PREPRINT 2010:35

Unitarization of linear representations of non-primitive posets

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Gothenburg Sweden 2010

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Gothenburg, June 2010

Preprint 2010:35
ISSN 1652-9715

Matematiska vetenskaper
Göteborg 2010

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Mathematics Subject Classification (2000). Primary 65N30; Secondary 65N15.

Keywords. unitarizations, partially ordered sets, subspaces, representations, semistable representations.

0. Introduction

The representation theory of partially ordered sets (posets) in linear vector spaces has been studied extensively and found to be of great importance for studying indecomposable representations of group and algebras, Cohen-Macaulay modules and many others algebraical objects (see [2, 3, 6, 7, 16] and many others). A representation of a given poset $\mathcal{P}$ in some vector space V is a collection $(V; V_i),\ i \in \mathcal{P}$ of vector subspaces $V_i \subset V$ such that $V_i \subset V_j$ as soon as $i \prec j$ in $\mathcal{P}$. Usually such representations are studied up to equivalence (which is given by linear bijections between two spaces that bijectively map the corresponded subspaces). M. Kleiner and L. Nazarova (see [6, 11]) completely classified all posets into three classes: finite type posets (posets that have finite number of indecomposable nonequivalent representation), tame posets (posets that have at most one-parametric family of indecomposable representations in each dimension) and wild posets (the classification problem of their indecomposable representations contains as a subproblem a problem of classification up to conjugacy classes a pair of two matrices).

It is also possible to develop a similar theory over Hilbert spaces. By representation we understand a collection $(H; H_i)$ of Hilbert subspaces in some Hilbert space H such that $H_i \subset H_j$ as soon as $i \prec j$. The equivalence between two system of Hilbert subspaces is given by unitary operator which bijectively maps corresponding subspaces. It turns out that in this case the classification problem becomes much more harder: even the poset $\mathcal{P} = \{a, b_1, b_2\}, b_1 \prec b_2$ becomes a $*$-wild

This work has been partially supported by the Scientific Program of National Academy of Sciences of Ukraine, Project No 0107U002333.

poset (it is impossible to classify all representation of this poset in a reasonable way see [9]). We add an "extra" relation

$$\alpha_1 P_1 + \ldots + \alpha_n P_n = \gamma I, \tag{0.1}$$

between the projections $P_i : H \mapsto H_i$ on corresponding subspaces for some weight $\chi = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n$ (this relation will be called *orthoscalarity condition*). When the system of subspaces is a so-called m-filtration, this relation plays an important role in different areas of mathematics (see [17, 8] and references therein) and this is actually one of the original motivations to investigate such representations of posets in Hilbert space.

The interconnection between linear and Hilbert representations of the posets is given by *unitarization* which asks whether for given linear representation (V, V_i) it is possible to provide a hermitian structure in V so that the linear relation (0.1) holds for some weight χ. In [4] for the case when $\mathcal{P}$ is a *primitive* poset we proved that a poset $\mathcal{P}$ is of finite orthoscalar type (has finitely many irreducible representations with orthoscalarity condition up to the unitary equivalence) if and only if it is of finite (linear) type. Also there were proved that each indecomposable representation of poset of finite (linear) type can be unitarized with some weight and for each representation we described all appropriated for unitarization weights.

In this paper we will prove the same for *non-primitive* posets. The approach given in [4] does not work longer for *non-pirmitive* case. We will use instead the notion of χ-stable representation from Geometric Invariant Theory of the product of the grassmannians of V (see [5, 17] and references therein). It turns out that for indecomposable representation χ-unitarizability is essentially the same as χ-stability. This gives a machinery to compute all appropriated for unitarization weights.

The main results of the paper are the following theorems.

Theorem 1. *A partially ordered set $\mathcal{P}$ has finite number of irreducible finite-dimensional Hilbert representations with orthoscalarity condition if and only if it does not contain subsets of the following form* $(1,1,1,1)$*,* $(2,2,2)$*,* $(1,3,3)$*,* $(1,2,5)$*,* $(N,4)$*, where* $(n_1, \ldots, n_s)$ *denotes the cardinal sum of linearly ordered set* $\mathcal{L}_1, \ldots, \mathcal{L}_s$*, whose orders equal* $n_1, \ldots, n_s$*, respectively, and* $(N,4)$ *is the set* $\{a_1, a_2, b_1, b_2, c_1, c_2, c_3, c_4\}$*, with the order* $a_1 \prec a_2, b_1 \prec b_2, b_1 \prec a_2, c_1 \prec c_2 \prec c_3 \prec c_4$*, and no other elements are comparable.*

Theorem 2. *Each indecomposable linear representation of the poset of finite type can be unitarized with some weight.*

Acknowledgments. We would like to thank Prof. Yu.S. Samoilenko and Prof. V.L. Ostrovskii for the statement of the problem and Prof. Ludmila Turowska for stimulating discussion. The second author thanks Chalmers University of Technology for hospitality and for the fruitful environment and Thorsten Weist for helpful remarks. The second author was partially supported by the Swedish Institute and by Ukrainian Grant for Young Scientists.

1. Preliminaries.

In this section we will briefly recall some basic facts concerning partially ordered sets, their representations and unitarization of linear representations.

1.1. Posets and Hasse quivers

Let $(\mathcal{P}, \prec)$ be a finite partially ordered set (or poset for short) which for us will be $\{a_1, \ldots, a_n\}$. By the width of the poset $\mathcal{P}$ we understand the cardinality of the largest antichain of $\mathcal{P}$, i.e. the cardinality of a subset of $\mathcal{P}$ where any two element are incomparable.

A poset $\mathcal{P}$ of the width s is called *primitive* and denoted by $(n_1, \ldots, n_s)$ if this poset is the cardinal sum of s linearly ordered sets $\mathcal{L}_1, \ldots, \mathcal{L}_s$ of orders $n_1, \ldots, n_s$. Otherwise the poset is called *non-primitive*.

We will use the standard graphic representations for the poset $\mathcal{P}$ called *Hasse quiver*. This representation associates to each elements $x \in \mathcal{P}$ a vertex x and a unique arrow $x \to y,\ y \in \mathcal{P}$ if $x \prec y$ and if there is no $z \in \mathcal{P}$ such that $x \prec z \prec y$. For example, let $\mathcal{P} = (N, 2) = \{a_1, a_2, b_1, b_2, c_1, c_2\}$ with the following order

$$a_1 \prec a_2, \quad b_1 \prec b_2, \quad c_1 \prec c_2, \quad b_1 \prec a_2,$$

then the corresponding Hasse quiver is the following:

$$\begin{tikzcd}[column sep=small] a_2 & b_2 & c_2 \\ a_1 \arrow[u] & b_1 \arrow[u] \arrow[ul] & c_1 \arrow[u] \end{tikzcd} .$$

1.2. Linear representations of posets. Indecomposability and Bricks

By a linear representation π of a given poset $\mathcal{P}$ in a complex vector space V we understand a rule that to each element $i \in \mathcal{P}$ associates a subspace $V_i \subseteq V$ in such a way that $i \prec j$ implies $V_i \subseteq V_j$. We will often think of $\mathcal{P}$ as a set $\{1, 2, \ldots, n\}$, where n is the cardinality of $\mathcal{P}$ and write $\pi = (V; V_1, \ldots, V_n)$ or $\pi = (V; V_i)$, notation $\pi(i) = V_i$, $\pi_0 = V$ also will be used.

By the dimension vector d_π of the representation π we understand a vector $d_\pi = (d_0; d_1, \ldots, d_n)$, where $d_0 = \dim V$, $d_i = \dim(V_i / \sum_{j \prec i} V_j)$. There is qudratic form $Q_{\mathcal{P}}$ on $\mathbb{Z}^{card(\mathcal{P})+1}$ given by

$$Q_{\mathcal{P}}(x_0, x_1, \ldots, x_n) = x_0^2 + \sum_{i \in \mathcal{P}} x_i^2 + \sum_{a, b \in \mathcal{P}, a \prec b} x_a x_b - \sum_{a \in \mathcal{P}} x_0 x_a.$$

Throughout the paper we denote by e_i the i-th coordinate vector $e_i = (\delta_{ij})$, by $e_{i_1 \ldots i_k}$ we understand the vector $e_{i_1} + \ldots + e_{i_k}$, and by $\langle x_1, \ldots, x_n \rangle$ the complex vector space spanned by vectors $x_1, \ldots, x_n \in V$. We will use a graphical picture for the representation of posets. For example the following picture describes the representation $\pi = (\mathbb{C}\langle e_1, e_2 \rangle; \langle e_1 \rangle, \langle e_2 \rangle, \langle e_{12} \rangle)$ for the poset $(1, 1, 1)$

$$
\begin{tikzcd}
 & \mathbb{C}^2 & \\
\langle e_1\rangle \arrow[ur] & \langle e_2\rangle \arrow[u] & \langle e_{12}\rangle \arrow[ul]
\end{tikzcd}
$$

In fact the set of all linear representations of a poset $\mathcal{P}$ forms the additive category $\mathrm{Rep}(\mathcal{P})$, where the set of morphisms $\mathrm{Mor}(\pi_1,\pi_2)$ between two representations $\pi_1 = (V; V_1,\dots,V_n)$ and $\pi_2 = (W; W_1,\dots,W_n)$ consists of linear maps $C : V \to W$, such that $C(V_i) \subset W_i$. Two representations π_1 and π_2 of $\mathcal{P}$ are isomorphic (or equivalent) if there exists an invertible morphism $C \in \mathrm{Mor}(\pi_1,\pi_2)$, i.e. there exist an invertible linear map $C : V \to W$ such that $C(V_i) = W_i$.

One can define a direct sum $\pi = \pi_1 \oplus \pi_2$ of two objects $\pi_1, \pi_2 \in \mathcal{P}$ in the following way:

$$\pi = (V \oplus W; V_1 \oplus W_1, \dots, V_n \oplus W_n).$$

Using the notion of direct sum it is natural to define *indecomposable* representations as the representations that are not isomorphic to the direct sum of two non-zero representations, otherwise representations are called *decomposable.* It is easy to show that a representation π is indecomposable if and only if there is no non-trivial idempotents in endomorphism ring $\mathrm{End}(\pi)$. A representation π is called *brick* if there is no non-trivial endomorphism of this representation (or equivalently when the ring $\mathrm{End}(\pi)$ is one-dimensional). It is obvious that if a representation is *brick* then it is *indecomposable.* But there exist *indecomposable* representations of posets which are not *brick*, for example

$$A_\alpha = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \quad \alpha \in \mathbb{R}\backslash\{0\},$$

the representation $\pi_\alpha = (V; V_1, V_2, V_3, V_4)$ of the poset $\mathcal{N} = (1,1,1,1)$

$$V = \mathbb{C}^2 \oplus \mathbb{C}^2; \quad V_1 = \mathbb{C}^2 \oplus 0, \quad V_2 = 0 \oplus \mathbb{C}^2,$$
$$V_3 = \left\{(x,x) \in \mathbb{C}^4 \mid x \in \mathbb{C}^2\right\}, \quad V_4 = \left\{(x, A_\alpha x) \in \mathbb{C}^4 \mid x \in \mathbb{C}^2\right\},$$

is indecomposable but is not brick.

Recall that a poset $\mathcal{P}$ is called a poset of *finite (linear) type* if there exist only finitely many non-isomorphic indecomposable representation of $\mathcal{P}$ in the category $\mathrm{Rep}(\mathcal{P})$. Result obtained by M.M.Kleiner [6] gives a complete description of the posets of finite type.

