CHALMERS | UNIVERSITY OF GOTHENBURG

PREPRINT 2008:43

On Convergence of Mixed Finite Element Methods for the Vlasov-Poisson-Fokker-Planck System

MOHAMMAD ASADZADEH

Department of Mathematical Sciences
Division of Mathematics
CHALMERS UNIVERSITY OF TECHNOLOGY
UNIVERSITY OF GOTHENBURG
Göteborg Sweden 2008

Preprint 2008:43

On Convergence of Mixed Finite Element Methods for the Vlasov-Poisson-Fokker-Planck System

Mohammad Asadzadeh

Department of Mathematical Sciences
Division of Mathematics
Chalmers University of Technology and University of Gothenburg
SE-412 96 Göteborg, Sweden
Göteborg, December 2008

Preprint 2008:43
ISSN 1652-9715

Matematiska vetenskaper
Göteborg 2008

ON CONVERGENCE OF MIXED FINITE ELEMENT METHODS FOR THE VLASOV-POISSON-FOKKER-PLANCK SYSTEM

MOHAMMAD ASADZADEH[1]

Abstract. In this paper we investigate the basic ingredients for global superconvergence strategy used for the mixed finite element approximations, in H^1 and $W^{1,\infty}$-norms (see [1]), for the solution of the Vlasov–Poisson–Fokker–Planck system. This study is an extension of the results in [2]-[5], to finite element schemes including discretizations of the Poisson term, where we also introduce results of an extension of the h-versions of the streamline diffusion (SD) and the discontinuous Galerkin (DG) methods to the corresponding hp-versions. Optimal convergence results presented in the paper relay on the estimates for the regularized Green's functions with memory terms where some interpolation postprocessing techniques play important roles, see [7].

1. Introduction

Our purpose is to study the global superconvergence in L_2 and maximum norms, for h and hp-versions of, the mixed finite element approximations for h and hp-versions of the streamline diffusion and discontinuous Galerkin methods for the solution of the deterministic, multi-dimensional Vlasov–Poisson–Fokker–Planck (VPFP) system of Coulomb particles. The mathematical problem is formulated as follows: given the initial distribution of particles $f_0(x,v) \geq 0$, in the phase–space variable $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$, $d = 1,2,3$, and the physical parameters $\beta \geq 0$ and $\sigma \geq 0$, find the distribution function $f(x,v,t)$ for $t > 0$, satisfying the nonlinear system of evolution equations

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + \operatorname{div}_v[(E-\beta v)f] = \sigma\Delta_v f, & \text{in } \mathbb{R}^{2d}\times(0,\infty), \\ f(x,v,0) = f_0(x,v), & \text{for } (x,v)\in\mathbb{R}^{2d}, \\ E(x,t) = \dfrac{\theta}{|\mathcal{S}|^{d-1}}\dfrac{x}{|x|^d} *_x \rho(x,t), & \text{for } (x,t)\in\mathbb{R}^d\times(0,\infty), \\ \rho(x,t) = \displaystyle\int_{\mathbb{R}^d} f(x,v,t)\,dv, & E=\theta\tilde{E}, \text{ and } \theta = \pm 1, \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^d$ is the position, $v \in \mathbb{R}^d$ is the velocity, and $t > 0$ is the time, $v\cdot\nabla_x = \sum_{i=1}^d v_i\partial/\partial x_i$. Finally $|\mathcal{S}|^{d-1} \sim 1/\omega_d$ is the surface area of the unit disc in $\mathbb{R}^d$, $\rho(x,t)$ is the spatial density and $*_x$ denotes the convolution in x. E and ρ can be interpreted as the electrical field, and charge, respectively. Here the macroscopic force field E can be taken as

$$E(x,t) = -\nabla_x\phi(x,t), \tag{1.2}$$

1991 *Mathematics Subject Classification.* 65M12, 65M15, 65M60, 82D10, 35L80.

Key words and phrases. Vlasov-Poisson-Fokker-Planck system, Green's functions, mixed finite element method, streamline diffusion method, discontinuous Galerkin method.

[1] Partially supported by the *Swedish Foundation of Strategic Research (SSF) in Gothenburg Mathematical Modeling Center (GMMC).*

with $\phi(x,t)$ being the internal potential field. For a gradient field, E is divergence free and with no viscosity: $\beta = 0$, the first equation in (1.1) would become

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = \sigma \Delta_v f, \tag{1.3}$$

which, with the rest of equations in (1.1), gives rise to the Vlasov–Fokker–Planck system. If in addition $\sigma = 0$, then we obtain the classical Vlasov–Poisson equation with $\phi(x,t)$ satisfying the Poisson equation

$$\Delta_x \phi(x,t) = \int_{\mathbb{R}^d} f(x,v,t)dv = \rho(x,t). \tag{1.4}$$

We shall concentrate on the following (modified) version of the VPFP equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = \nabla_v \cdot (\beta v f + \sigma \nabla_v f). \tag{1.5}$$

The mathematical study of the VPFP system has been considered by several authors in various settings, see e.g. [11] The deterministic approach is based on controlling the behavior of the trajectories, i.e., the solutions of the ordinary differential equations underlying the Vlasov–Poisson equation. Compared to the analytical studies the numerical analysis of the VPFP system is much less developed. In the deterministic approaches the dominant part of numerical studies are using method of characteristics: basically particle methods developed for the Vlasov–Poisson equation, see [10]. Probabilistic approaches commonly employ Mote-Carlo simulations. Concerning *hp* finite element strategy: In the classical finite element method (h-version) convergence order improvement relies on mesh refinement while keeping the approximation order within the elements at a fixed low value (suitable for problems with highly singular solutions that require small mesh parameter). Some studies on the h-version of the SD finite element method can be found, e.g., in [15] for advection-diffusion, Navier-Stokes and first order hyperbolic equations; in [16] for Euler and Navier-Stokes equations; in [2] for the Vlasov-Poisson and in [3], for the Fokker-Planck and Fermi equations and in [17] for conservation laws. On the other hand in the spectral method the accuracy improvement is accomplished by raising the order of approximation polynomial rather than mesh refinement (advantageous in approximating smooth solutions). However, most realistic problems have local behavior (are locally smooth or locally singular), therefore a more realistic numerical approach would be a combination of mesh refinement in the vicinity of singularities (with lower order polynomial approximations), and higher order polynomial approximations in high regularity regions (with larger, non-refined, mesh parameter). This strategy, which can be viewed as a generalized adaptive approach, is the *hp*-version of the finite element method. For some basic$p$ and *hp*-finite element studies see, e.g., [6], [18] and [19].

