{M'}\\ \text{on}\quad \{\,N_{\emptyset,+}^{[-n,n]^d}(L,T) > M\,\},&\end{aligned}$$

which together with (13) and (16) yields

$$\begin{aligned}\mathbf{P}_p[\,{}_{L+2n+1}C_{t+1}^{\emptyset,[-n,n]^d} &\supseteq x+[-n,n]^d \text{ some } 0\le t<T,\\ &\text{some } x\in\{L+n\}\times[0,L)^{d-1}\,] > (1-\delta)(1-\delta^{2^{-d}/d}).\end{aligned}$$

This gives (11) when δ is chosen appropriately. □

The next part of the program is to carry out a comparison with oriented percolation. For this, we start to combine (10) and (11) into one.

LEMMA 15 *If $\{C_t\}$ survives at $p>0$, then it satisfies the following condition: For all $\epsilon>0$ there exists $n, L \ge 1$ and $T>0$ such that*

$$\begin{aligned}&\mathbf{P}_p[\,{}_{2L+3n}C_t^{\emptyset,[-n,n]^d} \supseteq x+[-n,n]^d \text{ some } T\le t<2T,\\ (18)\qquad &\qquad\text{some } x\in[L+n,2L+n]\times[0,2L)^{d-1}\,] > 1-\epsilon\end{aligned}$$

Proof. We follow Proposition 2.20 in [9]. Let (x,τ) be the first (in time) space-time point with the property appearing in the probability (11), where x is choosen according to some deterministic ordering of $\mathbb{Z}^d$ and restart (B_t, C_t) at time $\tau + 1$. From (10), (11) and the fact that these probabilities are increasing in the background process, it follows that

$$\mathbf{P}_p[\,{}_{2L+3n}C_t^{\emptyset,[-n,n]^d} \supseteq x + [-n,n]^d \text{ some } T+1 \le t < 2T+2,$$
$$\text{some } x \in [L+n, 2L+n] \times [0,2L)^{d-1}\,] > (1-\epsilon)^2.$$

Replace $T+1$ with T and the proof is complete. □

Now we are ready for the fundamental step in the construction towards the comparison.

LEMMA 16 *Assume $\{C_t\}$ survives at $p > 0$ and $\epsilon > 0$. Then there exists $\delta > 0$, n, a, b with $n < a$ such that for all $(x,t) \in [-a,a]^d \times [0,b]$*

$$\mathbf{P}_{p-\delta}[\,\exists\,(y,s) \in [a,3a] \times [-a,a]^{d-1} \times [5b,6b] \textit{ such that there are}$$
$$\emptyset\textit{-active paths from } (x,t) + \Big([-n,n]^d \times \{0\}\Big)$$
$$\textit{to every point in } (y,s) + \Big([-n,n]^d \times \{0\}\Big)$$
$$\textit{that stays in } [-5a,5a]^d \times [0,6b]\,] > 1-\epsilon.$$

Proof. One can proceed exactly as in Proposition 2.22, page 52-53 in [9] to first obtain the statement with $p-\delta$ replaced by p and therefore we only outline this part of the argument. The main idea is to use Lemma 15 (or a "reflected" version of it) repeatedly (between 4 to 10 times) to steer things properly so that the desired event occurs. The existence of $\delta > 0$ is a consequence of the fact that the event in question depends only on the graphical representation in $[-5a,5a]^d \times [0,6b]$ and hence is continuous in p. □

Repeated use of the previous lemma together with appropriate stopping times and monotonicity in the background process yields:

LEMMA 17 *Assume $\{C_t\}$ survives at $p > 0$, $\epsilon > 0$ and $k \geq 1$. Then there exists $\delta > 0$, n, a, b with $n < a$ such that the following holds: For all $(x,t) \in [-a,a]^d \times [0,b]$, with $\mathbf{P}_{p-\delta}$-probability at least $1-\epsilon$, there exists a translate $(y,s) + [-n,n]^d \times \{0\}$ of $[-n,n]^d \times \{0\}$ such that*

a) $(y,s) \in ([-a,a] + 2ka) \times [-a,a]^{d-1} \times [5kb,(5k+1)b]$

b) There are $\emptyset$-active paths from $(x,t) + [-n,n]^d \times \{0\}$ to every point in $(y,s) + [-n,n]^d \times \{0\}$ that stays in the region

$$\mathcal{A} = \bigcup_{j=0}^{k-1} ([-5a,5a] + 2ja) \times [-5a,5a]^{d-1} \times ([0,6b] + 5jb)\,.$$

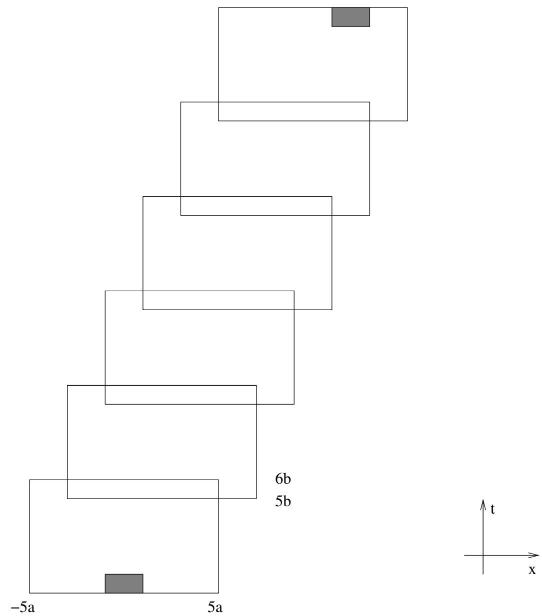

Figure 2.1: *The set $\mathcal{A}$.*

Our final step towards (8) is to use the previous lemma in a so called renormalization argument. The set $\mathcal{A}$ from Lemma 17 (see Figure 2.1) and its reflection with respect to the t-axis will consist of our building blocks. Given the conditions in Lemma 17, the distance c in Figure 2.2 is well defined. (Define it to be zero if the dashed vertical line is to the right of the left corner of the rectangle

R, see Figure 2.2.) It is easy to see that, if we choose $k > 5$, c will be bigger than $3a$, independent of the value of a. Fix such a k.

