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On unimodality and real-rootedness of polynomials in combinatorics

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Abstract

We present four papers. The common theme is real-rootedness and unimodality of polynomials occurring in combinatorics.

In the first paper we introduce a new class of labeled posets, *sign-graded posets*, which contains the class of graded naturally labeled posets. We prove that the Eulerian polynomial (W-polynomial) of a sign-graded poset has symmetric and unimodal coefficients. This verifies the motivating consequence of the Neggers-Stanley conjecture on real zeros for this class of posets. It also extends a recent result of Reiner and Welker who proved unimodality of the Eulerian polynomials of naturally labeled graded posets by associating to each graded poset a simplicial polytopal sphere. By proving that the Eulerian polynomial of a sign-graded poset has the right sign at -1 we are able to prove the Charney-Davis conjecture for these spheres (whenever they are flag).

In the second paper we refine a technique used in a paper by Schur, from 1914, on polynomials with real zeros. This amounts to an extension of a theorem of Wagner on Hadamard products of Pólya frequency sequences. We also apply our results to polynomials for which the Neggers-Stanley conjecture is known to hold. More precisely, we settle interlacing properties for E-polynomials of series-parallel posets and column-strict labeled Ferrers posets.

The third paper is a continuation of the second. It is concerned with linear operators that preserve the Pólya frequency property and real-rootedness. We apply our results to settle some conjectures and open problems in combinatorics proposed by Bóna, Brenti and Reiner-Welker.

In the last paper we provide the first counterexamples to the Neggers-Stanley conjecture on real zeros. This conjecture asserts that the polynomial whose coefficients count the linear extensions of a labeled partially ordered set by their number of descents has real zeros only. We provide a family of labeled posets such that for any integer $M > 0$ there is a labeled poset whose corresponding polynomial has more than M non-real zeros.

Keywords and phrases: Neggers-Stanley conjecture, partially ordered set, linear extension, Eulerian polynomial, real zeros, unimodality, Pólya frequency sequence

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[II] P. Brändén,
On operators on polynomials preserving real-rootedness and the Neggers-Stanley Conjecture, *Journal of Algebraic Combinatorics* **20** (2004), 119–130.

[III] P. Brändén,
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ON UNIMODALITY AND REAL-ROOTEDNESS OF POLYNOMIALS IN COMBINATORICS

PETTER BRÄNDÉN

Introduction

Polynomials with real zeros occur frequently and naturally in analysis, combinatorics, geometry and probability theory. It is a field with a rich history and mathematicians such as Cauchy, Fourier, Gauss, Hermite, Karlin, Laguerre, Newton, Pólya, Schur, Sturm and Szegö have all contributed to the existing theory. Recently, combinatorialists have taken greater interest in the subject of polynomials with real zeros and log-concave and unimodal sequences. The reader should consult the extensive surveys [11, 21] for examples and the rich variety of methods available.

Why should a combinatorialist want to know if the polynomials he/she is currently working with have real zeros only? One reason is that it entails several inequalities among the coefficients. The *Newton inequalities* are the following. Let $a_0 + a_1z + \cdots + a_dz^d \in \mathbb{R}[z]$ be a polynomial of degree d with real zeros only. Then, for $i = 1, 2, \ldots, d-1$:

$$\frac{a_i^2}{\binom{d}{i}^2} \geq \frac{a_{i-1}}{\binom{d}{i-1}} \frac{a_{i+1}}{\binom{d}{i+1}}.$$