In this paper we derive optimal error estimates for finite element approximation of (1.1) through the study of regularized Green's function, (see [12], [13], and [21]) for (1.4) combined with the SD and DG methods for (1.3) and (1.5). We also give optimal stability and convergence results for the *hp*-versions of the above approaches. We shall give the Green's function approach for the mixed finite element methods in some detail, however, to keep the presentation concise, we restrict both the SD, DG and *hp* approaches to mentioning the main results and, for the detailed proofs, refer the reader to follow the techniques in some current literature.

2. The continuous problem

With separate study of ϕ (and $\beta = 0$) we are left with the continuous problem called the Vlasov–Fokker–Planck system:

$$
\begin{cases} \partial_t f + v\cdot\nabla_x f + E\cdot\nabla_v f - \sigma\Delta_v f = 0, & f(x,v,0) = f_0(x,v), \\ E(x,t) = C_d \displaystyle\int_{\mathbb{R}^d} \frac{x-y}{|x-y|^d}\rho(y,t)\,dy, & \rho(x,t) = \displaystyle\int_{\mathbb{R}^d} f(x,v,t)\,dv. \end{cases} \tag{2.1}
$$

We split the study of problem (1.1) to solving the Poisson equation (1.4) for ϕ in order to determine the field E and then solve the following linear Fokker–Planck equation for f,

$$
f_t + v\cdot\nabla_x f + E\cdot\nabla_v f - \sigma\Delta_v f = g, \quad f(x,v,0) = f_0(x,v), \tag{2.2}
$$

where

$$
E(x,v,t) = \Big(E_i(x,v,t)\Big)_{i=1}^{d},
$$

is a given vector field and $f_0(x,v)$ and $g(x,v,t)$ are given functions. Existence, uniqueness, stability and regularity properties of the solution for the equation (2.2) are derived following 1D results in [7] for degenerate type equations.

In our studies $(x,v) \in \Omega := \Omega_x \times \Omega_v$, where Ω_x, $\Omega_v \subset \mathbb{R}^d$ are bounded simply connected domains and we let $\Omega_T := \Omega_x \times \Omega_v \times (0,T]$. With these assumptions and $\beta \neq 0$ we consider the VPFP problem of finding (f,ϕ) satisfying

$$
\begin{cases} \partial_t f + v\cdot\nabla_x f - \nabla_x\phi\cdot\nabla_v f = \nabla_v(\beta\, v f + \sigma\nabla_v f), & \text{in } \Omega_T, \\ f(x,v,0) = f_0(x,v), & \text{in } \Omega, \\ f(x,v,t) = 0, \qquad \text{on } \Gamma^- := \{(x,v)\in\partial\Omega_x\times\partial\Omega_v : \mathbf{n}\cdot G < 0\}. \end{cases} \tag{2.3}
$$

Here $G := (v, -\nabla_x\phi)$, and $\mathbf{n} = (\mathbf{n}(x), \mathbf{n}(v))$, with $\mathbf{n}(x)$ and $\mathbf{n}(v)$being outward unit normals to $\partial\Omega_x$ and $\partial\Omega_v$ at the point $(x,v) \in \partial\Omega_x \times \partial\Omega_v$. Further ϕ and f are associated through the Poisson equation

$$
-\Delta_x\phi(x,t) = \int_{\Omega_v} f(x,v,t)\,dv, \qquad (x,t)\in\Omega_x\times(0,T] := \Omega_T, \tag{2.4}
$$

where $\nabla_x\phi$ is uniformly bounded and $|\nabla_x\phi| \to 0$ as $x \to \partial\Omega_x$. We shall use the notation $\nabla f := (\nabla_x f, \nabla_v f)$ and

$$
G(f) := (v, -\nabla_x\phi) = \Big(v_1,\dots,v_d, -\frac{\partial\phi}{\partial x_1},\dots,-\frac{\partial\phi}{\partial x_d}\Big) = (G_1,\dots,G_{2d}),
$$

leading to the following useful divergent free drift coefficient:

$$
\operatorname{div} G(f) = \sum_{i=1}^{d}\frac{\partial G_i}{\partial x_i} + \sum_{i=d+1}^{2d}\frac{\partial G_i}{\partial v_{i-d}} = 0, \quad d = 1,2,3. \tag{2.5}
$$

3. Regularized Green's function

The Green's function plays a central role in the study of convergence of the finite element approximations for the elliptic equations and is usually considered as the solution of a dual problem. We apply this procedure to Poisson equation for ϕ by introducing its general framework below. To this end we recall the back-ward Gronwall's inequality:

Lemma 3.1. *Assume that ψ and φ are two non-negative functions defined on $[0,T]$. Then*

$$\psi(t)\le\varphi(t)+C\int_t^T\psi(s)\,ds\quad\Longrightarrow\quad\psi(t)\le C\Big\{\varphi(t)+\int_t^T\varphi(s)\,ds\Big\},\quad t\in(0,T).$$