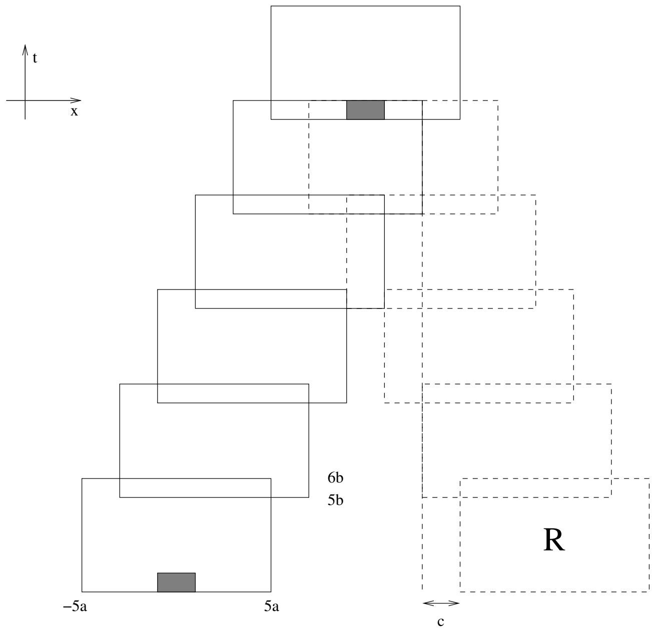

Figure 2.2: *The definition of c.*

THEOREM 18 *If $\{C_t\}$ survives at $p > 0$, then there are integers n,a and $\delta > 0$ such that*

$$\mathbf{P}_{p-\delta}[\, C_t^{\emptyset,[-n,n]^d} \textit{ survives in } \mathbb{Z} \times [-5a, 5a]^{d-1} \times [0, \infty)\,] > 0$$

Proof. The proof is a modification of Lemma 21 of [1]. Let $\eta > 0$ be given and take $\epsilon > 0$ such that $1 - \epsilon > 1 - \eta$ and let n, a, b and δ be as in Lemma 17. We will make an appropriate choice of η later. Construct a process $Z_n(i) = (X_n(i), Y_n(i))$, $i \geq 0$, $n \geq 0$, where $X_n(i) \in \{0, 1\}$ and $Y_n(i)$ is a point in $\mathbb{Z}^d \times [0, \infty)$. $Y_n(i)$ will be undefined when $X_n(i) = 0$. Start with $Z_0(0) = (1, 0)$, $X_0(i) = 0$, $i \neq 0$ and define inductively as follows: With $Z_k(i)$ already defined for $i \geq 0$, $0 \leq k \leq n$ let $X_{n+1}(i) = 1$ if for either $j = i$ or $j = i - 1$ it is the case that $X_n(j) = 1$ and there is a translation of $[-n, n]^d$ to the shaded area (see Figure 2.3 for the shaded regions) on the top of the corresponding

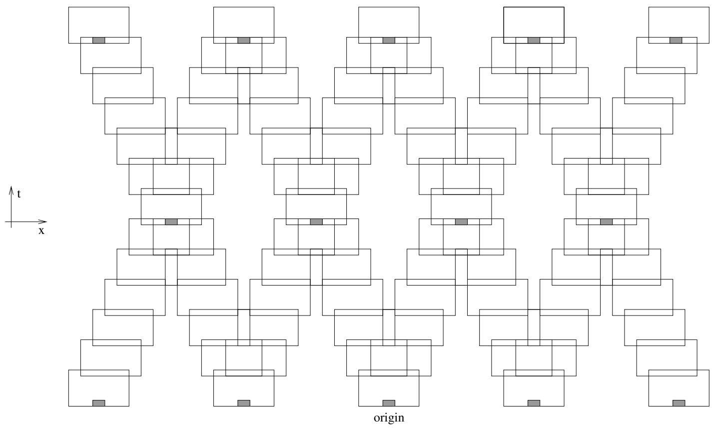

Figure 2.3: *Our building block $\mathcal{A}$ together with its reflection are translated in the x_1 and t direction. The shaded regions indicate where the paths start and stop in the definition of Z_n.*

block such that $Y_n(j) + [-n, n]^d$ is connected with $\emptyset$-active paths to every point in that translation. Furthermore, define $Y_{n+1}(i) = (x_{n+1}(i), t_{n+1}(i))$, where $t_{n+1}(i)$ is the earliest center of such a translation and $x_{n+1}(i)$ is chosen according to some fixed ordering of $\mathbb{Z}^d$. Note that if $X_n(i) = 1$ for infinitely many pairs (i, n), then $C_t^{\emptyset,[-n,n]^d}$ survives in $\mathbb{Z} \times [-5a, 5a]^{d-1} \times [0, \infty)$ so it remains to prove that the former has positive probability. Let $\mathcal{F}_n$ be the σ-algebra generating by $Z_k(i)$, where $i \geq 0$, $0 \leq k \leq n$ and note that from Lemma 17 we get

$$\mathbf{P}_p[\, X_{n+1}(i) = 1 \,|\, \mathcal{F}_n \,] > 1 - \eta \quad \text{on} \quad \{\, X_n(i-1) = 1 \text{ or } X_n(i) = 1 \,\}.$$