The Newton inequalities imply that the sequence $\{a_i\}_{i=0}^d$ is *log-concave* i.e., that for $i = 1, 2, \ldots, d-1$:

$$a_i^2 \geq a_{i-1}a_{i+1}.$$

A sequence $\{a_i\}_{i=0}^d$ is *unimodal* if there is an integer c, $0 \leq c \leq n$ such that

$$a_0 \leq a_1 \leq \cdots \leq a_c \geq a_{c+1} \geq \cdots \geq a_d$$

If the polynomial $a_0 + a_1z + \cdots + a_dz^d$ has real zeros only and the coefficients are nonnegative then $\{a_i\}_{i=0}^d$ has no internal zeros. The log-concavity property then implies that $\{a_i\}_{i=0}^d$ is unimodal. There is an infinite family of equalities among the coefficients of a polynomial whose zeros are all real and nonpositive. A sequence of real numbers $\{a_i\}_{i=0}^\infty$ is a *Pólya frequency sequence* if all minors of the matrix $(a_{i-j})_{i,j=0}^\infty$ are nonnegative. The following characterization was first proved by Edrei [14].

Theorem 1. *Let $f = a_0 + a_1x + \cdots + a_dx^d \in \mathbb{R}[x]$ be a polynomial with $a_d > 0$. Then the sequence $\{a_0, a_1, \ldots, a_d, 0, 0, \ldots\}$ is a Pólya frequency sequence if and only if all the zeros of f are real and nonpositive.*

For a treatment of Pólya frequency sequences occurring in combinatorics the reader should consult [10].

1. The Neggers-Stanley conjecture

One of the most famous conjectures on real zeros in combinatorics is the *Neggers-Stanley conjecture* (also known as the Poset conjecture). This conjecture asserts that certain polynomials associated to posets have real zeros only. Let P be a finite poset of cardinality p and let $\omega : P \to \{1, 2, \dots p\}$ be a bijection. The pair (P, ω) is a *labeled poset*.

A (P, ω)*-partition* is a map $\sigma : P \to \{1, 2, 3, \dots\}$ such that

- σ is order reversing, that is, if $x \leq y$ then $\sigma(x) \geq \sigma(y)$,
- if $x < y$ and $\omega(x) > \omega(y)$ then $\sigma(x) > \sigma(y)$.

The theory of (P, ω)-partitions was developed by Stanley in [22]. This theory encompasses topics including ordinary partitions, compositions, plane partitions, Young tableaux, symmetric functions and quasi-symmetric functions. The number of (P, ω)-partitions σ with largest part at most n is a polynomial of degree p in n called the *order polynomial* of (P, ω) and is denoted $\Omega(P, \omega; n)$. Indeed, if $e_k(P, \omega)$ denotes the number of surjective (P, ω)-partitions $\sigma : P \to \{1, 2, \dots, k\}$ we have

$$\Omega(P, \omega; n) = \sum_{k=1}^{p} e_k(P, \omega) \binom{n}{k}.$$

The (P, ω)*-Eulerian polynomial*, $W(P, \omega; t)$, is defined by

$$\sum_{n \geq 0} \Omega(P, \omega; n+1) t^n = \frac{W(P, \omega; t)}{(1-t)^{p+1}}.$$

The set, $\mathcal{L}(P, \omega)$, of permutations $\omega(x_1), \omega(x_2), \dots, \omega(x_p)$ where $x_1, x_2, \dots, x_p$ is a linear extension of P is called the *Jordan-Hölder set* of (P, ω). A *descent* in a permutation $\pi = \pi_1 \pi_2 \cdots \pi_p$ is an index i such that $\pi_i > \pi_{i+1}$. The number of descents in π is denoted $\text{des}(\pi)$. A fundamental result in the theory of (P, ω)-partitions, see [22], is that the (P, ω)-Eulerian polynomial can be written as

$$W(P, \omega; t) = \sum_{\pi \in \mathcal{L}(P, \omega)} t^{\text{des}(\pi)}.$$

The reason for the name (P, ω)-Eulerian polynomial is that when (P, ω) is a p element anti-chain we recover the traditional Eulerian polynomial $A_p(t)$, whose coefficients count the number of permutations in the symmetric group on p elements by the number of descents. The Neggers-Stanley conjecture is the following:

Conjecture 2 (Neggers-Stanley). *Let (P, ω) be a labeled poset. Then all zeros of $W(P, \omega; t)$ are real.*

It should be noted that the polynomial $E(P, \omega; t) = \sum_{k=1}^{p} e_k(P, \omega) t^k$ is related to $W(P, \omega; t)$ via

$$E(P, \omega; t) = t(1+t)^{p-1} W(P, \omega; \frac{t}{1+t}),$$

so $E(P, \omega; t)$ is real-rooted if and only if $W(P, \omega; t)$ is. Moreover, all real zeros of $E(P, \omega; t)$ are necessarily in the interval $[-1, 0]$.