We start by introducing a finite element structure on $\Omega_x\times\Omega_v$. Let $T_h^x=\{\tau_x\}$ and $T_h^v=\{\tau_v\}$ denote finite element subdivisions of Ω_x and Ω_v, with elements τ_x and τ_v, respectively. Then $\mathcal{T}_h=T_h^x\times T_h^v=\{\tau_x\times\tau_v\}=\{\tau\}$ will be a finite element partition of $\Omega=\Omega_x\times\Omega_v$ into triangular or quadrilaterals with quasi-uniform elements $\tau=\tau_x\times\tau_v$. Let $V_h\subset H_0^1(\Omega)$ be the corresponding finite element space of order r. For a given point $z:=(y,u)\in\Omega=\Omega_x\times\Omega_v$, let $\delta_h^z(p)\in V_h,\ p=(x,v)$ be the smoothed δ-function at z which satisfies

$$(\delta_h^z,g)=g(z),\quad g\in V_h. \tag{3.1}$$

Now, for $\varphi(t)\in C_0^\infty(0,T)$ with $\|\varphi\|_{L_1(0,T)}\le 1$, we define the Green's function $\mathcal{G}^z(t):=\mathcal{G}^z(p,t;z)\in L^2((0,T);H_0^1(\Omega)\cap H^2(\Omega))$, to be the solution of the equation

$$A(t)\mathcal{G}^z(t)+\int_t^T\mathcal{B}(s,t)\mathcal{G}^z(s)\,ds=\delta_h^z\varphi(t),\quad\text{in }\Omega_T, \tag{3.2}$$

where $\mathcal{B}$ is an integral kernel. Let l be any fixed hyperline direction and define the directional derivative

$$\partial_z\delta_h^z:=\lim_{\Delta z\| l,\ \Delta z\to 0}\frac{\delta_h^{z+\Delta z}-\delta_h^z}{|\Delta z|},\quad\text{satisfying }(\partial_z\delta_h^z,g)=\partial_z g(z),\quad g\in V_h. \tag{3.3}$$

We introduce the weight function $\mu(p)=\mu_z(p):=(|p-z|^2+\nu^2)^{-1}$, with $\nu:=\gamma h$ and $\gamma>0$, and define

$$\|w\|_{\mu^\alpha}^2:=\int_\Omega\mu^\alpha|w|^2\,dxdv,\quad\|w\|_{m,\mu^\alpha}^2:=\sum_{|k|\le m}\|D^kw\|_{\mu^\alpha}^2,\quad m=1,2,\dots,\alpha\in\mathbb{R}.$$

In this setting we have the estimate:

Lemma 3.2. *There is a constant C such that*

$$\|\mu^{-1}\partial_z\delta_h^z\|_0=\|\partial_z\delta_h^z\|_{\mu^{-2}}\le C.$$

Similarly we may define a Green's function for the derivatives, then we have a regularity requirement of the type $\partial_z\mathcal{G}^z(t)\in L^2((0,T);H_0^1(\Omega)\cap H^2(\Omega))$ such that

$$A(t)\partial_z\mathcal{G}^z(t)+\int_t^T\mathcal{B}(s,t)\partial_z\mathcal{G}^z(s)\,ds=\partial_z\delta_h^z\varphi(t),\quad\text{in }\Omega_T. \tag{3.4}$$

Let $\mathcal{G}_h^z(t)$ and $\partial_z\mathcal{G}_h^z(t)$ be finite element approximations of the regularized Green's functions $\mathcal{G}^z$ and $\partial_z\mathcal{G}^z$, respectively.

We discretize the Poisson equation for $\varphi=\varphi(x,t)$, see (1.2) and (1.4), using the above version of the regularized Green's functions approach. The discretization variable is $x\in\Omega_x$ and the t variable is tackled separately in a backward-Euler type time-discretization. Below we outline the main steps for the Regularized Green's functions procedure and refer to [12], [13], and [21], for the details. Consider the Dirichlet problem for the second order elliptic equation viz: find $\varphi\in H^1(\Omega_x)$ such that

$$\begin{cases}-\Delta_x\varphi=\rho(x,t), & (x,t)\in\Omega_x\times[0,T],\\ \varphi=g, & \text{on }\ \partial\Omega_x.\end{cases} \tag{3.5}$$

Note that in our case $\rho(x,t) = \displaystyle\int_{\Omega_v} f(x,v,t)\,dv$. Let $E = -\nabla_x\varphi$, the idea is to study a mixed form for (E,φ) given by

$$
\text{(3.6)} \qquad \begin{cases} E + \nabla_x\varphi = 0, & \text{in} \quad \Omega_x, \\ \text{div}\ E = \rho, & \text{in} \quad \Omega_x, \\ \varphi = g, & \text{on} \quad \partial\Omega_x. \end{cases}
$$

To this end we define the necessary measurement environments: Let

$$
\mathcal{H} := H(\text{div}, \Omega_x) = \{w \in \Big[L_2(\Omega_x)\Big]^d : \text{div}\ w \in L_2(\Omega_x)\},
$$

be a Hilbert space associated with the norm

$$
||w||^2_{\mathcal{H}} = ||w||^2_2 + ||\text{div}\, w||^2_2.
$$

Now the weak form of (3.6) reads as follows. Find $(E,\varphi) \in \mathcal{H} \times L_2(\Omega :_x)$ such that

$$
\text{(3.7)} \qquad \begin{cases} (E,w) - (\text{div}\ w, \varphi) = -< g, w\cdot\mathbf{n} >, & \forall w \in \mathcal{H}, \\ (\text{div}\ w, u) = (\rho, u), & \forall u \in L_2(\Omega_x), \end{cases}
$$

where $(\cdot,\cdot)$ is the usual inner product in either $\mathcal{H} = \Big[L_2(\Omega_x)\Big]^d$ or $L_2(\Omega_x)$ (the actual space at each usage will be obvious from the content) and $< \cdot,\cdot >$ is the inner product in $L_2(\partial\Omega_x)$ and $\mathbf{n}$ is the outward unit normal to $\partial\Omega_x$. The problems (3.5) and (3.7) are equivalent and the solubility of (3.7) is based on the $\inf - \sup$ condition