Theorem 3. *(see* [6]*, Theorem 1) A poset $\mathcal{P}$ is a set of finite type if and only if it does not contain subsets of the form* $(1,1,1,1)$*,* $(2,2,2)$*,* $(1,3,3)$*,* $(1,2,5)$ *and* $(N,4)$*, where*

$$(N,4) = \{a_1, a_2, b_1, b_2, c_1, c_2, c_3, c_4\},$$
$$a_1 \prec a_2,\ b_1 \prec b_2,\ b_1 \prec a_2,\ c_1 \prec c_2 \prec c_3 \prec c_4.$$

Remark 1. *The posets in previous theorem will be called critical henceforth.*

Kleiner described also all indecomposable representations of posets of finite type up to equivalence using the notion of *sincere* representations (see [7]).

Definition 4. *We call a representation* π sincere *if it is indecomposable and the components of the dimension vector* $d_\pi = (d_0; d_1, \dots, d_n)$ *satisfy* $d_i \neq 0$; *otherwise we say that the representation is* degenerated.

A poset is called *sincere* if it has at least one *sincere* representation. It is easy to see that any indecomposable representation of a poset of finite type actually is a sincere representation of its some sincere subposet. To describe all indecomposable representations of a fixed poset $\mathcal{P}$ of finite type one needs to describe all sincere representations of its all sincere subposets $\mathcal{R}$ (including itself if $\mathcal{P}$ is sincere) and to add zero spaces $V_i / \sum_{j \prec i} V_j$ if $i \notin \mathcal{R}$.

1.3. Unitary representations of posets

In the spirit of a number of previous articles we study representation theory of posets over Hilbert spaces. Denote by $\mathrm{Rep}(\mathcal{P}, \mathrm{H})$ a sub-category in $\mathrm{Rep}(\mathcal{P})$, defined as follows: its set of objects consists of finite-dimensional Hilbert spaces and two objects $\pi = (H, H_i)$ and $\tilde{\pi} = (\tilde{H}, \tilde{H}_i)$ are equivalent in $\mathrm{Rep}(\mathcal{P}, \mathrm{H})$ if there exists a unitary operator $U : H \to \tilde{H}$ such that $U(H_i) = \tilde{H}_i$ (unitary equivalent). Representation $\pi \in \mathrm{Rep}(\mathcal{P}, \mathrm{H})$ is called *irreducible* iff the C^*-algebra generated by set of orthogonal projections $\{P_i\}$ on the subspaces $\{H_i\}$ is irreducible. Let us remark that indecomposability of a representation π in $\mathrm{Rep}(\mathcal{P})$ implies irreducibility of $C^*(\{P_i, i \in \mathcal{P}\})$ but the converse is false.

The problem of classification all irreducible objects in the category $\mathrm{Rep}(\mathcal{P}, \mathrm{H})$ becomes much harder. Even for the primitive poset $\mathcal{P} = (1, 2)$ it is hopeless to describe in a reasonable way all its irreducible representations: indeed this lead us to classify up to unitary equivalence three subspace in a Hilbert space, two of which are orthogonal, but it is well-known due to [9] that such problem is $*$-wild. Hence it is natural to consider some additional relation.

Let us consider those objects $\pi \in \mathrm{Rep}(\mathcal{P}, \mathrm{H})$, $\pi = (H; H_1, \dots, H_n)$, for which the following linear relation holds:

$$\alpha_1 P_1 + \ldots + \alpha_n P_n = \gamma I, \tag{1.1}$$

where α_i, γ are some positive real numbers, and P_i are the orhoprojections on the subspaces H_i. These objects form a category which will be also denoted by $\mathrm{Rep}(\mathcal{P}, \mathrm{H})$. Such representations will be called *orthoscalar representations*.

1.4. Unitarization

Obviously there exists a forgetful functor from $\mathrm{Rep}(\mathcal{P}, \mathrm{H})$ to $\mathrm{Rep}(\mathcal{P})$ which maps each system of Hilbert spaces to its underlying system of vector spaces. We ask whether there exists "functor in reverse direction"?

Definition 5. *We say that a given representation* $\pi \in \mathrm{Rep}(\mathcal{P})$, $\pi = (V; V_1, \dots, V_n)$ *of the poset* $\mathcal{P}$ *can be unitarized with a weight (or is unitarizable)* $\chi = (\alpha_1, \dots, \alpha_n)$

if it is possible to choose hermitian structure $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ in V, so that the corresponding projections P_i onto subspace V_i satisfy the following relation:

$$\alpha_1 P_1 + \ldots + \alpha_n P_n = \lambda_\chi(\pi) I,$$

where $\lambda_\chi(\pi)$ is equal to $\frac{1}{\dim V}\sum_{i=1}^n \alpha_i \dim V_i$.

For a given linear representation π of the poset $\mathcal{P}$ by $\triangle_\pi^{\mathcal{P}}$ we denote the set of those weights χ that are appropriated for unitarization. And correspondingly we say that representation π can be *unitarized* if the set $\triangle_\pi^{\mathcal{P}}$ is nonempty.

In [4] we showed that each indecomposable non-degenerated representation of primitive poset of finite type can be unitarized and for each such representation we completely described the sets $\triangle_\pi^{\mathcal{P}}$. The approach provided in [4] does not work longer for *non-pirmitive* case. In this paper we will use rather different approach which comes from Geometric Invariant Theory and gives exact criteria for unitarization.

2. Balanced metric and stable representations of posets

Let $\pi = (V; V_1, \ldots, V_n)$ be a system of subspaces (in particular it can be a representation of some poset $\mathcal{P}$) in a complex vector space V and let $\chi = (\alpha_1, \ldots, \alpha_n)$ be some weight, i.e. the vector from $\mathbb{R}_+^n$. Denote by $\lambda_\chi(\pi)$ the number defined by

$$\lambda_\chi(\pi) = \frac{1}{\dim V}\sum_{i=1}^n \alpha_i \dim V_i.$$

If U is a subspace of V one can form another system of subspaces $\pi\cap U$ generated by π and U

$$\pi \cap U = (U; V_1\cap U, \ldots, V_n \cap U).$$

By $\lambda_\chi(\pi\cap U)$ we will understand the number given by

$$\lambda_\chi(\pi\cap U) = \frac{1}{\dim U}\sum_{i=1}^n \alpha_i \dim(V_i\cap U).$$

Definition 6. *We say that a system of subspace $\pi = (V; V_1, ..., V_n)$ is χ-stable if for each proper subspace $U \subset V$ the following inequality holds*

$$\lambda_\chi(\pi\cap U) < \lambda_\chi(\pi).$$

Suppose for a moment that for a system $\pi = (V; V_1, \ldots, V_n)$ we have chosen a sesquilinear form $\langle\cdot,\cdot\rangle$ on V so that

$$\chi_1 P_{V_1} + \ldots + \chi_n P_{V_n} = \lambda_\chi(\pi) I_V,$$

following [5] we will call such form χ-*balanced* metric. For a system π to possess a χ-balanced metric is essentially the same as to be unitarized with the weight χ. The list of necessarily restrictions on weight χ can be obtained by the following lemma

Lemma 7. *If the indecomposable system of subspaces π possesses χ-balanced metric then π is χ-stable.*

Proof. The proof of these statement can be obtained by taking the trace from linear equation. Indeed let $\pi = (V; V_1, \dots, V_n)$ possesses a χ-balanced metric, i.e. a metric $\langle \cdot, \cdot \rangle$ on V such that

$$\alpha_1 P_{V_1} + \ldots + \alpha_n P_{V_n} = \lambda_\chi(\pi) I_V. \tag{2.1}$$

By taking the trace from the left and right hand sides of (2.1) we can write $\lambda_\chi(\pi)$ as

$$\lambda_\chi(\pi) = \frac{tr(\alpha_1 P_{V_1} + \ldots + \alpha_n P_{V_n})}{tr(I_V)} = \frac{1}{tr(I_V)} \sum_{i=1}^{n} \alpha_i tr(P_{V_i}).$$

Let U be some proper subspace of V. Denote by P_U an orthogonal projection on U. Multiplying (2.1) by P_U from the left we obtain

$$\chi_1 P_{V_1} P_U + \ldots + \chi_n P_{V_n} P_U = \lambda_\chi(\pi) P_U,$$

then taking trace of the last we get

$$\lambda_\chi(\pi) = \frac{1}{tr(P_U)} \sum_{i=1}^{n} \alpha_i tr(P_{V_i} P_U).$$

Observe that $tr(P_{V_i} P_U) \geq tr(P_{V_i \cap U})$ (this can be proved using spectral theorem for the pair of projection and then by restriction to two-dimensional representation). This gives us

$$\lambda_\chi(\pi) = \frac{1}{tr(P_U)} \sum_{i=1}^{n} \alpha_i tr(P_{V_i} P_U) \geq \frac{1}{tr(P_U)} \sum_{i=1}^{n} \alpha_i tr(P_{V_i \cap U}) = \lambda_\chi(\pi \cap U).$$

It remains to prove that the inequality is strict. Indeed assume that $tr(P_{V_i} P_U) = tr(P_{V_i \cap U})$ for all i. Then it is easy to see the all P_{V_i} commutes with P_U (again using spectral theorem for the pair of two projections) hence the subspace U is invariant with respect to projections P_i which means that the representation π is decomposable. Therefore $\lambda_\chi(\pi \cap U) < \lambda_\chi(\pi)$ for all proper subspaces U, and hence π is χ-stable. $\square$

A natural question arises whether the reverse statement is true, i.e. does every χ-stable system π possesses χ-balanced metric?

When π is a collection of filtrations (recall that a filtration is a chain of subspaces $V_0 \subset \ldots \subset V_m$) this assertion was proved by Totaro ([17]) and Klyachko ([8]). If fact this can be proved for any configuration of subspaces and any weight χ. Here we will reproduce shortly what was done in [5].

Let V be a complex vector space and let $\chi \in \mathbb{N}^n$. Consider the product of Grassmanians

$$\mathrm{Gr}(k_1, V) \times \ldots \times \mathrm{Gr}(k_n, V).$$

Any system of subspaces $\pi = (V; V_1, \dots, V_n)$ of vector space V with dimension vector equal to $d = (\dim V; k_1, \dots, k_n)$ can be considered as a point of $\prod_{i=1}^n \mathrm{Gr}(k_i, V)$. We equip $\prod_{i=1}^n \mathrm{Gr}(k_i, V)$ with simplectic form δ, which is the skew bilinear form

$$\delta : \prod_{i=1}^n \mathrm{Gr}(k_i, V) \times \prod_{i=1}^n \mathrm{Gr}(k_i, V) \to \mathbb{C}$$
$$\delta : (\pi, \tilde{\pi}) \mapsto \sum_i \chi_i tr(A_i \tilde{A}_i^*),$$

where A_i and $\tilde{A}_i$ is a matrix representation of V_i and $\tilde{V}_i$ (their columns form an orthonormal bases for V_i and $\tilde{V}_i$ correspondingly), and $*$ is adjoint correspondingly to standart hermitian metric $\langle \cdot, \cdot \rangle$ on V.

As Lie group $SU(V)$ acts diagonally on $\prod_{i=1}^m \mathrm{Gr}(k_i, V)$ preserving symplectic form σ, the action is given by operating on V (via its linear representation). The corresponding moment map $\Phi : \prod_{i=1}^m \mathrm{Gr}(k_i, V) \to su^*(V)$. $su^*(V)$ the dual of Lie algebra of $SU(V)$ which is given by the algebra of traceless Hermitian matrices over V. This moment map is given by

$$\Phi(\pi) = \sum_i \chi_i A_i A_i^* - \lambda_\chi(\pi) I,$$

Assuming that π is χ-stable is possible to find (see [5]) such $g \in SL(V)$ that $\Phi(g \cdot \pi) = 0$. Then correspondently to hermitian metric $g\langle \cdot, \cdot \rangle = \langle g\cdot, g\cdot \rangle$ the following holds

$$\sum_i \chi_i A_i A_i^* = \lambda_\omega(\pi) I.$$

Taking into account that $A_i A_i^*$ is an orthogonal projection on V_i correspondinly to $g\langle \cdot, \cdot \rangle$ we get desirable result.