Also, our choice of k and the fact that events that depends on disjoint parts of the graphical representation are independent, we have that, conditioned on $\mathcal{F}_n$, the collection of variables $\{\, X_{n+1}(i) \,:\, i \geq 0 \,\}$ is one-dependent. Now, we are ready to make the construction above for a specific choice of η. Take $1/4 \leq p < 1$ so large that an oriented percolation process, $\{A_n\}$, on $\mathbb{N}$ with parameter p survives with positive probability when it starts with a single infection at the origin and choose η such that $1 - \eta > 1 - (1 - \sqrt{p})^3$. A result of Liggett, Schonmann and Stacey (1997) (see also Theorem B26 [9]) tells us that a $1 - d$ process with density $1 - \eta$ stochastically dominates a product measure with density p on $\mathbb{N}$. We can then conclude that $\{X_n\}$ dominates $\{A_n\}$. This completes the proof. □

We end with the following question:

Does the process obey a complete convergence theorem, i.e is it the case that for all $p \in [0,1]$ and $\beta, \eta \in \{0,1\}^{\mathbb{Z}^d}$

$$(\delta_\beta \times \delta_\eta) S_p(t) \to \alpha_p(\beta,\eta)\bar{\nu}_p + (1 - \alpha_p(\beta,\eta))\pi_p \times \delta_\emptyset \quad \text{as } t \to \infty,$$

where

$$\alpha_p(\beta,\eta) = \mathbf{P}_p[\, C_t^{\beta,\eta} \neq \emptyset \, \forall t \geq 0 \,].$$

We strongly believe this to be the case and plan to pursue some ideas that we have.

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PAPER II

Attractive nearest-neighbor spin systems on the integers in a randomly evolving environment

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3

PAPER II

ABSTRACT

We consider spin systems on $\mathbb{Z}$ (i.e. interacting particle systems on $\mathbb{Z}$ in which each coordinate only has two possible values and only one coordinate changes in each transition) whose rates are determined by another process, called a background process. A canonical example is the so called contact process in randomly evolving environment (CPREE), introduced and analysed by E. Broman and furthermore studied by J. Steif and the author, where the marginals of the background process independently evolve as 2-state Markov chains and determine the recovery rates for a contact process. We prove a generalization of a result by Liggett, that under certain conditions on the rates there are only two extremal stationary distributions.

Key words and phrases: spin systems, varying environment

Subject classification : 60K35.

3.1 Introduction

The contact process in a random environment, in which the rates are taken to be random variables and then fixed in time, has been studied the last twenty years, see for example [1], [3], [5], and [6]. However, recently in [2] they introduced a variant where the environment changes in time in a Markovian way. (See also [7] for further analysis concerning the CPREE.) More precisely, they consider the Markov process $\{(B_t, C_t)\}_{t\geq 0}$ on $\{0,1\}^{\mathbb{Z}^d} \times \{0,1\}^{\mathbb{Z}^d}$ described by the following rates at a site x:

transition	rate
$(0,0) \to (0,1)$	$\sum_{y\sim x} C(y)$
$(1,0) \to (1,1)$	$\sum_{y\sim x} C(y)$
$(0,1) \to (0,0)$	δ_0
$(1,1) \to (1,0)$	δ_1
$(0,0) \to (1,0)$	γp
$(0,1) \to (1,1)$	γp
$(1,0) \to (0,0)$	$\gamma(1-p)$
$(1,1) \to (0,1)$	$\gamma(1-p)$

where $\gamma, \delta_0, \delta_1 > 0$ with $\delta_1 \leq \delta_0$ and $p \in [0,1]$. In other words, the background process evolves independently for each site and determines the recovery rate for the right marginal in the following way: At a given site x and time t, the rate is δ_0 or δ_1 depending on whether $B_t(x) = 0$ or $B_t(x) = 1$. One says that the process evolves in a randomly *evolving* environment. In this paper we have the same structure: a background process influencing another interacting particle system, but here both processes are more general. We prove, under certain conditions on the rates, that we only have two extremal invariant distributions when $d = 1$.

3.2 The model and results

We consider the Markov process, $\{(\beta_t, \eta_t)\}_{t\geq 0}$ on $\{0,1\}^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}}$ described by the following rates at a site x:

transition	rate
$(\beta,\eta)\to(\beta,\eta_x)$	$c_0(x,\eta)$ if $\beta(x)=0$
$(\beta,\eta)\to(\beta,\eta_x)$	$c_1(x,\eta)$ if $\beta(x)=1$
$(\beta,\eta)\to(\beta_x,\eta)$	$b(x,\beta)$

Here $c_0(x,\eta)$, $c_1(x,\eta)$ and $b(x,\beta)$ are given rates where the first two satisfy

$$\begin{aligned} c_0(x,\eta) &\le c_1(x,\eta) \quad \text{if} \quad \eta(x)=0,\\ c_1(x,\eta) &\le c_0(x,\eta) \quad \text{if} \quad \eta(x)=1, \end{aligned} \tag{1}$$

and all three satisfy the following attractivity condition:

DEFINITION 1 *A spin system on* $\{0,1\}^{\mathbb{Z}}$*, with rates* $c(x,\eta)$ *is said to be attractive if whenever* $\eta \le \eta'$,

$$\begin{aligned} c(x,\eta) &\le c(x,\eta') \quad \textit{if} \quad \eta(x)=\eta'(x)=0, \quad \textit{and}\\ c(x,\eta) &\ge c(x,\eta') \quad \textit{if} \quad \eta(x)=\eta'(x)=1. \end{aligned} \tag{2}$$

Here, $\le$ refers to the usual partial ordering on $\{0,1\}^{\mathbb{Z}}$, i.e., $\eta\le\eta'$ if and only if $\eta(x)\le\eta'(x)$ for all $x\in\mathbb{Z}$. We also assume that the dynamics are translation invariant and that the rates $c_0(x,\eta)$, $c_1(x,\eta)$ only depend on η through

$$\{\eta(x-1),\eta(x),\eta(x+1)\}.$$

In other words, the rates for the system are completely described by $b(x,\beta)$ and the 16 parameters determining c_0 and c_1. To describe the values we will use the following notation:

$$c_i(001)=c_i(x,\eta) \quad \text{when} \quad \eta(x-1)=0,\, \eta(x)=0 \quad \text{and} \quad \eta(x+1)=1.$$