$$
\text{(3.8)} \qquad \inf_{E\in\mathcal{H}} \sup_{u\in L_2} \frac{(\text{div}E, u)}{||E||_{\mathcal{H}}||u||} \geq \lambda,
$$

due to Babuška and Brezzi, where λ is a positive constant. Now consider a quasi-uniform triangulation of Ω_x as $\Omega_x^h : \mathcal{T}_h^x = \{\tau\}$ and let $\mathcal{H}_h \times L_2^h \subset \mathcal{H} \times L_2$ be the associated pair of finite element spaces corresponding to the discretization $\mathcal{T}_h^x$ such that the discrete version of the Babuška-Brezzi condition holds true:

$$
\text{(3.9)} \qquad \inf_{E_h\in\mathcal{H}_h} \sup_{u_h\in L_2^h} \frac{(\text{div}E_h, u_h)}{||E_h||_{\mathcal{H}}||u_h||} \geq \tilde{\lambda},
$$

where $\tilde{\lambda}$ is independent of h. Now the mixed finite element method for (3.7) is formulated as follows: Find $(E_h, \varphi_h) \in \mathcal{H}_h \times L_2^h(\Omega_x^h)$ such that

$$
\text{(3.10)} \qquad \begin{cases} (E_h,w) - (\text{div}\ w, \varphi_h) = -< g, w\cdot\mathbf{n} >, & \forall w \in \mathcal{H}_h, \\ (\text{div}\ E_h, u) = (\rho, u), & \forall u \in L_2^h(\Omega_x). \end{cases}
$$

Subtracting (3.10) from (3.7), formulated in the subspaces $\mathcal{H}_h \times L_2^h \subset \mathcal{H} \times L_2$, we obtain the error equations for the mixed method as:

$$
\text{(3.11)} \qquad \begin{cases} (E - E_h, w) - (\text{div}\ w, \varphi - \varphi_h) = 0, & \forall w \in \mathcal{H}_h, \\ (\text{div}\ (E - E_h), u) = 0, & \forall u \in L_2^h(\Omega_x). \end{cases}
$$

There exist local projections Π_h and π_h

$$
\Pi_h = \Pi_h^k : H(\text{div},\ \Omega_x) \to \mathcal{H}_h, \qquad \text{and} \qquad \pi_h = \pi_h^{k-1} : L_2 \to L_2^h,
$$

with π_h^{k-1} denoting the local L_2-projection, such that

$$
\text{(3.12)} \qquad \text{div}\Pi_h^k = \pi_h^{k-1}\text{div},
$$

Now we state the main error estimates, due to [9], for the mixed finite element viz,

Theorem 3.1. *Using the BDM spaces we for have the following error estimates for the mixed finite element scheme* (3.10)*:*

$$\|E - E_h\|_2 \leq C\|E - \Pi_h^k E\|_2, \tag{3.13}$$

$$\|\varphi - \varphi_h\|_2 \leq Ch\|E - \Pi_h^k E\|_2 + Ch^{\min(2,k)}\|\rho - \pi_h^{k-1}\rho\|_2. \tag{3.14}$$

Proof. As we mentioned above a somewhat detailed and technical proof is based on an approach given by Brezzi et al [9] and therefore omitted. □

We sketch the procedure for deriving the projection errors on the right hand sides in (3.13) and (3.14). For complete proofs we refer to the work of Wang in [21]. To this approach we recall a version of the weight function ($\mu(p)$):

$$\sigma(x, x_0) = (|x - x_0|^2 + \theta^2)^{1/2}, \qquad \theta = Ch, \quad C > 0, \quad x_0 \in \Omega_x \tag{3.15}$$

introduced by [14] , satisfying the following properties

$$\max_{x\in\tau} \sigma(x, x_0) \leq C \min_{x\in\tau} \sigma(x, x_0), \qquad \forall\tau \in \mathcal{T}_h^x, \quad x_0 \in \Omega_x, \tag{3.16}$$

$$\int_{\Omega_x} \sigma^{-2}\, dx \leq C|\log h|, \qquad \text{and} \qquad |D^j\sigma^\alpha| \leq C\sigma^{\alpha-|j|}, \qquad \alpha \in \mathbb{R}. \tag{3.17}$$

For $\eta \in H^s(\Omega_x)$, η being a scalar or vector-valued function, we define the, σ^α, weighted L_2-norm by

$$\|D^j\eta\|_{\sigma^\alpha}^2 = \int_{\Omega_x} |D^j\eta^2\sigma^\alpha|\, dx.$$

Then the construction of Π_h^k and π_h^{k-1} yields the inequalities:

$$\|w - \Pi_h^k w\|_{\sigma^\alpha} \leq Ch^{k+1}\|D^{k+1}w\|_{\sigma^\alpha}, \qquad \forall w \in \left[H^{k+1}(\Omega_x)\right]^d, \tag{3.18}$$

$$\|u - \pi_h^{k-1}u\|_{\sigma^\alpha} \leq Ch^k\|D^k u\|_{\sigma^\alpha}, \qquad \forall u \in H^k(\Omega_x), \tag{3.19}$$

$$\|\sigma^2 w - \Pi_h^k\sigma^2 w\|_{\sigma^{-2}} \leq Ch\|E\|_2 + Ch^2(\|w\|_{\sigma^{-2}} + \|\nabla_\tau w\|_2), \quad \forall w \in S_h, \tag{3.20}$$

where ∇_τ indicates element-by-element gradient.