To conclude it remains to note that for $\chi \in \mathbb{R}_+^n$ one can find appropriated sequence of rational χ_n that tends to χ and that π is χ_n-stable (it is possible because stable condition is open) and then one can make use (for example) Shulman's lemma about representation of limit relation.

Summing up the following theorem holds ([5]).

Theorem 8. *Let χ be the weight. For the indecomposable system of subspaces π the following conditions are equivalent:*

(i) π *can be unitarized with weight* χ*;*
(ii) π *is* χ*-stable system.*

This theorem gives exact criteria of unitarization of any linear indecomposable representation of partially ordered set with the weight χ. On practice in order to check the χ-stability for a given representation $\pi = (V; V_1, \dots, V_n)$ of some poset with dimension vector $d = (\dim V_0; \dim V_1, \dots, \dim V_n)$ one can describe all possible subdimension vectors $d' = (\dim U; \dim(V_1 \cap U), \dots, \dim(V_n \cap U))$, $U \subset V$ and to check for these vector stability condition

$$\frac{1}{d'_0}\sum \chi_i d'_i < \frac{1}{d_0}\sum \chi_i d_i,$$

Let us remark that on subdimension vectors there exist a natural coordinate partial order. That is evident that to check the stability condition one should check inequality above for maximal vectors.

3. Proof of the Theorem 1

Sufficiently. Let $\mathcal{P}$ be the poset which does not contain any of critical posets. Assume that these posets could have infinite number of unitary inequivalent Hilbert space representations with fixed weight. If two Hilbert space representations with the same weight are unitary non-equivalent then they are linearly inequivalent (due to [10], Theorem 1). Hence each such poset has infinite number of indecomposable linear representation. But this contradicts Kleiner's theorem.

Necessity. Our aim is for each critical poset $\mathcal{P}$ to build infinite series of indecomposable pairwise nonequivalent Hilbert representations of this poset with the same weight. For the primitive case this can be done using the connection between Hilbert orthoscalar representations of the posets with the representations of some certain class of $*$-algebras that connected with star-shaped graphs (see for example [13]). But this approach does not work for the nonprimitive case (namely for the set $(N,4)$). Here we consider quite different approach which is based on unitarization.

Let $\mathcal{P}$ be a poset and let $I \subset \mathcal{P}$ (I can be empty). Define an extended poset $\tilde{\mathcal{P}}_I$ by adding to $\mathcal{P}$ an element $\tilde{p}$ subject to the relations $\tilde{p} \prec i$, $i \in I$. Let $\pi = (V; V_i)$ be a representation of the poset $\mathcal{P}$. Assume that there are two linearly independent vectors $v_1, v_2 \in V$ such that the following conditions are satisfied

$$\dim\Big(\big(\sum_{i\in I} V_i + \langle v_1 + \lambda v_2\rangle\big) \cap \big(\sum_{i\in I} V_i + \langle v_1 + \tilde{\lambda} v_2\rangle\big)\Big) = \dim\big(\sum_{i\in I} V_i\big),$$

where $\lambda \neq \tilde{\lambda} \in D$, and D is dense in $\mathbb{C}$. One can show that such vectors exist if $\dim(\sum_{i\in I} V_i) + 1 < \dim(V)$. We can define a family of representation $\tilde{\pi}_\lambda$ of the $\tilde{\mathcal{P}}_I$, by letting

$$\tilde{\pi}_\lambda(x) = \begin{cases} \pi(x), & x \neq \tilde{p}, \\ \sum_{i\in I} \pi(i) + \langle v_1 + \lambda v_2\rangle, & x = \tilde{p}. \end{cases}$$

The following proposition is straightforward.

Proposition 9. *Assume that $\pi = (V; V_i)$ is a brick representation of the poset $\mathcal{P}$ and $\tilde{\pi}_\lambda(\tilde{p})$ is the corresponding family of representations of $\mathcal{P}_I$ for appropriated choosen $v_1, v_2 \in V$. Then the following holds*

1. *The representation $\tilde{\pi}_\lambda$ is brick ($End(\tilde{\pi}_\lambda) \cong \mathbb{C}$) for each $\lambda \in \mathbb{C}$.*
2. *If $\lambda \neq \lambda'$ then $\tilde{\pi}_\lambda$ is not equivalent to $\tilde{\pi}_{\lambda'}$.*

3. If $\lambda \neq \lambda'$ and π_λ, $\pi_{\lambda'}$ are χ-stable then the corresponding systems of projection (after unitarization) are unitary inequivalent.

Using the construction above for each pre-critical poset (that is critical poset without one element) we will build its extended poset that coincide with critical poset and we will choose appropriated representation of pre-critical what allows to build extended representations of critical posets. These family of extended representation are given in dimension $d^{\mathcal{P}}$ which is the imaginary root of the corresponding quadratic form $Q_{\mathcal{P}}$, $Q_{\mathcal{P}}(d^{\mathcal{P}}) = 0$. We will prove that these representations are stable for all $\lambda \in \mathbb{C}, \lambda \neq 0, 1$ (see Appendix B for the description of subdimension vectors) with the same weight $\chi^{\mathcal{P}}$ which is defined by the dimension vector in the following way: $\chi_i^{\mathcal{P}} = d_i^{\mathcal{P}}$ and $\lambda_\chi = d_0^{\mathcal{P}}$.

1) Case $(1, 1, 1, 1)$.

For $\mathcal{P} = (1, 1, 1)$, $I = \emptyset$ the extended poset $\tilde{\mathcal{P}}_I$ is equal to $(1, 1, 1, 1)$ and for the representation $\pi = (\mathbb{C}^2; \langle e_1\rangle; \langle e_2\rangle; \langle e_1 + e_2\rangle)$ its extended representation has the following form.

$$\langle e_1\rangle \quad \langle e_2\rangle \quad \langle e_1 + e_2\rangle \quad \langle e_1 + \lambda e_2\rangle$$

The dimension vector is equal to $d^{(1,1,1,1)} = (2; 1; 1; 1; 1)$ and the representations are $(1, 1, 1, 1)$–stable.

2) Case $(2, 2, 2)$.

For $\mathcal{P} = (1, 2, 2)$, $I = \{a_1\}$ the extended poset $\tilde{\mathcal{P}}_I$ is equal to $(2, 2, 2)$ and for the representation $\pi = (\mathbb{C}^3; \langle e_{123}\rangle; \langle e_2\rangle, \langle e_1, e_2\rangle; \langle e_2\rangle, \langle e_2, e_3\rangle)$ its extended representation has the following form

$$\begin{tikzcd}
\langle e_{123}, e_1 + \lambda e_3\rangle & \langle e_1, e_2\rangle & \langle e_2, e_3\rangle \\
\langle e_{123}\rangle \arrow[u] & \langle e_1\rangle \arrow[u] & \langle e_3\rangle \arrow[u]
\end{tikzcd}$$

The dimension vector is equal to $d^{(2,2,2)} = (3; 1, 1; 1, 1; 1, 1)$ and the representations are $(1, 1, 1, 1, 1, 1)$–stable.

3) Case $(1, 3, 3)$.

For $\mathcal{P} = (1, 2, 3)$, $I = \{b_2\}$ the extended poset $\tilde{\mathcal{P}}_I$ is equal to $(1, 3, 3)$ and for the representation $\pi = (\mathbb{C}^4; \langle e_{123}, e_{24}\rangle; \langle e_4\rangle, \langle e_1, e_4\rangle; \langle e_3\rangle, \langle e_2, e_3\rangle, \langle e_1, e_2, e_3\rangle)$ its extended representation has the following form

$$\begin{tikzcd}
 & \langle e_1, e_4, e_2 + \lambda e_3\rangle & \langle e_1, e_2, e_3\rangle \\
 & \langle e_1, e_4\rangle \arrow[u] & \langle e_2, e_3\rangle \arrow[u] \\
\langle e_{123}, e_{24}\rangle & \langle e_4\rangle \arrow[u] & \langle e_3\rangle \arrow[u]
\end{tikzcd}$$

Dimension vector is equal to $d^{(1,3,3)} = (4; 2; 1, 1, 1; 1, 1, 1)$ and the representations are $(2, 1, 1, 1, 1, 1, 1)$–stable.

4) Case $(1, 2, 5)$.

For $\mathcal{P} = (1, 2, 4)$, $I = \{c_4\}$ the extended poset $\tilde{\mathcal{P}}_I$ is equal to $(1, 2, 5)$ and for the representation

$$\pi = (\mathbb{C}^6; \langle e_{123}, e_{245}, e_{16}\rangle; \langle e_5, e_6\rangle, \langle e_1, e_2, e_5, e_6\rangle; \\ \langle e_4\rangle, \langle e_3, e_4\rangle, \langle e_2, e_3, e_4\rangle, \langle e_1, e_2, e_3, e_4\rangle)$$

its extended representation has the following form

$$\begin{tikzcd}
 & & \langle e_1, e_2, e_3, e_4, e_5 + \lambda e_6\rangle \\
 & & \langle e_1, e_2, e_3, e_4\rangle \arrow[u] \\
 & & \langle e_2, e_3, e_4\rangle \arrow[u] \\
 & \langle e_1, e_2, e_5, e_6\rangle & \langle e_3, e_4\rangle \arrow[u] \\
\langle e_{123}, e_{245}, e_{16}\rangle & \langle e_5, e_6\rangle \arrow[u] & \langle e_4\rangle \arrow[u]
\end{tikzcd}$$

In this case the dimension vector is equal to $d^{(1,2,5)} = (6; 3; 2, 2; 1, 1, 1, 1, 1)$ and the representations are $(3, 2, 2, 1, 1, 1, 1, 1)$–stable.

5) Case $(N, 4)$.

For $\mathcal{P} = (1, 2, 4)$, $I = \{a_1, b_1\}$ its extended poset $\tilde{\mathcal{P}}_I$ is equal to $(N, 4)$ and for the representation

$$\pi = (\mathbb{C}^5; \langle e_{235}, e_{134}\rangle; \langle e_5\rangle, \langle e_1, e_2, e_5\rangle; \langle e_4\rangle, \langle e_3, e_4\rangle, \langle e_2, e_3, e_4\rangle, \langle e_1, e_2, e_3, e_4\rangle)$$

its extended representations has the following form

$$\begin{tikzcd}
 & & \langle e_1, e_2, e_3, e_4\rangle \\
 & & \langle e_2, e_3, e_4\rangle \arrow[u] \\
\langle e_{235}, e_{134}, e_5, e_3 + \lambda e_4\rangle & \langle e_1, e_2, e_5\rangle & \langle e_3, e_4\rangle \arrow[u] \\
\langle e_{235}, e_{134}\rangle \arrow[u] & \langle e_5\rangle \arrow[u] \arrow[ul] & \langle e_4\rangle \arrow[u]
\end{tikzcd}$$

The dimension vector is equal to $d^{(\mathcal{N}_4)} = (5; 2, 1; 1, 2; 1, 1, 1, 1)$ and the representations are $(2, 1, 1, 2, 1, 1, 1, 1)$–stable.

So, for each critical posets $\tilde{\mathcal{P}}$ we build the infinite family of pairwise inequivalent $d^{\mathcal{P}}$–stable brick representation π_λ in the dimension $d^{\mathcal{P}}$ (see appendix for the proof of stability). Hence each critical poset has infinite number of pairwise unitary inequivalent representations satysfying

$$d_1^{\mathcal{P}} P_1 + \ldots + d_n^{\mathcal{P}} P_n = d_0^{\mathcal{P}} I,$$

i.e. each critical posets has infinite Hilbert space representable type. This completes the proof of Theorem 1.

Remark 2. *Theorem 1 can be reformulated in the following way — a poset has finite orthoscalar Hilbert type if and only if it has finite linear type.*

4. Quite sincere representations and the proof of Theorem 2

4.1. Quite sincere representations

To describe all irreducible orthoscalar representations of posets of finite type we need a new notion of sincere representation which we will call quite sincerity.