We always refer to the left marginal as the *background process*. Furthermore, note that we can equivalently view our process on $\{0,1\}^{\mathbb{Z}\times\{0,1\}}$ and that the conditions (1) and (2) just means that the whole process is attractive on that space. (Definition 1 can of course be generalizad to $\{0,1\}^S$ where S is countable.) The attractivity can be used to show (via monotonicity) the existence of two extremal stationary distrubutions ν_0 and ν_1 defined by

$$\nu_0=\lim_{t\to\infty}\delta_0 S(t) \qquad \nu_1=\lim_{t\to\infty}\delta_1 S(t),$$

where δ_0 and δ_1 denote the point masses corresponding to the elements $\eta \equiv 0$ and $\eta \equiv 1$ in $\{0,1\}^{\mathbb{Z}\times\{0,1\}}$ and $\{S(t)\}_{t\geq 0}$ denotes the semigroup associated to $\{(\beta_t, \eta_t)\}_{t\geq 0}$. The main result here is that, under mild conditions on the background process and the rates c_0, c_1 these are the only extremal stationary distributions. Let $\mathcal{I}$ denote the set of stationary distributions for the process and let $\mathcal{I}_e$ denote its extreme points. Furthermore, for technical reasons seen later, define

$$\begin{aligned} C_1 = \{\, & c_i(100) + c_j(110),\; c_i(001) + c_j(011), \\ & c_i(011) + c_j(110),\; c_i(100) + c_j(001),\; i = 0,1,\; j = 0,1\,\} \end{aligned}$$

and let

$$C = \min\left(C_1\right).$$

Before we state our main reult, we want to emphasize that the case when $c_0 = c_1$, i.e. no background process, has been studied before by Liggett. The proof of our main result follows the ideas of his proof.

THEOREM 1 (LIGGETT) *Suppose $c_0 = c_1 := c$ and that c satisfies*

$$c(x,\eta) + c(x,\eta_x) > 0 \quad \textit{whenever} \quad \eta(x-1) \neq \eta(x+1). \tag{3}$$

Then $\mathcal{I}_e = \{\nu_0, \nu_1\}$.

For a proof, see Theorem 3.13, page 152 in [4].

THEOREM 2 *Suppose that the background process is ergodic and $C > 0$. Then $\mathcal{I}_e = \{\nu_0, \nu_1\}$.*

Remark: In the case when $c_0 = c_1$ the condition of Theorem 2 is stronger than the condition of Theorem 1. However, if (3) holds and $C = 0$, then it can be seen more directly that $\mathcal{I}_e = \{\nu_0, \nu_1\}$. (See the beginning of Theorem 3.13, page 152 in [4].)

Remark: If c_i is symmetric under reflections, i.e satisfies

$$\begin{aligned} c_i(100) &= c_i(001) \\ c_i(110) &= c_i(011), \quad i = 0,1 \end{aligned}$$

then $C > 0$ if and only if $c_i(001) > 0$ and $c_i(011) > 0$, $i = 0, 1$.

Remark: Note that we are not assuming independence or even nearest-neighbor interaction between coordinates in the background process.

Example: To see that the conclusion may fail if we drop the ergodic assumption on the background process, let $b(x, \beta)$, in addition to being attractive and translation invariant, be nearest neighbor with $b(000) = b(111) = 0$ and satisfiy

$$b(x, \beta) + b(x, \beta_x) > 0 \qquad \text{whenever } \beta(x-1) \neq \beta(x+1).$$

Let $c_0 = c_1$ be the rates corresponding to a supercritical contact process on $\mathbb{Z}$. Then

$$\mathcal{I}_e = \{ \delta_0 \times \delta_0, \, \delta_0 \times \bar{\nu}, \, \delta_1 \times \delta_0, \, \delta_1 \times \bar{\nu} \},$$

where δ_0, δ_1 are the point masses corresponding to the elements $\eta \equiv 0$ and $\eta \equiv 1$ in $\{0,1\}^{\mathbb{Z}}$ respectively and $\bar{\nu}$ denotes the upper invariant measure for the contact process. If we take the same background process, but instead a subcritical contact process, we further see that the ergodic assumption is not necessary for having only two extremal stationary distributions.

In the proof, we make extensive use of a maximal type coupling which we now describe. Denote

$$U = \{0,1\}^{\mathbb{Z}}, \quad V = \{ (\eta, \gamma, \xi) \in U^3 : \eta \leq \gamma \leq \xi \} \quad \text{and} \quad W = U \times V.$$

The coupled process $(\beta_t, \eta_t, \gamma_t, \xi_t)$, which we now define, lives on W and its flip rates are described as follows: First, let flips of the type

$$(\beta, \eta, \gamma, \xi) \to (\beta_x, \eta, \gamma, \xi)$$

occur at rate $b(x, \beta)$.

	(0,0,0,0)	(0,0,0,1)	(0,0,1,1)	(0,1,1,1)
(0,0,0,0)	–	$c_0(x,\xi) - c_0(x,\gamma)$	$c_0(x,\gamma) - c_0(x,\eta)$	$c_0(x,\eta)$
(0,0,0,1)	$c_0(x,\xi)$	–	$c_0(x,\gamma) - c_0(x,\eta)$	$c_0(x,\eta)$
(0,0,1,1)	$c_0(x,\xi)$	$c_0(x,\gamma) - c_0(x,\xi)$	–	$c_0(x,\eta)$
(0,1,1,1)	$c_0(x,\xi)$	$c_0(x,\gamma) - c_0(x,\xi)$	$c_0(x,\eta) - c_0(x,\gamma)$	–

