CHALMERS | UNIVERSITY OF GOTHENBURG

PREPRINT 2010:13

Σ-Convergence

GABRIEL NGUETSENG
NILS SVANSTEDT

Department of Mathematical Sciences
Division of Mathematics
CHALMERS UNIVERSITY OF TECHNOLOGY
UNIVERSITY OF GOTHENBURG
Gothenburg Sweden 2010

Preprint 2010:13

Σ-Convergence

Gabriel Nguetseng and Nils Svanstedt

Department of Mathematical Sciences
Division of Mathematics
Chalmers University of Technology and University of Gothenburg
SE-412 96 Gothenburg, Sweden
Gothenburg, March 2010

Preprint 2010:13
ISSN 1652-9715

Matematiska vetenskaper
Göteborg 2010

Σ-CONVERGENCE

GABRIEL NGUETSENG AND NILS SVANSTEDT

Abstract. We discuss two new concepts of convergence in L^p-spaces, the so-called weak Σ-convergence and strong Σ-convergence, which are intermediate between classical weak convergence and strong convergence. We also introduce the concept of Σ-convergence for Radon measures. Our basic tool is the classical Gelfand representation theory. Apart from being a natural generalization of well-known two-scale convergence theory, the present study lays the foundation of the mathematical framework that is needed to undertake a systematic study of deterministic homogenization problems beyond the usual periodic setting. A few homogenization problems are worked out by way of illustration.

1. INTRODUCTION

To systematically pass to the limit in a product of two weakly convergent sequences one classically requires that (at least) one of the two sequences converges strongly. More precisely, let Ω be an open set in the N-dimensional numerical space $\mathbb{R}^N$ $(N \geq 1)$, let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^p(\Omega)$ (Ω provided with Lebesgue measure) and let $(v_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^{p'}(\Omega)$, where $1 < p < \infty$ and $\frac{1}{p'} = 1 - \frac{1}{p}$. It is a classical fact that if $u_\varepsilon \to u$ in $L^p(\Omega)$ (strong) and $v_\varepsilon \to v$ in $L^{p'}(\Omega)$-weak as $\varepsilon \to 0$, then $u_\varepsilon v_\varepsilon \to u_0 v_0$ in $L^1(\Omega)$-weak.

However, in a great number of situations arising in mathematical analysis it is often crucial to investigate the limiting behaviors of products of the preceding form in spite of the fact that none of the two sequences is allowed to strongly converge. For example in homogenization theory [36, 14, 37, 3, 35, 38, 2, 21] it is frequent to have to compute limits such as

$$\lim_{0<\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\, \psi\left(x, \frac{x}{\varepsilon}\right) dx, \tag{1.1}$$

where $u_\varepsilon \to u$ in $L^p(\Omega)$-weak as $\varepsilon \to 0$, and $\psi \in L^{p'}(\Omega; \mathcal{C}_{per}(Y))$ with $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^N$, $\mathcal{C}_{per}(Y)$ being the space of those continuous complex functions f on $\mathbb{R}^N$ that are Y-periodic, i.e., that satisfy $f(y+k) = f(y)$ for $y \in \mathbb{R}^N$ and $k \in \mathbb{Z}^N$ ($\mathbb{Z}$ denotes the integers), $\mathcal{C}_{per}(Y)$ provided with the supremum

2000 *Mathematics Subject Classification.* 46J10, 35B40, 28A33.

Key words and phrases. Homogenization, Homogenization algebras, Sigma-convergence, Gelfand transformation.

norm. It is of interest to recall here that

$$\psi^{\varepsilon} \to \widetilde{\psi} \text{ in } L^{p'}(\Omega)\text{-weak as } \varepsilon \to 0, \tag{1.2}$$

where $\psi^{\varepsilon}(x) = \psi\left(x, \frac{x}{\varepsilon}\right)$ and $\widetilde{\psi}(x) = \int_Y \psi(x,y)\,dy$ for $x \in \Omega$ (see, e.g., [26]). Furthermore, unless ψ is constant with respect to the periodicity variable $y = (y_1, ..., y_N)$ (this is a quite trivial occurrence), it is hopeless to try to get strong convergence in (1.2) (see, e.g., [3]). Thus, it is beyond the classical resources of mathematical analysis to compute the limit in (1.1).

It was precisely to overcome such difficulties that the first author introduced in 1989 basic ideas on two-scale convergence (see [27]). Shortly after, the direction pointed out by further pioneering papers (see [28, 1]) on two-scale convergence initiated a great activity that increased in interest over the years. See, e.g., [23] and the references therein.

Without going to deeply into details, let us recall the main ideas underlying two-scale convergence theory. To begin, for the benefit of the reader it should be reminded that a sequence $(u_\varepsilon)_{\varepsilon>0}$ in $L^p(\Omega)$ $(1 \le p < \infty)$ is said to weakly two-scale converge in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega; L^p_{per}(Y))$ ($L^p_{per}(Y)$ stands for the space of Y-periodic complex functions in $L^p_{loc}(\mathbb{R}^N_y)$) if as $\varepsilon \to 0$,

$$\int_\Omega u_\varepsilon(x)\,\psi^\varepsilon(x)\,dx \to \int\int_{\Omega\times Y} u_0(x,y)\,\psi(x,y)\,dxdy$$

for all $\psi \in L^{p'}(\Omega; \mathcal{C}_{per}(Y))$. A sequence $(v_\varepsilon)_{\varepsilon>0}$ in $L^q(\Omega)$ $(1 \le q < \infty)$ is said to strongly two-scale converge in $L^q(\Omega)$ to some $v_0 \in L^q(\Omega; L^q_{per}(Y))$ if for all $\eta > 0$ and $f \in L^q(\Omega; \mathcal{C}_{per}(Y))$ satisfying $\|v_0 - f\|_{L^q(\Omega\times Y)} \le \frac{\eta}{2}$, one can find some $\alpha > 0$ such that $\|v_\varepsilon - f^\varepsilon\|_{L^q(\Omega)} \le \eta$ provided $0 < \varepsilon \le \alpha$.

If $(u_\varepsilon)_{\varepsilon>0}$ and $(v_\varepsilon)_{\varepsilon>0}$ are as above (with the respective assigned two-scale convergence properties), it can be shown that when $\varepsilon \to 0$, the sequence $(u_\varepsilon)_{\varepsilon>0}$ weakly converges to $\widetilde{u}_0$ in $L^p(\Omega)$ (with $\widetilde{u}_0(x) = \int_Y u_0(x,y)\,dy$, $x \in \Omega$) whereas $(v_\varepsilon)_{\varepsilon>0}$ weakly converges to $\widetilde{v}_0$ (defined as $\widetilde{u}_0$) in $L^q(\Omega)$ and further, there is no reason for our assuming that one of those two sequences is strongly convergent. Nevertheless, letting $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and assuming that $r \ge 1$, it can be shown that when $\varepsilon \to 0$, the sequence $(u_\varepsilon v_\varepsilon)_{\varepsilon>0}$ weakly converges in $L^r(\Omega)$ to the function $z(x) = \int_Y u_0(x,y)\,v_0(x,y)\,dy \quad (x \in \Omega)$.

As might be expected, strong two-scale convergence implies weak two-scale convergence. The function u_0 (resp. v_0) above is unique and is referred to as the weak (resp. strong) two-scale limit of the sequence (u_ε) (resp. (v_ε)). One of the major results in two-scale convergence theory is the so-called two-scale compactness theorem ([27], [23, Theorem 7], [26, Theorem 1]): from any bounded sequence $(u_{\varepsilon_n})_{n\in\mathbb{N}}$ in $L^p(\Omega)$ $(1 < p < \infty)$, where $0 < \varepsilon_n \to 0$ as $n \to \infty$, one can extract a subsequence that weakly two-scale converges in $L^p(\Omega)$. The two-scale compactness theorem is the corner stone of a by now well-known homogenization approach, the so-called two-scale convergence method (see, e.g., [11, 12, 13, 25, 39, 23]).

In fact, weak two-scale convergence is intended to supply the deficiency of usual weak convergence (observe that the former implies the latter) whereas strong two-scale convergence is fitted to temper the stiffness of usual strong convergence (indeed, the latter implies the former). For further results concerning two-scale convergence we refer to [9, 40, 23] and the references therein.

The present study is intended to generalize the two-scale convergence theory to nonperiodic settings, so true is it that two-scale convergence is strictly relevant to periodic structures. It goes without saying that such an undertaking requires appropriate materials, the usual material for two-scale convergence theory being obsolete in the forthcoming general framework. In this connection a fundamental role will be played by so-called homogenization algebras. One of our main tools will be the classical Gelfand representation theory (see, e.g., [22, 15]). Most of the main results proved here are stated (without proofs) in some articles by the first author with reference to an unpublished paper [29] as regards the proofs. Algebras with mean values were first introduced in [41] but a complete theory adapted for e.g. homogenization theory in the present form was first introduced in [29].

The rest of the paper is organized as follows. Section 2 deals with homogenization algebras introduced earlier in [30]. Several concrete examples of homogenization algebras are considered. The special case of almost periodic homogenization algebras is discussed. In Section 3 we discuss weak Σ-convergence and strong Σ-convergence in L^p. It is of great interest to stress here that all the main results achieved in two-scale convergence theory carry over mutatis mutandis to Σ-convergence theory. Thus, it is no wonder that the Σ-convergence method is a mere adaptation of the two-scale convergence method. In Section 4 we introduce the concept of Σ-convergence of measures. Finally, in Section 5 we show how Σ-convergence theory is applied to study homogenization problems beyond the usual periodic setting.

Except where otherwise stated, vector spaces are considered over $\mathbb{C}$ (the complex numbers) and scalar functions are assumed to take complex values. We will mostly follow the standard notation. For example if X and F denote a locally compact space and a Banach space, respectively, we write $\mathcal{C}(X;F)$ for the space of continuous mappings of X into F, $\mathcal{B}(X;F)$ for the space of bounded continuous functions of X into F, and $\mathcal{K}(X;F)$ for the space of compactly supported continuous functions of X into F. The norm in $\mathcal{B}(X;F)$ will be the supremum norm $\|u\|_\infty = \sup_{x\in X}\|u(x)\|$, where $\|.\|$ stands for the norm in F. $\mathcal{K}(X;F)$ is provided with the usual inductive limit topology. For shortness we will write $\mathcal{C}(X)$ for $\mathcal{C}(X;\mathbb{C})$, $\mathcal{B}(X)$ for $\mathcal{B}(X;\mathbb{C})$ and $\mathcal{K}(X)$ for $\mathcal{K}(X;\mathbb{C})$. Likewise we will put $L^p(X)$ for $L^p(X;\mathbb{C})$, and $L^p_{loc}(X)$ for $L^p_{loc}(X;\mathbb{C})$. We generally refer to [4, 5, 18] for integration theory.

2. HOMOGENIZATION ALGEBRAS

2.1. **Preliminaries.** Let N be a positive integer. For any real $\varepsilon > 0$, we set

$$H_\varepsilon(x) = \left(\frac{x_1}{\varepsilon^{\alpha_1}}, ..., \frac{x_N}{\varepsilon^{\alpha_N}}\right), \; x = (x_1, ..., x_N) \in \mathbb{R}^N, \tag{2.1}$$

where $(\alpha_i)_{1 \le i \le N}$ is a given family of positive integers. This gives a family $\mathcal{H} = (H_\varepsilon)_{\varepsilon>0}$ of mappings of $\mathbb{R}^N$ into $\mathbb{R}^N$ with the following properties:

$(\mathrm{H})_1$ $\lim_{\varepsilon\to0} |H_\varepsilon(x)| = +\infty$ for any $x \in \mathbb{R}^N$ with $x \neq \omega$, where $|.|$ and ω denote the Euclidean norm and the origin in $\mathbb{R}^N$, respectively.

$(\mathrm{H})_2$ $\lim_{\varepsilon\to0} \left|H_{\frac{1}{\varepsilon}}(x)\right| = 0$ for all $x \in \mathbb{R}^N$.

For $u \in L^1_{loc}(\mathbb{R}^N_y)$ ($\mathbb{R}^N_y$ denotes the space $\mathbb{R}^N$ of variables $y = (y_1, ..., y_N)$), we will put for simplicity

$$u^\varepsilon(x) = u(H_\varepsilon(x)) \qquad (x \in \mathbb{R}^N).$$

Now, the family $\mathcal{H} = (H_\varepsilon)_{\varepsilon>0}$ generates a mean value on $\mathbb{R}^N$ as follows. Let $\Pi^\infty = \Pi^\infty(\mathbb{R}^N_y; \mathcal{H})$ be the space of those functions $u \in \mathcal{B}(\mathbb{R}^N_y)$ for which a complex number $\widetilde{u}$ exists such that $u^\varepsilon \to \widetilde{u}$ in $L^\infty(\mathbb{R}^N_x)$-weak $*$ as $\varepsilon \to 0$. This yields a linear operator M from $\mathcal{B}(\mathbb{R}^N_y)$ to $\mathbb{C}$ whose domain is $D(M) = \Pi^\infty$ and whose value at $u \in D(M)$ is $M(u) = \widetilde{u}$ (the above limit).

It is not hard to check that Π^∞ is a closed vector subspace of $\mathcal{B}(\mathbb{R}^N)$ containing the constants. Furthermore, the following properties are trivial: $M(u) \ge 0$ for $u \in \Pi^\infty$ with $u \ge 0$, $M(1) = 1$. Finally, Π^∞ is translation invariant, i.e., we have $\tau_a u \in \Pi^\infty$ whenever $u \in \Pi^\infty$ and $a \in \mathbb{R}^N_y$ (where $\tau_a u(y) = u(y-a)$ for $y \in \mathbb{R}^N$), and further $M(\tau_a u) = M(u)$. This follows immediately by a simple adaptation of the proof of [31, Theorem 4.1]. Thus, M is a mean value on $\mathbb{R}^N$ (see Definition 2.1 of [31]). Specifically, M is *the mean value on* $\mathbb{R}^N$ *for* $\mathcal{H}$.

2.2. **Definition and basic properties of a homogenization algebra.** Let the basic notation be as above.

Definition 2.1. *We term a homogenization algebra (or an H-algebra) on* $\mathbb{R}^N$ *(for* $\mathcal{H}$*), any closed subalgebra* A *of* $\mathcal{B}(\mathbb{R}^N_y)$ *with the following properties:*

(HA)$_1$ *A with the supremum norm is separable.*

(HA)$_2$ *A contains the constants.*

(HA)$_3$ *If $u \in A$, then $\overline{u} \in A$ ($\overline{u}$ the complex conjugate of u).*

(HA)$_4$ $A \subset D(M) = \Pi^\infty$.

In the sequel the H-algebra A is assumed to be equipped with the supremum norm. Thus, A is a commutative $\mathcal{C}^*$-algebra with identity. We denote the spectrum of A by $\Delta(A)$ (the set of all nonzero multiplicative linear forms on A), the latter being endowed with the Gelfand topology, i.e., the relative weak $*$ topology on A' (topological dual of A). As is classical (see, e.g., [22, p.71], [15, p.304]), $\Delta(A)$ is a compact space. The Gelfand transformation on A will be denoted by $\mathcal{G}$. For the benefit of the reader we recall that $\mathcal{G}$ is defined to be the mapping of A into $\mathcal{C}(\Delta(A))$ such that $\mathcal{G}(u)(s) = \langle s, u\rangle$ for $s \in \Delta(A)$ and $u \in A$, where the brackets stand for the duality pairing between A' and A. One classical result on which we will greatly lean is the so-called commutative Gelfand-Naimark theorem [22, p.277], which says that $\mathcal{G}$ is an isometric isomorphism of the $\mathcal{C}^*$-algebra A onto the $\mathcal{C}^*$-algebra $\mathcal{C}(\Delta(A))$. It results from this that the space $\mathcal{C}(\Delta(A))$ is separable, thanks to $(\mathrm{HA})_1$. We deduce using a classical result (see, e.g., [6, TGX. 24]) that the compact space $\Delta(A)$ is metrizable.

Except where otherwise stated, $\Delta(A)$ is provided with the so-called *M-measure* for A, denoted below by β. It is worth reminding that β is the positive Radon measure on $\Delta(A)$, of total mass 1, such that

$$M(u) = \int_{\Delta(A)} \mathcal{G}(u)(s)\, d\beta(s) \qquad\qquad (u \in A).$$

We refer to [30] for more detail about β.

The next proposition includes a few other useful properties of H-algebras.

Proposition 2.1. *Let $p \in \mathbb{R}$, $p > 0$. For $u \in A$, we have $|u|^p \in A$ with $\mathcal{G}(|u|^p) = |\mathcal{G}(u)|^p$ and $M(|u|^p) = \int_{\Delta(A)} |\mathcal{G}(u)(s)|^p\, d\beta(s)$.*

Proof. For p and u as stated, it is clear that $|\mathcal{G}(u)|^p$ lies in $\mathcal{C}(\Delta(A))$. Therefore, we may consider $v \in A$ such that $\mathcal{G}(v) = |\mathcal{G}(u)|^p$. For $y \in \mathbb{R}^N$, it follows $v(y) = \mathcal{G}(v)(\delta_y) = |\mathcal{G}(u)(\delta_y)|^p = |u(y)|^p$, where δ_y denotes the Dirac measure on $\mathbb{R}^N$ at y. Hence the proposition follows readily. ∎

We turn now our attention to a concept of degeneracy.

Definition 2.2. *The H-algebra A is said to be nondegenerate if the only function $u \in A$ verifying $u \geq 0$ and $M(u) = 0$ is the zero function in $\mathcal{B}(\mathbb{R}^N)$. Otherwise A is termed degenerate.*

Proposition 2.2. *The following two assertions are equivalent.*
(i) A is nondegenerate.
(ii) $Supp\beta = \Delta(A)$.

Proof. Suppose (i) holds. We claim that (ii) is true. Otherwise let r be some point in $\Delta(A)$ lying off $Supp\beta$ (the support of β). By Urysohn's lemma we may consider some $\varphi \in \mathcal{C}(\Delta(A))$ such that $\varphi \geq 0$, $\varphi(r) = 1$ and $\varphi = 0$ on $Supp\beta$. Clearly $\beta(\varphi) \equiv \int_{\Delta(A)} \varphi(s)\, d\beta(s) = 0$. Therefore, letting $u = \mathcal{G}^{-1}(\varphi)$, it follows $M(u) = 0$. Since $u \geq 0$ (indeed, it is a classical fact that $\mathcal{G}$ and $\mathcal{G}^{-1}$ are order preserving), we see by (i) that $u = 0$. Hence $\varphi(s) = \mathcal{G}(u)(s) = 0$ for any $s \in \Delta(A)$, a contradiction and so (ii) is true. Reciprocally, assume (ii) and let $u \in A$ with $u \geq 0$ and $M(u) = 0$. Then $\varphi = \mathcal{G}(u) \geq 0$ and $\beta(\varphi) = 0$. Consequently $\varphi = 0$ on $Supp\beta$ (see, e.g., [4, p. 69]); hence $\varphi(s) = 0$ for all $s \in \Delta(A)$, according to (ii). Therefore $u = 0$ and so (i) follows. ∎

2.3. **Almost periodic H-algebras.** Our purpose is to present typical examples of H-algebras. First of all, we recall that by an almost periodic continuous function on $\mathbb{R}^N$ is meant any $u \in \mathcal{B}(\mathbb{R}^N)$ whose translates $\{\tau_a u : a \in \mathbb{R}^N\}$ (recall that $\tau_a u(y) = u(y-a)$ for $y \in \mathbb{R}^N$) form a relatively compact set in $\mathcal{B}(\mathbb{R}^N)$. The space of such functions is commonly denoted by $AP(\mathbb{R}^N)$, and is a Banach space under the supremum norm. Specifically, $AP(\mathbb{R}^N)$ with the supremum norm and the usual algebra operations in $\mathcal{B}(\mathbb{R}^N)$ is a commutative $\mathcal{C}^*$-algebra with identity. On the other hand, given $u \in AP(\mathbb{R}^N)$, it can be shown that the closed convex hull of $\{\tau_a u : a \in \mathbb{R}^N\}$ in $\mathcal{B}(\mathbb{R}^N)$ contains one and only one constant $m(u)$ called the mean of u (see [20, p.94] and [31]). This yields a mapping $u \to m(u)$ of $AP(\mathbb{R}^N)$ into $\mathbb{C}$, which is linear, positive, translation invariant, and which attains the value 1 on the constant function 1. Therefore, this determines a mean value m on $\mathbb{R}^N$ with $D(m) = AP(\mathbb{R}^N)$, called the mean value (on $\mathbb{R}^N$) for $AP(\mathbb{R}^N)$. Interesting enough, M (the mean value on $\mathbb{R}^N$ for $\mathcal{H}$) is an extension of m, as shown below.