In the sequel we give a Galerkin approach for regularized Green's function that concerns L_1-error estimates useful in the L_∞ estimates as well as asymptotic error expansions. The conventional approach is based on an split of the procedure to solving two system of equations for the first and second regularized Green's functions, respectively. We start with the first system:

Let $(\Gamma_1, \gamma_1) = (\Gamma_1(x, x_0), \gamma_1(x, x_0))$ be the first regularized Greens function at $x_0 \in \Omega_x$ defined by the following system of equations associated to (3.5):

$$\begin{cases} \Gamma_1 + \nabla\gamma_1 = 0, & \text{in} \quad \Omega_x, \\ \operatorname{div} \Gamma_1 = \delta_1^h, & \text{in} \quad \Omega_x, \\ \gamma_1 = 0, & \text{on} \quad \partial\Omega_x, \end{cases} \tag{3.21}$$

where $\delta_1^h = \delta_1^h(x, x_0)$ is regularized Dirac δ-function at $x_0 \in \Omega_x$ given by

$$\delta_1^h(x, x_0) = \begin{cases} \frac{1}{|D|}, & x \in D, \\ 0, & \text{else}, \end{cases} \tag{3.22}$$

where $D \subset \tau$ satisfies the following properties:

(i) diam $D = \omega h_\tau$, for some ω (see also [21]),
(ii) there exists a ball B with radius rh_τ such that $B \subset D$,
(iii) D is star-shaped with respect to B.

Now we use the notation $U_h := L_2^h$. Let $u \in U_h$, then there is an element $\tau \in \mathcal{T}_h^x$ and a point $x_0 \in \tau$ such that

$$\|u\|_\infty = |u(x_0)|. \tag{3.23}$$

Using the mean value theorem and an inverse estimate one can show that (see [12]), for an appropriate choice of D

$$\|u\|_\infty \le C|(u, \delta_1^h)|, \qquad u \in U_h, \quad C > 1, \quad (C \sim d = \mathrm{diam}(\Omega_x)). \tag{3.24}$$

Proposition 3.1 (Scott [20]). *For the Green's function γ_1 we have the following classical estimates, due Scott:*

$$\max\Big(\|\gamma_1\|_2, \|D^2\gamma_1\|_{\sigma^2}\Big) \le C|\log h|^{1/2}, \tag{3.25}$$

$$\|D^2\gamma_1\|_p \le Ch^{1-p}|\log h|^{2-p}, \qquad p = 1, 2. \tag{3.26}$$

Proposition 3.2 (Wang [21]). *Now (Γ_1^h, γ_1^h) be the mixed finite element approximation of (Γ_1, γ_1) in BDM space stated above. Then* (3.13) *and* (3.26) *implies that,*

$$\|\Gamma_1 - \Gamma_1^h\|_p \le C|\log h|^{2-p}, \qquad p = 1, 2, \tag{3.27}$$

$$\|\gamma_1 - \gamma_1^h\|_2 \le Ch|\log h|^{1/2}. \tag{3.28}$$

Next we outline the second Green's function approach which is, mostly, applied in L_∞-estimates. Let $\Big(\Gamma_2(x.x_0), \gamma_2(x.x_0)\Big)$ be defined as functions satisfying the following system of equations:

$$\begin{cases} \Gamma_2 + \nabla\gamma_2 = \tilde{\delta}_2^h, & \text{in} \quad \Omega_x, \\ \mathrm{div}\,\Gamma_2 = 0, & \text{in} \quad \Omega_x, \\ \gamma_2 = 0, & \text{on} \quad \partial\Omega_x, \end{cases} \tag{3.29}$$

where $\tilde{\delta}_2^h = \tilde{\delta}_2^h(x, x_0)$ is a vector of dimension d with all zero components except one (either one) being the regularized Dirac δ-function δ_2^h at x_0 (with x_0 and D as in the definition of $\delta_1^h(\cdot, x_0)$) satisfying

(iv) $\tilde{\delta}_2^h(\cdot, x_0) \in C^1(\Omega_x)$, $\mathrm{supp}\delta_2^h \Subset D$,
(v) $\delta_2^h \ge 0$, $\int_{\Omega_x} \delta_2^h\, dx = 1$,
(vi) $\|\delta_2^h\|_\infty \le Ch^{-d-|i|}$, $j = 0, 1$.

Now we gather the corresponding estimates for the second regularized Green's system (3.29) in the following proposition

Proposition 3.3. *let (Γ_2^h, γ_2^h) be the mixed finite element approximation of (Γ_2, γ_2) in BDM space stated above. Then by the H^2-regularity assumption of the domain Ω_x we have*

$$\|\nabla\gamma_2\|_2 \le Ch^{-1}. \tag{3.30}$$

Further, once again, using (3.13) *and* (3.26) *we have the following estimates*

$$\|\nabla^p\gamma_2\|_1 \le Ch^{1-p}|\log h|^{1/p}, \qquad p = 1, 2 \tag{3.31}$$

and

$$\|\Gamma_2 - \Gamma_2^h\|_p \le Ch^{1-p}|\log h|^{(2-p)/2}, \qquad p = 1, 2 \tag{3.32}$$

$$\|\gamma_2 - \gamma_2^h\|_1 \le C. \tag{3.33}$$

Proof. The proof is straightforward following similar techniques as in Section 5 in Wang, [21]. □

As a consequence of these regularized Green's function approaches, we finally get the *projection error estimates* for the potential φ gathered in the following theorem:

Theorem 3.2. *Let (E, φ) and (E_h, φ_h) be the exact solution for (3.7) and the mixed finite element approximations in BDM space, respectively, and assume that $\varphi \in W^{1,\infty}(\Omega_x)$. Then*

$$\|\varphi_h - \pi_h^{k-1}\varphi\|_\infty \le \begin{cases} Ch|\log h|\Big(\|E - \Pi_h^1 E\|_\infty + |\log h|^{-1/2}\|\rho - \pi_h^0\rho\|_2\Big), & k = 1, \\ C\Big(\|E - \Pi_h^1 E\|_2 + h|\log h|^{1/2}\|\rho - \pi_h^0\rho\|_2\Big), & k = 1, \\ Ch|\log h|\Big(\|E - \Pi_h^k E\|_\infty + h\|\rho - \pi_h^{k-1}\rho\|_\infty\Big), & k > 1. \end{cases}$$

$$\|E - \Pi_h E\|_\infty \le C|\log h|^{1/2}\Big(\|E - \Pi_h^1 E\|_\infty + h|\log h|^{\delta_{1k}/2}\|\rho - \pi_h^{k-1}\rho\|_\infty\Big).$$

where δ_{1k} is the Kronecker function. An improved version of the above estimate for sufficiently smooth $\partial\Omega$ and $k > 1$ is given by

$$\|E - \Pi_h E\|_\infty \le C\Big(|\log h|^{1/2}\|E - \Pi_h^1 E\|_\infty + h\|\rho - \pi_h^{k-1}\rho\|_\infty\Big). \tag{3.34}$$