The following is for linear representations of posets.

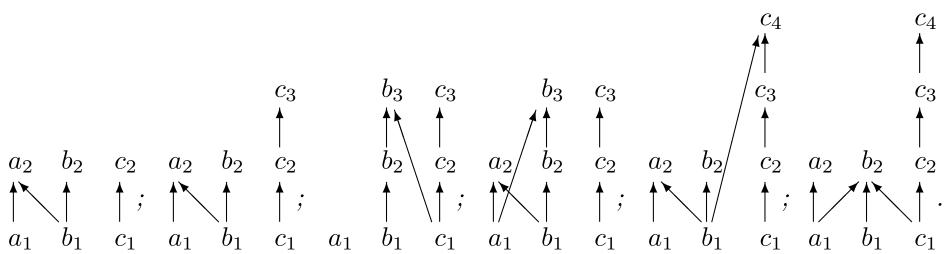
Definition 10. *Let $\mathcal{P} = \{1, \ldots, n\}$ be a poset. We call a representation π of $\mathcal{P}$* quite sincere *if it is indecomposable and the following conditions holds for all $i = 1, \ldots, n$:*

- $\pi(i) \neq 0$;
- $\pi(i) \neq \pi(0)$;
- $\pi(i) \neq \pi(j)$ *as* $i \prec j$.

Definition 11. *We say that the poset is* quite sincere *if it has at least one* quite sincere *representation.*

The following theorem describes all quite sincere posets of finite type and gives all their quite sincere representations.

Theorem 12. *The set of quite sincere posets of finite type consists of four primitive posets* $(1,1,1)$, $(1,2,2)$, $(1,2,3)$, $(1,2,4)$ *and following non-primitive posets:*



$\mathcal{P}_1$ $\mathcal{P}_2$ $\mathcal{P}_3$ $\mathcal{P}_4$ $\mathcal{P}_5$ $\mathcal{P}_5^*$

Complete list of all quite sincere representations of these posets are given in the Appendix A.

Proof. Let π be a quite sincere representation of a poset $\mathcal{P}$. The following two possibilities can occur

- π is a sincere representation of $\mathcal{P}$ with $\pi(i) \neq \pi(0)$;
- there is $k \in \mathcal{P}$ such that $\pi(k) = \sum_{i \prec k} \pi(i)$.

In the first case the representation (and obviously the corresponding poset) is in Kleiner's list of sincere posets and their representations [7]. In the second the representation π generates an indecomposable representation of poset the $\mathcal{P}_1 = \mathcal{P} \setminus k$ which we denote by π_1. It is clear that π_1 is a quite sincere representation of $\mathcal{P}_1$ and it again satisfies one of the two above possibilities. Proceeding in this way we will obtain a sincere representation of some poset $\mathcal{P}_k$ with condition $\pi(i) \neq \pi(0)$ which is in Kleiner's list.

Summing up we have the following algorithm for calculation of all quite sincere posets and their quite sincere representations:

1. all sincere posets which have a sincere representation such that $\pi(i) \neq \pi(0)$ are quite sincere and Kleiner's list gives all. These posets are $(1,1,1)$, $(1,2,2)$, $(1,2,3)$, $(1,2,4)$, $\mathcal{P}_2$ with representations listed in Appendix A (for $\mathcal{P}_2$ marked with *);
2. Let $\mathcal{P}$ be a quite sincere poset and and $I \subset \mathcal{P}$ such that $\sum_{i\in I} \pi(i) \neq \pi(0)$. Let $\tilde{\mathcal{P}}_I$ be the corresponding extended poset defined as in Section 3. Let $\tilde{\pi}_I$ be a representation of $\tilde{\mathcal{P}}_I$ given by $\tilde{\pi}_I(j) = \pi(j)$ for all $j \in \mathcal{P}$ and $\tilde{\pi}_I(\tilde{p}) = \sum_{i\in I} \pi(i)$. This is evident that these representations are quite sincere representations of corresponding posets $\tilde{\mathcal{P}}_I$.

In this way by induction one can obtain all quite sincere posets and all their quite sincere representations. The above procedure terminates because the dimensions of representations are bounded. □

Remark 3. *Let us remark that the unitarization of a quite sincere representation is equivalent to unitarization of all indecomposable representation of poset of finite type.*

4.2. Proof of Theorem 2

As it was mentioned before in order to prove that all primitive posets of finite type can be unitarized it is enough to see that all quite sincere posets and their quite sincere representations are unitarized. All such representations are listed in Appendix A (second column in the table). The fact that all quite sincere representation of primitive poset (from Appendix A) can be unitarized with some weight is due to [4].

To prove that quite sincere representation 1)-6) of the poset $\mathcal{P}_2$ from Appendix A can be unitarized we show their stability with the the weight which is equal to the dimension of representation. For this one can describe all possible subdimension for each representation and then check the stability condition. Description of all possible subdimension is straightforward (this is done in Appendix D) and this is a routine to check the stability condition for all these vectors.

Representations of $\mathcal{P}_1, \mathcal{P}_3 - \mathcal{P}_5$ and representations 7), 8) of $\mathcal{P}_2$ can be obtained by adding one (or several) subspace(s) to either a representation of primitive posets or one of the representation 1) $-$ 6) of $\mathcal{P}_2$. Their unitarization follows from the following observation.

Lemma 13. *Let $\pi = (V; V_i)$ be a χ-stable system of subspaces. Then for each subspace $U \subset V$ there exists a weight $\tilde{\chi}$ such that the system of subspaces $\tilde{\pi}$ generated by π and U,*

$$\tilde{\pi} = (V; V_1, \dots, V_n, U)$$

is $\tilde{\chi}$-stable.

Proof. Let $K \subset V$ be a subspace of V that the difference $\lambda_\chi(\pi) - \lambda_\chi(\pi \cap K)$ is minimal. Let $R = \lambda_\chi(\pi) - \lambda_\chi(\pi \cap K)$. Since π is stable, $R > 0$ and there exist $\epsilon > 0$ that $R - \epsilon > 0$. Define $\tilde{\chi}$ in the folowing way

$$\tilde{\chi}_i = \chi_i, \quad i = 1, \dots, n, \quad \tilde{\chi}_{n+1} = R - \epsilon.$$

Our claim is that $\tilde{\pi}$ is $\tilde{\chi}$-stable. Indeed, let $M \subset V$ be an arbitrary subspace of V then we have

$$\begin{aligned} \frac{1}{\dim M} \sum_{i=1}^{n+1} \tilde{\chi}_i \dim(\tilde{\pi}(i) \cap M) &= \frac{1}{\dim M} \sum_{i=1}^{n} \chi_i \dim(V_i \cap M) + \frac{\tilde{\chi}_{n+1} \dim(U \cap M)}{\dim M} \\ &\leq \frac{1}{\dim V} \sum_{i=1}^{n} \chi_i \dim V_i - R + \frac{(R-\epsilon) \dim(U \cap M)}{\dim M} \\ &< \frac{1}{\dim V} \sum_{i=1}^{n} \chi_i \dim V_i < \frac{1}{\dim V} \sum_{i=1}^{n+1} \tilde{\chi}_i \dim \tilde{\pi}(i). \end{aligned}$$

Hence $\tilde{\pi}$ is $\tilde{\chi}$-stable. □

Corollary 14. *If indecomposable system of subspaces $\pi = (V; V_1, \dots, V_n)$ is unitarizable then for arbitrary collections of subspaces $U_j \subset V$, $j = 1, \dots, m$ the system $\pi = (V; V_1, \dots, V_n, U_1, \dots, U_m)$ is also unitarizable.*

Remark 4. *Let us note that when a preliminary version of this article was ready the authors were informed that the same result was independently obtained in* [19].

It remains to prove that the representation of $\mathcal{P}_5^*$ from the Appendix A is unitarized. Since it is dual to representation of $\mathcal{P}_5$, this follows from the following lemma.

Lemma 15. *Let $\pi = (V; V_i)$ be unitarizable with the weight χ and let $\pi' = (V; V_i')$ be indecomposable dual system (each V_i' is a complement to V_i) assume also that the dimension vector of π is a real root, i.e. $Q_{\mathcal{P}}(d_\pi) = 1$. Then π' is also unitarizable with the weight χ.*

Proof. As π is unitarizable with χ, we have $\sum \chi_i P_{V_i} = \lambda_\pi I$ for appropriated choice of scalar product. It is not hard to check the dimension vector d_π' is also a real root. Undecomposability of π' implies that it is linearly equivalent to the system

$(V;\mathrm{Im}(I-P_{V_i}))$ (because there exist only one indecomposable representation with dimension vector d'_π). The latter system is obviously unitarized with the weight χ due to $\sum\chi_i(I-P_{V_i})=(\sum\chi_i-\lambda_\pi)I$. $\square$

Now Theorem 2 is completely proved.

Appendix A. Quite sincere representation and weights appropriated for unitarization.

In this appendix you can find a complete description of all quite sincere representations of finite posets and the weights appropriated for the unitarization. To simplify the notation we will denote by $V_{i_1,\ldots,i_j}$ vector space spanned by vectors $e_{i_1},\ldots,e_{i_j}$.

Poset	**Representation** $\pi=(V;\pi(a_i);\pi(b_i);\pi(c_i))$	**Weight** χ
$(1,1,1)$	$(\mathbb{C}^2;V_1;V_2;V_{12})$	$(1,1,1)$
$(1,2,2)$	1) $(\mathbb{C}^3;V_{123};V_1,V_{1,2};V_3,V_{2,3})$	$(1,1,1,1,1)$
	2) $(\mathbb{C}^3;V_{12,13};V_1,V_{1,2};V_3,V_{2,3})$	$(2,1,1,1,1)$
$(1,2,3)$	1) $(\mathbb{C}^4;V_{123,24};V_4,V_{1,4};V_3,V_{2,3},V_{1,2,3})$	$(2,1,1,1,1,1)$
	2) $(\mathbb{C}^4;V_{124,13};V_4,V_{1,2,4};V_3,V_{2,3},V_{1,2,3})$	$(2,1,2,1,1,1)$
	3) $(\mathbb{C}^4;V_{123,24};V_{1,4},V_{1,2,4};V_3,V_{2,3},V_{1,2,3})$	$(2,2,1,1,1,1)$
$(1,2,4)$	1) $(\mathbb{C}^5;V_{134,235};V_5,V_{1,2,5};$	
	$V_4,V_{3,4},V_{2,3,4},V_{1,2,3,4})$	$(2,1,2,1,1,1,1)$
	2) $(\mathbb{C}^5;V_{123,245};V_{1,5},V_{1,2,5};$	
	$V_4,V_{3,4},V_{2,3,4},V_{1,2,3,4})$	$(2,2,1,1,1,1,1)$
	3) $(\mathbb{C}^5;V_{124,235};V_{1,5},V_{1,2,3,5};$	
	$V_4,V_{3,4},V_{2,3,4},V_{1,2,3,4})$	$(2,2,2,1,1,1,1)$
	4) $(\mathbb{C}^5;V_{124,23,15};V_5,V_{1,2,5};$	
	$V_4,V_{3,4},V_{2,3,4},V_{1,2,3,4})$	$(3,1,2,1,1,1,1)$
	5) $(\mathbb{C}^5;V_{13,234,45};V_{1,5},V_{1,2,5};$	
	$V_4,V_{3,4},V_{2,3,4},V_{1,2,3,4})$	$(3,2,1,1,1,1,1)$
	6) $(\mathbb{C}^5;V_{12,234,45};V_{1,5},V_{1,2,3,5};$	
	$V_4,V_{3,4},V_{2,3,4},V_{1,2,3,4})$	$(3,2,2,1,1,1,1)$
	7) $(\mathbb{C}^6;V_{123,245,16};V_{5,6},V_{1,2,5,6};$	