Conjecture 2 was formulated for naturally labeled posets by Neggers [18] in 1978. (A labeling is natural if $x < y$ implies $\omega(x) < \omega(y)$). It was later generalized to its current form by Stanley in 1986. It has been proved for some special cases by Brenti [10] and Wagner [25].

In [6] we provide the first counterexamples to Conjecture 2. In fact, we prove that for any integer $M > 0$ there is a labeled poset (P, ω) such that $W(P, \omega; t)$ has more than M non-real zeros. Inspired by these counterexamples Stembridge [24] shortly afterwards conducted a systematic computer search for counterexamples similar in nature to those in [6]. Stembridge found counterexamples that are naturally labeled.

Although Conjecture 2 is refuted in its generality the log-concavity and unimodality consequences of real-rootedness still remain open. A poset P is *graded* if all maximal chains in P have the same length r. Here r is called the *rank* of P. The first non-trivial result on unimodality of (P, ω)-Eulerian polynomials was produced by Gasharov [17]. He proved that if (P, ω) is a naturally labeled poset of rank at most 2, then $W(P, \omega; t)$ is unimodal.

Reiner and Welker [20] later proved that the (P, ω)-Eulerian polynomial of any graded naturally labeled poset has unimodal coefficients. Their proof uses the theory of f- and h-vectors of simplicial complexes. Let Δ be a simplicial complex and let $f_i = f_i(\Delta)$ denote the number of faces of Δ of dimension i. The *f-vector* of Δ is the vector $(f_0, f_1, \ldots, f_{d-1})$. Let $f_{-1} = 1$. Define the numbers $h_0, h_1, \ldots, h_d$ by:

$$\sum_{i=0}^{d} h_i x^i (1+x)^{d-i} = \sum_{i=0}^{d} f_{i-1} x^i. \tag{1}$$

The vector $(h_0, h_1, \ldots, h_d)$ is called the *h-vector* of Δ. Reiner and Welker [20] associated to any graded naturally labeled poset (P, ω) a simplicial polytopal sphere, $\Delta_{eq}(P)$, whose h-vector are the coefficients of the (P, ω)-Eulerian polynomial. There is a famous characterization of h-vectors of simplicial polytopal spheres due to Stanley [23] and Billera-Lee [2, 3] known as the g-theorem. The g-theorem implies, in particular, that such h-vectors are symmetric and unimodal. Hence, unimodality of the (P, ω)-Eulerian polynomials of graded naturally labeled posets follows from the g-theorem and Reiner and Welkers construction.

In [7] we study a class of labeled posets which generalizes graded posets. Let (P, ω) be a labeled poset and let $E = E(P) = \{(x, y) \in P \times P : x \prec y\}$ be the edges of the Hasse-diagram of P. We associate a labeling $\epsilon : E \to \{-1, 1\}$ to (P, ω) as follows

$$\epsilon(x, y) = \begin{cases} 1 \text{ if } & \omega(x) < \omega(y), \\ -1 \text{ if } & \omega(x) > \omega(y). \end{cases}$$

We say that (P, ω) is *sign-graded*, if for every maximal chain $x_0 \prec x_1 \prec \cdots \prec x_n$ the sum

$$\sum_{i=1}^{n} \epsilon(x_{i-1}, x_i)$$

is the same. The common value of the above sum is called the *rank* of (P, ω). Note that if ϵ is identically equal to 1, i.e., if (P, ω) is naturally labeled, then a sign-graded poset with respect to ϵ is just a graded poset. Our main theorem in [7] is the following.