Proposition 2.3. *We have* $AP(\mathbb{R}^N) \subset \Pi^\infty$ *and* $m(u) = M(u)$ *for all* $u \in AP(\mathbb{R}^N)$.

Proof. To begin, let Γ be the algebra of all functions $u : \mathbb{R}^N \to \mathbb{C}$ of the form

$$u(y) = \sum_k c_k \exp(2i\pi k \cdot y) \qquad (y \in \mathbb{R}^N),$$

where k ranges over a finite subset of $\mathbb{R}^N$ (depending on u), and the dot denotes the usual Euclidean inner product in $\mathbb{R}^N$. Each such u is called a trigonometric polynomial on $\mathbb{R}^N$. We have $\Gamma \subset AP(\mathbb{R}^N)$ and further Γ is dense in $AP(\mathbb{R}^N)$ (see, e.g., [20, chap.5], [22, chap.10]). Thus, the proposition is proved if we can check that for each $u \in \Gamma$, we have $u^\varepsilon \to m(u)$ in $L^\infty(\mathbb{R}^N)$-weak $*$ as $\varepsilon \to 0$. Clearly it is enough to verify this for $u = \gamma_k$ $(k \in \mathbb{R}^N)$, where $\gamma_k(y) = \exp(2i\pi k \cdot y)$ for $y \in \mathbb{R}^N$. In other words, the whole problem reduces to showing that, given any arbitrary $f \in L^1(\mathbb{R}^N_x)$ (f independent of ε), we have as $\varepsilon \to 0$,

$$\int \gamma_k^\varepsilon f dx \to m(\gamma_k) \int f dx$$

for all $k \in \mathbb{R}^N$. This is trivial if $k = \omega$ (the origin in $\mathbb{R}^N$), because $m(1) = 1$. So assume that $k \neq \omega$. Recalling that $m(\gamma_k) = 0$ in this case, we see that the proposition is proved once we have verified that $\lim_{\varepsilon \to 0} \mathcal{F}f(-H_\varepsilon(k)) = 0$, where $\mathcal{F}f$ stands for the Fourier transform of f. But this follows immediately by the Riemann-Lebesgue lemma. ∎

Thus, $AP(\mathbb{R}^N)$ is a closed subalgebra of $\mathcal{B}(\mathbb{R}^N)$ verifying properties $(\mathrm{HA})_2$-$(\mathrm{HA})_4$. Unfortunately $AP(\mathbb{R}^N)$ fails to carry out $(\mathrm{HA})_1$ and hence we are led to restrict ourselves to some specific subalgebras.

Let $\mathcal{R}$ be a countable subgroup of the additive group $\mathbb{R}_y^N$. We define

$$AP_{\mathcal{R}}\left(\mathbb{R}^N\right) = \left\{u \in AP\left(\mathbb{R}^N\right) : Sp\left(u\right) \subset \mathcal{R}\right\}$$

with $Sp\left(u\right) = \left\{k \in \mathbb{R}^N : M\left(\overline{\gamma}_k u\right) \neq 0\right\}$ (spectrum of u). Note that the spectrum of any function in $AP\left(\mathbb{R}^N\right)$ is a countable set, and so the definition of $AP_{\mathcal{R}}\left(\mathbb{R}^N\right)$ makes sense. Now, let $\Gamma_{\mathcal{R}}$ be the set of all functions of the form $\sum_k c_k\gamma_k$ with $c_k \in \mathbb{C}$ and $\gamma_k\left(y\right) = \exp\left(2i\pi k{\cdot}y\right)$ $\left(y \in \mathbb{R}^N\right)$, where k ranges over an arbitrary finite subset of $\mathcal{R}$. The set $\Gamma_{\mathcal{R}}$ is a subalgebra of $AP\left(\mathbb{R}^N\right)$, and $AP_{\mathcal{R}}\left(\mathbb{R}^N\right)$ coincides with the closure of $\Gamma_{\mathcal{R}}$ in $\mathcal{B}\left(\mathbb{R}^N\right)$ (see, e.g., [20, p.93, Proposition 5.4]). Hence, recalling Proposition 2.3, it becomes an elementary exercise to verify that $AP_{\mathcal{R}}\left(\mathbb{R}^N\right)$ is a homogenization algebra on $\mathbb{R}^N$ (for $\mathcal{H}$). We will refer to $AP_{\mathcal{R}}\left(\mathbb{R}^N\right)$ as the almost periodic H-algebra attached to $\mathcal{R}$.

Before going any further, let us recall a classical notion we will need. If G is a locally compact Abelian group, we denote its dual by $\widehat{G}$, i.e., $\widehat{G}$ is the group of all continuous homomorphisms of G into the unit circle $\mathbb{U} = \{\xi \in \mathbb{C} : |\xi| = 1\}$. With the topology of compact convergence on G, $\widehat{G}$ is a locally compact Abelian group. Points in $\widehat{G}$ are the so-called continuous characters of G. If $\gamma \in \widehat{G}$ and $y \in G$, it is customary to denote $\gamma\left(y\right)$ by $\langle\gamma, y\rangle$ or $\langle y, \gamma\rangle$.

Having made this point, let us keep in mind that the countable subgroup $\mathcal{R}$ of $\mathbb{R}^N$ introduced above is naturally provided with the discrete topology. Consequently, its dual group $\widehat{\mathcal{R}}$ is compact (see, e.g., [22, p.122]). We will also need the (group) homomorphism $\varphi : \mathbb{R}^N \to \widehat{\mathcal{R}}$ defined at each $y \in \mathbb{R}^N$ by

$$\langle\varphi\left(y\right), k\rangle = \gamma_k\left(y\right) = \exp\left(2i\pi k{\cdot}y\right) \qquad \left(k \in \mathcal{R}\right).$$

The function φ maps continuously $\mathbb{R}^N$ into $\widehat{\mathcal{R}}$ and, on the other hand, $\varphi\left(\mathbb{R}^N\right)$ is dense in $\widehat{\mathcal{R}}$ (this is a classical result; use, e.g., [16, p.98, (22.11.5)] if need be). Finally, the canonical isomorphism of $\mathcal{R}$ onto $\widehat{\widehat{\mathcal{R}}}$ (dual group of $\widehat{\mathcal{R}}$) will be denoted by $\boldsymbol{\psi}$. It is well to recall that $\boldsymbol{\psi}$ is given by $\langle\boldsymbol{\psi}\left(k\right), \gamma\rangle = \langle\gamma, k\rangle$ for $k \in \mathcal{R}$, $\gamma \in \widehat{\mathcal{R}}$. We are now in a position to prove the following result.

Proposition 2.4. *Let $A = AP_{\mathcal{R}}\left(\mathbb{R}^N\right)$. Then, the compact space $\Delta\left(A\right)$ can be provided with a group operation under which it is an Abelian group and further the Haar measure on $\Delta\left(A\right)$ is precisely the M-measure β.*

Proof. For each function of the form $u = \sum_k c_k\gamma_k$ $\left(c_k \in \mathbb{C}\right)$, where k ranges over a finite subset of $\mathcal{R}$ depending solely on u, let $T\left(u\right) = \sum_k c_k\boldsymbol{\psi}\left(k\right)$. This defines a linear mapping $T : \Gamma_{\mathcal{R}} \to \mathcal{C}(\widehat{\mathcal{R}})$ such that $\left\|T\left(u\right)\right\|_\infty = \left\|u\right\|_\infty$ and

$$T\left(u\right)\left(\varphi\left(y\right)\right) = u\left(y\right) \qquad \left(y \in \mathbb{R}^N\right) \tag{2.2}$$

for all $u \in \Gamma_{\mathcal{R}}$. Thanks to the fact that $\Gamma_{\mathcal{R}}$ is dense in A, we see that we can extend T by continuity to a continuous linear mapping, still denoted by T, of A into $\mathcal{C}(\widehat{\mathcal{R}})$. Moreover, the latter is an isometric homomorphism of the $\mathcal{C}^*$-algebra A into the $\mathcal{C}^*$-algebra $\mathcal{C}(\widehat{\mathcal{R}})$, and (2.2) holds for all $u \in A$. By using the classical property that $\widehat{\widehat{\mathcal{R}}}$ is total in $\mathcal{C}(\widehat{\mathcal{R}})$, it can be shown without difficulty that T is surjective and therefore an isometric isomorphism of the $\mathcal{C}^*$-algebra A onto the $\mathcal{C}^*$-algebra $\mathcal{C}(\widehat{\mathcal{R}})$. This being so, let L be the mapping of $\mathcal{C}(\Delta(A))$ into $\mathcal{C}(\widehat{\mathcal{R}})$ defined by $L(f) = T\left(\mathcal{G}^{-1}(f)\right)$ for $f \in \mathcal{C}(\Delta(A))$, where $\mathcal{G}$ is the Gelfand transformation on A. This mapping is clearly an isometric isomorphism of the $\mathcal{C}^*$-algebra $\mathcal{C}(\Delta(A))$ onto the $\mathcal{C}^*$-algebra $\mathcal{C}(\widehat{\mathcal{R}})$. Consequently, according to [22, p.90, Theorem 4.1.4], there exists a homeomorphism h of $\widehat{\mathcal{R}}$ onto $\Delta(A)$ such that $L(f)(t) = f(h(t))$ $(t \in \widehat{\mathcal{R}})$ for any $f \in \mathcal{C}(\Delta(A))$. Now, for s_1, s_2, $s \in \Delta(A)$, put $s_1 + s_2 = h(t_1 t_2)$ and $-s = h\left(t^{-1}\right)$ (observe that $\widehat{\mathcal{R}}$ is a multiplicative group), where $t_i = h^{-1}(s_i)$ $(i = 1, 2)$ and $t = h^{-1}(s)$. This defines a binary relation $+$ under which $\Delta(A)$ is an Abelian topological group (with the Gelfand topology) and h is a group homomorphism of $\widehat{\mathcal{R}}$ onto $\Delta(A)$. It remains to verify that the Haar measure on $\Delta(A)$ coincides with β (the M-measure for A). Clearly it amounts to verifying that β is translation invariant. For this purpose, introduce the mapping $j : \mathbb{R}^N \to \Delta(A)$ defined by $j(y) = \delta_y$ (Dirac measure at $y \in \mathbb{R}^N$). We need to show that j is a group homomorphism. It suffices to check that $j = h \circ \varphi$ (usual composition). Fix freely $y \in \mathbb{R}^N$. Letting $\widehat{u} = \mathcal{G}(u)$, we have $\widehat{u}(h(\varphi(y))) = L(\widehat{u})(\varphi(y)) = T(u)(\varphi(y)) = u(y) = \widehat{u}(j(y))$ for any $u \in A$. Hence $j(y) = h(\varphi(y))$ and so j is a group homomorphism, as claimed. With this in mind, let $u \in A$ and $a \in \mathbb{R}^N$. Then, clearly $\left(\tau_{j(a)}\mathcal{G}(u)\right)(j(y)) = \mathcal{G}(\tau_a u)(j(y))$ for all $y \in \mathbb{R}^N$. By the density of $j\left(\mathbb{R}^N\right)$ in $\Delta(A)$ (this is a classical result), it follows $\mathcal{G}(\tau_a u) = \tau_{j(a)}\mathcal{G}(u)$ for all $a \in \mathbb{R}^N$ and all $u \in A$. Therefore, using the fact that M is translation invariant, we deduce $\beta(\tau_s f) = \beta(f)$ for all $s \in j\left(\mathbb{R}^N\right)$ where f is freely fixed in $\mathcal{C}(\Delta(A))$. Hence, the translation invariance of β (i.e., $\beta(\tau_s f) = \beta(f)$ for $f \in \mathcal{C}(\Delta(A))$, $s \in \Delta(A)$) follows from the facts that $j\left(\mathbb{R}^N\right)$ is dense in $\Delta(A)$ and the mapping $s \to \beta(\tau_s f)$ sends continuously $\Delta(A)$ into $\mathbb{C}$. This completes the proof. ∎

As a direct consequence of the above proposition, there is the following corollary.

Corollary 2.1. *The H-algebra* $A = AP_{\mathcal{R}}\left(\mathbb{R}^N\right)$ *is nondegenerate (see Definition 2.2).*

Proof. Considering that the support of a Haar measure on a locally compact group is just the said group (this is a classical result), we see that the corollary follows immediately by Proposition 2.4 and use of Proposition 2.2. ∎

Remark 2.1. *In the course of the proof of Proposition 2.4 we have found that* $\Delta(A) = \widehat{\mathcal{R}}$ *(up to a topological group isomorphism), where* $A = AP_{\mathcal{R}}\left(\mathbb{R}^N\right)$.

The basic case of periodic H-algebras. Let $A=\mathcal{C}_{per}(Y)$ (see Section 1) with $Y=\left(-\frac{1}{2},\frac{1}{2}\right)^N$. It is an easy exercise to check that A is an H-algebra. We have here $M(u)=\int_Y u(y)\,dy$ for $u\in A$. Now, we observe that this H-algebra is only a particular almost periodic H-algebra. More precisely, we have $\mathcal{C}_{per}(Y)=AP_{\mathcal{R}=\mathbb{Z}^N}\left(\mathbb{R}^N\right)$, as is easily verified. Hence, according to Remark 2.1, $\Delta(A)=\mathbb{T}^N$ (the N-torus) with $A=\mathcal{C}_{per}(Y)$, of course; indeed $\widehat{\mathcal{R}}=\mathbb{T}^N\equiv(\mathbb{R}/\mathbb{Z})^N$ for $\mathcal{R}=\mathbb{Z}^N$ (see, e.g., [20]). Let us stress that the above equality between $\Delta(A)$ and $\mathbb{T}^N$ actually proceeds from an identification by means of a (topological) group isomorphism. In this connection, let π be the isometric isomorphism of $\mathcal{C}_{per}(Y)$ onto $\mathcal{C}\left(\mathbb{T}^N\right)$ such that $\pi(u)(p(y))=u(y)$ $\left(y\in\mathbb{R}^N\right)$ for $u\in\mathcal{C}_{per}(Y)$, where p denotes the canonical homomorphism of $\mathbb{R}^N$ onto $\mathbb{T}^N$. Then, for any $u\in A=\mathcal{C}_{per}(Y)$, we have

$$\int_Y u(y)\,dy=\int_{\Delta(A)}\mathcal{G}(u)(s)\,d\beta(s)=\int_{\mathbb{T}^N}\pi(u)(z)\,dz,$$

where dz denotes Haar measure on the compact group $\mathbb{T}^N$.

Remark 2.2. *More generally, let $\{b_1,...,b_N\}$ be a (nonnecessarily orthogonal) basis of $\mathbb{R}^N$ (viewed as an N-dimensional vector space over $\mathbb{R}$). Let S be the set of all $k\in\mathbb{R}^N$ of the form $k=\sum_{i=1}^N t_ib_i$ $(t_i\in\mathbb{Z})$, and let*

$$Y=\left\{y\in\mathbb{R}^N: y=\sum_{i=1}^N r_ib_i,\ -\frac{1}{2}\leq r_i\leq\frac{1}{2}\right\}.$$

A continuous complex function u on $\mathbb{R}^N$ is said to be Y-periodic if $u(y+k)=u(y)$ for all $y\in\mathbb{R}^N$ and all $k\in S$. We define $P_Y\left(\mathbb{R}^N\right)$ to be the space of all such functions. There is no serious difficulty in showing that $P_Y\left(\mathbb{R}^N\right)=AP_{\mathcal{R}=S^*}\left(\mathbb{R}^N\right)$ where $S^*=\left\{l\in\mathbb{R}^N: l{\cdot}k\in\mathbb{Z}\text{ for all }k\in S\right\}$ (the dot denotes the usual Euclidean inner product in $\mathbb{R}^N$). Thus, $P_Y\left(\mathbb{R}^N\right)$ is an H-algebra on $\mathbb{R}^N$ (for $\mathcal{H}$). It can be shown that the above development regarding $\mathcal{C}_{per}(Y)$ carries over mutatis mutandis to the present general setting.

2.4. **Further examples of homogenization algebras.** The space A in each of the following examples has proved to be an H-algebra on $\mathbb{R}^N$ for $\mathcal{H}$ (see, e.g., [30]).

Example 2.1. *Put $A=\mathcal{B}_\infty\left(\mathbb{R}^N_y\right)$, where $\mathcal{B}_\infty\left(\mathbb{R}^N_y\right)$ denotes the space of those continuous complex functions on $\mathbb{R}^N_y$ that converge (to a finite number) at infinity. We have here $M(u)=\lim_{|y|\to\infty}u(y)$ for $u\in A$, and it is evident that A is a degenerate H-algebra.*

Example 2.2. *Let $A=\mathcal{B}_{\infty,per}(Y)$ be the closure in $\mathcal{B}\left(\mathbb{R}^N_y\right)$ of the space of functions of the form $u=\sum\varphi_iu_i$ with a summation of finitely many terms, where $\varphi_i\in\mathcal{B}_\infty\left(\mathbb{R}^N_y\right)$, $u_i\in\mathcal{C}_{per}(Y)$ with $Y=\left(-\frac{1}{2},\frac{1}{2}\right)^N$. This is an H-algebra.*

Example 2.3. *More generally, let $\mathcal{R}$ be a countable subgroup of the additive group $\mathbb{R}^N$. Define $\mathcal{B}_{\infty,\mathcal{R}}\left(\mathbb{R}^N\right)$ to be the closure in $\mathcal{B}\left(\mathbb{R}_y^N\right)$ of the space of functions $u=\sum_{finite}\varphi_i u_i$ with $\varphi_i \in \mathcal{B}_\infty\left(\mathbb{R}^N\right)$, $u_i \in AP_{\mathcal{R}}\left(\mathbb{R}^N\right)$. The space $A=\mathcal{B}_{\infty,\mathcal{R}}\left(\mathbb{R}^N\right)$ is an H-algebra.*

Remark 2.3. *The H-algebras of examples 2.2 and 2.3 are degenerate.*

Example 2.4. *Let A_1 be an H-algebra on $\mathbb{R}^{N-1}$, and let $\mathcal{B}_\infty\left(\mathbb{R};A_1\right)$ be the space of all continuous functions $u:\mathbb{R}\to A_1$ such that $\lim_{\tau\to\infty}\left\|u\left(\tau\right)-\varsigma\right\|_\infty=0$, where $\varsigma\in A_1$ (ς depending on u). The space $A=\mathcal{B}_\infty\left(\mathbb{R};A_1\right)$ is an H-algebra on $\mathbb{R}^N$.*

2.5. **The spaces $\mathfrak{X}_A^p\left(\mathbb{R}_y^N\right)$ $(1\leq p<\infty)$.** The present and next subsections are concerned with function spaces of great interest in deterministic homogenization theory.