	$V_4, V_{3,4}, V_{2,3,4}, V_{1,2,3,4})$	$(3,2,2,1,1,1,1)$
	8) $(\mathbb{C}^6; V_{125,234,46}; V_{1,6}, V_{1,2,3,6};$	
	$V_5, V_{4,5}, V_{3,4,5}, V_{1,2,3,4,5})$	$(3,2,2,1,1,1,2)$
	9) $(\mathbb{C}^6; V_{125,134,46}; V_{1,6}, V_{1,2,3,6};$	
	$V_5, V_{4,5}, V_{2,3,4,5}, V_{1,2,3,4,5})$	$(3,2,2,1,1,2,1)$
	10) $(\mathbb{C}^6; V_{125,234,46}; V_{1,6}, V_{1,2,3,6};$	
	$V_5, V_{3,4,5}, V_{2,3,4,5}, V_{1,2,3,4,5})$	$(3,2,2,1,2,1,1)$
	11) $(\mathbb{C}^6; V_{135,124,46}; V_{1,6}, V_{1,2,3,6};$	
	$V_{4,5}, V_{3,4,5}, V_{2,3,4,5}, V_{1,2,3,4,5})$	$(3,2,2,2,1,1,1)$
$\mathcal{P}_1$	$(\mathbb{C}^3; V_{123}, V_{23,1}; V_1, V_{1,2}; V_3, V_{2,3})$	$(1,0.1,1,1,1,1)$
$\mathcal{P}_2$	1) $(\mathbb{C}^4; V_{14}, V_{1,2,4}; V_4, V_{4,123}; V_3, V_{2,3}, V_{1,2,3})^*$	$(1,1,1,1,1,1,1)$
	2) $(\mathbb{C}^4; V_{14}, V_{1,2,4}; V_4, V_{4,12,23}; V_3, V_{2,3}, V_{1,2,3})^*$	$(1,1,1,2,1,1,1)$
	3) $(\mathbb{C}^5; V_{1,25}, V_{1,2,3,5}; V_5, V_{123,24,5};$	
	$V_{3,4}, V_{2,3,4}, V_{1,2,3,4})^*$	$(2,1,1,2,2,1,1)$
	4) $(\mathbb{C}^5; V_{1,25}, V_{1,2,3,5}; V_5, V_{13,234,5};$	
	$V_4, V_{2,3,4}, V_{1,2,3,4})^*$	$(2,1,1,2,1,2,1)$
	5) $(\mathbb{C}^5; V_{1,25}, V_{1,2,3,5}; V_5, V_{123,24,5};$	
	$V_4, V_{3,4}, V_{1,2,3,4})^*$	$(2,1,1,2,1,1,2)$
	6) $(\mathbb{C}^5; V_{15,4}, V_{1,2,4,5}; V_5, V_{123,24,5};$	
	$V_3, V_{2,3}, V_{1,2,3})^*$	$(2,1,1,2,1,1,1)$
	7) $(\mathbb{C}^4; V_{123,24}, V_{13,2,4}; V_4, V_{1,4};$	
	$V_3, V_{2,3}, V_{1,2,3})$	$(2,0.1,1,1,1,1,1)$
	8) $(\mathbb{C}^4; V_{124,13}, V_{12,13,4}; V_4, V_{1,2,4};$	
	$V_3, V_{2,3}, V_{1,2,3})$	$(2,0.1,1,2,1,1,1)$
$\mathcal{P}_3$	$(\mathbb{C}^4; V_{123,24}; V_4, V_{1,4}, V_{1,3,4}; V_3, V_{2,3}, V_{1,2,3})$	$(2,1,1,0.1,1,1,1)$
$\mathcal{P}_4$	$(\mathbb{C}^4; V_{14}, V_{1,2,4}; V_4, V_{4,123}, V_{1,23,4};$	
	$V_3, V_{2,3}, V_{1,2,3})$	$(1,1,1,1,0.1,1,1,1)$
$\mathcal{P}_5$	$(\mathbb{C}^5; V_{15,4}, V_{1,2,4,5}; V_5, V_{123,24,5};$	
	$V_3, V_{2,3}, V_{1,2,3}, V_{1,2,3,5})$	$(2,1,1,2,1,1,1,0.1)$

$\mathcal{P}_5^*$	$(\mathbb{C}^5; V_5, V_{1,2,5}; V_{134,235}, V_{13,23,4,5};$ $V_4, V_{3,4}, V_{2,3,4}, V_{1,2,3,4})$	$(1, 2, 2, 0.1, 1, 1, 1, 1)$

Remark 5. *An interesting phenomena that each quite sincere representation can be unitarized with the weight that equal to the dimension vector (for primitive case) or with the weight that an arbitrary closed to the dimension vector (for non-primitive poset) and this weight in some sense is the most stable weight.*

Appendix B. The sets $\triangle_\pi^{\mathcal{P}}$ and several examples.

It is routine to describe the set $\triangle_\pi^{\mathcal{P}}$ for an arbitrary linear representation π of $\mathcal{P}$. Instead we will give an algorithm of its description.

Proposition 16. $\triangle_\pi^{\mathcal{P}}$ *is convex.*

Proof. One can see that if $\chi \in \triangle_\pi^{\mathcal{P}}$ then $(1-t)\chi \in \triangle_\pi^{\mathcal{P}}$ for each $t \in [0, 1]$, because if π is stable with χ then π is also stable with $(1-t)\chi$. Hence $\triangle_\pi^{\mathcal{P}}$ is convex and connected. □

Stability conditions for the system of subspaces $\pi = (V; V_1, \dots, V_n)$ define some matrix $A_\pi \in M_{m+n,n}(\mathbb{R})$. Namely this matrix is defined in the following way. For any vector of the form $d = (d_0; d_1, \dots, d_n) \in \mathbb{Z}^{n+1}$ with $d_0 > 0$ by $n(d)$ we denote normalized vector $n(d) = \left(\frac{d_1}{d_0}, \dots, \frac{d_n}{d_0}\right)$. Let $\dim(\pi) = (\dim \pi_0; \dim \pi_i)$ be dimension vector of π and $Sub(\pi) = \{d_{\pi,i} \mid i \in \{1, \dots, m\}\}$ be the set of maximal subdimension vectors for π. Then the matrix A_π has the following form

$$A_\pi = \begin{pmatrix} n(d_{\pi,1}) - n(\dim(\pi)) \\ \vdots \\ n(d_{\pi,m}) - n(\dim(\pi)) \end{pmatrix} \oplus -I_n, \quad \text{where} \quad A_1 \oplus A_2 = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$

The set $\triangle_\pi^{\mathcal{P}}$ thus can be defined as the set of those $\chi = (\alpha_1, \dots, \alpha_n)$ that $A_\pi \chi < 0$. Using the standard methods concerning to systems of linear inequalities (see for example [14]) these sets can be described in terms of extreme points and extremal rays.

Definition 17. *Let $P \subset \mathbb{R}^n$ be some subset. A point $x \in P$ is called an extreme point of P if for all $x_1, x_2 \in P$ and every $0 < \mu < 1$ such that $x = \mu x_1 + (1-\mu)x_2$, we have $x = x_1 = x_2$.*

The set of extreme points can be determined by the set of those points $x_0 \in \mathbb{R}^n$ that $A'_\pi x_0 = 0$ for some $(n \times n)$ full rank submatrix A'_π (see [14]), hence this set contains the only element $x_0 = 0$.

Definition 18. *1. A set $C \subset \mathbb{R}^n$ is a cone if for every pair of points $x_1, x_2 \in C$ we have $\lambda_1 x_1 + \lambda_2 x_2 \in C$ for all $\lambda_1, \lambda_2 \geq 0$.*

2. A half-line $y = \{\lambda x \mid \lambda \geq 0,\ x \in \mathbb{R}^n\}$ is an extremal ray of C if $y \in C$ and $-y \notin C$ and if for all $y_1, y_2 \in C$ and $0 < \mu < 1$ with $y = (1-\mu)y_1 + \mu y_2$ we have $y = y_1 = y_2$.

Obviously $\triangle_\pi^{\mathcal{P}}$ is a cone. The following proposition describes how to determine all externmal rays of $\triangle_\pi^{\mathcal{P}}$.

Proposition 19. *(see [14]) $x \in \overline{\triangle_\pi^{\mathcal{P}}}$ is an extremal ray if and only if there exist $rank(A_\pi) - 1$ linear independent row vectors $a_1, \ldots, a_{rank(A_\pi)-1}$ of A_π such that*

$$\begin{pmatrix} a_1 \\ \vdots \\ a_{rank(A_\pi)-1} \end{pmatrix} \cdot x = \vec{0}.$$

Proposition 20. *(see [14]) The sets $\triangle_\pi^{\mathcal{P}}$ are described in the following way*

$$\triangle_\pi^{\mathcal{P}} = cone(A),$$

where A is the set of extremal rays, $cone(X)$ is an open cone of finite set $X = \{x_1, \ldots, x_n\}$ defined by

$$cone(x_1, \ldots, x_n) = \left\{ \sum_{i=1}^{n} \mu_i x_i \mid \mu_i > 0 \right\}.$$

We consider two examples:

1). Let $\mathcal{P} = (1,1,1)$, and $\pi = (\mathbb{C}^2; V_1, V_2, V_{12})$. Matrix A_π is given by

$$A_\pi = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \oplus -I_3.$$

The set $\triangle_\pi^{(1,1,1)}$ has the only extreme point $(0,0,0)$ and three extremal rays $(1,1,0),\ (1,0,1),\ (0,1,1)$. Hence the whole set is given by

$$\triangle_{(\mathbb{C}^2; V_1, V_2, V_{12})}^{(1,1,1)} = \{(\alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3) \mid \alpha_i \in \mathbb{R}_+\};$$

2). Let us take the poset $\mathcal{P} = (N, 2) = \{a_1, a_2, b_1, b_2, c_1, c_2\}$, $a_1 \prec a_2$, $b_1 \prec b_2$, $b_1 \prec a_2$, $c_1 \prec c_2$. It has the only quite sincere representation π:

$$\begin{tikzcd}
\langle e_1, e_{123} \rangle & \langle e_1, e_2 \rangle & \langle e_2, e_3 \rangle \\
\langle e_{123} \rangle \arrow[u] & \langle e_1 \rangle \arrow[u] \arrow[ul] & \langle e_3 \rangle \arrow[u]
\end{tikzcd}$$

The normalized dimension vector of π is equal to $\left(\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3}\right)$. The set of maximal subdimension vectors is

$$\begin{aligned}Sub(\pi) = \{&(1;0,0;0,0;1,1),\ (1;0,0;0,1;0,1),\ (1;0,1;0,0;0,1),\ (1;0,1;1,1;0,0),\\ &(1;1,1;0,0;0,0),\ (2;0,1;0,1;1,2),\ (2;0,1;1,1;1,1),\ (2;0,1;1,2;0,1),\\ &(2;1,1;0,1;1,1),\ (2;1,2;1,1;0,1)\}.\end{aligned}$$

The corresponding matrix A_π has the following form

$$A_\pi = \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{6} \\ \frac{1}{6} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & -\frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \end{pmatrix} \oplus -I_6.$$

The set $\triangle_\pi^{(N,2)}$ has nine extremal rays

$$\begin{gathered}(0,0,0,1,1,1),\ (0,0,1,0,0,1),\ (0,1,0,0,1,0),\ (0,1,0,1,0,1),\\ (0,2,1,1,3,1),\ (1,0,0,0,0,1),\ (2,0,0,1,1,2),\ (1,0,0,1,1,1),\\ (1,0,1,0,1,0).\end{gathered}$$

Hence the whole set is given by

$$\begin{aligned}\triangle_\pi^{(N,2)} = \{&(\alpha_6+2\alpha_7+\alpha_8+\alpha_9, \alpha_3+\alpha_4+2\alpha_5, \alpha_2+\alpha_5+\alpha_9,\\ &\alpha_1+\alpha_4+\alpha_5+\alpha_7+\alpha_8, \alpha_1+\alpha_3+3\alpha_5+\alpha_7+\alpha_8+\alpha_9,\\ &\alpha_2+\alpha_4+\alpha_5+\alpha_6+2\alpha_7+\alpha_8) \mid \alpha_i \in \mathbb{R}_+\}.\end{aligned}$$

Remark 6. *One can see that* $(1,1,1,1,1,1) \notin \triangle_\pi^{(N,2)}$ *(this vector is semistable but is not stable) which means that given linear representation of the poset* $(N,2)$ *can not be obtained as the spectral filtration of three partial reflections* $A_i = A_i^* = A_i^3$ *sum of which is zero* $A_1+A_2+A_3=0$.