Table 3.1: *Transition rates when the background process is in state 0.*

	(1,0,0,0)	(1,0,0,1)	(1,0,1,1)	(1,1,1,1)
(1,0,0,0)	–	$c_1(x,\xi) - c_1(x,\gamma)$	$c_1(x,\gamma) - c_1(x,\eta)$	$c_1(x,\eta)$
(1,0,0,1)	$c_1(x,\xi)$	–	$c_1(x,\gamma) - c_1(x,\eta)$	$c_1(x,\eta)$
(1,0,1,1)	$c_1(x,\xi)$	$c_1(x,\gamma) - c_1(x,\xi)$	–	$c_1(x,\eta)$
(1,1,1,1)	$c_1(x,\xi)$	$c_1(x,\gamma) - c_1(x,\xi)$	$c_1(x,\eta) - c_1(x,\gamma)$	–

Table 3.2: *Transition rates when the background process is in state 1.*

Then, let the other three marginals flip according to Table 3.1 and 3.2. These tables should be interpreted as follows. For example, when $\beta_t(x) = 0$, $\eta_t(x) = 0$, $\gamma_t(x) = 0$ and $\xi_t(x) = 1$, $\xi_t(x)$ will flip alone at rate $c_0(x, \xi_t)$, $\gamma_t(x)$ will flip alone at rate $c_0(x, \gamma_t) - c_0(x, \eta_t)$ and $\eta_t(x)$ and $\gamma_t(x)$ flip together at rate $c_0(x, \eta_t)$. Note that the pairs $\{(\beta_t, \eta_t)\}$, $\{(\beta_t, \gamma_t)\}$, $\{(\beta_t, \xi_t)\}$ each evolve as the original Markov process and that the second, third and fourth marginals try to flip together as much as possible. Also, observe that the background process is not allowed to flip together with any of the other processes.

The proof of Theorem 2 consists of several lemma concerning certain functionals of the process. For $m \leq n$, let $f_{m,n}(\beta, \eta, \gamma, \xi)$ be the number of intervals of zeros and ones in γ between m and n, counted only where η and ξ differ. Furthermore, let $m \leq x_1 < x_2 < \ldots < x_k \leq n$ be all those x's between m and n for which $\eta(x) = 0$ and $\xi(x) = 1$. For $l \geq 1$, define

$$g^l_{m,n}(\beta, \eta, \gamma, \xi) = \text{number of } i \text{ such that } i \geq 1,\ i + l + 1 \leq k \text{ and}$$
$$\gamma(x_i) \neq \gamma(x_{i+1}) = \gamma(x_{i+2}) = \ldots = \gamma(x_{i+l}) \neq \gamma(x_{i+l+1}).$$

In other words, $g^l_{m,n}(\beta, \eta, \gamma, \xi)$ is the number of interior intervals of zeros and ones of length l in γ between m and n, counted only where η and ξ differ. For example if,

$$\begin{array}{ccccccccccccc|c}
\cdots & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & \cdots & \xi \\
\cdots & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & \cdots & \gamma \\
\cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \eta \\
\cdots & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & \cdots & \beta \\
 & m & & & & & & & & & & n & &
\end{array}$$

then $f_{m,n} = 4$, $g^2_{m,n} = 1$, $g^3_{m,n} = 1$ and $g^l_{m,n} = 0$ when $l \notin \{2, 3\}$. Let

$$K = \max\left(\max_{\eta} c_0(x, \eta),\ \max_{\eta} c_1(x, \eta)\right)$$

and denote the set of stationary distributions and the generator of the coupled process by $\tilde{\mathcal{I}}$ and $\tilde{\Omega}$ respectively. Furthermore, for a given set $\mathcal{A}$, denote the set of extreme points by $\mathcal{A}_e$. The first lemma concerns certain basic properties of $f_{m,n}$ and $g^l_{m,n}$.

LEMMA 3

a) $f_{m,n}, g^l_{m,n}$ *are increasing when* n *increases or* m *decreases.*

b) $f_{m,n} \le 2 + \sum_{l=1}^{\infty} g^l_{m,n}$.

c) $\sum_{l=1}^{\infty} l g^l_{m,n} \le n - m + 1$.

If $\nu \in \tilde{\mathcal{I}}$,

d) $C \int g^1_{m,n}\, d\nu \le K \int [f_{m-1,n} + f_{m,n+1} - 2f_{m,n}]\, d\nu$, *for* $m \le n$

e) $C \int g^{l+1}_{m,n}\, d\nu \le 12Kl \int g^l_{m,n}\, d\nu$, *for* $m \le n$, $l \ge 1$.

Proof. We follow Lemma 3.7, page 148-150, in [4]. a), b) and c) follow directly from the definition. For d) and e) assume $\nu \in \tilde{\mathcal{I}}$. Note that $f_{m,n}$ and $g^l_{m,n}$ are cylinder functions so that

$$\int \tilde{\Omega} f_{m,n}\, d\nu = \int \tilde{\Omega} g^l_{m,n}\, d\nu = 0. \tag{4}$$

For cylinder function f, the generator has the form

$$\begin{aligned}
&\tilde{\Omega} f(\beta, \eta, \gamma, \xi) = \\
&\quad \sum_{(\beta, \bar{\eta}, \bar{\gamma}, \bar{\xi})} c(\beta, \eta, \gamma, \xi, \bar{\eta}, \bar{\gamma}, \bar{\xi}) \left(f(\beta, \bar{\eta}, \bar{\gamma}, \bar{\xi}) - f(\beta, \eta, \gamma, \xi)\right) + \\
&\quad \sum_{x} b(x, \beta) \left(f(\beta_x, \eta, \gamma, \xi) - f(\beta, \eta, \gamma, \xi)\right)
\end{aligned} \tag{5}$$

where the first sum is over all possible transitions when the second, third or fourth marginal flip. (Recall that the first marginal is not allowed to flip together with any of the others.) Here, since both $f_{m,n}$ and $g^l_{m,n}$ do not depend on β, the second sum is zero, so our task is to calculate the first part. Let $(\beta, \eta, \gamma, \xi)$ be fixed and note that the only way $f_{m,n}$ can increase because of a flip is if $f_{m-1,n} = f_{m,n} + 1$ or $f_{m,n+1} = f_{m,n} + 1$. In the first case the flip must occur at $x = m$ and in the second at $x = n$. The rate for such a flip is at most K so the positive terms in (5) are bounded above by