For each real $p\geq 1$, we first of all introduce the space $\Xi^p\left(\mathbb{R}_y^N\right)$ of those functions $u\in L_{loc}^p\left(\mathbb{R}_y^N\right)$ for which the sequence $\left(u^\varepsilon\right)_{0<\varepsilon\leq 1}$ (u^ε defined in subsection 2.1) is bounded in $L_{loc}^p\left(\mathbb{R}_x^N\right)$. This is clearly a vector subspace of $L_{loc}^p\left(\mathbb{R}_y^N\right)$. Let

$$\|u\|_{\Xi^p}=\sup_{0<\varepsilon\leq 1}\left(\int_{B_N}\left|u\left(H_\varepsilon\left(x\right)\right)\right|^p dx\right)^{\frac{1}{p}}\qquad\left(u\in\Xi^p\left(\mathbb{R}_y^N\right)\right),$$

where B_N denotes the open unit ball of $\mathbb{R}_x^N$. This defines a norm on $\Xi^p\left(\mathbb{R}_y^N\right)$, which makes the latter a Banach space (the verification is a routine exercise left to the reader).

Now, let A be an H-algebra on $\mathbb{R}^N$ (for $\mathcal{H}$). For each real $p\geq 1$, we define $\mathfrak{X}_A^p\left(\mathbb{R}_y^N\right)$ (or simply $\mathfrak{X}_A^p$, or even $\mathfrak{X}^p$ when there is no danger of confusion) as being the closure of A in $\Xi^p\left(\mathbb{R}_y^N\right)$. Provided with the Ξ^p-norm, $\mathfrak{X}_A^p$ is a Banach space.

Let us turn to the proofs of some fundamental results that were pointed out earlier in [30].

Proposition 2.5. *The mean value M on $\mathbb{R}^N$ for $\mathcal{H}$ (see subsection 2.1) viewed as defined on A, extends by continuity to a (unique) continuous linear form on $\mathfrak{X}_A^p$ still denoted M. Furthermore, given $u\in\mathfrak{X}_A^p$ and a fixed bounded open set Ω in $\mathbb{R}_x^N$, we have $u^\varepsilon\to M\left(u\right)$ in $L^p\left(\Omega\right)$-weak as $\varepsilon\to 0$, where u^ε is considered as defined on Ω.*

Proof. For $\psi\in A$, we have

$$\left|\int_{B_N}\psi\left(H_\varepsilon\left(x\right)\right)dx\right|\leq\left|B_N\right|^{\frac{1}{p'}}\|\psi\|_{\Xi^p}\qquad\left(0<\varepsilon\leq 1\right),$$

where $\left|B_N\right|$ stands for the measure of B_N (with respect to Lebesgue measure on $\mathbb{R}^N$). As $\varepsilon\to 0$, it follows $\left|M\left(\psi\right)\right|\leq\left|B_N\right|^{-\frac{1}{p}}\|\psi\|_{\Xi^p}$, from which we deduce the first part of the proposition by extension by continuity. Now, let u and

Ω be as stated above. If $u\in A$, then it is evident that $u^{\varepsilon}\rightarrow M\left(u\right)$ in $L^{p}\left(\Omega\right)$-weak as $\varepsilon\rightarrow 0$. So, in what follows we assume that u is an arbitrarily given function in $\mathfrak{X}_{A}^{p}$. Let $\varphi\in L^{p'}\left(\Omega\right)$ ($\frac{1}{p'}=1-\frac{1}{p}$), φ assumed to be a nonzero function. Fix freely $\eta>0$. Thanks to the density of A in $\mathfrak{X}_{A}^{p}$, we may consider some $\psi\in A$ such that

$$\left(\int_{\Omega}\left|u^{\varepsilon}-\psi^{\varepsilon}\right|^{p}dx\right)^{\frac{1}{p}}\leq\frac{\eta}{3\left\|\varphi\right\|_{L^{p'}(\Omega)}}\qquad\left(0<\varepsilon\leq 1\right)$$

and

$$\left|M\left(u-\psi\right)\int_{\Omega}\varphi dx\right|\leq\frac{\eta}{3}\quad\text{(use the first part of Proposition 2.5).}$$

On the other hand, as pointed out above, there is some real $0<r\leq 1$ such that

$$\left|\int_{\Omega}\psi^{\varepsilon}\varphi dx-M\left(\psi\right)\int_{\Omega}\varphi dx\right|\leq\frac{\eta}{3}$$

for all $0<\varepsilon\leq r$. Hence, by writing

$$\int_{\Omega}u^{\varepsilon}\varphi dx-M\left(u\right)\int_{\Omega}\varphi dx=\int_{\Omega}\left(u^{\varepsilon}-\psi^{\varepsilon}\right)\varphi dx+\int_{\Omega}\psi^{\varepsilon}\varphi dx$$
$$-M\left(\psi\right)\int_{\Omega}\varphi dx+M\left(\psi-u\right)\int_{\Omega}\varphi dx,$$

we see immediately that

$$\left|\int_{\Omega}u^{\varepsilon}\varphi dx-M\left(u\right)\int_{\Omega}\varphi dx\right|\leq\eta$$

for all $0<\varepsilon\leq r$. The proposition follows thereby. ∎

Proposition 2.6. *The Gelfand transformation $\mathcal{G}:A\rightarrow\mathcal{C}\left(\Delta\left(A\right)\right)$ extends by continuity to a (unique) continuous linear mapping of $\mathfrak{X}_{A}^{p}$ into $L^{p}\left(\Delta\left(A\right)\right)$ still denoted by $\mathcal{G}$.*

Proof. Let $u\in A$. Then,

$$\int_{B_{N}}\left|u\left(H_{\varepsilon}\left(x\right)\right)\right|^{p}dx\leq\left\|u\right\|_{\Xi^{p}}^{p}\qquad\left(0<\varepsilon\leq 1\right).$$

Letting $\varepsilon\rightarrow 0$, it follows $M\left(\left|u\right|^{p}\right)^{\frac{1}{p}}\leq\left|B_{N}\right|^{-\frac{1}{p}}\left\|u\right\|_{\Xi^{p}}$, hence

$\left\|\mathcal{G}\left(u\right)\right\|_{L^{p}(\Delta(A))}\leq\left|B_{N}\right|^{-\frac{1}{p}}\left\|u\right\|_{\Xi^{p}}$, according to Proposition 2.1. The proposition follows by extention by continuity, A being dense in $\mathfrak{X}_{A}^{p}$. ∎

Remark 2.4. *The mapping $\mathcal{G}:\mathfrak{X}_{A}^{p}\rightarrow L^{p}\left(\Delta\left(A\right)\right)$ derived from Proposition 2.6 is referred to as the canonical mapping of $\mathfrak{X}_{A}^{p}$ into $L^{p}\left(\Delta\left(A\right)\right)$.*

The preceding proposition has three important corollaries.

Corollary 2.2. *We have $M\left(u\right)=\int_{\Delta(A)}\mathcal{G}\left(u\right)d\beta$ for $u\in\mathfrak{X}_{A}^{p}$, where M and $\mathcal{G}$ denote the extension mappings constructed in Propositions 2.5-2.6, respectively.*

Proof. This is straightforward by the said propositions and use of the definition of the measure β (see subsection 2.2). ∎

Corollary 2.3. *Let* $1 < p,\, q < +\infty$ *with* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \leq 1$. *If* $u \in \mathfrak{X}^p = \mathfrak{X}^p_A$ *and* $v \in \mathfrak{X}^q$, *then* $uv \in \mathfrak{X}^r$ *and* $\mathcal{G}(uv) = \mathcal{G}(u)\,\mathcal{G}(v)$.

Proof. This follows readily by Proposition 2.6 and use of Hölder's inequality. ∎

Corollary 2.4. *The following assertions are true for* $1 \leq p < \infty$:

(i) If $u \in \mathfrak{X}^p$, *then* $\overline{u} \in \mathfrak{X}^p$ *and* $\mathcal{G}(\overline{u}) = \overline{\mathcal{G}(u)}$.

(ii) If $u \in \mathfrak{X}^p$, *then* $|u|^p \in \mathfrak{X}^1$ *and* $\mathcal{G}(|u|^p) = |\mathcal{G}(u)|^p$.

(iii) If $\psi \in A$ *and* $u \in \mathfrak{X}^p$, *then* $\psi u \in \mathfrak{X}^p$ *and* $\mathcal{G}(\psi)\,\mathcal{G}(u) = \mathcal{G}(\psi u)$.

(iv) If $u \in \mathfrak{X}^1$ *and further* u *is real valued, then* $\mathcal{G}(u)$ *is real valued. If moreover* $u \geq 0$ *a.e. (almost everywhere), then* $\mathcal{G}(u) \geq 0$ *a.e.*

(v) If $u \in \mathfrak{X}^1 \cap L^\infty$, *then* $\mathcal{G}(u) \in L^\infty(\Delta(A))$ *and*

$$\|\mathcal{G}(u)\|_{L^\infty(\Delta(A))} \leq \|u\|_{L^\infty}.$$

Proof. (i) follows by Proposition 2.6 and use of the equality $\overline{\mathcal{G}(u)} = \mathcal{G}(\overline{u})$ for $u \in A$. We turn now to the proof of (ii). Let $u \in \mathfrak{X}^p$. Choose some sequence (u_n) in A such that $u_n \to u$ in $\Xi^p(\mathbb{R}^N)$ as $n \to \infty$. By taking $a = u_n(y)$ and $b = u(y)$ (where the integer $n > 0$ and the point $y \in \mathbb{R}^N$ are arbitrarily fixed) in the simple inequality

$$\left||a|^p - |b|^p\right| \leq p\,|a-b|\,(|a| + |b|)^{p-1} \qquad (a, b \in \mathbb{C})$$

and then using an obvious Hölder's inequality, we get

$$\left\||u_n|^p - |u|^p\right\|_{\Xi^1} \leq c\,\|u_n - u\|_{\Xi^p}$$

with $c = p\sup_{m>0}\left\||u_m| - |u|\right\|^{p-1}_{\Xi^p} < \infty$. We deduce that $|u_n|^p \to |u|^p$ in Ξ^1 as $n \to \infty$, hence $|u|^p \in \mathfrak{X}^1$, since $|u_n|^p \in A$ (Proposition 2.1). On the other hand, according to Proposition 2.6, we have in the $L^1(\Delta(A))$-norm,

$$\mathcal{G}(|u_n|^p) \to \mathcal{G}(|u|^p) \text{ and } |\mathcal{G}(u_n)|^p \to |\mathcal{G}(u)|^p \text{ as } n \to \infty.$$

Therefore the rest of (ii) follows by Proposition 2.1, once again. Assertion (iii) being straightforward, let us next verify (iv). For this purpose, fix freely $u \in \mathfrak{X}^1$. Suppose u is real valued. Then, by (i) we have $\mathcal{G}(u) = \overline{\mathcal{G}(u)}$ and so $\mathcal{G}(u)$ is real valued too. Suppose further that $u \geq 0$ a.e. Let $\psi \in A$ with $\psi \geq 0$. Then $\psi u \in \mathfrak{X}^1$ with $\psi u \geq 0$ a.e., hence $M(\psi u) \geq 0$ (use Proposition 2.5). Consequently

$$\int_{\Delta(A)} \mathcal{G}(\psi)\,\mathcal{G}(u)\,d\beta \geq 0,$$

as is straightforward by (iii) and use of Corollary 2.2. Thus, $\int_{\Delta(A)} \varphi\mathcal{G}(u)\,d\beta \geq 0$ for all $\varphi \in \mathcal{C}(\Delta(A))$ with $\varphi \geq 0$. This shows that $\mathcal{G}(u) \geq 0$ a.e. (see, e.g., [5, p.47, Corol.3]). We will finally establish (v). Let $u \in \mathfrak{X}^1 \cap L^\infty$. Since $|u| \leq \|u\|_{L^\infty}$ a.e., we have $|\psi u| \leq \|u\|_{L^\infty}|\psi|$ a.e. for all $\psi \in A$. Thus

$M(|\psi u|) \leq \|u\|_{L^\infty} M(|\psi|)$ for all $\psi \in A$ (see Proposition 2.5). We deduce by Corollary 2.2 and use of parts (ii) and (iii) that

$$\left| \int_{\Delta(A)} \mathcal{G}(\psi) \mathcal{G}(u) \, d\beta \right| \leq \|u\|_{L^\infty} \int_{\Delta(A)} |\mathcal{G}(\psi)| \, d\beta$$

for all $\psi \in A$, or equivalently,

$$\left| \int_{\Delta(A)} \varphi \mathcal{G}(u) \, d\beta \right| \leq \|u\|_{L^\infty} \|\varphi\|_{L^1(\Delta(A))}$$

for all $\varphi \in \mathcal{C}(\Delta(A))$. Hence (v) follows. ∎

Remark 2.5. *Let $A = \mathcal{C}_{per}(Y)$ with $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^N$ (see subsection 2.3). Then $\mathfrak{X}^p_A = L^p_{per}(Y)$ ($1 \leq p < \infty$), where the right-hand side denotes the space of Y-periodic functions in $L^p_{loc}(\mathbb{R}^N)$. Indeed, this follows immediately by two facts: 1) the space $\Xi^p(\mathbb{R}^N)$ is continuously embedded in $L^p_{loc}(\mathbb{R}^N)$; 2) the space $L^p_{per}(Y)$ is continuously embedded in $\Xi^p(\mathbb{R}^N)$, as is straightforward by* [26, Lemma 1].

2.6. **Sobolev spaces $W^{m,p}(\Delta(A))$.** Let A be an H-algebra on $\mathbb{R}^N$ (for $\mathcal{H}$). Before we can define so-called Sobolev spaces on $\Delta(A)$, we need to introduce the notion of a partial derivative on $\Delta(A)$. This will be achieved by carrying over the usual derivatives on $\mathbb{R}^N$. Specifically, for any integer $m \geq 1$, let

$$A^m = \left\{ \psi \in \mathcal{C}^m\left(\mathbb{R}^N_y\right) : D^\alpha_y \psi \in A \text{ for } \alpha \in \mathbb{N}^N, \ |\alpha| \leq m \right\}$$

and

$$\|\psi\|_m = \sup_{|\alpha| \leq m} \left\| D^\alpha_y \psi \right\|_\infty \qquad (\psi \in A^m),$$

where $D^\alpha_y = \frac{\partial^{|\alpha|}}{\partial y_1^{\alpha_1} \ldots \partial y_N^{\alpha_N}}$. Provided with the norm $\|\cdot\|_m$, A^m is a Banach space. Furthermore, put

$$A^\infty = \underset{m \geq 1}{\cap} A^m.$$

We provide A^∞ with the locally convex topology defined by the family of norms $\|\cdot\|_m$ ($m \geq 1$), which makes it a Fréchet space. Finally, set

$$\mathcal{D}^m(\Delta(A)) = \left\{ \varphi \in \mathcal{C}(\Delta(A)) : \mathcal{G}^{-1}(\varphi) \in A^m \right\} \quad (m \geq 1)$$

$$\mathcal{D}(\Delta(A)) = \left\{ \varphi \in \mathcal{C}(\Delta(A)) : \mathcal{G}^{-1}(\varphi) \in A^\infty \right\}.$$

Remark 2.6. *$\mathcal{D}^m(\Delta(A)) = \mathcal{G}(A^m)$ and $\mathcal{D}(\Delta(A)) = \mathcal{G}(A^\infty)$.*

We are now is a position to define partial derivatives on $\Delta(A)$.

Definition 2.3. *By the partial derivative of index i ($1 \leq i \leq N$) on $\Delta(A)$ we shall understand the unbounded linear operator ∂_i from $\mathcal{C}(\Delta(A))$ to $\mathcal{C}(\Delta(A))$ defined as $D(\partial_i) = \mathcal{D}^1(\Delta(A))$ ($D(\partial_i)$ stands for the domain of ∂_i), $\partial_i \varphi = \left(\mathcal{G} \circ \frac{\partial}{\partial y_i} \circ \mathcal{G}^{-1}\right) \varphi$ for $\varphi \in \mathcal{D}^1(\Delta(A))$.*

More generally, the partial derivative of index $\alpha \in \mathbb{N}^N$ on $\Delta(A)$ is defined to be the unbounded linear operator ∂^α from $\mathcal{C}(\Delta(A))$ to $\mathcal{C}(\Delta(A))$ such that $D(\partial^\alpha) = \mathcal{D}^{|\alpha|}(\Delta(A))$ and $\partial^\alpha \varphi = \left(\mathcal{G} \circ D_y^\alpha \circ \mathcal{G}^{-1}\right)\varphi$ for $\varphi \in \mathcal{D}^{|\alpha|}(\Delta(A))$. We equip $\mathcal{D}^m(\Delta(A))$ with the norm $\|\varphi\|_m = \sup_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_\infty \quad (\varphi \in \mathcal{D}^m(\Delta(A)))$, and $\mathcal{D}(\Delta(A))$ with the family of norms $\|\cdot\|_m$ $(m \geq 1)$. It is easily seen that $\mathcal{D}^m(\Delta(A))$ is a Banach space and $\mathcal{D}(\Delta(A))$ is a Fréchet space. Furthermore, $\mathcal{G}$ maps A^m isometrically onto $\mathcal{D}^m(\Delta(A))$ and A^∞ isomorphically onto $\mathcal{D}(\Delta(A))$.

The topological dual of $\mathcal{D}(\Delta(A))$ is denoted by $\mathcal{D}'(\Delta(A))$. We assume that $\mathcal{D}'(\Delta(A))$ is provided with the strong dual topology. Each $T \in \mathcal{D}'(\Delta(A))$ is called a distribution on $\Delta(A)$; the value of T at some $\varphi \in \mathcal{D}(\Delta(A))$ is denoted by $\langle T, \varphi \rangle$. The derivative of index $\alpha \in \mathbb{N}^N$ of T is defined to be the distribution $\partial^\alpha T$ on $\Delta(A)$ given by $\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \varphi \rangle$ for $\varphi \in \mathcal{D}(\Delta(A))$. It is an easy exercise to verify that the transformation $T \to \partial^\alpha T$ maps continuously and linearly $\mathcal{D}'(\Delta(A))$ into itself.

In passing we wish to draw attention to one basic result.

Proposition 2.7. *For any* $\varphi \in \mathcal{D}^m(\Delta(A))$ $(m \geq 1)$ *and any multi-index* α *with* $1 \leq |\alpha| \leq m$, *we have* $\int_{\Delta(A)} \partial^\alpha \varphi(s)\, d\beta(s) = 0$.