Appendix C. Missing details in the proof of Theorem 1.

Lemma 21. *The following lists contains all possible subdimension for extended representations used in the proof of Theorem 1.*

Poset	**Subdimensions** $d = (\dim U; \dim(V_i \cap U))$	**Subspace** U
$(1,1,1,1)$	$(1;1;0;0;0)$	$\langle e_1 \rangle$
	$(1;0;1;0;0)$	$\langle e_2 \rangle$
	$(1;0;0;1;0)$	$\langle e_1 + e_2 \rangle$
	$(1;0;0;0;1)$	$\langle e_1 + \lambda e_2 \rangle$
$(2,2,2)$	$(1;0,0;0,0;1,1)$	$\langle e_3 \rangle$
	$(1;0,0;0,1;0,1)$	$\langle e_2 \rangle$
	$(1;0,1;0,0;0,1)$	$\langle e_2 + (\lambda - 1)e_3 \rangle$
	$(1;0,0;1,1;0,0)$	$\langle e_1 \rangle$
	$(1;0,1;0,1;0,0)$	$\langle (\lambda - 1)e_1 + \lambda e_2 \rangle$
	$(1;1,1;0,0;0,0)$	$\langle e_1 + e_2 + e_3 \rangle$
	$(2;0,1;0,1;1,2)$	$\langle e_3, e_2 \rangle$
	$(2;0,1;1,1;1,1)$	$\langle e_3, e_1 \rangle$
	$(2;0,1;1,2;0,1)$	$\langle e_2, e_1 \rangle$
	$(2;1,1;0,1;1,1)$	$\langle e_3, e_1 + e_2 + e_3 \rangle$
	$(2;1,1;1,1;0,1)$	$\langle e_1, e_1 + e_2 + e_3 \rangle$
	$(2;1,2;0,1;0,1)$	$\langle e_1 + \lambda e_3, e_1 + e_2 + e_3 \rangle$
$(1;3;3)$	$(1;0;0,0,0;1,1,1)$	$\langle e_3 \rangle$
	$(1;0;0,0,1;0,1,1)$	$\langle e_2 + \lambda e_3 \rangle$
	$(1;0;0,1,1;0,0,1)$	$\langle e_1 \rangle$
	$(1;0;1,1,1;0,0,0)$	$\langle e_4 \rangle$
	$(1;1;0,0,0;0,0,1)$	$\langle e_1 + e_2 + e_3 \rangle$
	$(1;1;0,0,1;0,0,0)$	$\langle \lambda e_1 + e_2 + \lambda e_3 + (1 - \lambda)e_4 \rangle$
	$(2;0;0,0,1;1,2,2)$	$\langle e_3, e_2 \rangle$
	$(2;0;0,1,1;1,1,2)$	$\langle e_3, e_1 \rangle$
	$(2;0;0,1,2;0,1,2)$	$\langle e_2 + \lambda e_3, e_1 \rangle$
	$(2;0;1,1,1;1,1,1)$	$\langle e_4, e_3 \rangle$
	$(2;0;1,1,2;0,1,1)$	$\langle e_4, e_2 + \lambda e_3 \rangle$

	$(2;0;1,2,2;0,0,1)$	$\langle e_4, e_1\rangle$
	$(2;1;0,0,1;1,1,2)$	$\langle e_3, e_1+e_2+e_3\rangle$
	$(2;1;0,1,1;0,1,2)$	$\langle e_1, e_1+e_2+e_3\rangle$
	$(2;1;1,1,1;0,1,1)$	$\langle e_4, e_2\rangle$
	$(2;1;1,1,2;0,0,1)$	$\langle \lambda e_1+e_2+\lambda e_3+(1-\lambda)e_4, e_4\rangle$
	$(2;1;0,1,2;0,1,1)$	$\langle \lambda e_1+e_2+\lambda e_3+(1-\lambda)e_4, e_2+\lambda e_3\rangle$
	$(2;1;0,1,1;1,1,1)$	$\langle e_3, e_1+e_2-e_4\rangle$
	$(2;2;0,0,1;0,0,1)$	$\langle e_2+e_4, e_1+e_2+e_3\rangle$
	$(3;1;0,1,2;1,2,3)$	$\langle e_3, e_2, e_1\rangle$
	$(3;1;1,1,2;1,2,2)$	$\langle e_4, e_3, e_2\rangle$
	$(3;1;1,2,2;1,1,2)$	$\langle e_4, e_3, e_1\rangle$
	$(3;1;1,2,3;0,1,2)$	$\langle e_4, e_2+\lambda e_3, e_1\rangle$
	$(3;2;0,1,2;1,1,2)$	$\langle e_3, e_2+e_4, e_1+e_2+e_3\rangle$
	$(3;2;1,1,2;0,1,2)$	$\langle e_4, e_2, e_1+e_2+e_3\rangle$
$(1,2,5)$	$(1;0;0,0;1,1,1,1,1)$	$\langle e_4\rangle$
	$(1;0;0,1;0,0,1,1,1)$	$\langle e_2\rangle$
	$(1;0;1,1;0,0,0,0,1)$	$\langle e_5+2e_6\rangle$
	$(1;1;0,0;0,0,0,1,1)$	$\langle e_1+e_2+e_3\rangle$
	$(1;1;0,1;0,0,0,0,0)$	$\langle e_1+e_6\rangle$
	$(2;0;0,0;1,2,2,2,2)$	$\langle e_4, e_3\rangle$
	$(2;0;0,1;1,1,2,2,2)$	$\langle e_4, e_2\rangle$
	$(2;0;0,2;0,0,1,2,2)$	$\langle e_2, e_1\rangle$
	$(2;0;1,1;1,1,1,1,2)$	$\langle e_5+\lambda e_6, e_4\rangle$
	$(2;0;1,2;0,0,1,1,2)$	$\langle e_5+\lambda e_6, e_2\rangle$
	$(2;0;2,2;0,0,0,0,1)$	$\langle e_6, e_5\rangle$
	$(2;1;0,0;1,1,1,2,2)$	$\langle e_4, e_1+e_2+e_3\rangle$
	$(2;1;0,1;0,1,1,2,2)$	$\langle e_3, e_1+e_2+e_3\rangle$
	$(2;1;0,1;1,1,1,1,1)$	$\langle e_4, e_2+e_4+e_5\rangle$
	$(2;1;0,2;0,0,1,1,1)$	$\langle e_2, e_1+e_6\rangle$

$(2;1;1,1;0,0,0,1,2)$	$\langle e_5+\lambda e_6, e_1+e_2+e_3\rangle$
$(2;1;1,1;0,0,1,1,1)$	$\langle e_5, e_2+e_4+e_5\rangle$
$(2;1;1,2;0,0,0,1,1)$	$\langle e_6, e_1\rangle$
$(2;2;0,1;0,0,0,1,1)$	$\langle e_1+e_6, e_1+e_2+e_3\rangle$
$(3;0;0,1;1,2,3,3,3)$	$\langle e_4, e_3, e_2\rangle$
$(3;0;0,2;1,1,2,3,3)$	$\langle e_4, e_2, e_1\rangle$
$(3;0;1,1;1,2,2,2,3)$	$\langle e_5+\lambda e_6, e_4, e_3\rangle$
$(3;0;1,2;1,1,2,2,3)$	$\langle e_5+\lambda e_6, e_4, e_2\rangle$
$(3;0;1,3;0,0,1,2,3)$	$\langle e_5+\lambda e_6, e_2, e_1\rangle$
$(3;0;2,2;1,1,1,1,2)$	$\langle e_6, e_5, e_4\rangle$
$(3;0;2,3;0,0,1,1,2)$	$\langle e_6, e_5, e_2\rangle$
$(3;1;0,1;1,2,2,3,3)$	$\langle e_4, e_3, e_1+e_2+e_3\rangle$
$(3;1;0,2;0,1,2,3,3)$	$\langle e_3, e_2, e_1\rangle$
$(3;1;1,1;1,1,1,2,3)$	$\langle e_5+\lambda e_6, e_4, e_1+e_2+e_3\rangle$
$(3;1;1,2;0,1,1,2,3)$	$\langle e_5+\lambda e_6, e_3, e_1+e_2+e_3\rangle$
$(3;1;1,2;1,1,2,2,2)$	$\langle e_5, e_4, e_2\rangle$
$(3;1;1,3;0,0,1,2,2)$	$\langle e_6, e_2, e_1\rangle$
$(3;1;2,2;0,0,1,1,2)$	$\langle e_6, e_5, e_2+e_4+e_5\rangle$
$(3;1;2,3;0,0,0,1,2)$	$\langle e_6, e_5, e_1\rangle$
$(3;2;0,1;1,1,1,2,2)$	$\langle e_4, e_2+e_4+e_5, e_1+e_2+e_3\rangle$
$(3;2;0,2;0,1,1,2,2)$	$\langle e_3, e_1+e_6, e_1+e_2+e_3\rangle$
$(3;2;0,2;1,1,1,1,2)$	$\langle e_4, e_2+e_4+e_5, e_1+e_6\rangle$
$(3;2;1,2;0,0,1,2,2)$	$\langle e_6, e_1, e_1+e_2+e_3\rangle$
$(3;3;0,1;0,0,0,1,2)$	$\langle e_2+e_4+e_5, e_1+e_6, e_1+e_2+e_3\rangle$
$(4;1;0,2;1,2,3,4,4)$	$\langle e_4, e_3, e_2, e_1\rangle$
$(4;1;1,2;1,2,3,3,4)$	$\langle e_5+\lambda e_6, e_4, e_3, e_2\rangle$
$(4;1;1,3;1,1,2,3,4)$	$\langle e_5+\lambda e_6, e_4, e_2, e_1\rangle$
$(4;1;2,2;1,2,2,2,3)$	$\langle e_6, e_5, e_4, e_3\rangle$
$(4;1;2,3;1,1,2,2,3)$	$\langle e_6, e_5, e_4, e_2\rangle$