$$K\left[f_{m-1,n} + f_{m,n+1} - 2f_{m,n}\right].$$

Furthermore, there are $g^1_{m,n}$ sites x where a flip decreases $f_{m,n}$ by two. At such an x, $\gamma(x) = 0$ or $\gamma(x) = 1$. Assume $\gamma(x) = 1$. Then we necessarely have $\gamma(x-1) = \eta(x-1)$ and $\gamma(x+1) = \eta(x+1)$. Therefore, the flip rate at x becomes

$$c_0(x,\gamma) + c_0(x,\eta) = \begin{cases} c_0(010) + c_0(000) & \text{if } \gamma(x-1) = 0, \gamma(x+1) = 0, \\ c_0(011) + c_0(001) & \text{if } \gamma(x-1) = 0, \gamma(x+1) = 1, \\ c_0(110) + c_0(100) & \text{if } \gamma(x-1) = 1, \gamma(x+1) = 0, \\ c_0(111) + c_0(101) & \text{if } \gamma(x-1) = 1, \gamma(x+1) = 1, \end{cases}$$

when $\beta(x) = 0$ and

$$c_1(x,\gamma) + c_1(x,\eta) = \begin{cases} c_1(010) + c_1(000) & \text{if } \gamma(x-1) = 0, \gamma(x+1) = 0, \\ c_1(011) + c_1(001) & \text{if } \gamma(x-1) = 0, \gamma(x+1) = 1, \\ c_1(110) + c_1(100) & \text{if } \gamma(x-1) = 1, \gamma(x+1) = 0, \\ c_1(111) + c_1(101) & \text{if } \gamma(x-1) = 1, \gamma(x+1) = 1, \end{cases}$$

when $\beta(x) = 1$. Also the attractivity condition gives

$$\begin{aligned} c_i(010) &\geq \max\{\, c_i(011),\ c_i(110)\,\} \\ c_i(101) &\geq \max\{\, c_i(001),\ c_i(100)\,\}, \quad i = 0, 1 \end{aligned}$$

and so the rates above are bounded below by $C/2$. By symmetry, the same argument works if $\gamma(x) = 0$ and so we can conclude that the negative terms in (5) are bounded above by $-Cg^1_{m,n}$. We get the estimate

$$\tilde{\Omega} f_{m,n} \leq K\left[f_{m-1,n} + f_{m,n+1} - 2f_{m,n}\right] - Cg^1_{m,n}$$

which via (4) gives d). For e), note that $g^l_{m,n}$ can only decrease via flips at no more than $lg^l_{m,n}$ sites or their neighbors. The rate for such a flip is bounded by $2K$ and $g^l_{m,n}$ can at most decrease by two. The negative terms in the generator are therefore bounded below by $-12Klg^l_{m,n}$. Furthermore, $g^l_{m,n}$ can increase at no fewer than $g^{l+1}_{m,n}$ pair of sites. These pair of sites are the endpoints of an interval of length $l+1$. To get a lower bound on the flip rate for such endpoints, let $x < y$ denote such a pair and suppose $\gamma(x) = \gamma(y) = 1$. Then we have $\gamma(x-1) = \eta(x-1)$ and $\gamma(y+1) = \eta(y+1)$. The flip rate at x is at least $c_i(100)$ if $\gamma(x-1) = \eta(x-1) = 1$, $\beta(x) = i$ and at least $c_i(011)$ if $\gamma(x-1) = \eta(x-1) = 0$, $\beta(x) = i$. In a similar fashion, the flip rate at y is at least $c_i(001)$ if $\gamma(y-1) = \eta(y-1) = 1$, $\beta(y) = i$ and at least $c_i(110)$ if $\gamma(y-1) = \eta(y-1) = 0$, $\beta(y) = i$. In either case the sum of the flip rates for the pair is always at least C. The same statement holds if $\gamma(x) = \gamma(y) = 0$ and so we obtain that the positive terms in the generator expression are bounded below by $Cg^{l+1}_{m,n}$. Hence, we get the estimate

$$\tilde{\Omega} g_{m,n} \geq C g^{l+1}_{m,n} - 12Klg^l_{m,n}.$$

Equation (4) then finally gives us

$$C \int g^{l+1}_{m,n} d\nu \leq 12Kl \int g^l_{m,n}\, d\nu$$

and the proof is complete. □

Denote

$$\begin{aligned}
A_1 &= \{\, (\beta,\eta,\gamma,\xi) \in W : \gamma \equiv \eta \,\},\\
A_2 &= \{\, (\beta,\eta,\gamma,\xi) \in W : \gamma \equiv \xi \,\},\\
A_3 &= \{\, (\beta,\eta,\gamma,\xi) \in W \setminus A_1 \cup A_2 : \exists x \in \mathbb{Z} \text{ such that}\\
&\qquad \gamma(y) = \eta(y) \text{ when } y \leq x \text{ and } \gamma(y) = \xi(y) \text{ when } y > x \,\},\\
A_4 &= \{\, (\beta,\eta,\gamma,\xi) \in W \setminus A_1 \cup A_2 : \exists x \in \mathbb{Z} \text{ such that}\\
&\qquad \gamma(y) = \xi(y) \text{ when } y \leq x \text{ and } \gamma(y) = \eta(y) \text{ when } y > x \,\},
\end{aligned}$$

LEMMA 4 *Assume $C > 0$. Then*

$$\begin{aligned}
&a)\ \nu \in \tilde{\mathcal{I}} &&\Longrightarrow &&\nu\,(A_1 \cup A_2 \cup A_3 \cup A_4) = 1,\\
&b)\ \nu \in \tilde{\mathcal{I}}_e &&\Longrightarrow &&\nu\,(A_i) = 1 \ \textit{for some } i.
\end{aligned}$$