Proof. Clearly it is enough to assume that $m = 1$. Thus, the problem reduces to verifying that $\int_{\Delta(A)} \partial_i \varphi d\beta = 0$ for $\varphi \in \mathcal{D}^1(\Delta(A))$ and $1 \leq i \leq N$. We will need the equality

$$M(g * u) = M(u) \int g(y)\, dy$$

for $u \in \Pi^\infty$ and $g \in L^1\left(\mathbb{R}^N\right)$ (see [31, Proposition 4.1]), where $*$ denotes the convolution on $\mathbb{R}^N$. So, letting $\psi = \mathcal{G}^{-1}(\varphi)$, where φ is as above, we see that the proposition is proved if we can check that $M\left(\frac{\partial \psi}{\partial y_i}\right) = 0$, $1 \leq i \leq N$. But this is straightforward. Indeed, let $f \in \mathcal{D}\left(\mathbb{R}^N\right) = \mathcal{C}_0^\infty\left(\mathbb{R}^N\right)$ with $\int f(y)\, dy = 1$. By the above equality, we have $M\left(\frac{\partial \psi}{\partial y_i}\right) = M\left(f * \frac{\partial \psi}{\partial y_i}\right)$. Recalling that $f * \frac{\partial \psi}{\partial y_i} = \psi * \frac{\partial f}{\partial y_i}$, and appealing to the above equality, once again, we get on the other hand $M\left(f * \frac{\partial \psi}{\partial y_i}\right) = M(\psi) \int \frac{\partial f}{\partial y_i} dy = 0$. Hence the proposition follows. ∎

Throughout the rest of the study it is assumed that A^∞ is dense in A (this amounts to saying that $\mathcal{D}(\Delta(A))$ is dense in $\mathcal{C}(\Delta(A))$). It is worth noting that this hypothesis is always satisfied in practice. Then, it becomes possible to identify any given function $u \in L^1(\Delta(A))$ with the distribution $T_u \in \mathcal{D}'(\Delta(A))$ defined by

$$\langle T_u, \varphi \rangle = \int_{\Delta(A)} u(s)\, \varphi(s)\, d\beta(s) \qquad (\varphi \in \mathcal{D}(\Delta(A))).$$

Hence $L^p(\Delta(A)) \subset \mathcal{D}'(\Delta(A))$ $(1 \leq p \leq \infty)$ with continuous embedding. Consequently, given a real $p \geq 1$ and an integer $m \geq 1$, we may define

$$W^{m,p}(\Delta(A)) = \{u \in L^p(\Delta(A)) : \partial^\alpha u \in L^p(\Delta(A)) \text{ for } |\alpha| \leq m\},$$

where the partial derivatives $\partial^\alpha u$ are computed in the distribution sense on $\Delta(A)$, of course. Provided with the norm

$$\|u\|_{W^{m,p}(\Delta(A))} = \left(\sum_{|\alpha|\leq m} \|\partial^\alpha u\|^p_{L^p(\Delta(A))}\right)^{\frac{1}{p}} \qquad (u \in W^{m,p}(\Delta(A))),$$

$W^{m,p}(\Delta(A))$ is a Banach space (in particular $W^{m,2}(\Delta(A))$ is a Hilbert space).

However, in practice the appropriate space is not the whole $W^{m,p}(\Delta(A))$ but its closed subspace

$$W^{m,p}(\Delta(A))/\mathbb{C} = \left\{u \in W^{m,p}(\Delta(A)) : \int_{\Delta(A)} u d\beta = 0\right\}$$

equipped with the seminorm

$$\|u\|_{W^{m,p}(\Delta(A))/\mathbb{C}} = \left(\sum_{|\alpha|=m} \|\partial^\alpha u\|^p_{L^p(\Delta(A))}\right)^{\frac{1}{p}} \qquad (u \in W^{m,p}(\Delta(A))/\mathbb{C}).$$

Unfortunately, $W^{m,p}(\Delta(A))/\mathbb{C}$ so topologized is in general non-separated and non-complete (see subsection 2.7).

Definition 2.4. *We define $W^{m,p}_{\#}(\Delta(A))$ as separated completion of $W^{m,p}(\Delta(A))/\mathbb{C}$, and J to be the canonical mapping of $W^{m,p}(\Delta(A))/\mathbb{C}$ into $W^{m,p}_{\#}(\Delta(A))$.*

We refer to, e.g., [7, chap.II, §3, n° 7], [8, chap.I, §1, n° 4] and [18, pp.61-62], for the basic notions involved in the above definition.

Remark 2.7. *$W^{m,p}_{\#}(\Delta(A))$ is a Banach space and further the following classical assertions hold true.*

1) J is linear

2) $J(W^{m,p}(\Delta(A))/\mathbb{C})$ is dense in $W^{m,p}_{\#}(\Delta(A))$

3) $\|J(u)\|_{W^{m,p}_{\#}(\Delta(A))} = \|u\|_{W^{m,p}(\Delta(A))/\mathbb{C}}$ $(u \in W^{m,p}(\Delta(A))/\mathbb{C})$

4) If F is a Banach space and if L is a continuous linear mapping of $W^{m,p}(\Delta(A))/\mathbb{C}$ into F, then there exists a unique continuous linear mapping L' of $W^{m,p}_{\#}(\Delta(A))$ into F such that $L = L' \circ J$.

The preceding remark leads us immediately to the following proposition.

Proposition 2.8. *Let the distribution derivative ∂^α ($\alpha \in \mathbb{N}^N$, $|\alpha| \geq 1$) be viewed as a mapping of $W^{m,p}(\Delta(A))/\mathbb{C}$ into $L^p(\Delta(A))$. Then there exists a unique continuous linear mapping, still denoted by ∂^α, of $W^{m,p}_{\#}(\Delta(A))$*

into $L^p(\Delta(A))$ *such that* $\partial^\alpha J(v) = \partial^\alpha v$ *for* $v \in W^{m,p}(\Delta(A))/\mathbb{C}$. *Furthermore,*

$$\|u\|_{W_{\#}^{m,p}(\Delta(A))} = \left(\sum_{|\alpha|=m} \|\partial^\alpha u\|_{L^p(\Delta(A))}^p \right)^{\frac{1}{p}}$$

for $u \in W_{\#}^{m,p}(\Delta(A))$.

2.7. **Sobolev spaces $H^m(\Delta(A))$ with A an almost periodic H-algebra.** We consider here the particular case where A is an almost periodic H-algebra (see subsection 2.3). So we have here

$$A = AP_{\mathcal{R}}\left(\mathbb{R}^N\right),$$

where $\mathcal{R}$ is a countable subgroup of $\mathbb{R}^N$ (viewed as an additive group). In this setting, we suppose $p = 2$, so that the Sobolev spaces under consideration are $H^m(\Delta(A)) = W^{m,2}(\Delta(A))$ (integers $m \geq 1$). In this context we will be able to point out a few interesting results by means of Fourier analysis.

To begin, we observe that A^∞ is dense in A (indeed, $\Gamma_{\mathcal{R}}$ is dense in A, as is pointed out in subsection 2.3) and so we are justified in introducing the preceding Sobolev spaces. Now, we recall that $\Delta(A)$ is here a compact Abelian group and β is nothing but the Haar measure on $\Delta(A)$ (Proposition 2.4). The dual group of $\Delta(A)$ is the discrete group

$$\widehat{\Delta(A)} = \{\widehat{\gamma}_k : k \in \mathcal{R}\} \ \ (\text{with } \widehat{\gamma}_k = \mathcal{G}(\gamma_k),\ \gamma_k(y) = \exp(2i\pi k \cdot y) \ \ \left(y \in \mathbb{R}^N\right))$$

which may be identified with $\mathcal{R}$ (the reader is referred to subsection 2.3 and in particular to Remark 2.1). Thus, the Fourier transform of a function $u \in L^1(\Delta(A))$ may be viewed as a mapping,

$$k \to a_k(u) = \int_{\Delta(A)} u(s)\overline{\widehat{\gamma}_k(s)}d\beta(s),$$

of $\mathcal{R}$ into $\mathbb{C}$. The complex numbers $a_k(u)$ ($k \in \mathcal{R}$) are the so-called Fourier coefficients of $u \in L^1(\Delta(A))$. According to a classical result (see, e.g., [20, p.56]), $\widehat{\Delta(A)}$ is an orthonormal basis of the Hilbert space $L^2(\Delta(A))$. Therefore we have, for any $u \in L^2(\Delta(A))$,

$$u = \sum_{k \in \mathcal{R}} a_k(u)\widehat{\gamma}_k \qquad (\text{in the } L^2(\Delta(A))\text{-norm}), \tag{2.3}$$

hence

$$\|u\|_{L^2(\Delta(A))}^2 = \sum_{k \in \mathcal{R}} |a_k(u)|^2.$$

At the present time, for $k = (k_1, ..., k_N) \in \mathcal{R}$ and $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}^N$, it is not hard to see that

$$\partial^\alpha \widehat{\gamma}_k = (2i\pi)^{|\alpha|} k^\alpha \widehat{\gamma}_k, \tag{2.4}$$

where $k^\alpha = k_1^{\alpha_1} k_2^{\alpha_2} ... k_N^{\alpha_N}$. Hence

$$a_k(\partial^\alpha u) = (2i\pi)^{|\alpha|} k^\alpha a_k(u) \tag{2.5}$$

for any $u \in H^m(\Delta(A))$. Having made these preliminaries, let us turn now to the proof of the following proposition.

Proposition 2.9. *The following assertions are true:*
(i) $\|\cdot\|_{H^m(\Delta(A))/\mathbb{C}}$ *is a norm on* $H^m(\Delta(A))/\mathbb{C}$.
(ii) $\mathcal{D}(\Delta(A))$ *is dense in* $H^m(\Delta(A))$.

Proof. (i) Let $u \in H^m(\Delta(A))/\mathbb{C}$ with $\|u\|_{H^m(\Delta(A))/\mathbb{C}} = 0$. Then $\partial^\alpha u = 0$ for all $\alpha \in \mathbb{N}^N$ with $|\alpha| = m$. Fix freely $k = (k_1, ..., k_N) \in \mathcal{R}$ with $k \neq \omega$ (ω the origin in $\mathbb{R}^N$). Consider an integer $1 \leq n \leq N$ such that $k_n \neq 0$, and let $\alpha = (\alpha_j) \in \mathbb{N}^N$ with $\alpha_n = m$, $\alpha_j = 0$ if $j \neq n$. Then $k^\alpha = k_n^m \neq 0$; hence $a_k(u) = 0$, according to (2.5); and so $u = 0$ (use (2.3)), since $a_\omega = 0$. This shows (i).

(ii) Consider a sequence of nonempty finite sets $\mathcal{R}_n \subset \mathcal{R}$ (n ranging over the positive integers) such that

$$\mathcal{R}_n \subset \mathcal{R}_{n+1},\ \mathcal{R} = \bigcup_{n\geq 1} \mathcal{R}_n.$$

Let $u \in H^m(\Delta(A))$. For each integer $n \geq 1$, put

$$u_n = \sum_{k\in\mathcal{R}_n} a_k(u)\,\widehat{\gamma}_k.$$

This gives a sequence $(u_n)_{n\geq 1}$ with $u_n \in \mathcal{D}(\Delta(A))$ and further, thanks to (2.4)-(2.5),

$$\partial^\alpha u_n = \sum_{k\in\mathcal{R}_n} a_k(\partial^\alpha u)\,\widehat{\gamma}_k \qquad (|\alpha| \leq m).$$

Hence, by (2.3) it follows that $u_n \to u$ in $H^m(\Delta(A))$ as $n \to \infty$, which shows (ii). ∎

As an immediate consequence of this, there is the following corollary.

Corollary 2.5. *The space* $\mathcal{D}(\Delta(A))/\mathbb{C} = \left\{\varphi \in \mathcal{D}(\Delta(A)) : \int_{\Delta(A)} \varphi d\beta = 0\right\}$ *is dense in* $H^m(\Delta(A))/\mathbb{C}$.

Thus, according to part (i) of Proposition 2.9, $H^m(\Delta(A))/\mathbb{C}$ is a separated preHilbert space; so that $H^m_\#(\Delta(A)) = W^{m,2}_\#(\Delta(A))$ in the present setting coincides with the completion of $H^m(\Delta(A))/\mathbb{C}$. As we will see in a little while, $H^m(\Delta(A))/\mathbb{C}$ is not necessarily complete. For simplicity we assume in the sequel that $m = 1$. We will need one preliminary result.

Lemma 2.1. *The following two assertions are equivalent.*
(i) There exists a constant $c > 0$ *such that*

$$\|u\|_{L^2(\Delta(A))} \leq c\,\|u\|_{H^1(\Delta(A))/\mathbb{C}}$$

for all $u \in H^1(\Delta(A))/\mathbb{C}$.
(ii) $\mathcal{R}$ *is a discrete subgroup of* $\mathbb{R}^N$ *(see* [6, TGVII.2]*).*

Proof. Let $u \in H^1(\Delta(A))/\mathbb{C}$. It is clear that

$$\|u\|^2_{L^2(\Delta(A))} = \sum_{\omega \neq k \in \mathcal{R}} |a_k(u)|^2$$

and

$$\|u\|^2_{H^1(\Delta(A))/\mathbb{C}} = 4\pi^2 \sum_{\omega \neq k \in \mathcal{R}} |k|^2 |a_k(u)|^2 ,$$

where $|k|$ is the Euclidean norm of k and ω the origin in $\mathbb{R}^N$. Thus, assuming (i) implies at once

$$\sum_{\omega \neq k \in \mathcal{R}} |a_k(u)|^2 \leq 4\pi^2 c^2 \sum_{\omega \neq k \in \mathcal{R}} |a_k(u)|^2 |k|^2 \tag{2.6}$$

and that for any $u \in H^1(\Delta(A))/\mathbb{C}$. Hence

$$|k| \geq r > 0 \qquad (\omega \neq k \in \mathcal{R}) \tag{2.7}$$

with $r = \frac{1}{2\pi c}$, which means that $\mathcal{R}$ is a discrete subgroup of $\mathbb{R}^N$, and so (i) implies (ii). Conversely suppose (ii) holds. This amounts to saying that (2.7) holds for some suitable constant $r > 0$. Immediately we see that if u lies in $H^1(\Delta(A))/\mathbb{C}$, then (2.6) holds with $c = \frac{1}{2\pi r}$. Hence (i) follows. This completes the proof. ∎

We are now able to justify our allegation about the completeness of $H^1(\Delta(A))/\mathbb{C}$.

Proposition 2.10. *$H^1(\Delta(A))/\mathbb{C}$ (with the norm $\|\cdot\|_{H^1(\Delta(A))/\mathbb{C}}$) is complete if and only if $\mathcal{R}$ is a discrete subgroup of $\mathbb{R}^N$.*

Proof. $H^1(\Delta(A))/\mathbb{C}$ being a closed vector subspace of $H^1(\Delta(A))$, by the open mapping theorem (see, e.g., [10, p.19]) we see that $H^1(\Delta(A))/\mathbb{C}$ with the norm $\|\cdot\|_{H^1(\Delta(A))/\mathbb{C}}$ is complete if and only if the two norms $\|\cdot\|_{H^1(\Delta(A))/\mathbb{C}}$ and $\|\cdot\|_{H^1(\Delta(A))}$ are equivalent. But this happens if and only if condition (i) of Lemma 2.1 is fulfilled. Therefore the proposition follows by the same lemma. ∎

Thus, if for example $\mathcal{R} = \mathbb{Q}^N$ ($\mathbb{Q}$ the rationals), then the norm $\|\cdot\|_{H^1(\Delta(A))/\mathbb{C}}$ on $H^1(\Delta(A))/\mathbb{C}$ is not complete and hence the latter space is not a Hilbert space. Consequently, in general the passage to the completion is necessary.

3. Σ-CONVERGENCE IN L^p $(1 \leq p < \infty)$

Throughout the present section, Ω denotes an open set in $\mathbb{R}^N_x$ (Ω independent of $\varepsilon > 0$) and $\mathcal{H} = (H_\varepsilon)_{\varepsilon > 0}$ is as above (see (2.1)). The letter E will denote a family of positive real numbers admitting 0 as an accumulation point. In the particular case where $E = (\varepsilon_n)_{n \in \mathbb{N}}$ with $0 < \varepsilon_n \leq 1$ and $\varepsilon_n \to 0$ as $n \to \infty$, we will refer to E as a fundamental sequence. For $\psi \in L^1_{loc}(\Omega \times \mathbb{R}^N_y)$, it is customary to put

$$\psi^\varepsilon(x) = \psi(x, H_\varepsilon(x)) \qquad (x \in \Omega) \tag{3.1}$$

whenever the right-hand side makes sence. This will be the case if in particular ψ lies in $\mathcal{K}\left(\overline{\Omega};L^{\infty}\left(\mathbb{R}_{y}^{N}\right)\right)$ ($\overline{\Omega}$ the closure of Ω in $\mathbb{R}_{x}^{N}$) or $L^{p}\left(\Omega;A\right)$ ($1\leq p\leq\infty$), where A is any closed vector subspace of $\mathcal{B}\left(\mathbb{R}_{y}^{N}\right)$ equipped with the supremum norm (see [26], and observe that Lemma 2 and Proposition 3 therein, together with their proofs, remain rigorously valid when Ω is unbounded provided $\mathcal{C}$ is replaced with $\mathcal{K}$).

Finally, in the sequel A denotes a given H-algebra on $\mathbb{R}^{N}$ for $\mathcal{H}$ with the assumption that A^{∞} is dense in A. The basic notation attached to A is as before (see section 2).

3.1. **The weak Σ-convergence in $L^{p}\left(\Omega\right)$.** Let $1\leq p<\infty$.

Definition 3.1. *A sequence $\left(u_{\varepsilon}\right)_{\varepsilon\in E}$, $u_{\varepsilon}\in L^{p}\left(\Omega\right)$, is said to be weakly Σ-convergent in $L^{p}\left(\Omega\right)$ to some $u_{0}\in L^{p}\left(\Omega;L^{p}\left(\Delta\left(A\right)\right)\right)=L^{p}\left(\Omega\times\Delta\left(A\right)\right)$ if as $E\ni\varepsilon\rightarrow0$, we have*

$$\int_{\Omega}u_{\varepsilon}\left(x\right)\psi^{\varepsilon}\left(x\right)dx\rightarrow\int\int_{\Omega\times\Delta\left(A\right)}u_{0}\left(x,s\right)\widehat{\psi}\left(x,s\right)dxd\beta\left(s\right) \tag{3.2}$$

for all $\psi\in L^{p'}\left(\Omega;A\right)$ $\left(\frac{1}{p'}=1-\frac{1}{p}\right)$, where $\widehat{\psi}=\mathcal{G}\circ\psi$ (usual composition).

Remark 3.1. *$\widehat{\psi}$ is the function in $L^{p'}\left(\Omega;\mathcal{C}\left(\Delta\left(A\right)\right)\right)$ given by $\widehat{\psi}\left(x\right)=\mathcal{G}\left(\psi\left(x\right)\right)$ for $x\in\Omega$.*

We will briefly express the above notion of convergence by writing $u_{\varepsilon}\rightarrow u_{0}$ in $L^{p}\left(\Omega\right)$-weak Σ.

Before we proceed any further, let us prove a result from which we will next derive one fundamental example of a weakly Σ-convergent sequence in $L^{p}\left(\Omega\right)$.