	$(4;1;2,4;0,0,1,2,3)$	$\langle e_6, e_5, e_2, e_1\rangle$
	$(4;2;0,2;1,2,2,3,3)$	$\langle e_4, e_3, e_2+e_4+e_5, e_1+e_2+e_3\rangle$
	$(4;2;1,2;1,1,2,3,3)$	$\langle e_6, e_4, e_1, e_1+e_2+e_3\rangle$
	$(4;2;1,3;0,1,2,3,3)$	$\langle e_6, e_3, e_2, e_1\rangle$
	$(4;2;1,3;1,1,2,2,3)$	$\langle e_5, e_4, e_2, e_1+e_6\rangle$
	$(4;2;2,3;0,0,1,2,3)$	$\langle e_6, e_5, e_2+e_4+e_5, e_1\rangle$
	$(4;3;0,2;1,1,1,2,3)$	$\langle e_4, e_2+e_4+e_5, e_1+e_6, e_1+e_2+e_3\rangle$
	$(4;3;1,2;0,0,1,2,3)$	$\langle e_6, e_2+e_4+e_5, e_1, e_1+e_2+e_3\rangle$
	$(5;2;1,3;1,2,3,4,5)$	$\langle e_5+\lambda e_6, e_4, e_3, e_2, e_1\rangle$
	$(5;2;2,3;1,2,3,3,4)$	$\langle e_6, e_5, e_4, e_3, e_2\rangle$
	$(5;2;2,4;1,1,2,3,4)$	$\langle e_6, e_5, e_4, e_2, e_1\rangle$
	$(5;3;1,3;1,2,2,3,4)$	$\langle e_4, e_3, e_2+e_4+e_5, e_1+e_6, e_1+e_2+e_3\rangle$
	$(5;3;2,3;0,1,2,3,4)$	$\langle e_6, e_5, e_2+e_4+e_5, e_1, e_1+e_2+e_3\rangle$
$(N,4)$	$(1;0,0;0,0;1,1,1,1)$	$\langle e_4\rangle$
	$(1;0,0;0,1;0,0,1,1)$	$\langle e_2\rangle$
	$(1;0,1;0,0;0,1,1,1)$	$\langle e_3+\lambda e_4\rangle$
	$(1;0,1;1,1;0,0,0,0)$	$\langle e_5\rangle$
	$(1;1,1;0,0;0,0,0,1)$	$\langle e_1+e_3+e_4\rangle$
	$(2;0,1;0,0;1,2,2,2)$	$\langle e_4, e_3\rangle$
	$(2;0,1;0,1;1,1,2,2)$	$\langle e_4, e_2\rangle$
	$(2;0,1;0,2;0,0,1,2)$	$\langle e_2, e_1\rangle$
	$(2;0,1;1,1;1,1,1,1)$	$\langle e_5, e_4\rangle$
	$(2;0,1;1,2;0,0,1,1)$	$\langle e_5, e_2\rangle$
	$(2;0,2;1,1;0,1,1,1)$	$\langle e_5, e_3+\lambda e_4\rangle$
	$(2;1,1;0,0;1,1,1,2)$	$\langle e_4, e_1+e_3+e_4\rangle$
	$(2;1,1;0,1;0,1,1,2)$	$\langle e_1, e_1+e_3+e_4\rangle$
	$(2;1,2;0,0;0,1,1,2)$	$\langle e_3+\lambda e_4, e_1+e_3+e_4\rangle$
	$(2;1,2;1,1;0,0,1,1)$	$\langle e_5, e_2+e_3+e_5\rangle$
	$(2;2,2;0,0;0,0,0,1)$	$\langle e_2+e_3+e_5, e_1+e_3+e_4\rangle$

$(3;0,2;0,1;1,2,3,3)$	$\langle e_4, e_3, e_2\rangle$
$(3;0,2;0,2;1,1,2,3)$	$\langle e_4, e_2, e_1\rangle$
$(3;0,2;1,1;1,2,2,2)$	$\langle e_5, e_4, e_3\rangle$
$(3;0,2;1,2;1,1,2,2)$	$\langle e_5, e_4, e_2\rangle$
$(3;0,2;1,3;0,0,1,2)$	$\langle e_5, e_2, e_1\rangle$
$(3;1,2;0,1;1,2,2,3)$	$\langle e_4, e_3, e_1\rangle$
$(3;1,2;0,2;0,1,2,3)$	$\langle e_2, e_1, e_1+e_3+e_4\rangle$
$(3;1,2;1,1;1,1,2,2)$	$\langle e_5, e_4, e_2+e_3+e_5\rangle$
$(3;1,2;1,2;0,1,2,2)$	$\langle e_5, e_3, e_2\rangle$
$(3;1,3;1,1;0,1,2,2)$	$\langle e_5, e_3+\lambda e_4, e_2+e_3+e_5\rangle$
$(3;2,2;0,1;1,1,1,2)$	$\langle e_4, e_2+e_3+e_5, e_1+e_3+e_4\rangle$
$(3;2,3;0,1;0,1,1,2)$	$\langle e_3+\lambda e_4, e_2+e_3+e_5, e_1+e_3+e_4\rangle$
$(3;2,3;1,1;0,0,1,2)$	$\langle e_5, e_2+e_3+e_5, e_1+e_3+e_4\rangle$
$(4;1,3;0,2;1,2,3,4)$	$\langle e_4, e_3, e_2, e_1\rangle$
$(4;1,3;1,2;1,2,3,3)$	$\langle e_5, e_4, e_3, e_2\rangle$
$(4;1,3;1,3;1,1,2,3)$	$\langle e_5, e_4, e_2, e_1\rangle$
$(4;2,3;0,2;1,2,2,3)$	$\langle e_4, e_3, e_2+e_3+e_5, e_1\rangle$
$(4;2,3;1,2;1,1,2,3)$	$\langle e_5, e_4, e_2+e_3+e_5, e_1+e_3+e_4\rangle$
$(4;2,4;1,2;0,1,2,3)$	$\langle e_5, e_3+\lambda e_4, e_2+e_3+e_5, e_1+e_3+e_4\rangle$

It is routine to check that for the corresponding representations stability conditions (where weight is taken to be dimension) holds for all maximal subdimension listed above, hence representation are unitarizable.

Appendix D. Missing details in the proof of Theorem 2.

Lemma 22. *The following lists contain all possible subdimension for representations 1)-6) of the poset $\mathcal{P}_2$.*

No.	**Subdimensions** $d = (\dim U; \dim(V_i \cap U))$	**Subspace** U
1)	$(1;0,0;0,0;1,1,1)$	$\langle e_3\rangle$
	$(1;0,0;0,1;0,0,1)$	$\langle e_1+e_2+e_3\rangle$

	$(1;0,1;0,0;0,1,1)$	$\langle e_2\rangle$
	$(1;0,1;1,1;0,0,0)$	$\langle e_4\rangle$
	$(1;1,1;0,0;0,0,0)$	$\langle e_1+e_4\rangle$
	$(2;0,1;0,0;1,2,2)$	$\langle e_3,e_2\rangle$
	$(2;0,1;0,1;1,1,2)$	$\langle e_3,e_1+e_2+e_3\rangle$
	$(2;0,1;1,1;1,1,1)$	$\langle e_4,e_3\rangle$
	$(2;0,1;1,2;0,0,1)$	$\langle e_4,e_1+e_2+e_3\rangle$
	$(2;0,2;0,0;0,1,2)$	$\langle e_2,e_1\rangle$
	$(2;0,2;1,1;0,1,1)$	$\langle e_4,e_2\rangle$
	$(2;1,1;0,0;1,1,1)$	$\langle e_3,e_1+e_4\rangle$
	$(2;1,2;0,0;0,1,1)$	$\langle e_2,e_1+e_4\rangle$
	$(2;1,2;1,1;0,0,1)$	$\langle e_4,e_1\rangle$
	$(3;0,2;0,1;1,2,3)$	$\langle e_3,e_2,e_1\rangle$
	$(3;0,2;1,1;1,2,2)$	$\langle e_4,e_3,e_2\rangle$
	$(3;0,2;1,2;1,1,2)$	$\langle e_4,e_3,e_1+e_2+e_3\rangle$
	$(3;1,2;0,1;1,2,2)$	$\langle e_3,e_2,e_1+e_4\rangle$
	$(3;1,2;1,1;1,1,2)$	$\langle e_4,e_3,e_1\rangle$
	$(3;1,2;1,2;0,1,2)$	$\langle e_4,e_1,e_1+e_2+e_3\rangle$
	$(3;1,3;1,1;0,1,2)$	$\langle e_4,e_2,e_1\rangle$
2)	$(1;0,0;0,0;1,1,1)$	$\langle e_3\rangle$
	$(1;0,0;0,1;0,1,1)$	$\langle e_2+e_3\rangle$
	$(1;0,1;0,0;0,1,1)$	$\langle e_2\rangle$
	$(1;0,1;0,1;0,0,1)$	$\langle e_1+e_2\rangle$
	$(1;0,1;1,1;0,0,0)$	$\langle e_4\rangle$
	$(1;1,1;0,0;0,0,0)$	$\langle e_1+e_4\rangle$
	$(2;0,1;0,1;1,2,2)$	$\langle e_3,e_2\rangle$
	$(2;0,1;0,2;0,1,2)$	$\langle e_2+e_3,e_1+e_2\rangle$
	$(2;0,1;1,1;1,1,1)$	$\langle e_4,e_3\rangle$
	$(2;0,1;1,2;0,1,1)$	$\langle e_4,e_2+e_3\rangle$

	$(2;0,2;0,1;0,1,2)$	$\langle e_2, e_1\rangle$
	$(2;0,2;1,1;0,1,1)$	$\langle e_4, e_2\rangle$
	$(2;0,2;1,2;0,0,1)$	$\langle e_4, e_1+e_2\rangle$
	$(2;1,1;0,1;1,1,1)$	$\langle e_3, e_1+e_4\rangle$
	$(2;1,2;0,1;0,1,1)$	$\langle e_2, e_1+e_4\rangle$
	$(2;1,2;1,1;0,0,1)$	$\langle e_4, e_1\rangle$
	$(3;0,2;0,2;1,2,3)$	$\langle e_3, e_2, e_1\rangle$
	$(3;0,2;1,2;1,2,2)$	$\langle e_4, e_3, e_2\rangle$
	$(3;0,2;1,3;0,1,2)$	$\langle e_4, e_2+e_3, e_1+e_2\rangle$
	$(3;1,2;0,2;1,2,2)$	$\langle e_3, e_2, e_1+e_4\rangle$
	$(3;1,2;1,2;1,1,2)$	$\langle e_4, e_3, e_1\rangle$
	$(3;1,3;1,2;0,1,2)$	$\langle e_4, e_2, e_1\rangle$
3)	$(1;0,0;0,1;0,1,1)$	$\langle e_2+e_4\rangle$
	$(1;0,1;0,0;1,1,1)$	$\langle e_3\rangle$
	$(1;0,1;0,1;0,0,1)$	$\langle e_1+e_2+e_3\rangle$
	$(1;0,1;1,1;0,0,0)$	$\langle e_5\rangle$
	$(1;1,1;0,0;0,0,1)$	$\langle e_1\rangle$
	$(2;0,1;0,0;2,2,2)$	$\langle e_4, e_3\rangle$
	$(2;0,1;0,1;1,2,2)$	$\langle e_4, e_2\rangle$
	$(2;0,1;0,2;0,1,2)$	$\langle e_2+e_4, e_1+e_2+e_3\rangle$
	$(2;0,1;1,2;0,1,1)$	$\langle e_5, e_2+e_4\rangle$
	$(2;0,2;0,0;1,2,2)$	$\langle e_3, e_2\rangle$
	$(2;0,2;0,1;1,1,2)$	$\langle e_3, e_1+e_2+e_3\rangle$
	$(2;0,2;1,1;1,1,1)$	$\langle e_5, e_3\rangle$
	$(2;0,2;1,2;0,0,1)$	$\langle e_5, e_1+e_2+e_3\rangle$
	$(2;1,1;0,1;1,1,1)$	$\langle e_4, e_2+e_5\rangle$
	$(2;1,2;0,0;1,1,2)$	$\langle e_3, e_1\rangle$
	$(2;1,2;0,1;0,1,2)$	$\langle e_1, e_1+e_2+e_3\rangle$
	$(2;1,2;1,1;0,1,1)$	$\langle e_5, e_2\rangle$