Proof. The proof for all these estimates can easily be reconstructed from the results in [21]. Further estimates of these type are given by Brezzi et al in [9]. □

4. The streamline diffusion method

The streamline diffusion (SD) method is a finite element method constructed for convection dominated convection–diffusion problems which (i) is higher order accurate and (ii) has good stability properties. The (SD) method was introduced by Hughes and Brooks [15] for the stationary problems. The mathematical analysis for this method in two settings (streamline diffusion and discontinuous Galerkin) are developed for ,e.g., two–dimensional incompressible Euler and Navier–Stokes equations in [16], for multi–dimensional Vlasov–Poisson equation in [2], for hyperbolic conservations laws in [17], and for the two–dimensional Fermi and Fokker–Planck in [3]. Here is the SD strategy:

Let $0 = t_0 < t_1 < \cdots < t_M = T$ be a partition of the time interval $I = [0, T]$ into subintervals $I_m = (t_m, t_{m+1})$, $m = 0, 1, \dots, M-1$. Let $\mathcal{C}_h$ be the corresponding subdivision of $Q_T = \Omega \times [0, T]$ into elements $K := \tau \times I_m$, with the mesh parameter $h = \operatorname{diam} K$ and $P_k(K) = P_k(\tau_x) \times P_k(\tau_v) \times P_k(I_m)$ the set of polynomials in (x, v, t) of degree at most k on K. In the study of SD–method for the VPFP system given by (2.3), the trial functions are continuous in the x and v variables, but may assumed to be discontinuous in time. Below we introduce the basis finite element space

$$V_h = \Big\{g \in \mathcal{H}_0 : g\Big|_K \in P_k(\tau) \times P_k(I_m); \quad \forall K = \tau \times I_m \in \mathcal{C}_n,\ k = 0, 1, \dots\Big\},$$

where $\mathcal{H}_0 = \prod_{m=0}^{M-1} H_0^1(S_m)$ with $H_0^1 = \Big\{g \in H^1 : g \equiv 0 \text{ on } \partial\Omega_v^h\Big\}$ and the slabs $S_m = \Omega \times I_m$, $m = 0, 1, \dots, M-1$. Further $(f,g)_m = (f,g)_{S_m}$, $\|g\|_m = (g,g)_m^{1/2}$, $< f, g >_m = (f(\cdot,\cdot,t_m), (g(\cdot,\cdot,t_m))_\Omega$ and $|g|_m =< g, g >_m^{1/2}$. We also present the jump $[g] = g_+ - g_-$, where for $t \in I$ and $(x,v) \in \partial\Omega_x \times \Omega_v^h$,

$$g_\pm = \lim_{s\to 0\pm} g(x,v,t+s), \quad (x,v) \in \text{ Int } (\Omega_x)\times\Omega_v^h, \qquad g_\pm = \lim_{s\to 0\pm} g(x+sv,v,t+s),$$

and the boundary integrals defined by

$$< f_+, g_+ >_{\Gamma^-} = \int_{\Gamma^-} f_+ g_+ \,(G^h\cdot\mathbf{n})\, d\nu, \quad < f_+, g_+ >_{\Gamma_m^-(\Gamma_I^-)} = \int_{I_m(I)} < f_+, g_+ >_{\Gamma^-} \, d\nu,$$

with $G^h := G(f^h)$ defined as above. We use the discrete version of (2.5): div $G(f^h) = 0$, and for a given appropriate function $\tilde f$, define the trilinear form B by

$$\begin{aligned} B\big(G(\tilde f); f, g\big) =& \big(f_t + G(\tilde f)\nabla f, g + h(g_t + G(f^h)\nabla g)\big)_{Q_T} - h\sigma\big(\Delta_v f, g_t + G(f^h)\nabla g\big)_{Q_T} \\ &+ \sigma\big(\nabla_v f, \nabla_v g\big)_{Q_T} + \sum_{m=1}^{M-1} \langle [f], g_+\rangle_m + \langle f_+, g_+\rangle_0 - \langle f_+, g_+\rangle_{\Gamma_I^-}, \end{aligned}$$

and the bilinear form K by

$$K\big(f,g\big) = \big(\nabla_v(\beta v f), g + h(g_t + G(f^h)\nabla g)\big)_{Q_T}.$$

Note that both B and K depend implicitly on f^h (hence on h) through the term $G(f^h)$. We also define the linear form L

$$L(g) = \langle f_0, g_+\rangle_0.$$

Using this notation we can formulate the SD-problem in the following concise form: *find $f^h \in V_h$ such that*

$$B\big(G(f^h); f^h, g\big) - K\big(f^h, g\big) = L(g), \qquad \forall g \in V_h. \tag{4.1}$$

We shall give our stability and convergence estimates for (4.1) in a triple norm defined by

$$\begin{aligned} |||g|||^2 = \frac{1}{2}\Big[& 2\sigma\|\nabla_v g\|_{Q_T}^2 + |g|_M^2 + |g|_0^2 + \sum_{m=1}^{M-1} |[g]|_m^2 + 2h\|g_t + G(f^h)\nabla g\|_{Q_T}^2 \\ &+ \int_{\partial\Omega\times I} g^2 \big|G^h\cdot\mathbf{n}\big|\, d\nu\, ds\Big]. \end{aligned}$$