Proof. We follow Lemma 3.10, page 150-151 in [4]. b) follows from a) since A_i is closed for the coupled process in the sense that

$$P^{(\beta,\eta,\gamma,\xi)}[\,(\beta_t,\eta_t,\gamma_t,\xi_t)\in A_i\,]=1\quad\forall t>0$$

whenever $(\beta,\eta,\gamma,\xi)\in A_i$. To prove a), suppose $\nu\in\tilde{\mathcal{I}}$. Since

$$\cup_{i=1}^4 A_i=\{\,g^l_{m,n}=0\,\forall m\leq n,\,l\geq 1\,\}$$

we obtain that

$$\int g^l_{m,n}d\nu=0 \text{ for all } m\leq n, l\geq 1 \tag{6}$$

is equivalent to

$$\nu\,(A_1\cup A_2\cup A_3\cup A_4)=1$$

and so we are done if (6) holds. For this, note that

$$f_{m-1,n}\leq f_{m,n}+1\qquad\text{and}\qquad f_{m,n+1}\leq f_{m,n}+1$$

and so parts d) and e) of Lemma 3 gives us

$$M=\sup_{m\leq n}\int g^l_{m,n}\,d\nu<\infty,\quad\forall l\geq 1. \tag{7}$$

Let $L\geq 1$. From part b) of the same lemma, we get

$$\frac{1}{n-m}\int f_{m,n}\,d\nu\leq\frac{2}{n-m}+\frac{1}{n-m}\int\sum_{l\geq 1}g^l_{m,n}\,d\nu.$$

Split the sum and now use part c) of the lemma together with (7) to obtain that for any L

$$\frac{1}{n-m}\int f_{m,n}\,d\nu\leq\frac{2}{n-m}+\frac{ML}{n-m}+\frac{1}{L}\left(1+\frac{1}{n-m}\right),$$

and so

$$\limsup_{n-m\to\infty}\frac{1}{n-m}\int f_{m,n}\,d\nu\leq\frac{1}{L}.$$

Since $L\geq 1$ was arbitrary we can conclude

$$\lim_{n-m\to\infty}\frac{1}{n-m}\int f_{m,n}\,d\nu=0. \tag{8}$$

Now, for $N \geq 1$, part d) of Lemma 3 gives us

$$C \sum_{m=-N+1}^{0} \sum_{n=0}^{N-1} \int g_{m,n}^{1}\, d\nu \leq$$
$$\leq K \sum_{m=-N+1}^{0} \sum_{n=0}^{N-1} \int [f_{m-1,n} + f_{m,n+1} - 2f_{m,n}]\, d\nu. \tag{9}$$

After some cancellations in the sum to the right, we get

$$\sum_{m=-N+1}^{0} \sum_{n=0}^{N-1} \int [f_{m-1,n} + f_{m,n+1} - 2f_{m,n}]\, d\nu \leq$$
$$\leq \sum_{m=-N+1}^{0} \int f_{m,N}\, d\nu + \sum_{n=0}^{N-1} \int f_{-N,n}\, d\nu$$

and together with (8) and (9) we obtain

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{m=-N+1}^{0} \sum_{n=0}^{N-1} \int g_{m,n}^{1}\, d\nu = 0.$$

This implies $\int g_{m,n}^{1}\, d\nu = 0$ for all $m \leq n$ and part e) of the lemma gives $\int g_{m,n}^{l}\, d\nu = 0$ for all $l \geq 1$ and we are done with the proof. □

We are soon ready for the proof of Theorem 2. However, in the proof we make use of a 5-variant coupling $\{(\beta_t, \eta_t, \gamma_{1,t}, \gamma_{2,t}, \xi_t)\}$ of the one used so far. This coupling is also of maximal type and evolves on

$$X = \{\, (\beta, \eta, \gamma_1, \gamma_2, \xi) \in U^5 : \; \eta \leq \gamma_1 \leq \xi,\, \eta \leq \gamma_2 \leq \xi\}$$

in a way such that $\{(\beta_t, \eta_t, \gamma_{1,t}, \xi_t)\}$ and $\{(\beta_t, \eta_t, \gamma_{2,t}, \xi_t)\}$ evolve exactly as the previous described coupling. We can therefore apply all we have done so far to each of these processes. The last tool we need is to have existence of an extremal stationary distribution for the 5-variant coupled process, given extremal stationary distributions for the $\{(\beta_t, \eta_t)\}$ process. For a stochastic variable X and a distribution μ, let $X \sim \mu$ denote that X is distributed according to μ. Also, let $\mathcal{I}^5$ denote the set of stationary distributions for the 5-variant coupled process on X.

LEMMA 5 *Given μ, $\mu' \in \mathcal{I}_e$ there exists $\nu((\beta,\eta,\gamma_1,\gamma_2,\xi)\in\cdot)\in\mathcal{I}_e^5$ such that $(\beta,\eta)\sim\nu_0$, $(\beta,\gamma_1)\sim\mu$, $(\beta,\gamma_2)\sim\mu'$ and $(\beta,\xi)\sim\nu_1$.*

Proof. For any measure μ let μ_{ij} denote the projection to the ith and jth coordinate. Construct a coupling on $(\{0,1\}^{\mathbb{Z}}\times\{0,1\}^{\mathbb{Z}})^4$ of four $\{\beta_t,\eta_t\}$-processes such that the background processes agree as much as possible as well as the right marginals. Note that our 5-variant coupling above can be identified with such a coupling started with all the background processes equal. Starting the coupling with

$$\delta_{(\emptyset,\emptyset)}\times\mu\times\mu'\times\delta_{(\mathbb{Z},\mathbb{Z})}$$

and taking a suitable subsequence of Cesaro averages gives us a stationary distribution ρ for the coupling and by projecting to the first, second, fourth, sixth and eighth coordinate we get a probability measure $\tilde{\nu}\in\mathcal{I}^5$ with

$$\tilde{\nu}((\beta,\eta,\gamma_1,\gamma_2,\xi)\in U^5:\ \eta\le\gamma_1\le\xi,\ \eta\le\gamma_2\le\xi)=1.$$