Theorem 3. *Suppose that (P, ω) is sign-graded of rank r and let $d = p - 1 - r$. Then*

$$W(P, \omega; t) = \sum_{i=0}^{\lfloor d/2 \rfloor} a_i(P, \omega) t^i (1+t)^{d-2i},$$

where $a_i(P,\omega)$ are nonnegative integers.

The polynomials $t^i(1+t)^{d-2i}$, $i = 0, 1, \dots, \lfloor d/2 \rfloor$ are all unimodal and symmetric with the same center of symmetry $d/2$. From the nonnegativity of $a_i(P,\omega)$ symmetry and unimodality thus follow. This extends the theorem of Reiner and Welker to sign-graded posets. Also, it generalizes the well known fact that the Eulerian polynomials $A_n(t)$ can be expanded as in Theorem 3, see for example [15].

As a bonus, Theorem 3 also answers a question raised by Reiner and Welker in [20]. They asked whether the (P,ω)-Eulerian polynomials of naturally labeled graded posets have predictable signs at -1. This is related to the Charney-Davis conjecture. (A simplicial complex Δ is *flag* if the minimal non-faces of Δ have cardinality two.)

Conjecture 4 (Charney-Davis, [13]). *Let Δ be a flag simplicial homology $(d-1)$-sphere, where d is even. Then the h-vector, $h(\Delta, t)$, of Δ satisfies*

$$(-1)^{d/2} h(\Delta, -1) \geq 0.$$

As the h-polynomial of, $\Delta_{eq}(P)$, associated to a graded naturally labeled poset (P,ω) is the (P,ω)-Eulerian polynomial we have by Theorem 3 that

$$(-1)^{d/2} h(\Delta_{eq}, -1) = a_{d/2}(P,\omega) \geq 0,$$

so the Charney-Davis conjecture holds for $\Delta_{eq}(P)$ (whenever it is flag). After the completion of the first version of [7] we were informed that Gal [16] has conjectured that if Δ is a flag simplicial homology $(d-1)$-sphere, then its h-vector admits an expansion

$$h(\Delta, t) = \sum_{i=0}^{\lfloor d/2 \rfloor} a_i(\Delta) t^i (1+t)^{d-2i},$$

where $a_i(\Delta)$ are nonnegative integers. This would imply the Charney-Davis conjecture and Theorem 3 can be seen as supporting evidence for Gal's conjecture.

So what is left of the Neggers-Stanley conjecture? Log-concavity and unimodality of the polynomials $W(P,\omega;t)$ and $E(P,\omega;t)$ is still open in general. Partial unimodality for $W(P,\omega;t)$ for naturally labeled posets was recently obtained by Athanasiadis [1] and partial unimodality for $E(P,\omega;t)$ for naturally labeled posets was recently obtained by Björner and Farley [4]. It should be mentioned that real-rootedness is still open for sign-graded posets and in particular for the important class of naturally labeled graded posets.

2. Linear operators on polynomials preserving real-rootedness

An important tool in proving that polynomials have real zeros only is to use results on linear operators that preserve real-rootedness. There are a some classical results in this area worth mentioning. The *Laguerre-Pólya class* of entire functions consists of all entire functions Ψ that have a representation of the form

$$\Psi(z) = cz^\kappa e^{-a^2z^2+bz} \prod_{k=1}^{\infty} (1 + t_k z) e^{-\lambda t_k z}, \tag{2}$$

where $a, b, c, t_k \in \mathbb{R}$, $\kappa \in \mathbb{N}$, $\lambda \in \{0, 1\}$ and $\sum_{k=1}^{\infty} |t_k|^{\lambda+1} < \infty$. Note that the Laguerre-Pólya class includes all polynomials with real zeros. In fact, the members of the Laguerre-Pólya class are exactly those entire functions that are uniform limits on compact subsets of $\mathbb{C}$ of polynomials with real zeros. Within the Laguerre-Pólya class, those functions Ψ for which $\lambda = a = 0$, $b \geq 0$ and $t_k \geq 0$ in (2) are said to be of *type I.* The functions of type I in the Laguerre-Pólya class are exactly those entire functions which are uniform limits on compact subsets of $\mathbb{C}$ of polynomials with real negative zeros.