Lemma 4.1 (Stability I). *We have that*

$$\forall g \in \mathcal{H}_0, \qquad B\big(G(f^h); g, g\big) \ge \frac{1}{2}|||g|||^2.$$

Lemma 4.2 (Stability II). *For any constant $C_1 > 0$ we have for any $g \in \mathcal{H}_0$,*

$$\|g\|_{\Omega_T}^2 \le \left[\frac{1}{C_1}\|g_t + G(f^h)\nabla g\|_{Q_T}^2 + \sum_{m=1}^{M} |g_-|_m^2 + \int_{\partial\Omega\times I} g^2\big|G^h\cdot\mathbf{n}\big|\, d\nu\, ds\right] he^{C_1 h}.$$

For the proofs follow the argument in [2]-[3] (be constructive). Let $\tilde{f}^h \in V_h$ be an interpolant of f with the interpolation error denoted by $\eta = f - \tilde{f}^h$ and set $\xi = f^h - \tilde{f}^h$, so we have $e = f - f^h = \eta - \xi$. The objective in the error estimates is to dominate $|||\xi|||$ by the known interpolation estimates for $|||\eta|||$. Our main result in this section is as follows:

Theorem 4.1. *Assume that $f^h \in V_h$ and $f \in H^{k+1}(Q_T)$, $k \geq 1$, are the solutions of* (4.1) *and* (2.3)*, respectively, such that*

$$\|\nabla f\|_\infty + \|G(f)\|_\infty + \|\nabla \eta\|_\infty \leq C. \tag{4.2}$$

Then there exists a constant C such that

$$|||f - f^h||| \leq Ch^{k+\frac{1}{2}} \|f\|_{k+1,\Omega_T}.$$

In the proof of Theorem 4.1 we use two results estimating the forms B and K. Combining these results, with the estimates of the previous section for ϕ as a generalized Green's function, gives superconvergence for the SD estimate for VPFP. The discontinuous Galerkin counterpart assumes discontinuities, even, in x and v and follows similar pattern, however somewhat lengthy procedure where, in addition to the sum over jumps in the time direction, we also have a sum over the jumps over the enter-element boundaries, (see the formulation below).

5. Discontinuous Galerkin and hp results

Theorem 5.1. *Under the conditions of theorem 3, the discontinuous Galerkin approximation for solutions of* (2.3)*, satisfies*

$$|||f - f^h|||_{DG} \leq Ch^{k+\frac{1}{2}} \|f\|_{k+1,\Omega_T},$$

where

$$|||f - f^h|||_{DG} = |||f - f^h||| + \sum_{K \in \mathcal{C}_h} \int_{\partial K_-(G'')} [u]^2 |G^h \cdot n|,$$

with $\partial K_-(G'') = \{(x, v, t) \in \partial K_-(G') : n_t(x, v, t) = 0\}$ controls an additional term corresponding to the sum of the jump-discontinuities over enter-element boundaries.

As for the *hp* version (p is the degree of polynomial in spectral approximation, see the definition of $\mathcal{A}_p$ below. The accuracy of *hp* is measured in powers of (h/p) with small mesh parameter h and high spectral degree p, see [19]). Assume a partition $\mathcal{P}$ of Ω_T into open patches P which are image of the canonical cube $\hat{P} = (-1, 1)^{2d+1}$, under smooth bijections F_P: $\forall P \in \mathcal{P}; \quad P = F_P(\hat{P})$. For each P a mesh $\hat{\mathcal{T}}_P$ is obtained by subdividing $\hat{P}$ into quadrilaterals labeled $\hat{\tau}$ affine equivalent to $\hat{P}$,

$$\forall P \in \mathcal{P}; \quad \mathcal{T}_P := \{\tau | \tau = F_P(\hat{\tau}), \; \hat{\tau} \in \hat{\mathcal{T}}_P\}.$$

Each $\hat{\tau}$ is an image of $\hat{P}$ under affine mapping $A_{\hat{\tau}} : \hat{P} \to \hat{\tau}$. Let $\mathcal{T} := \cup_{P \in \mathcal{P}} \mathcal{T}_P, \quad F_\tau = F_P \circ A_{\hat{\tau}}$ and define $F_{\mathcal{P}} = \{F_P : \; P \in \mathcal{P}\}$ and

$$\mathcal{A}_p = \text{span}\{(\hat{x}, \hat{v})^\alpha : 0 \leq \alpha_i \leq p, \; 1 \leq i \leq 2d+1\}, \qquad (\hat{x}, \hat{v}) \in \hat{P}.$$

We skip the details and, with these notations, state a patch-wise optimal *hp* convergence result for the VPFP system.

Theorem 5.2. *The hp-estimate with piecewise polynomials of degree p for the SD method for solutions of* (2.3)*, satisfies*

$$\|f - f\|^2_{SD,P} \leq C \sum_{\tau \in \mathcal{T}_P} h_\tau^{2s_\tau+1} \frac{\Phi(p_\tau, s_\tau)}{p_\tau} \|\hat{f}\|^2_{s_\tau+1,\hat{\tau}} \leq \left(\frac{h}{p}\right)^{2s_\tau+1} \|\hat{f}\|^2_{s_\tau+1,\hat{\tau}}, \quad \tau \in \mathcal{T},$$

where $\Phi(p_\tau, s_\tau) = \max(\Phi_1(p_\tau, s_\tau), \Phi_2(p_\tau, s_\tau))$, *and with parameters* $\alpha_p = \frac{1}{p(p+1)}$ *and* $\beta_{|m|_k} = \frac{(p-s+|m|_k)!}{(p+s-|m|_k)!}$, *we have*

$$\Phi_1(p, s) = \mathcal{N} \sum_{i=1}^{\mathcal{N}} 2^{i-1} \sum_{|m|_{i-1} \leq i-1} \alpha_p^{|m|_{i-1}+1} \beta_{|m|_{i-1}},$$

$$\Phi_2(p, s) = \mathcal{N} \sum_{i=1}^{\mathcal{N}} \sum_{j=1}^{\mathcal{N}} 2^j \sum_{\substack{|m|_{j-1} \leq j-1 \\ m_i=1}} \alpha_p^{|m|_{j-1}} \beta_{|m|_{j-1}}.$$