Proposition 3.1. *Let $u\in L^{p}\left(\Omega;A\right)$. We have $u^{\varepsilon}\rightarrow\widetilde{u}$ in $L^{p}\left(\Omega\right)$-weak as $\varepsilon\rightarrow0$, where u^{ε} is defined as in (3.1) and $\widetilde{u}\left(x\right)=M\left(u\left(x\right)\right)$ for $x\in\Omega$.*

Proof. Let $\mathcal{K}\left(\overline{\Omega}\right)\otimes A$ denote the space of complex functions ψ on $\overline{\Omega}\times\mathbb{R}_{y}^{N}$ of the form

$$\psi\left(x,y\right)=\sum\varphi_{i}\left(x\right)w_{i}\left(y\right)\qquad\left(x\in\overline{\Omega},\ y\in\mathbb{R}^{N}\right)$$

with a summation of finitely many terms, $\varphi_{i}\in L^{p}\left(\Omega\right)$, $w_{i}\in A$. Having regard to axiom $(\mathrm{HA})_{4}$ of Definition 2.1, it is clear that the claimed convergence property holds true if u is taken in $\mathcal{K}\left(\overline{\Omega}\right)\otimes A$, hence in $\mathcal{K}\left(\overline{\Omega};A\right)$, thanks to the fact that $\mathcal{K}\left(\overline{\Omega}\right)\otimes A$ is dense in $\mathcal{K}\left(\overline{\Omega};A\right)$ (see, e.g., [4, p.46]). Therefore the proposition follows by the density of $\mathcal{K}\left(\overline{\Omega};A\right)$ in $L^{p}\left(\Omega;A\right)$ (the way of proceeding is a routine exercise left to the reader). ∎

This yields the claimed fundamental example through the next result.

Corollary 3.1. *Let $u\in L^{p}\left(\Omega;A\right)$. Then, the sequence $\left(u^{\varepsilon}\right)_{\varepsilon>0}$ is weakly Σ-convergent in $L^{p}\left(\Omega\right)$ to $\widehat{u}=\mathcal{G}\circ u$.*

Proof. For each $\psi \in L^{p'}(\Omega; A)$, we have $u\psi \in L^1(\Omega; A)$; hence the corollary follows readily by Proposition 3.1. ∎

The next result is very simple and the proof is therefore omitted.

Proposition 3.2. *Suppose a sequence $(u_\varepsilon)_{\varepsilon \in E}$, $u_\varepsilon \in L^p(\Omega)$, is weakly Σ-convergent in $L^p(\Omega)$ to $u_0 \in L^p(\Omega \times \Delta(A))$. Then:*
(i) $u_\varepsilon \to \widetilde{u}_0$ in $L^p(\Omega)$-weak as $E \ni \varepsilon \to 0$, where

$$\widetilde{u}_0(x) = \int_{\Delta(A)} u_0(x,s)\, d\beta(s) \qquad (x \in \Omega).$$

(ii) If E is a fundamental sequence, then $(u_\varepsilon)_{\varepsilon \in E}$ is bounded in $L^p(\Omega)$.

Now, for each real number $r \geq 1$, let $\mathfrak{X}_A^{r,\infty} = \mathfrak{X}_A^r \cap L^\infty(\mathbb{R}_y^N)$. Equipped with the L^∞-norm, $\mathfrak{X}_A^{r,\infty}$ is a Banach space (note that $L^\infty(\mathbb{R}^N)$ is continuously embedded in $\Xi^r(\mathbb{R}^N)$). For future purposes we wish to show that if a sequence $(u_\varepsilon)_{\varepsilon \in E}$ is weakly Σ-convergent in $L^p(\Omega)$ to $u_0 \in L^p(\Omega \times \Delta(A))$, then as $E \ni \varepsilon \to 0$, (3.2) holds for $\psi \in \mathcal{K}\left(\overline{\Omega}; \mathfrak{X}_A^{p',\infty}\right)$ provided $1 < p < \infty$. It may be remarked in passing that if $\psi \in \mathcal{K}\left(\overline{\Omega}; \mathfrak{X}_A^{p',\infty}\right)$, then $\psi \in \mathcal{K}\left(\overline{\Omega}; L^\infty(\mathbb{R}^N)\right)$ and therefore ψ^ε is well-defined by (3.1). We will also need the following obvious remark.

Remark 3.2. *Given $\zeta_0 \in \mathbb{C}$ and a sequence of complex numbers $(\zeta_\varepsilon)_{\varepsilon \in E}$, we have $\zeta_\varepsilon \to \zeta_0$ as $E \ni \varepsilon \to 0$ if and only if, for any sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ with $0 < \varepsilon_n \leq 1$, $\varepsilon_n \in E$, $\varepsilon_n \to 0$ as $n \to \infty$, we have $\zeta_{\varepsilon_n} \to \zeta_0$ as $n \to \infty$.*

Having made this point, let us now concentrate on proving the claimed result.

Proposition 3.3. *Assume that $1 < p < \infty$. Suppose a sequence $(u_\varepsilon)_{\varepsilon \in E}$, $u_\varepsilon \in L^p(\Omega)$, is weakly Σ-convergent in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega \times \Delta(A))$. Then, as $E \ni \varepsilon \to 0$, we have (3.2) for all $\psi \in \mathcal{K}\left(\overline{\Omega}; \mathfrak{X}_A^{p',\infty}\right)$.*

Proof. According to Remark 3.2, we may assume without loss of generality that E is a fundamental sequence. According to part (ii) of Proposition 3.3, it follows that the sequence $(u_\varepsilon)_{\varepsilon \in E}$ is bounded in $L^p(\Omega)$. With this in mind, let us begin by showing that (3.2) holds for $\psi \in \mathcal{K}(\overline{\Omega}) \otimes \mathfrak{X}_A^{p',\infty}$. But then it clearly suffices to verify that (3.2) holds true for each ψ of the form

$$\psi(x,y) = \varphi(x)\, v(y) \quad \left(x \in \overline{\Omega},\, y \in \mathbb{R}^N\right),\ \varphi \in \mathcal{K}(\overline{\Omega}),\ v \in \mathfrak{X}_A^{p',\infty}.$$

Let ψ be as above. Let $\eta > 0$. In view of the density of A in $\mathfrak{X}_A^{p'}$, we may consider some $w \in A$ such that $\|v - w\|_{\Xi^{p'}} \leq \eta$. Let

$$f(x,y) = \varphi(x)\, w(y) \qquad \left(x \in \overline{\Omega},\, y \in \mathbb{R}^N\right),$$

which gives a function $f \in \mathcal{K}(\overline{\Omega}; A)$. Now, we can write

$$\begin{aligned}\int_\Omega u_\varepsilon\psi^\varepsilon dx - \int\int_{\Omega\times\Delta(A)} u_0\widehat{\psi}d\beta dx &= \int_\Omega u_\varepsilon\left(\psi^\varepsilon - f^\varepsilon\right)dx \\ &+ \int_\Omega u_\varepsilon f^\varepsilon dx - \int\int_{\Omega\times\Delta(A)} u_0\widehat{f}d\beta dx \\ &+ \int\int_{\Omega\times\Delta(A)} u_0\left(\widehat{f}-\widehat{\psi}\right)d\beta dx,\end{aligned}$$

the object being to establish that the left-hand side goes to zero as $E \ni \varepsilon \to 0$. First, by Hölder's inequality we have

$$\left|\int_\Omega u_\varepsilon\left(\psi^\varepsilon - f^\varepsilon\right)dx\right| \leq \left\|u_\varepsilon\right\|_{L^p(\Omega)}\left\|\psi^\varepsilon - f^\varepsilon\right\|_{L^{p'}(\Omega)}.$$

On the other hand,

$$\left\|\psi^\varepsilon - f^\varepsilon\right\|_{L^{p'}(\Omega)} \leq \left\|\varphi\right\|_\infty\left(\int_K\left|v^\varepsilon - w^\varepsilon\right|^{p'}dx\right)^{\frac{1}{p'}},$$

where K is a compact set in $\overline{\Omega}$ containing the support of φ. But

$$\left(\int_K\left|v^\varepsilon - w^\varepsilon\right|^{p'}dx\right)^{\frac{1}{p'}} \leq c\left(K\right)\left\|v-w\right\|_{\Xi^{p'}} \qquad \left(\varepsilon\in E\right),$$

where the constant $c\left(K\right) > 0$ depends solely on K. From all that we deduce

$$\left|\int_\Omega u_\varepsilon\left(\psi^\varepsilon - f^\varepsilon\right)dx\right| \leq c\eta \qquad \left(\varepsilon\in E\right),$$

where c is a positive real number independent of both η and ε. In another connection, again by Hölder's inequality and use of Proposition 2.6, we have

$$\left|\int\int_{\Omega\times\Delta(A)} u_0\left(\widehat{f}-\widehat{\psi}\right)d\beta dx\right| \leq c\left\|u_0\right\|_{L^p(\Omega\times\Delta(A))}\left\|\varphi\right\|_{L^{p'}(\Omega)}\left\|v-w\right\|_{\Xi^{p'}},$$

hence

$$\left|\int\int_{\Omega\times\Delta(A)} u_0\left(\widehat{f}-\widehat{\psi}\right)d\beta dx\right| \leq c\eta,$$

where c is a positive real independent of both η and ε. Considering that

$$\int_\Omega u_\varepsilon f^\varepsilon dx \to \int\int_{\Omega\times\Delta(A)} u_0\widehat{f}d\beta dx$$

as $E \ni \varepsilon \to 0$, we have in the end

$$\lim_{E\ni\varepsilon\to 0}\left|\int_\Omega u_\varepsilon\psi^\varepsilon dx \to \int\int_{\Omega\times\Delta(A)} u_0\widehat{\psi}d\beta dx\right| \leq c\eta,$$

where c is a positive real independent of η. Therefore the desired result follows by the arbitrariness of η. Finally, if ψ is considered in $\mathcal{K}\left(\overline{\Omega};\mathfrak{X}_A^{p',\infty}\right)$, then, based on the density of $\mathcal{K}\left(\overline{\Omega}\right)\otimes\mathfrak{X}_A^{p',\infty}$ in $\mathcal{K}\left(\overline{\Omega};\mathfrak{X}_A^{p',\infty}\right)$, the same line of argument as followed before shows that we again arrive at (3.2), thereby completing the proof. ∎

As a consequence of this, there is the following corollary.

Corollary 3.2. *For $u \in \mathcal{K}\left(\overline{\Omega}; \mathfrak{X}_A^{p,\infty}\right)$ $(1 < p < \infty)$, the sequence $(u^\varepsilon)_{\varepsilon>0}$ is weakly Σ-convergent in $L^p(\Omega)$ to $\widehat{u}$.*

Proof. Endeed, this follows immediately by combining Proposition 3.3 with Corollary 3.1. ∎

The next result is the corner-stone of Σ-convergence theory.

Theorem 3.1. *Assume that $1 < p < \infty$. Suppose E is a fundamental sequence and let a sequence $(u_\varepsilon)_{\varepsilon\in E}$ be bounded in $L^p(\Omega)$. Then, a subsequence E' can be extracted from E such that $(u_\varepsilon)_{\varepsilon\in E'}$ is weakly Σ-convergent in $L^p(\Omega)$.*

Proof. For any $\varepsilon \in E$, put

$$F_\varepsilon(\psi) = \int_\Omega u_\varepsilon(x)\,\psi(x, H_\varepsilon(x))\,dx \qquad \left(\psi \in L^{p'}(\Omega; A)\right),$$

where $\frac{1}{p'} = 1 - \frac{1}{p}$. This yields a sequence $(F_\varepsilon)_{\varepsilon\in E}$ in $\left[L^{p'}(\Omega; A)\right]'$ (topological dual of $L^{p'}(\Omega; A)$) which is bounded (in the latter space). Hence, observing that $L^{p'}(\Omega; A)$ is a separable Banach space (thanks to the separability of A, as stated in point $(\mathrm{AH})_1$ of Definition 2.1 !), we can extract a subsequence E' from E in such a way that, as $E' \ni \varepsilon \to 0$, $F_\varepsilon \to F_0$ in $\left[L^{p'}(\Omega; A)\right]'$-weak $*$, that is, $F_\varepsilon(\psi) \to F_0(\psi)$ for any $\psi \in L^{p'}(\Omega; A)$. The next point is to characterize the functional F_0. However, as will presently become apparent, it is more appropriate to characterize the closely connected functional $G_0 : L^{p'}(\Omega; \mathcal{C}(\Delta(A))) \to \mathbb{C}$ given by $G_0(\varphi) = F_0\left(\mathcal{G}^{-1} \circ \varphi\right)$, $\varphi \in L^{p'}(\Omega; \mathcal{C}(\Delta(A)))$. Prior to this, let $\psi \in \mathcal{K}(\Omega; A)$. Clearly

$$|F_\varepsilon(\psi)| \le c\left(\int_\Omega |\psi(x, H_\varepsilon(x))|^{p'} \chi_K(x)\,dx\right)^{\frac{1}{p'}} \qquad (\varepsilon \in E'),$$

where c is a positive constant (independent of ε and ψ, as well), K is a compact set in Ω containing the support of ψ, and χ_K is the characteristic function of K in Ω. By letting $E' \ni \varepsilon \to 0$ and applying Proposition 3.1 (with $u(x,y) = |\psi(x,y)|^{p'}$, $x \in \Omega$, $y \in \mathbb{R}^N$), we get

$$|F_0(\psi)| \le c\left\|\widehat{\psi}\right\|_{L^{p'}(\Omega\times\Delta(A))}$$

and that for any $\psi \in \mathcal{K}(\Omega; A)$, where it is worth recalling that $\widehat{\psi} = \mathcal{G} \circ \psi$, and further $\widehat{\psi}$ has support in K. Thus,

$$|G_0(\varphi)| \le c\,\|\varphi\|_{L^{p'}(\Omega\times\Delta(A))}$$

for all $\varphi \in \mathcal{K}(\Omega; \mathcal{C}(\Delta(A))) = \mathcal{K}(\Omega \times \Delta(A))$. Based on the density of $\mathcal{K}(\Omega; \mathcal{C}(\Delta(A)))$ in $L^{p'}\left(\Omega; L^{p'}(\Delta(A))\right) = L^{p'}(\Omega \times \Delta(A))$, we can extend

G_0 by continuity to an element of $\left[L^{p'}(\Omega \times \Delta(A))\right]' = L^p(\Omega \times \Delta(A))$. Hence there exists $u_0 \in L^p(\Omega \times \Delta(A))$ such that

$$G_0(\varphi) = \int\int_{\Omega \times \Delta(A)} u_0(x,s)\,\varphi(x,s)\,dxd\beta(s)$$

for all $\varphi \in \mathcal{K}(\Omega; \mathcal{C}(\Delta(A)))$. Thus,

$$F_0(\psi) = \int\int_{\Omega \times \Delta(A)} u_0(x,s)\,\widehat{\psi}(x,s)\,dxd\beta(s)$$

for all $\psi \in \mathcal{K}(\Omega; A)$ and therefore for all $\psi \in L^{p'}(\Omega; A)$, thanks to the density of $\mathcal{K}(\Omega; A)$ in $L^{p'}(\Omega; A)$. The theorem follows. ∎

Remark 3.3. *The above compactness theorem is the main reason for requiring a homogenization algebra to be separable.*

3.2. **The strong Σ-convergence in $L^p(\Omega)$.** The concept of strong Σ-convergence in $L^p(\Omega)$ leans on the density of $L^p(\Omega; \mathcal{C}(\Delta(A)))$ in $L^p(\Omega \times \Delta(A))$.

Let $1 \leq p < \infty$.

Definition 3.2. *A sequence $(u_\varepsilon)_{\varepsilon \in E}$, $u_\varepsilon \in L^p(\Omega)$, is said to be strongly Σ-convergent in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega \times \Delta(A))$ if the following condition is fulfilled:*

$$\text{(SSC)} \quad \begin{cases} \text{Given } \eta > 0 \text{ and } v \in L^p(\Omega; A) \\ \text{such that } \|u_0 - \widehat{v}\|_{L^p(\Omega \times \Delta(A))} \leq \frac{\eta}{2} \\ \text{(with } \widehat{v} = \mathcal{G} \circ v\text{), there is some } \alpha > 0 \text{ such that} \\ \|u_\varepsilon - v^\varepsilon\|_{L^p(\Omega)} \leq \eta \text{ provided } E \ni \varepsilon \leq \alpha. \end{cases}$$

We express this by writing $u_\varepsilon \to u_0$ in $L^p(\Omega)$-strong Σ.

Let us verify the unicity of u_0 in Definition 3.2.

Proposition 3.4. *If a sequence $(u_\varepsilon)_{\varepsilon \in E}$, $u_\varepsilon \in L^p(\Omega)$, is strongly Σ-convergent in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega \times \Delta(A))$, then u_0 is unique.*

Proof. In the above notation, suppose we have $u_\varepsilon \to u_0^1$ and $u_\varepsilon \to u_0^2$ in $L^p(\Omega)$-strong Σ. Let $\eta > 0$. The space $\mathcal{K}(\Omega; A)$ being dense in $L^p(\Omega; A)$, we may choose $v_i \in \mathcal{K}(\Omega; A)$ such that $\left\|u_0^i - \widehat{v}_i\right\|_{L^p(\Omega \times \Delta(A))} \leq \frac{\eta}{6}$, $i = 1, 2$. According to Definition 3.2, this yields some $\alpha > 0$ such that $\|u_\varepsilon - v_i^\varepsilon\|_{L^p(\Omega)} \leq \frac{\eta}{3}$ $(i = 1, 2)$ for all $E \ni \varepsilon \leq \alpha$. It follows $\|v_2^\varepsilon - v_1^\varepsilon\|_{L^p(\Omega)} \leq \frac{2\eta}{3}$ for $E \ni \varepsilon \leq \alpha$. Observing that

$$\|v_2^\varepsilon - v_1^\varepsilon\|_{L^p(\Omega)} = \left(\int_\Omega |v_2(x, H_\varepsilon(x)) - v_1(x, H_\varepsilon(x))|^p \chi_K(x)\,dx\right)^{\frac{1}{p}},$$

where K is a compact set in Ω containing the supports of v_1 and v_2, we see that we can pass to the limit, as $E \ni \varepsilon \to 0$, in the preceding inequality (use Proposition 3.1) and obtain

$$\|\widehat{v}_2 - \widehat{v}_1\|_{L^p(\Omega \times \Delta(A))} \leq \frac{2\eta}{3}$$

Consequently, by writing $u_0^2 - u_0^1 = u_0^2 - \widehat{v}_2 + \widehat{v}_2 - \widehat{v}_1 + \widehat{v}_1 - u_0^1$, we get $\left\| u_0^2 - u_0^1 \right\|_{L^p(\Omega\times\Delta(A))} \leq \eta$. Hence $u_0^2 = u_0^1$, since η is arbitrary. ∎

Before we can present one fundamental example of a strongly Σ-convergent sequence, we require a preliminary lemma.