Here it is important to note that the set

$$\{\,(\beta_1,\eta,\beta_2,\gamma_1,\beta_3,\gamma_2,\beta_4,\xi)\in U^8:\ \beta_1\le\beta_2\le\beta_4,\ \beta_1\le\beta_3\le\beta_4,$$
$$\eta\le\gamma_1\le\xi,\ \eta\le\gamma_2\le\xi\}$$

is closed under the evolution of the coupling and that the first, third, fifth and seventh coordinate are equal under ρ. Furthermore, it is clear that $\tilde{\nu}$ satisfies

$$\tilde{\nu}_{12}=\nu_0,\quad \tilde{\nu}_{13}=\mu\quad \tilde{\nu}_{14}=\mu'\quad\text{and}\quad \tilde{\nu}_{15}=\nu_1.$$

Define

$$\mathcal{B}=\{\,\nu\in\mathcal{I}^5:\ \nu_{12}=\nu_0,\ \nu_{13}=\mu,\ \nu_{14}=\mu',\ \nu_{15}=\nu_1\,\}.$$

$\mathcal{B}$ is non-empty by the above and is compact and convex. Hence, by the Krein-Milman theorem, $\mathcal{B}$ can be written as the closed convex hull of its extreme points. Therefore, since $\mathcal{B}\neq\emptyset$, we have $\mathcal{B}_e\neq\emptyset$. Hence, the proof is complete if $\mathcal{B}_e\subset\mathcal{I}_e^5$. Assume $\nu\in\mathcal{B}_e$ and let $\nu=\alpha\rho+(1-\alpha)\sigma$, where $0<\alpha<1$ and $\rho,\sigma\in\mathcal{I}^5$. If ρ, $\sigma\in\mathcal{B}$ we get $\nu=\rho=\sigma$ and we are done. In order to see this, let (i,j) be one of the pairs $(1,2)$, $(1,3)$, $(1,4)$ or $(1,5)$. Since

$\nu_{ij} = \alpha\rho_{ij} + (1-\alpha)\sigma_{ij}$, where $\rho_{ij}, \sigma_{ij} \in \mathcal{I}$, and the left hand side is an element of $\{\nu_0, \mu, \mu', \nu_1\} \subseteq \mathcal{I}_e$, we obtain

$$\nu_0 = \rho_{12} = \sigma_{12} \qquad \mu = \rho_{13} = \sigma_{13}$$
$$\mu' = \rho_{14} = \sigma_{14} \qquad \nu_1 = \rho_{15} = \sigma_{15}$$

and so $\rho, \sigma \in \mathcal{B}$. $\square$

Proof of Theorem 2. Let $\mu_1 \in \mathcal{I}_e$. Since $\nu_0 \le \mu \le \nu_1$ for every stationary distribution μ, we can assume $\nu_0 \ne \nu_1$. Let $\mu_2 = \mu_1 \circ \theta_x^{-1}$, where θ_x is a translation by $x \in \mathbb{Z}$. Since the dynamics are translation invariant and $\mu_1 \in \mathcal{I}_e$, we get that $\mu_2 \in \mathcal{I}_e$. Let ρ be an extremal stationary distribution for the 5-variant coupling mentioned above with

$$(\beta, \eta) \sim \nu_0 \qquad (\beta, \gamma_1) \sim \mu_1$$
$$(\beta, \gamma_2) \sim \mu_2 \qquad (\beta, \xi) \sim \nu_1$$

Such a measure exists by Lemma 5. Let ρ_1 and ρ_2 be the distributions obtained from the projections

$$(\beta, \eta, \gamma_1, \gamma_2, \xi) \to (\beta, \eta, \gamma_1, \xi)$$
$$(\beta, \eta, \gamma_1, \gamma_2, \xi) \to (\beta, \eta, \gamma_2, \xi)$$

respectively. Since $\rho_1, \rho_2 \in \tilde{\mathcal{I}}_e$, Lemma 4 gives

$$\rho_1(A_i) = 1 \quad \text{some } 1 \le i \le 4 \quad \text{and} \quad \rho_2(A_i) = 1 \quad \text{some } 1 \le i \le 4.$$

However, γ_1 and γ_2 are just translations of each other so there is an i such that $\rho_1(A_i) = \rho_2(A_i) = 1$. It follows that

$$\rho\Big((\beta, \eta, \gamma_1, \gamma_2, \xi) : \sum_x |\gamma_1(x) - \gamma_2(x)| < \infty\Big) = 1.$$

Also, $(\gamma_{1,t}, \gamma_{2,t})$ has the property that

$$P^{(\gamma,\gamma)}[\gamma_{1,t} = \gamma_{2,t}] = 1 \quad \text{and} \quad P^{(\gamma_1,\gamma_2)}[\gamma_{1,t} = \gamma_{2,t}] > 0$$

whenever $\sum_x |\gamma_1(x) - \gamma_2(x)| < \infty$ and so since ρ is stationary, we must in fact have

$$\rho\Big((\beta, \eta, \gamma_1, \gamma_2, \xi) : \gamma_1 = \gamma_2\Big) = 1.$$

This implies $\mu_1 = \mu_2$ i.e, μ_1 is translation invariant. Therefore i equals 1 or 2. If $i = 1$, $\mu_1(U \times (\cdot)) = \nu_0(U \times (\cdot))$ and since the background process has a unique stationary distribution we must also have $\mu_1((\cdot) \times U) = \nu_0((\cdot) \times U)$. But since $\nu_0 \leq \mu_1$ this yields $\mu_1 = \nu_0$. If $i = 2$ we get in a similar way that $\mu_1 = \nu_1$. $\Box$

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