A proof for can be obtained following the outlines in [5]*, using Stirling's formula (under certain assumption) to show:*

$$\Phi(p_\tau, s_\tau) \leq C p_\tau^{-2s_\tau}.$$

Similar estimates hold for the hp DG approximation including additional terms corresponding to the sum of the jump-discontinuities over enter-element boundaries. To summarize we have a convergence of order $\mathcal{O}(h/p)^{s+1/2}$ in $H^{s+1}(\Omega_T)$ which is an optimal result improving the classical convergence rate for hyperbolic problems by $\mathcal{O}(h/p)^{1/2}$.

Conclusion

This article is a survey introducing some of the recent techniques on convergence analysis for the finite element approaches. The objective is twofold (i) To concisely demonstrate the mathematical structure in some finite element procedures employed for the numerical solution of general partial differential equations (PDE) and (ii) To follow basic steps in deriving convergence rates for a deterministic numerical approach for a rather complex PDE: the Vlasov-Poisson-Fokker-Planck system. While introducing a kind of strategy for the numerical studies in the interdisciplinary research, we have avoided all the proofs and technical details. Extensions towards analytical aspects as well as simulations are equally important and involve challenging projects.

References

[1] R. A. Adams, *Solve Spaces*, Academic Press, New York, 1975.

[2] M. Asadzadeh, *Streamline diffusion methods for The Vlasov-Poisson equation*, Math. Model. Numer. Anal., **24** (1990), no. 2, 177–196.

[3] M. Asadzadeh, *Streamline diffusion methods for Fermi and Fokker-Planck equations*, Transport Theory Statist. Phys., **26** (1997), no. 3, 319–340.

[4] M. Asadzadeh and P. Kowalczyk, *Convergence of streamline diffusion methods for the Vlasov–Poisson– Fokker-Planck equations*, Numer Methods Partial Differential Eqs., 21 (2005), 472-495.

[5] M. Asadzadeh and A. Sopasakis, *Convergence of a hp Streamline Diffusion Scheme for Vlasov-Fokker-Planck system*, Math. Mod. Meth. Appl. Sci., 17(2007), 1159-1182.

[6] I. Babuška, C, E. Baumann and J. T. Oden, *A discontinuous hp-finite element method for diffusion problems: 1-D analysis*, Comp. Math. , **37** (1999), 103–122.

[7] M. S. Baouendi and P. Grisvard, *Sur une équation d' évolution changeant de type*, J. Funct. Anal., (1968), 352–367.

[8] F. Brezzi, G. Manzini, D. Marini, P. Pietra and A. Russo, *Discontinuous Galerkin approximations for elliptic problems*, Numer. Meth. Partial Diff. Equs., **16** (2000), no. 4, 365–378.

[9] F. Brezzi, J. Douglas and L. D. Marini, *Two families of mixed finite elements for second order elliptic problems.* Numer. Math. 47 (1985), no. 2, 217–235.

[10] G. H. Cottet and P. A. Raviart, *On particle-in-cell methods for the Vlasov-Poisson equations*, Trans. Theory Statist. Phys., **15** (1986), 1–31.

[11] P. Degond and P. A. Raviart,*Global existence of smooth solutions for the Vlasov-Fokker-Planck equation in 1 and 2 space dimensions*, Ann. Scient. Éc. Norm. Sup., 4^e série, **19** (1986), 519–542.

[12] R. Duràn, R. H. Nochetto and J. Wang, *Sharp maximum norm error estimates for finite element approximations of the Stokes problem in 2-D.* Math. Comp. 51 (1988), no. 184, 491–506.

[13] R. E. Ewing, Y. Liu, J. Wang, and S. Zhang, *L^∞-error estimates and superconvergence in maximum norm of mixed finite element methods for non-Fickian flows in porous media.* Int. J. Numer. Anal. Model. 2 (2005), no. 3, 301–328.

[14] J. Frehse and R. Rannacher, *Eine L^1-Fehlerabshtzung für diskret Grundlsungen in der Methode fer finite Elemente.* Bonn. Math. Schrift **89** (1976), 92–114.

[15] T.J. Hughes, and A. Brooks, *A multidimensional upwind scheme with no crosswind diffusion*, in ADM, **34**, Finite Element Methods for Convection Dominated Flows, T.J. Hughs (ed,), ASME, New York (1979).

[16] C. Johnson and J. Saranen, *Streamline diffusion methods for the incompressible Euler and Navier-Stokes equations*, Math. Comp. **47** (1986), 1–18.

[17] A. Szepessy, *Convergence of a streamline diffusion finite element method for scalar conservation laws with boundary conditions*, RAIRO, Mode'l Math. Anal. Numer., **25** (1991), no. 6, 749–782.

[18] P. Houston, C. Schwab and E. Süli, *Stabilized hp-finite element methods for first order hyperbolic problems*, SIAM, J. Numer. Anal. **37** (2000), 1618–1643.

[19] C. Schwab, *p- and hp-Finite Element Methods.* Theory and Applications in Solid and Fluid Mechanics. Oxford Science Publication, (1998).

[20] R. Scott, *Optimal L^∞ estimates for the finite element method on irregular meshes.* Math. Comp. 30 (1976), no. 136, 681–697.

[21] J. Wang, *Asymptotic expansions and L^∞-error estimates for mixed finite element methods for second order elliptic problems.* Numer. Math. 55 (1989), no. 4, 401–430.

Department of Mathematical Sciences, Chalmers University of Technology, and Department of Mathematical Scineces, Gothenburg University, SE–412 96, Göteborg, Sweden

E-mail address: mohammad@chalmers.se