Lemma 3.1. *We have*

$$\lim_{\varepsilon\to 0} \|\Phi^\varepsilon\|_{L^p(\Omega)} = \left\| \widehat{\Phi} \right\|_{L^p(\Omega\times\Delta(A))} \qquad (\Phi \in L^p(\Omega; A)).$$

Proof. The first step is to recall that the lemma is true with $\mathcal{K}(\Omega; A)$ in place of $L^p(\Omega; A)$, as is straightforward by Proposition 3.1 and use of a routine argument (see the proof of Theorem 3.1). Now, fix freely $\Phi \in L^p(\Omega; A)$. Let $\eta > 0$. By a density argument, we may consider some $\psi \in \mathcal{K}(\Omega; A)$ such that

$$\|\Phi - \psi\|_{L^p(\Omega;A)} \equiv \left(\int_\Omega \|\Phi(x) - \psi(x)\|_\infty^p \, dx \right)^{\frac{1}{p}} \leq \frac{\eta}{2}.$$

With this in mind, we have on the other hand

$$\begin{aligned} \left| \|\Phi^\varepsilon\|_{L^p(\Omega)} - \left\| \widehat{\Phi} \right\|_{L^p(\Omega\times\Delta(A))} \right| &\leq \left| \|\Phi^\varepsilon\|_{L^p(\Omega)} - \|\psi^\varepsilon\|_{L^p(\Omega)} \right| \\ &+ \left| \|\psi^\varepsilon\|_{L^p(\Omega)} - \left\| \widehat{\psi} \right\|_{L^p(\Omega\times\Delta(A))} \right| \\ &+ \left| \left\| \widehat{\psi} \right\|_{L^p(\Omega\times\Delta(A))} - \left\| \widehat{\Phi} \right\|_{L^p(\Omega\times\Delta(A))} \right|. \end{aligned}$$

It follows

$$\begin{aligned} \left| \|\Phi^\varepsilon\|_{L^p(\Omega)} - \left\| \widehat{\Phi} \right\|_{L^p(\Omega\times\Delta(A))} \right| &\leq \|\Phi^\varepsilon - \psi^\varepsilon\|_{L^p(\Omega)} \\ &+ \left| \|\psi^\varepsilon\|_{L^p(\Omega)} - \left\| \widehat{\psi} \right\|_{L^p(\Omega\times\Delta(A))} \right| \\ &+ \left\| \widehat{\Phi} - \widehat{\psi} \right\|_{L^p(\Omega\times\Delta(A))}. \end{aligned}$$

But the first and third terms on the right are majorized by $\|\Phi - \psi\|_{L^p(\Omega;A)}$. Hence

$$\left| \|\Phi^\varepsilon\|_{L^p(\Omega)} - \left\| \widehat{\Phi} \right\|_{L^p(\Omega\times\Delta(A))} \right| \leq \eta + \left| \|\psi^\varepsilon\|_{L^p(\Omega)} - \left\| \widehat{\psi} \right\|_{L^p(\Omega\times\Delta(A))} \right|.$$

From which the lemma follows in an abvious way. ∎

We are now able to give the claimed example.

Example 3.1. *Let $u \in L^p(\Omega; A)$. Then, the sequence $(u^\varepsilon)_{\varepsilon>0}$ is strongly Σ-convergent in $L^p(\Omega)$ to $\widehat{u}$. Indeed, for any arbitrary $v \in L^p(\Omega; A)$, we have $\|u^\varepsilon - v^\varepsilon\|_{L^p(\Omega)} \to \|\widehat{u} - \widehat{v}\|_{L^p(\Omega\times\Delta(A))}$ as $\varepsilon \to 0$. We deduce immediately that the sequence $(u^\varepsilon)_{\varepsilon>0}$ and the function $\widehat{u}$ satisfy condition (SSC) of Definition 3.2.*

The remainder of the present subsection is concerned with a series of results of practical interest as regards homogenization theory. To begin, there is the following proposition whose proof is an easy verification left to the reader.

Proposition 3.5. *Suppose a sequence* $(u_\varepsilon)_{\varepsilon\in E}$*,* $u_\varepsilon \in L^p(\Omega)$*, is strongly* Σ*-convergent in* $L^p(\Omega)$ *to some* $u_0 \in L^p(\Omega \times \Delta(A))$*. Assume further that* $u_0 \in L^p(\Omega; \mathcal{C}(\Delta(A)))$*. Let* $v_0 \in L^p(\Omega; A)$*,* $v_0 = \mathcal{G}^{-1} \circ u_0$*. Then* $\|u_\varepsilon - v_0^\varepsilon\|_{L^p(\Omega)} \to 0$ *as* $E \ni \varepsilon \to 0$*.*

The next proposition and its corollary are likely to help us have a clear idea of the somewhat abstract concept of strong Σ-convergence.

Proposition 3.6. *Suppose a sequence* $(u_\varepsilon)_{\varepsilon\in E}$*,* $u_\varepsilon \in L^p(\Omega)$*, is strongly* Σ*-convergent in* $L^p(\Omega)$ *to some* $u_0 \in L^p(\Omega \times \Delta(A))$*. Then*
(i) $u_\varepsilon \to u_0$ *in* $L^p(\Omega)$*-weak* Σ*;*
(ii) $\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega\times\Delta(A))}$ *as* $E \ni \varepsilon \to 0$*.*

Proof. (i): Let $\psi \in L^{p'}(\Omega; A)$. Fix a real $c > 0$ with $\|\psi\|_{L^{p'}(\Omega;A)} \leq c$. Now, fix freely $\eta > 0$ and choose $v \in L^p(\Omega; A)$ such that $\|u_0 - \widehat{v}\|_{L^p(\Omega\times\Delta(A))} \leq \frac{\eta}{4c}$. By hypothesis there is some α_0 such that $\|u_\varepsilon - v^\varepsilon\|_{L^p(\Omega)} \leq \frac{\eta}{2c}$ for $E \ni \varepsilon \leq \alpha_0$. On the other hand, recalling that $v^\varepsilon \to \widehat{v}$ in $L^p(\Omega)$-weak Σ (Corollary 3.1), we may consider some $\alpha_1 > 0$ such that

$$\left| \int_\Omega v^\varepsilon \psi^\varepsilon dx - \int\int_{\Omega\times\Delta(A)} \widehat{v}\widehat{\psi} dx d\beta \right| \leq \frac{\eta}{4}$$

provided $0 < \varepsilon \leq \alpha_1$. Hence, by writing

$$\begin{aligned} \int_\Omega u_\varepsilon \psi^\varepsilon dx - \int\int_{\Omega\times\Delta(A)} u_0 \widehat{\psi} dx d\beta &= \int\int_{\Omega\times\Delta(A)} (\widehat{v} - u_0)\, \widehat{\psi} dx d\beta \\ &+ \int_\Omega (u_\varepsilon - v^\varepsilon)\, \psi^\varepsilon dx \\ &+ \int_\Omega v^\varepsilon \psi^\varepsilon dx - \int\int_{\Omega\times\Delta(A)} \widehat{v}\widehat{\psi} dx d\beta, \end{aligned}$$

one quickly arrives at

$$\left| \int_\Omega u_\varepsilon \psi^\varepsilon dx - \int\int_{\Omega\times\Delta(A)} u_0 \widehat{\psi} dx d\beta \right| \leq \eta$$

for $E \ni \varepsilon \leq \alpha = \min(\alpha_0, \alpha_1)$, which shows (i).
(ii): Let $\eta > 0$. Choose $v \in L^p(\Omega; A)$ such that $\|u_0 - \widehat{v}\|_{L^p(\Omega\times\Delta(A))} \leq \frac{\eta}{6}$. This yields a real $\alpha_0 > 0$ such that $\|u_\varepsilon - v^\varepsilon\|_{L^p(\Omega)} \leq \frac{\eta}{3}$ provided $E \ni \varepsilon \leq \alpha_0$. Thus, we have

$$\left| \|u_0\|_{L^p(\Omega\times\Delta(A))} - \|\widehat{v}\|_{L^p(\Omega\times\Delta(A))} \right| \leq \frac{\eta}{6}$$

and

$$\left|\|u_\varepsilon\|_{L^p(\Omega)} - \|v^\varepsilon\|_{L^p(\Omega)}\right| \leq \frac{\eta}{3} \qquad (E \ni \varepsilon \leq \alpha_0).$$

On the other hand, according to Lemma 3.1, there is some $\alpha_1 > 0$ such that $\left|\|v^\varepsilon\|_{L^p(\Omega)} - \|\widehat{v}\|_{L^p(\Omega\times\Delta(A))}\right| \leq \frac{\eta}{2}$ for $0 < \varepsilon \leq \alpha_1$. Hence, by the obvious inequality

$$\left|\|u_\varepsilon\|_{L^p(\Omega)} - \|u_0\|_{L^p(\Omega\times\Delta(A))}\right| \leq \left|\|u_\varepsilon\|_{L^p(\Omega)} - \|v^\varepsilon\|_{L^p(\Omega)}\right|$$
$$+ \left|\|v^\varepsilon\|_{L^p(\Omega)} - \|\widehat{v}\|_{L^p(\Omega\times\Delta(A))}\right|$$
$$+ \left|\|\widehat{v}\|_{L^p(\Omega\times\Delta(A))} - \|u_0\|_{L^p(\Omega\times\Delta(A))}\right|$$

we obtain readily

$$\left|\|u_\varepsilon\|_{L^p(\Omega)} - \|u_0\|_{L^p(\Omega\times\Delta(A))}\right| \leq \eta$$

for $E \ni \varepsilon \leq \alpha = \min(\alpha_0, \alpha_1)$, thereby proving (ii). ∎

Corollary 3.3. *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a sequence in $L^2(\Omega)$. In order that this sequence strongly Σ-converge in $L^2(\Omega)$ to $u_0 \in L^2(\Omega\times\Delta(A))$, it is necessary and sufficient that the following two conditions be satisfied:*

(i) $u_\varepsilon \to u_0$ in $L^2(\Omega)$-weak Σ;

(ii) $\|u_\varepsilon\|_{L^2(\Omega)} \to \|u_0\|_{L^2(\Omega\times\Delta(A))}$ as $E \ni \varepsilon \to 0$.

Proof. In view of Proposition 3.6, we only have to show the sufficiency. So, assuming (i)-(ii), consider any arbitrary $v \in L^2(\Omega; A)$, and use

$$\|u_\varepsilon - v^\varepsilon\|^2_{L^2(\Omega)} = \|u_\varepsilon\|^2_{L^2(\Omega)} - \int_\Omega u_\varepsilon \overline{v}^\varepsilon dx - \int_\Omega \overline{u}_\varepsilon v^\varepsilon dx + \|v^\varepsilon\|^2_{L^2(\Omega)}$$

to see that when $E \ni \varepsilon \to 0$, $\|u_\varepsilon - v^\varepsilon\|_{L^2(\Omega)}$ tends to $\|u_0 - \widehat{v}\|_{L^2(\Omega\times\Delta(A))}$. Hence, it follows that condition (SSC) of Definition 3.2 is satisfied. This Proves the corollary. ∎

We turn now to one result of very practical interest.

Proposition 3.7. *Suppose a real $q \geq 1$ is such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \leq 1$. Suppose a sequence $(u_\varepsilon)_{\varepsilon\in E}$ is strongly Σ-convergent in $L^p(\Omega)$ to some $u_0 \in L^p(\Omega\times\Delta(A))$, and a sequence $(v_\varepsilon)_{\varepsilon\in E}$ is weakly Σ-convergent in $L^q(\Omega)$ to some $v_0 \in L^q(\Omega\times\Delta(A))$. Then $u_\varepsilon v_\varepsilon \to u_0 v_0$ in $L^r(\Omega)$-weak Σ as $E \ni \varepsilon \to 0$.*

Proof. We may assume without loss of generality that E is a fundamental sequence. The result is that $(v_\varepsilon)_{\varepsilon\in E}$ is bounded in $L^q(\Omega)$ (Proposition 3.2). This being so, fix freely $\psi \in L^{r'}(\Omega; A)$ $\left(\frac{1}{r'} = 1 - \frac{1}{r}\right)$ and let $c > 0$ with

$$c \geq \max\left\{\|v_0\|_{L^q(\Omega\times\Delta(A))}\|\psi\|_{L^{r'}(\Omega;A)},\ \|\psi\|_{L^{r'}(\Omega;A)} \sup_{\varepsilon\in E}\|v_\varepsilon\|_{L^q(\Omega)}\right\}.$$

On the other hand, let $\eta > 0$. Having regard to the strong Σ-convergence of $(u_\varepsilon)_{\varepsilon\in E}$, introduce $f \in L^p(\Omega; A)$ such that $\left\| u_0 - \widehat{f} \right\|_{L^p(\Omega\times\Delta(A))} \leq \frac{\eta}{6c}$, and keep in mind that this infers the existence of some $\alpha_0 > 0$ such that $\|u_\varepsilon - f^\varepsilon\|_{L^p(\Omega)} \leq \frac{\eta}{3c}$ for $E \ni \varepsilon \leq \alpha_0$. Finally, noting that $f\psi \in L^{q'}(\Omega; A)$, and using the weak Σ-convergence of $(v_\varepsilon)_{\varepsilon\in E}$, we may consider some $\alpha_1 > 0$ such that

$$\left| \int_\Omega v_\varepsilon f^\varepsilon \psi^\varepsilon dx - \int\int_{\Omega\times\Delta(A)} v_0 \widehat{f\psi} dx d\beta \right| \leq \frac{\eta}{2}$$

for all $E \ni \varepsilon \leq \alpha_1$. Hence, by writing

$$\begin{aligned} \int_\Omega u_\varepsilon v_\varepsilon \psi^\varepsilon dx - \int\int_{\Omega\times\Delta(A)} u_0 v_0 \widehat{\psi} dx d\beta &= \int_\Omega (u_\varepsilon - f^\varepsilon) v_\varepsilon \psi^\varepsilon dx \\ &+ \int\int_{\Omega\times\Delta(A)} \left(\widehat{f} - u_0\right) v_0 \widehat{\psi} dx d\beta \\ &+ \int_\Omega v_\varepsilon f^\varepsilon \psi^\varepsilon dx - \int\int_{\Omega\times\Delta(A)} v_0 \widehat{f\psi} dx d\beta, \end{aligned}$$

one easily arrives at

$$\left| \int_\Omega u_\varepsilon v_\varepsilon \psi^\varepsilon dx - \int\int_{\Omega\times\Delta(A)} u_0 v_0 \widehat{\psi} dx d\beta \right| \leq \eta$$

provided $E \ni \varepsilon \leq \alpha = \min(\alpha_0, \alpha_1)$. This shows the proposition. ∎

This proposition has one useful corollary.

Corollary 3.4. *Let E be a fundamental sequence. Let $(u_\varepsilon)_{\varepsilon\in E}$ be a sequence in $L^p(\Omega)$ with $1 < p < \infty$, and $(v_\varepsilon)_{\varepsilon\in E}$ be a sequence in $L^{p'}(\Omega) \cap L^\infty(\Omega)$ $\left(\frac{1}{p'} = 1 - \frac{1}{p}\right)$ such that:*

(i) $u_\varepsilon \to u_0$ in $L^p(\Omega)$-weak Σ;
(ii) $v_\varepsilon \to v_0$ in $L^{p'}(\Omega)$-strong Σ;
(iii) $(v_\varepsilon)_{\varepsilon\in E}$ is bounded in $L^\infty(\Omega)$.
Then $u_\varepsilon v_\varepsilon \to u_0 v_0$ in $L^p(\Omega)$-weak Σ.

Proof. According to Proposition 3.7, we have $u_\varepsilon v_\varepsilon \to u_0 v_0$ in $L^1(\Omega)$-weak Σ. Thus, as $E \ni \varepsilon \to 0$,

$$\int_\Omega u_\varepsilon v_\varepsilon \psi^\varepsilon dx \to \int\int_{\Omega\times\Delta(A)} u_0 v_0 \widehat{\psi} dx d\beta \qquad (\psi \in \mathcal{K}(\Omega; A)).$$

On the other hand, observe that the sequence $(u_\varepsilon v_\varepsilon)_{\varepsilon\in E}$ is bounded in $L^p(\Omega)$. Hence, thanks to Theorem 3.1, we can extract E' from E such that the sequence $(u_\varepsilon v_\varepsilon)_{\varepsilon\in E'}$ weakly Σ-converges in $L^p(\Omega)$ to some $z_0 \in L^p(\Omega \times \Delta(A))$. Thus, as $E' \ni \varepsilon \to 0$,

$$\int_\Omega u_\varepsilon v_\varepsilon \psi^\varepsilon dx \to \int\int_{\Omega\times\Delta(A)} z_0 \widehat{\psi} dx d\beta \qquad (\psi \in \mathcal{K}(\Omega; A)).$$

From all that we deduce

$$\int\int_{\Omega\times\Delta(A)}(z_0-u_0v_0)\,\varphi dxd\beta=0$$

for all $\varphi\in\mathcal{K}(\Omega\times\Delta(A))$ (see Remark 4.1). Hence $z_0=u_0v_0$ almost everywhere in $\Omega\times\Delta(A)$. The corollary follows thereby. ∎

We conclude the present subsection by showing that strong Σ-convergence generalizes usual strong convergence. Specifically, we have

Proposition 3.8. *Suppose a sequence $(u_\varepsilon)_{\varepsilon\in E}$ is strongly convergent in $L^p(\Omega)$ to some $u_0\in L^p(\Omega)$. Then $u_\varepsilon\to u_0$ in $L^p(\Omega)$-strong Σ.*

Proof. Let us begin by observing that the function $u_0\in L^p(\Omega)$ may as well be viewed as a function in $L^p(\Omega;A)$ (resp. $L^p(\Omega\times\Delta(A))$) depending on the sole variable $x\in\Omega$. Having made this point, let $v\in L^p(\Omega;A)$. By applying Lemma 3.1 with $\Phi=u_0-v$, we see that if $\eta>0$ is freely fixed, then some $\alpha>0$ exists such that $\|u_0-v^\varepsilon\|_{L^p(\Omega)}\leq\|u_0-\widehat{v}\|_{L^p(\Omega\times\Delta(A))}+\frac{\eta}{4}$ and $\|u_\varepsilon-u_0\|_{L^p(\Omega)}\leq\frac{\eta}{4}$ for $E\ni\varepsilon\leq\alpha$. Hence $\|u_\varepsilon-v^\varepsilon\|_{L^p(\Omega)}\leq\frac{\eta}{2}+\|u_0-\widehat{v}\|_{L^p(\Omega\times\Delta(A))}$. We deduce that condition (SSC) of Definition 3.2 is satisfied by $(u_\varepsilon)_{\varepsilon\in E}$ and u_0, thereby proving the proposition. ∎

4. THE VAGUE Σ-CONVERGENCE OF RADON MEASURES

Let the basic notation and hypotheses be as in the preceding section (see in particular the beginning of Section 3). On the other hand, the space of all complex Radon measures on a locally compact space Z will be denoted by $\mathcal{M}(Z)$. Thus, $\mathcal{M}(Z)$ is nothing else than the topological dual of $\mathcal{K}(Z)$ (provided with the usual inductive limit topology). Also, the notion of a σ-compact locally compact space is worth recalling. By this is meant any locally compact space which can be expressed as the union of a countable family of compact subspaces.

Definition 4.1. *A sequence $(\mu_\varepsilon)_{\varepsilon\in E}$ of Radon measures on Ω is said to be vaguely Σ-convergent to some $\mu_0\in\mathcal{M}(\Omega\times\Delta(A))$ if as $E\ni\varepsilon\to0$,*

$$\int_\Omega\psi(x,H_\varepsilon(x))\,d\mu_\varepsilon(x)\to\int\int_{\Omega\times\Delta(A)}\widehat{\psi}(x,s)\,d\mu_0(x,s)$$

for all $\psi\in\mathcal{K}(\Omega;A)$. We express this by writing $\mu_\varepsilon\to\mu_0$ in $\mathcal{M}(\Omega)$-vague Σ.