	$(2;2,2;0,0;0,0,1)$	$\langle e_2+e_5,e_1\rangle$
	$(3;0,2;0,1;2,3,3)$	$\langle e_4,e_3,e_2\rangle$
	$(3;0,2;1,1;2,2,2)$	$\langle e_5,e_4,e_3\rangle$
	$(3;0,2;1,3;0,1,2)$	$\langle e_5,e_2+e_4,e_1+e_2+e_3\rangle$
	$(3;1,2;0,1;2,2,3)$	$\langle e_4,e_3,e_1\rangle$
	$(3;1,2;0,2;1,2,3)$	$\langle e_2+e_4,e_1,e_1+e_2+e_3\rangle$
	$(3;1,2;1,2;1,2,2)$	$\langle e_5,e_4,e_2\rangle$
	$(3;1,3;0,1;1,2,3)$	$\langle e_3,e_2,e_1\rangle$
	$(3;1,3;1,1;1,2,2)$	$\langle e_5,e_3,e_2\rangle$
	$(3;1,3;1,2;1,1,2)$	$\langle e_5,e_3,e_1+e_2+e_3\rangle$
	$(3;2,3;0,1;1,1,2)$	$\langle e_3,e_2+e_5,e_1\rangle$
	$(3;2,3;1,1;0,1,2)$	$\langle e_5,e_2,e_1\rangle$
	$(4;1,3;0,2;2,3,4)$	$\langle e_4,e_3,e_2,e_1\rangle$
	$(4;1,3;1,2;2,3,3)$	$\langle e_5,e_4,e_3,e_2\rangle$
	$(4;1,3;1,3;1,2,3)$	$\langle e_5,e_4,e_2,e_1+e_2+e_3\rangle$
	$(4;2,3;0,2;2,2,3)$	$\langle e_4,e_3,e_2+e_5,e_1\rangle$
	$(4;2,4;1,2;1,2,3)$	$\langle e_5,e_3,e_2,e_1\rangle$
4)	$(1;0,0;0,0;1,1,1)$	$\langle e_4\rangle$
	$(1;0,0;0,1;0,1,1)$	$\langle e_2+e_3+e_4\rangle$
	$(1;0,1;0,0;0,1,1)$	$\langle e_3\rangle$
	$(1;0,1;0,1;0,0,1)$	$\langle e_1+e_3\rangle$
	$(1;0,1;1,1;0,0,0)$	$\langle e_5\rangle$
	$(1;1,1;0,0;0,0,1)$	$\langle e_1\rangle$
	$(2;0,1;0,1;1,2,2)$	$\langle e_4,e_2+e_3+e_4\rangle$
	$(2;0,1;0,2;0,1,2)$	$\langle e_2+e_3+e_4,e_1+e_3\rangle$
	$(2;0,1;1,1;1,1,1)$	$\langle e_5,e_4\rangle$
	$(2;0,1;1,2;0,1,1)$	$\langle e_5,e_2+e_3+e_4\rangle$
	$(2;0,2;0,0;0,2,2)$	$\langle e_3,e_2\rangle$
	$(2;0,2;1,2;0,0,1)$	$\langle e_5,e_1+e_3\rangle$

	$(2;1,1;0,0;1,1,2)$	$\langle e_4, e_1\rangle$
	$(2;1,2;0,1;0,1,2)$	$\langle e_3, e_1\rangle$
	$(2;1,2;1,1;0,1,1)$	$\langle e_5, e_2\rangle$
	$(2;2,2;0,0;0,0,1)$	$\langle e_2+e_5, e_1\rangle$
	$(3;0,2;0,1;1,3,3)$	$\langle e_4, e_3, e_2\rangle$
	$(3;0,2;0,2;1,2,3)$	$\langle e_4, e_2+e_3+e_4, e_1+e_3\rangle$
	$(3;0,2;1,2;1,2,2)$	$\langle e_5, e_4, e_2+e_3+e_4\rangle$
	$(3;0,2;1,3;0,1,2)$	$\langle e_5, e_2+e_3+e_4, e_1+e_3\rangle$
	$(3;1,2;0,1;1,2,3)$	$\langle e_4, e_3, e_1\rangle$
	$(3;1,2;0,2;0,2,3)$	$\langle e_3, e_2+e_3+e_4, e_1\rangle$
	$(3;1,2;1,1;1,2,2)$	$\langle e_5, e_4, e_2\rangle$
	$(3;1,2;1,2;0,2,2)$	$\langle e_5, e_2, e_2+e_3+e_4\rangle$
	$(3;1,3;0,1;0,2,3)$	$\langle e_3, e_2, e_1\rangle$
	$(3;1,3;1,1;0,2,2)$	$\langle e_5, e_3, e_2\rangle$
	$(3;1,3;1,2;0,1,2)$	$\langle e_5, e_3, e_1\rangle$
	$(3;2,2;0,1;1,1,2)$	$\langle e_4, e_2+e_5, e_1\rangle$
	$(3;2,3;1,1;0,1,2)$	$\langle e_5, e_2, e_1\rangle$
	$(4;1,3;0,2;1,3,4)$	$\langle e_4, e_3, e_2, e_1\rangle$
	$(4;1,3;1,2;1,3,3)$	$\langle e_5, e_4, e_3, e_2\rangle$
	$(4;1,3;1,3;1,2,3)$	$\langle e_5, e_4, e_2+e_3+e_4, e_1+e_3\rangle$
	$(4;2,3;1,2;1,2,3)$	$\langle e_5, e_4, e_2, e_1\rangle$
	$(4;2,4;1,2;0,2,3)$	$\langle e_5, e_3, e_2, e_1\rangle$
5)	$(1;0,0;0,0;1,1,1)$	$\langle e_4\rangle$
	$(1;0,1;0,0;0,1,1)$	$\langle e_3\rangle$
	$(1;0,1;0,1;0,0,1)$	$\langle e_1+e_2+e_3\rangle$
	$(1;0,1;1,1;0,0,0)$	$\langle e_5\rangle$
	$(1;1,1;0,0;0,0,1)$	$\langle e_1\rangle$
	$(2;0,1;0,0;1,2,2)$	$\langle e_4, e_3\rangle$
	$(2;0,1;0,2;0,0,2)$	$\langle e_1+e_4, e_1+e_2+e_3\rangle$

	$(2;0,1;1,1;1,1,1)$	$\langle e_5, e_4\rangle$
	$(2;0,2;0,1;0,1,2)$	$\langle e_3, e_1+e_2+e_3\rangle$
	$(2;0,2;1,1;0,1,1)$	$\langle e_5, e_3\rangle$
	$(2;0,2;1,2;0,0,1)$	$\langle e_5, e_1+e_2+e_3\rangle$
	$(2;1,1;0,1;1,1,2)$	$\langle e_4, e_1\rangle$
	$(2;1,2;0,0;0,1,2)$	$\langle e_3, e_1\rangle$
	$(2;1,2;0,1;0,0,2)$	$\langle e_1, e_1+e_2+e_3\rangle$
	$(2;1,2;1,1;0,0,1)$	$\langle e_5, e_2\rangle$
	$(2;2,2;0,0;0,0,1)$	$\langle e_2+e_5, e_1\rangle$
	$(3;0,2;1,1;1,2,2)$	$\langle e_5, e_4, e_3\rangle$
	$(3;0,2;1,3;0,0,2)$	$\langle e_5, e_1+e_4, e_1+e_2+e_3\rangle$
	$(3;1,2;0,1;1,2,3)$	$\langle e_4, e_3, e_1\rangle$
	$(3;1,2;0,2;1,1,3)$	$\langle e_4, e_1, e_1+e_2+e_3\rangle$
	$(3;1,2;1,2;1,1,2)$	$\langle e_5, e_4, e_1\rangle$
	$(3;1,3;0,1;0,1,3)$	$\langle e_3, e_2, e_1\rangle$
	$(3;1,3;1,2;0,1,2)$	$\langle e_5, e_3, e_1+e_2+e_3\rangle$
	$(3;2,2;0,1;1,1,2)$	$\langle e_4, e_2+e_5, e_1\rangle$
	$(3;2,3;0,1;0,1,2)$	$\langle e_3, e_2+e_5, e_1\rangle$
	$(3;2,3;1,1;0,0,2)$	$\langle e_5, e_2, e_1\rangle$
	$(4;1,3;0,2;1,2,4)$	$\langle e_4, e_3, e_2, e_1\rangle$
	$(4;1,3;1,2;1,2,3)$	$\langle e_5, e_4, e_3, e_2\rangle$
	$(4;1,3;1,3;1,1,3)$	$\langle e_5, e_4, e_1, e_1+e_2+e_3\rangle$
	$(4;2,3;0,2;1,2,3)$	$\langle e_4, e_3, e_2+e_5, e_1\rangle$
	$(4;2,3;1,2;1,1,3)$	$\langle e_5, e_4, e_2, e_1\rangle$
	$(4;2,4;1,2;0,1,3)$	$\langle e_5, e_3, e_2, e_1\rangle$
6)	$(1;0,0;0,0;1,1,1)$	$\langle e_3\rangle$
	$(1;0,0;0,1;0,0,1)$	$\langle e_1+e_2+e_3\rangle$
	$(1;0,1;0,0;0,1,1)$	$\langle e_2\rangle$
	$(1;0,1;1,1;0,0,0)$	$\langle e_5\rangle$

$(1;1,1;0,0;0,0,0)$	$\langle e_4\rangle$
$(2;0,1;0,0;1,2,2)$	$\langle e_3, e_2\rangle$
$(2;0,1;0,1;1,1,2)$	$\langle e_3, e_1+e_2+e_3\rangle$
$(2;0,1;1,1;1,1,1)$	$\langle e_5, e_3\rangle$
$(2;0,1;1,2;0,0,1)$	$\langle e_5, e_1+e_2+e_3\rangle$
$(2;0,2;0,0;0,1,2)$	$\langle e_2, e_1\rangle$
$(2;0,2;1,1;0,1,1)$	$\langle e_5, e_2\rangle$
$(2;0,2;1,2;0,0,0)$	$\langle e_5, e_2+e_4\rangle$
$(2;1,1;0,0;1,1,1)$	$\langle e_4, e_3\rangle$
$(2;1,2;0,1;0,1,1)$	$\langle e_4, e_2\rangle$
$(2;1,2;1,1;0,0,1)$	$\langle e_5, e_1\rangle$
$(2;2,2;0,0;0,0,0)$	$\langle e_4, e_1+e_5\rangle$
$(3;0,2;0,1;1,2,3)$	$\langle e_3, e_2, e_1\rangle$
$(3;0,2;1,1;1,2,2)$	$\langle e_5, e_3, e_2\rangle$
$(3;0,2;1,2;1,1,2)$	$\langle e_5, e_3, e_1+e_2+e_3\rangle$
$(3;0,2;1,3;0,0,1)$	$\langle e_5, e_2+e_4, e_1+e_2+e_3\rangle$
$(3;1,2;0,1;1,2,2)$	$\langle e_4, e_3, e_2\rangle$
$(3;1,2;1,1;1,1,2)$	$\langle e_5, e_3, e_1\rangle$
$(3;1,2;1,2;0,1,2)$	$\langle e_5, e_1, e_1+e_2+e_3\rangle$
$(3;1,3;1,1;0,1,2)$	$\langle e_5, e_2, e_1\rangle$
$(3;1,3;1,2;0,1,1)$	$\langle e_5, e_4, e_2\rangle$
$(3;2,2;0,1;1,1,1)$	$\langle e_4, e_3, e_1+e_5\rangle$
$(3;2,3;0,1;0,1,1)$	$\langle e_4, e_2, e_1+e_5\rangle$
$(3;2,3;1,1;0,0,1)$	$\langle e_5, e_4, e_1\rangle$
$(4;1,3;1,2;1,2,3)$	$\langle e_5, e_3, e_2, e_1\rangle$
$(4;1,3;1,3;1,1,2)$	$\langle e_5, e_3, e_2+e_4, e_1+e_2+e_3\rangle$
$(4;2,3;0,2;1,2,2)$	$\langle e_4, e_3, e_2, e_1+e_5\rangle$
$(4;2,3;1,2;1,1,2)$	$\langle e_5, e_4, e_3, e_1\rangle$
$(4;2,4;1,2;0,1,2)$	$\langle e_5, e_4, e_2, e_1\rangle$

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