Remark 4.1. *It is an elementary exercise to verify that the transformation $\psi\to\widehat{\psi}=\mathcal{G}\circ\psi$ is a topological isomorphism of $\mathcal{K}(\Omega;A)$ onto $\mathcal{K}(\Omega\times\Delta(A))\equiv\mathcal{K}(\Omega;\mathcal{C}(\Delta(A)))$, each of the two spaces being endowed with the appropriate inductive limit topology. Consequently, for fixed $\varepsilon\in E$, it is easily seen that to each $\mu\in\mathcal{M}(\Omega)$ there is attached a unique $T_\varepsilon(\mu)\in\mathcal{M}(\Omega\times\Delta(A))$ such that*

$$\left\langle T_\varepsilon(\mu),\widehat{\psi}\right\rangle=\int_\Omega\psi(x,H_\varepsilon(x))\,d\mu(x)$$

for all $\psi \in \mathcal{K}(\Omega; A)$*, where the brackets denote the duality pairing between* $\mathcal{M}(\Omega \times \Delta(A))$ *and* $\mathcal{K}(\Omega \times \Delta(A))$*. This yields a transformation* $\mu \to T_\varepsilon(\mu)$ *that maps linearly* $\mathcal{M}(\Omega)$ *into* $\mathcal{M}(\Omega \times \Delta(A))$*. Thus, to say that a sequence* $(\mu_\varepsilon)_{\varepsilon \in E}$ *in* $\mathcal{M}(\Omega)$ *is vaguely* Σ*-convergent amounts to saying that as* $E \ni \varepsilon \to 0$*, the sequence of Radon measures* $T_\varepsilon(\mu_\varepsilon)$ $(\varepsilon \in E)$ *on* $\Omega \times \Delta(A)$ *is convergent in the weak* $*$ *topology on* $\mathcal{M}(\Omega \times \Delta(A))$*.*

The usefulness of the following lemma will come to light in a short while.

Lemma 4.1. *Let* Z *be a locally compact space, and let* $\mathcal{P} \subset \mathcal{M}(Z)$*. The following two assertions are equivalent:*

(i) $\mathcal{P}$ *is bounded in the weak* $*$ *topology on* $\mathcal{M}(Z)$*, i.e.,* $\sup_{\mu \in \mathcal{P}} |\mu(\varphi)| < +\infty$ *for each* $\varphi \in \mathcal{K}(Z)$*.*

(ii) $\mathcal{P}$ *is locally bounded in norm, i.e.,* $\sup_{\mu \in \mathcal{P}} |\mu|(K) < +\infty$ *for each compact set* $K \subset Z$*.*

Proof. According to [4, p.60, Proposition 15], assertion (i) is equivalent to the following:

(iii) For any compact set $H \subset Z$, there exists a constant $c_H \geq 0$ such that $\sup_{\mu \in \mathcal{P}} |\mu(\varphi)| \leq c_H \|\varphi\|_\infty$ for all $\varphi \in \mathcal{K}_H(Z) = \{f \in \mathcal{K}(Z) : Supp f \subset H\}$. Thus, the problem reduces to proving the equivalence (ii)⇔(iii). The (ii)⇒(iii) part being evident, we need only to concentrate on the proof of (iii)⇒(ii). So, assume (iii), and fix freely a compact set $K \subset Z$. Let U be a relatively compact open neighbourhood of K, and put $H = \overline{U}$ (closure of U). Then, in vue of (iii), we have $|\mu|(f) \leq c_H \|f\|_\infty$ for any $\mu \in \mathcal{P}$ and for all $f \in \mathcal{K}_H(Z)$ with $f \geq 0$, where c_H is a nonnegative constant. With this in mind, let $\mu \in \mathcal{P}$. Considering that χ_U (the characteristic function of U) is lower semicontinuous on Z, we have

$$|\mu|(U) = \sup_{f \in \mathcal{K}_+(Z),\ f \leq \chi_U} |\mu|(f),$$

where $\mathcal{K}_+(Z)$ is the set of all $\varphi \in \mathcal{K}(Z)$ with $\varphi \geq 0$. But each f such that $f \in \mathcal{K}_+(Z)$ and $f \leq \chi_U$ belongs to $\mathcal{K}_H(Z)$ and satisfies $\|f\|_\infty \leq 1$. Therefore $\sup_{\mu \in \mathcal{P}} |\mu|(U) \leq c_H$ and hence $\sup_{\mu \in \mathcal{P}} |\mu|(K) \leq c_H$, which shows (ii). ∎

Our goal now is to establish a Σ-compactness result similar to Theorem 3.1. Specifically, assuming that E is a fundamental sequence, we want to show that from any sequence $(\mu_\varepsilon)_{\varepsilon \in E}$ in $\mathcal{M}(\Omega)$ which is bounded in the weak $*$ topology on $\mathcal{M}(\Omega)$, one can extract a subsequence that is vaguely Σ-convergent. Actually, this will arise as a consequence of a more general result, viz.

Theorem 4.1. *Let* Z *be a metrizable* σ*-compact locally compact space. Let* $(\mu_n)_{n \in \mathbb{N}}$ *be an ordinary sequence of Radon measures on* Z*. Assume that this sequence is bounded in the weak* $*$ *topology on* $\mathcal{M}(Z)$*. Then one can extract a subsequence* $(\mu_{k_n})_{n \in \mathbb{N}}$ *from* $(\mu_n)_{n \in \mathbb{N}}$ *such that* $\mu_{k_n} \to \mu$ *in* $\mathcal{M}(Z)$*-weak* $*$ *when* $n \to +\infty$*.*

Proof. We achieve this in two steps.

Step 1. Let U be a relatively compact open set in Z. The aim here is to verify that any subsequence $\left(\mu_{t_n}\right)_{n\in\mathbb{N}}$ extracted from $(\mu_n)_{n\in\mathbb{N}}$ contains a subsequence $\left(\mu_{t'_n}\right)_{n\in\mathbb{N}}$ such that

$$\mu_{t'_n}|_U \to \nu' \text{ in } \mathcal{M}(U)\text{-weak } * \text{ as } n \to +\infty. \tag{4.1}$$

To this end, let $K = \overline{U}$ and put B' for the closed unit ball in $\mathcal{M}(K)$ (strong dual of $\mathcal{C}(K)$). Provided with the relative weak $*$ topology on $\mathcal{M}(K)$, B' is a metrizable compact space (see, e.g., [17, p.426]). Having made this point, let $\left(\mu_{t_n}\right)_{n\in\mathbb{N}}$ be any arbitrary subsequence extracted from $(\mu_n)_{n\in\mathbb{N}}$. For each integer $n \geq 0$, put $\nu_n = \mu_{t_n}|_K$. Then $\nu_n \in \mathcal{M}(K)$ and further $\sup_n \|\nu_n\| = \sup_n |\nu_n|(K) = \sup_n \left|\mu_{t_n}\right|(K) < +\infty$ (use Lemma 4.1), where n runs through $\mathbb{N}$. Thus, we may assume without loss of generality that the sequence $(\nu_n)_{n\in\mathbb{N}}$ is contained in B'. Hence we can extract a subsequence $(\nu_{r_n})_{n\in\mathbb{N}}$ from $(\nu_n)_{n\in\mathbb{N}}$ such that as $n \to +\infty$, $\nu_{r_n} \to \nu$ in $\mathcal{M}(K)$-weak $*$, whence $\nu_{r_n}|_U \to \nu|_U = \nu'$ in $\mathcal{M}(U)$-weak $*$. Therefore, (4.1) follows by letting $t'_n = t_{r_n}$ $(n \in \mathbb{N})$ and noting that $\nu_n|_U = \mu_{t_n}|_U$.

Step 2. Let $(U_i)_{i\in\mathbb{N}}$ be a sequence of open sets in Z such that $\overline{U}_i \subset U_{i+1}$, $\overline{U}_i$ compact and $\cup_{i\in\mathbb{N}} U_i = Z$. By suitably applying the result of Step 1 we are readily led to two sequences $(\nu_i)_{i\in\mathbb{N}}$ $(\nu_i \in \mathcal{M}(U_i))$ and $\left(\mu_{t_n^{(i)}}\right)_{(i,n)\in\mathbb{N}\times\mathbb{N}}$ in $\mathcal{M}(Z)$ framed as follows: $\left(\mu_{t_n^{(0)}}\right)_{n\in\mathbb{N}}$ is a subsequence extracted from $(\mu_n)_{n\in\mathbb{N}}$ in such a way that $\mu_{t_n^{(0)}}|_{U_0} \to \nu_0$ in $\mathcal{M}(U_0)$-weak $*$ as $n \to +\infty$; for $i \geq 1$, $\left(\mu_{t_n^{(i)}}\right)_{n\in\mathbb{N}}$ is a subsequence extracted from $\left(\mu_{t_n^{(i-1)}}\right)_{n\in\mathbb{N}}$ in such a way that $\mu_{t_n^{(i)}}|_{U_i} \to \nu_i$ in $\mathcal{M}(U_i)$-weak $*$ as $n \to +\infty$. Hence, by the usual diagonal process, it is immediate that the sequence $\left(\mu_{k_n}\right)_{n\in\mathbb{N}}$ with $k_n = t_n^{(n)}$ is a subsequence extracted from $(\mu_n)_{n\in\mathbb{N}}$ so that for each $i \in \mathbb{N}$, $\mu_{k_n}|_{U_i} \to \nu_i$ in $\mathcal{M}(U_i)$-weak $*$ as $n \to +\infty$. Furthermore, it is clear that $\nu_i = \nu_{i+1}|_{U_i}$ $(i \in \mathbb{N})$, hence a (unique) Radon measure μ on Z such that $\mu|_{U_i} = \nu_i$ for any $i \in \mathbb{N}$ (this is a classical property). Since each $\varphi \in \mathcal{K}(Z)$ lies in $\mathcal{K}(U_i)$ for some suitable index i, we deduce that $\mu_{k_n} \to \mu$ in $\mathcal{M}(Z)$-weak $*$ as $n \to +\infty$, thereby proving the theorem. ∎

This leads to the Σ-compactness result for measures, as claimed.

Corollary 4.1. *We assume that E is a fundamental sequence. Then, from any sequence $(\mu_\varepsilon)_{\varepsilon\in E}$ in $\mathcal{M}(\Omega)$ which is bounded in the weak $*$ topology on $\mathcal{M}(\Omega)$, one can extract a subsequence that is vaguely Σ-convergent.*

Proof. Let us observe that $\Omega \times \Delta(A)$ is a metrizable σ-compact locally compact space. Hence, considering Remark 4.1 and Theorem 4.1, the corollary if proved if we can show that the sequence $(T_\varepsilon(\mu_\varepsilon))_{\varepsilon\in E}$ is bounded in the weak $*$ topology on $\mathcal{M}(\Omega \times \Delta(A))$. This is straightforward. If $\psi \in$

$\mathcal{K}(\Omega;A)$, and if K is a compact set in Ω containing the support of ψ, then $\left|\left\langle T_{\varepsilon}(\mu_{\varepsilon}), \widehat{\psi}\right\rangle\right| \leq c\sup_{x\in\Omega}\left\|\widehat{\psi}(x)\right\|_{\infty}$ for all $\varepsilon\in E$, where $c=\sup_{r\in E}|\mu_r|(K)$ is finite, according to Lemma 4.1. The corollary is proved. ∎

We will end with a few remarks.
(1) Suppose a sequence $(u_{\varepsilon})_{\varepsilon\in E}$ in $L^p(\Omega)$ $(1\leq p<\infty)$ is such that, as $E\ni\varepsilon\to 0$,

$$\int_{\Omega} u_{\varepsilon}(x)\,\psi(x,H_{\varepsilon}(x))\,dx \to \int\int_{\Omega\times\Delta(A)} u_0(x,s)\,\widehat{\psi}(x,s)\,dxd\beta(s)$$

for all $\psi\in\mathcal{K}(\Omega;A)$, where $u_0\in L^p(\Omega\times\Delta(A))$. It is clear that the sequence $(u_{\varepsilon})_{\varepsilon\in E}$ is not weakly Σ-convergent in $L^p(\Omega)$. However, each function u_{ε} being viewed as a Radon measure on Ω, the above sequence is vaguely Σ-convergent: More precisely, we have $u_{\varepsilon}dx\to u_0(dx\otimes d\beta)$ in $\mathcal{M}(\Omega)$-vague Σ. We deduce that the vague Σ-convergence is a natural generalization of weak Σ-convergence in $L^p(\Omega)$.
(2) Suppose a sequence $(\mu_{\varepsilon})_{\varepsilon\in E}$ is vaguely Σ-convergent in $\mathcal{M}(\Omega)$ to some $\mu_0\in\mathcal{M}(\Omega\times\Delta(A))$. Then, as $E\ni\varepsilon\to 0$, we have $\mu_{\varepsilon}\to\widetilde{\mu}_0$ in $\mathcal{M}(\Omega)$-weak $*$, where $\widetilde{\mu}_0(\varphi)=\int\int_{\Omega\times\Delta(A)}\varphi(x)\,d\mu_0(x,s)$, $\varphi\in\mathcal{K}(\Omega)$.

5. APPLICATION OF Σ-CONVERGENCE

5.1. **Preliminaries.** In the present section we are concerned with showing how Σ-convergence arises in the homogenization of partial differential equations. To illustrate this, we find it more convenient to focus attention on the rather simple case of an elliptic linear differential operator of order two, in divergence form. Specifically, let

$$-\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(a_{ij}^{\varepsilon}\frac{\partial u_{\varepsilon}}{\partial x_j}\right)=f \text{ in } \Omega,\ u_{\varepsilon}\in H_0^1(\Omega)=W_0^{1,2}(\Omega)\,, \tag{5.1}$$

where $\varepsilon>0$, Ω is a fixed bounded open set in $\mathbb{R}_x^N$, $f\in H^{-1}(\Omega)=W^{-1,2}(\Omega)$, $a_{ij}^{\varepsilon}(x)=a_{ij}\left(\frac{x}{\varepsilon}\right)$ $(x\in\Omega)$ with $a_{ij}\in L^{\infty}\left(\mathbb{R}_y^N\right)$, $a_{ji}=\overline{a}_{ij}$, and the classical ellipticity condition: there is a constant $\alpha>0$ such that

$$\operatorname{Re}\sum_{i,j=1}^{N}a_{ij}(y)\,\xi_j\overline{\xi}_i\geq\alpha\,|\xi|^2 \qquad \left(\xi\in\mathbb{C}^N\right)$$

for almost all $y\in\mathbb{R}^N$. For each real number $\varepsilon>0$, (5.1) uniquely determines u_{ε}, so that we have in hand a generalized sequence $(u_{\varepsilon})_{\varepsilon>0}$ in $H_0^1(\Omega)$.

The purpose of homogenization in the present case is to investigate the limit behaviour, as $\varepsilon\to 0$, of u_{ε} provided the coefficients a_{ij} satisfy a suitable hypothesis with respect to the so-called local variable $y=(y_1,...,y_N)$. It is common in homogenization to require the a_{ij}'s to satisfy the *periodicity hypothesis*, which means that the functions a_{ij} $(1\leq i,j\leq N)$ are periodic, say with period 1 in each coordinate, i.e., for every $k\in\mathbb{Z}^N$, one has $a_{ij}(y+k) = a_{ij}(y)$ $(1 \leq i,j \leq N)$ almost everywhere in $y \in \mathbb{R}^N$. Σ-convergence, which coincides in the present setting with well-known two-scale convergence, has proved to be an efficient tool for studying the periodic homogenization of linear as well as nonlinear boundary value problems and initial boundary value problems, including (5.1). We refer for example to [1, 26, 23] (see in particular the references in [23]).

However, the periodicity hypothesis is only one thing among many other hypotheses under which we can consider the homogenization of say (5.1). There is no doubt that in a great number of physical situations the periodicity hypothesis is inappropriate and should be therefore substituted by a realistic hypothesis. We claim that Σ-convergence theory allows to tackle homogenization problems beyond the classical periodic setting. Before we can concentrate on the proof of this assertion as regards (5.1), let us exhibit a few concrete examples of *nonperiodicity hypotheses* on a_{ij} under which it is possible to successfully study the homogenization of (5.1).

Example 5.1. *Let* $Y' = \left(-\frac{1}{2}, \frac{1}{2}\right)^{N-1}$ *with* $N \geq 2$, *and let* $L^2_{per}(Y')$ *be the usual Hilbert space of* Y'*-periodic functions in* $L^2_{loc}\left(\mathbb{R}^{N-1}_{y'}\right)$ *(see section 1). We may replace the periodicity hypothesis on* a_{ij} $(1 \leq i,j \leq N)$ *by*

$$a_{ij} \in \mathcal{B}_\infty\left(\mathbb{R}; L^2_{per}(Y')\right) \qquad (1 \leq i,j \leq N), \tag{5.2}$$

where $\mathcal{B}_\infty\left(\mathbb{R}; L^2_{per}(Y')\right)$ *denotes the space of all continuous functions* $y_N \to u(y_N)$ *of* $\mathbb{R}$ *into* $L^2_{per}(Y')$ *such that* $u(y_N)$ *converges in* $L^2_{per}(Y')$ *as* $|y_N| \to \infty$.

Example 5.2. *More generally, instead of (5.2) we may consider* $a_{ij} \in \mathcal{C}\left(\mathbb{R}; L^2_{per}(Y')\right)$ *with* $a_{ij}(.,y_N) \to z^+_{ij}$ *in* $L^2_{per}(Y')$ *as* $y_N \to +\infty$ *and* $a_{ij}(.,y_N) \to z^-_{ij}$ *in* $L^2_{per}(Y')$ *as* $y_N \to -\infty$, $1 \leq i,j \leq N$, *where* $a_{ij}(.,y_N)$ *stands for the function* $y' = (y_1,...,y_{N-1}) \to a_{ij}(y',y_N)$ *(for fixed* $y_N \in \mathbb{R}$*) of* $\mathbb{R}^{N-1}$ *into* $\mathbb{C}$, *and* z^+_{ij}, z^-_{ij} *are two functions in* $L^2_{per}(Y')$ *that are in general different.*

Example 5.3. *(Almost periodicity hypothesis). Let* $\left(L^2, l^\infty\right)\left(\mathbb{R}^N\right)$ *be the so-called amalgam of* L^2 *and* l^∞ *on* $\mathbb{R}^N$ [19], *i.e.,* $\left(L^2, l^\infty\right)\left(\mathbb{R}^N\right)$ *is the space of all* $u \in L^2_{loc}\left(\mathbb{R}^N\right)$ *such that*

$$\|u\|_{2,\infty} = \sup_{k \in \mathbb{Z}^N} \left(\int_{k+Y} |u(y)|^2\, dy\right)^{\frac{1}{2}} < \infty$$

with $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^N$. *This is a Banach space under the norm* $\|\cdot\|_{2,\infty}$. *We define* $L^2_{AP}\left(\mathbb{R}^N\right)$ *to be the space of all functions* $u \in \left(L^2, l^\infty\right)\left(\mathbb{R}^N\right)$ *such that the set* $\left\{\tau_h u : h \in \mathbb{R}^N\right\}$ *(with* $\tau_h u(y) = u(y-h)$ *for* $y \in \mathbb{R}^N$*) has a compact closure in* $\left(L^2, l^\infty\right)\left(\mathbb{R}^N\right)$. *Such functions are termed almost periodic in the sense of Stepanoff* [19]. *This being so, we may as well replace the periodicity hypothesis by*

$$a_{ij} \in L^2_{AP}\left(\mathbb{R}^N\right) \qquad (1 \leq i,j \leq N). \tag{5.3}$$

Example 5.4. *Let* $L^2_{\infty,per}(Y)$ *denote the closure in* $\left(L^2, l^\infty\right)\left(\mathbb{R}^N\right)$ *of the space of all finite sums* $\sum_{finite}\varphi_i u_i$ $\left(\varphi_i \in \mathcal{B}_\infty\left(\mathbb{R}^N_y\right),\ u_i \in \mathcal{C}_{per}(Y)\right)$*, where* $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^N$*,* $\mathcal{C}_{per}(Y)$ *defined in section 1 and* $\mathcal{B}_\infty\left(\mathbb{R}^N_y\right)$ *defined in Example 2.1. We may as well consider the homogenization of (5.1) under the hypothesis*

$$a_{ij} \in L^2_{\infty,per}(Y) \qquad (1 \leq i, j \leq N) \tag{5.4}$$

in place of the periodicity hypothesis.

Example 5.5. *More generally, in place of (5.4) we may consider*

$$a_{ij} \in L^2_{\infty,AP}\left(\mathbb{R}^N\right) \qquad (1 \leq i, j \leq N), \tag{5.5}$$

where $L^2_{\infty,AP}\left(\mathbb{R}^N\right)$ *denotes the closure in* $\left(L^2, l^\infty\right)\left(\mathbb{R}^N\right)$ *of all finite sums* $\sum_{finite}\varphi_i u_i$ $\left(\varphi_i \in \mathcal{B}_\infty\left(\mathbb{R}^N_y\right),\ u_i \in AP\left(\mathbb{R}^N\right)\right)$ *(see section 2 for the definition of* $AP\left(\mathbb{R}^N\right)$*).*

Remark 5.1. *Hypothesis (5.5) generalizes (5.3) and (5.4), as well.*

Example 5.6. *We may as well consider the homogenization of (5.1) under the following hypothesis, where* $1 \leq i, j \leq N$*:*
a_{ij} is constant on each cell $k + Y$ $\left(k \in \mathbb{Z}^N\right)$ *with* Y *as above,*
and further, as $|k| \to \infty$*,* $\int_{k+Y} a_{ij}(y)\,dy$ *tends to a finite limit in* $\mathbb{C}$*.*

The study of the homogenization problem for (5.1) under any of the hypotheses stated in the preceding examples reduces to an abstract setting that we will now look into.

5.2. **The abstract homogenization problem for (5.1).** The main purpose of the present subsection is to investigate the limit behaviour, as $\varepsilon \to 0$, of u_ε (the solution of (5.1)) under the abstract hypothesis

$$a_{ij} \in \mathfrak{X}^2_A\left(\mathbb{R}^N_y\right) \qquad (1 \leq i, j \leq N), \tag{5.6}$$

where A is an H-algebra on $\mathbb{R}^N$ with the property that A^∞ is dense in A (see Section 2). We also require A to be $W^{1,2}$-proper in the following sense:

(P)$_1$ $\mathcal{D}(\Delta(A))$ is dense in $H^1(\Delta(A)) = W^{1,2}(\Delta(A))$.

(P)$_2$ Given an open set $\Omega \subset \mathbb{R}^N_x$, a fundamental sequence E and a sequence $(v_\varepsilon)_{\varepsilon\in E}$ which is bounded in $H^1(\Omega)$, a subsequence E' can be extracted from E such that as $E' \ni \varepsilon \to 0$, $v_\varepsilon \to v_0$ in $H^1(\Omega)$-weak and $\frac{\partial v_\varepsilon}{\partial x_j} \to \frac{\partial v_0}{\partial x_j} + \partial_j v_1$ in $L^2(\Omega)$-weak Σ $(1 \leq j \leq N)$, where $v_1 \in L^2\left(\Omega; H^1_\#(\Delta(A))\right)$.

The aim now is to show that the homogenization of (5.1) under (5.6) is possible provided the H-algebra A has the preceding properties. To this end, let

$$\mathbb{F}^1_0 = H^1_0(\Omega) \times L^2\left(\Omega; H^1_\#(\Delta(A))\right)$$

with the norm

$$\|\mathbf{v}\|_{\mathbb{F}_0^1} = \left(\|v_0\|^2_{H_0^1(\Omega)} + \|v_1\|^2_{L^2\left(\Omega;H^1_\#(\Delta(A))\right)}\right)^{\frac{1}{2}}, \ \mathbf{v} = (v_0, v_1) \in \mathbb{F}_0^1,$$

which makes it a Hilbert space ($\|\cdot\|_{H_0^1(\Omega)}$ stands for the usual gradient norm). By combining property $(\mathrm{P})_1$ with (parts (2) and (3) of) Remark 2.7, it follows readily that

$$F_0^\infty = \mathcal{D}(\Omega) \times [\mathcal{D}(\Omega) \otimes J(\mathcal{D}(\Delta(A))/\mathbb{C})] \text{ is dense in } \mathbb{F}_0^1, \tag{5.7}$$

where $\mathcal{D}(\Delta(A))/\mathbb{C}$ denotes the space of $\varphi \in \mathcal{D}(\Delta(A))$ such that $\int_{\Delta(A)} \varphi(s)\, d\beta(s) = 0$.

We also need the sesquilinear form $\widehat{a}_\Omega(.,.)$ on $\mathbb{F}_0^1 \times \mathbb{F}_0^1$ given by

$$\widehat{a}_\Omega(\mathbf{u}, \mathbf{v}) = \sum_{i,j=1}^N \int\int_{\Omega\times\Delta(A)} \widehat{a}_{ij} \left(\frac{\partial u_0}{\partial x_j} + \partial_j u_1\right) \overline{\left(\frac{\partial v_0}{\partial x_i} + \partial_i v_1\right)} dx d\beta$$

for $\mathbf{u} = (u_0, u_1)$ and $\mathbf{v} = (v_0, v_1)$ in $\mathbb{F}_0^1$, where $\widehat{a}_{ij} = \mathcal{G}(a_{ij}) \in L^\infty(\Delta(A))$ (see part (v) of Corollary 2.4). There is no difficulty in verifying that the sesquilinear form $\widehat{a}_\Omega(.,.)$ is Hermitian, continuous and coercive (use corollary 2.4, and note also that $\int_{\Delta(A)} \partial_i v d\beta = 0$ for $v \in H^1_\#(\Delta(A))$, as is straightforward by Proposition 2.7 and use of Remark 2.7). Consequently, if l denotes the continuous antilinear form on $\mathbb{F}_0^1$ given by $l(\mathbf{v}) = \langle f, \overline{v}_0\rangle$ for $\mathbf{v} = (v_0, v_1) \in \mathbb{F}_0^1$, then the variational problem

$$\begin{cases} \mathbf{u} = (u_0, u_1) \in \mathbb{F}_0^1 : \\ \widehat{a}_\Omega(\mathbf{u}, \mathbf{v}) = l(\mathbf{v}) \text{ for all } \mathbf{v} \in \mathbb{F}_0^1 \end{cases} \tag{5.8}$$

has one and only one solution.

We are now in a position to prove the following homogenization theorem.

Theorem 5.1. *Under the preceding hypotheses, let $\mathbf{u} = (u_0, u_1)$ be uniquely defined by (5.8), and for each real $\varepsilon > 0$, let u_ε be the unique solution of (5.1). Then, as $\varepsilon \to 0$,*

$$u_\varepsilon \to u_0 \text{ in } H_0^1(\Omega)\text{-weak}, \tag{5.9}$$

$$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \partial_j u_1 \text{ in } L^2(\Omega)\text{-weak } \Sigma \quad (1 \le i,j \le N). \tag{5.10}$$

Proof. For fixed $\varepsilon > 0$, we have

$$\sum_{i,j=1}^N \int_\Omega a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j} \overline{\frac{\partial v}{\partial x_i}} dx = \langle f, \overline{v}\rangle \tag{5.11}$$

for all $v \in H_0^1(\Omega)$. By taking in particular $v = u_\varepsilon$ and making use of the properties of the matrix $(a_{ij})_{1\le i,j\le N}$, we see that the sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $H_0^1(\Omega)$. Consequently, given an arbitrary fundamental sequence E, appeal to the $W^{1,2}$-properness of A (see in particular property $(\mathrm{P})_2$) yields a subsequence E' from E and some $\mathbf{u} = (u_0, u_1) \in \mathbb{F}_0^1$ such that, as

$E' \ni \varepsilon \to 0$, we have (5.9)-(5.10). Thus, the theorem is proved if we can check that $\mathbf{u}$ verifies the variational equation in (5.8) (attention is drawn to Remark 3.2). For this purpose, take in (5.11) the particular function $v = \Phi_\varepsilon$ with

$$\Phi_\varepsilon(x) = \psi_0(x) + \varepsilon\psi_1\left(x, \frac{x}{\varepsilon}\right) \qquad (x \in \Omega),$$

where $\psi_0 \in \mathcal{D}(\Omega)$, $\psi_1 \in \mathcal{D}(\Omega) \otimes (A^\infty/\mathbb{C})$ with $A^\infty/\mathbb{C} = \{\psi \in A^\infty : M(\psi) = 0\}$. Clearly $\Phi_\varepsilon \in \mathcal{D}(\Omega)$. Furthermore, it is an easy exercise to show that, as $\varepsilon \to 0$, we have $\Phi_\varepsilon \to \psi_0$ in $H_0^1(\Omega)$-weak, and $\frac{\partial\Phi_\varepsilon}{\partial x_i} \to \frac{\partial\psi_0}{\partial x_i} + \partial_i\widehat{\psi}_1$ in $L^2(\Omega)$-strong Σ $(1 \leq i \leq N)$. From the latter convergence result together with (5.10) (where $E' \ni \varepsilon \to 0$) we deduce using Corollary 3.4 that, as $E' \ni \varepsilon \to 0$,

$$\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial \Phi_\varepsilon}{\partial x_i} \to \left(\frac{\partial u_0}{\partial x_j} + \partial_j u_1\right)\left(\frac{\partial \psi_0}{\partial x_i} + \partial_i\widehat{\psi}_1\right) \text{ in } L^2(\Omega)\text{-weak } \Sigma$$

for $1 \leq i, j \leq N$. We can now pass to the limit (as $E' \ni \varepsilon \to 0$) in (5.11) using Proposition 3.3 (it is clear that a_{ij} may be viewed as a function in $\mathcal{C}\left(\overline{\Omega}; \mathfrak{X}_A^{2,\infty}\right) = \mathcal{K}\left(\overline{\Omega}; \mathfrak{X}_A^{2,\infty}\right)$). The result is that

$$\widehat{a}_\Omega(\mathbf{u}, \mathbf{\Phi}) = l(\mathbf{\Phi}) \text{ for all } \mathbf{\Phi} \in F_0^\infty.$$

Thanks to (5.7), it follows that $\mathbf{u}$ is the solution of (5.8). Hence the theorem follows. ∎

At the present time, for each $1 \leq j \leq N$, let

$$\text{(5.12)} \qquad \begin{cases} \chi^j \in H^1_\#(\Delta(A)) : \\ \widehat{a}\left(\chi^j, v\right) = \sum_{k=1}^N \int_{\Delta(A)} \widehat{a}_{kj}(s)\, \overline{\partial_k v}(s)\, d\beta(s) \\ \text{for all } v \in H^1_\#(\Delta(A)), \end{cases}$$

where $\widehat{a}(.,.)$ is the sesquilinear form on $H^1_\#(\Delta(A)) \times H^1_\#(\Delta(A))$ given by

$$\widehat{a}(u,v) = \sum_{i,j=1}^N \int_{\Delta(A)} \widehat{a}_{ij}(s)\, \partial_j u(s)\, \overline{\partial_i v}(s)\, d\beta(s), \quad u, v \in H^1_\#(\Delta(A)).$$

For obvious reasons, (5.12) uniquely determines χ^j. Let then

$$q_{ij} = \int_{\Delta(A)} \widehat{a}_{ij}(s)\, d\beta(s) - \sum_{k=1}^N \int_{\Delta(A)} \widehat{a}_{ik}(s)\, \partial_k \chi^j(s)\, d\beta(s), \; 1 \leq i, j \leq N.$$

It can be shown that the matrix $(q_{ij})_{1 \leq i,j \leq N}$ has the usual symmetry and ellipticity properties (proceed as in [26]). Finally, the limit function u_0 in (5.9) is the (unique) weak solution of

$$-\sum_{i,j=1}^N q_{ij} \frac{\partial^2 u_0}{\partial x_i \partial x_j} = f \text{ in } \Omega, \quad u_0 \in H_0^1(\Omega),$$

as is immediate by a simple adaptation of the analogous result in the periodic setting (see, e.g., [26]).

5.3. **Concluding remarks.** Each structure hypothesis on a_{ij} $(1 \leq i, j \leq N)$ exhibited above (see Examples 5.1-5.6) can be reduced to (5.6) for a suitable $W^{1,2}$-proper H-algebra A. By way of illustration, the appropriate H-algebras for Examples 5.1, 5.3, 5.4 and 5.5 are respectively the H-algebra of Example 2.4 with $A_1 = \mathcal{C}_{per}(Y')$, $Y' = \left(-\frac{1}{2}, \frac{1}{2}\right)^{N-1}$, the H-algebra $A = AP_{\mathcal{R}}\left(\mathbb{R}^N\right)$ for a suitable $\mathcal{R}$ (see subsection 2.3), the H-algebra $A = \mathcal{B}_{\infty,per}(Y)$ (Example 2.2), and the H-algebra $A = \mathcal{B}_{\infty,\mathcal{R}}\left(\mathbb{R}^N\right)$ (Example 2.3) for a suitable $\mathcal{R}$. For further details see [30, 32].
Thus, Σ-convergence theory seems to be the right tool that is needed to extend homogenization theory beyond the usual periodic setting and thereby brigde the gap between classical periodic homogenization and stochastic homogenization. For the sake of clearness we have chosen a simple PDE to illustrate the large part Σ-convergence is destined to play in homogenization. For the homogenization by Σ-convergence of rather sophisticated PDE's we refer, e.g., to [33, 34, 24].

References

[1] Allaire, G., Homogenization and two-scale convergence, SIAM J. Math. Anal. **23** (1992) 1482-1518.

[2] Bakhvalov, N. and G. Panasenko, Homogenization: Averaging processes in periodic media (Mathematics and its Applications: BVil 36). Kluwer Acad. Pub., Dordrecht, 1989.

[3] Bensoussan, A., Lions, J.L. and G. Papanicolaou, Asymptotic Analysis for Periodic Structures, North Holland, Amsterdam, 1978.

[4] Bourbaki, N., Intégration, Chap. 1-4, Hermann, Paris, 1966.

[5] Bourbaki, N., Intégration, Chap. 5, Hermann, Paris, 1967.

[6] Bourbaki, N., Topologie Générale, Chap. V-X, Hermann, Paris, 1974.

[7] Bourbaki, N., Topologie Générale, Chap. I-IV, Hermann, Paris, 1971.

[8] Bourbaki, N., Espaces Vectoriels Topologiques, Chap. I-II, Hermann, Paris, 1965.

[9] Bourgeat, A., Mikelic, A. and S. Wright, Stochastic two-scale convergence in the mean and applications. J. Reine Angew. Math. **456** (1994) 19-51.

[10] Brézis, H., Analyse Fonctionnelle, Masson, Paris, 1983.

[11] Casado-Diaz, J., Two-scale convergence for nonlinear Dirichlet problems in perforated domains, Proc. Roy. Soc. Edinburgh Sect. A, **130** (2000), 249-276

[12] Clark, G.W and L.A. Packer, Two-scale homogenization of implicit degenerate evolution equations, J. Math. Anal. Appl. **214** (1997) 420-438

[13] Clark, G.W and L.A. Packer, Two-scale homogenization of nonlinear degenerate evolution equations, J. Math. Anal. Appl. **238** (1999) 316-328

[14] De Giorgi, E., Sulla convergenza di alcune succession di integrali del tipo dell'area, Rend. Mat. **8** (1975) 277-294

[15] Dieudonné, J., Eléments d'Analyse, T.II, Chap. XII-XV, Gauthier-Villars, Paris, 1968

[16] Dieudonné, J., Eléments d'Analyse, T.VI, Chap. XXII, Gauthier-Villars, Paris, 1975

[17] Dunford, N. and J.T. Schwartz, Linear Operators, Part I, Interscience Pub., Inc., New York, 1957

[18] Edwards, R.D., Functional Analysis, New York et al., Holt-Rinehart-Winston, 1965.

[19] Fournier, J.J.F and J. Stewart, Amalgams of L^p and l^q, Bull. Amer. Math. Soc. **13** (1985) 1-21.

[20] Guichardet, A., Analyse Harmonique Commutative, Dunod, Paris, 1968.

[21] Jikov, V.V., Kozlov, SM and O.A. Oleinik, Homogenization of Differential Operators and Integral Functionals, Springer-Verlag, Berlin, 1994.
[22] Larsen, R., Banach Algebras, Marcel Dekker, New York, 1973.
[23] Lukkassen, D., Nguetseng, G. and P. Wall, Two-scale convergence, Int. J. Pure Appl. Math., **2** (2002) 35-86
[24] Lukkassen, D., Nguetseng, G., Nnang, H. and P. Wall, Reiterated homogenization of nonlinear monotone operators in a general deterministic setting, J. Func. Sp. Appl. **7** (2009) 121-152.
[25] Nandakumaran, A.K. and M. Rajesh, Homogenization of a nonlinear degenerate parabolic differential equation, Electron. J. Diff. Equns, **17** (2001) 1-19
[26] Nguetseng, G., Sigma-convergence of parabolic differential operators, Multiple scales problems in biomathematics, mechanics, physics and numerics, Gakuto Intern. Ser. Math. Sc. Appl., **31** (2009) 93-132.
[27] Nguetseng, G., A general convergence result for a functional related to the theory of homogenization, SIAM J. Math. Anal., **20** (1989) 608-623.
[28] Nguetseng, G., Asymptotic analysis for a stiff variational problem arising in mechanics, SIAM J. Math. Anal. **21** (1990) 1394-1414.
[29] Nguetseng, G., Almost periodic homogenization: asymptotic analysis of a second order elliptic equation, Preprint.
[30] Nguetseng, G., Homogenization Structures and Applications I, Zeit. Anal. Anwend. **22** (2003) 73-107.
[31] Nguetseng, G., Mean value on locally compact Abelian groups, Acta Sci. Math. (Szeged) **69** (2003) 203-221.
[32] Nguetseng, G., Homogenization Structures and Applications II, Zeit. Anal. Anwend. **23** (2004) 483-508.
[33] Nguetseng, G. and H. Nnang, Homogenization of nonlinear monotone operators beyond the periodic setting, Electron. J. Diff. Equns **36** (2003) 1-24.
[34] Nguetseng, G. and J.L. Woukeng, Sigma-convergence of nonlinear parabolic operators, Nonlinear Anal., **66** (2007) 968-1004.
[35] Sanchez-Palencia, E., Nonhomogeneous Media and Vibration Theory, Springer-Verlag, Berlin, 1980.
[36] Spagnolo, S., Sulla convergenza di soluzioni di equazioni paraboliche e ellitiche, Ann. Scul. Norm. Sup. Pisa **22** (1968) 571-597
[37] Spagnolo, S., Convergence in energy for elliptic operators, in Numerical Solutions of Partial Differential Equations III, Acad. Press, New York, 1976.
[38] Tartar, L., Cours Peccot, Collège de France, Paris, 1977.
[39] Zhikov, V.V., On an extension and application of the two-scale convergence method, Sb. Math. **191** (2000) 973-1014.
[40] Zhikov, V.V., On two-scale convergence, J. Math. Sc., **120** (2004) 1328-1352
[41] V. V. Zhikov and E. V. Krivenko, Homogenization of singularly perturbed elliptic operators, Matem. Zametki 33:4, (1983), 571-582- (Engl. transl.: Math. Notes 33 (1983), 294-300

Gabriel Nguetseng, University of Yaounde 1, Department of Mathematics, P. O. Box 812 Yaounde, Cameroon
E-mail address: `nguetseng@uy1.uninet.cm`

Nils Svanstedt, University of Gothenburg, Department of Mathematical Sciences, SE-412 96 Gothenburg, Sweden
E-mail address: `nilss@chalmers.se`