Theorem 5 (Hermite-Poulain-Pólya)**.** *Let $\Psi = \sum_{k=0}^{\infty} a_k z^k$ be a formal power series with coefficients in $\mathbb{R}$ and let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be the linear operator defined by*

$$T(f) = a_0 f(x) + a_1 f'(x) + a_2 f''(x) + \cdots ,$$

Then T preserves real-rootedness if and only if Ψ is in the Laguerre-Pólya class.

A *multiplier-sequence* is a sequence of real numbers $\Gamma = \{\gamma_i\}_{i=0}^{\infty}$ such that the corresponding linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ defined by

$$T(x^i) = \gamma_i x^i,$$

preserves real-rootedness. Such sequences were characterized by Pólya and Schur [19].

Theorem 6 (Pólya-Schur)**.** *Let $\Gamma = \{\gamma_i\}_{i=0}^{\infty}$ be a sequence of real numbers, and let*

$$\Psi(z) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k,$$

be the exponential generating function of Γ.

Then Γ is a multiplier-sequence if and only if $\Psi(z)$ or $\Psi(-z)$ is a function of type I in the Laguerre-Pólya class.

The *diamond product* (of polynomials) was introduced by Wagner in [25, 26] and is defined by

$$f \Diamond g = \sum_{k \geq 0} \frac{f^{(k)}(x)}{k!} \frac{g^{(k)}(x)}{k!} x^k (x+1)^k.$$

This product is important in the theory of (P, ω)-partitions. The disjoint union of two labeled poset (P, ω) and (Q, μ) is the labeled poset $(P \sqcup Q, \omega \sqcup \mu)$ where $P \sqcup Q$ is the disjoint union of the posets P and Q and $\omega \sqcup \mu$ is any labeling of $P \sqcup Q$ such that

$$(\omega \sqcup \mu)(x) < (\omega \sqcup \mu)(y)$$

if $\omega(x) < \omega(y)$ or $\mu(x) < \mu(y)$. The importance of the diamond product comes from the fact that

$$E(P \sqcup Q, \omega \sqcup \mu; x) = E(P, \omega; x) \Diamond E(P, \mu; x).$$

Hence, a sufficient condition for the Neggers-Stanley conjecture to respect disjoint unions is the following theorem due to Wagner [26].

Theorem 7. *Suppose that f and g are polynomials whose zeros all are in the interval $[-1, 0]$. Then so is the polynomial $f \Diamond g$.*

An equivalent form of Theorem 7 was conjectured by Brenti [10].

In [8, 9] we study linear differential operators of the form

$$T = \sum_{k=0}^{N} Q_k(x) D^k, \tag{3}$$

where $D = d/dx$ and $Q_k(x)$ is a polynomial with real coefficients. The *symbol* of T is the polynomial $F_T(x,z) = \sum_{k=0}^{N} Q_k(x)z^k \in \mathbb{R}[x,z]$. In [9] we give sufficient conditions on $F_T(x,z)$ for T to preserve real-rootedness. We say that (3) is *monotone* if for all $f \in \mathbb{R}[x]$ we have $\deg T(f) = \deg f + \deg Q_0$ and the leading coefficient of $T(f)$ has the same sign as the leading coefficient of f. In [9] we prove theorems similar in flavor to the following.

Theorem 8. *Let $T = \sum_{k=0}^{N} Q_k(x)D^k$ be monotone and suppose that*

(i) *For all $\xi \in \mathbb{R}$ the polynomial $F_T(\xi, z)$ is real-rooted in z,*
(ii) *$T(\alpha x + \beta)$ is real- and simple-rooted for all $\alpha, \beta \in \mathbb{R}$ not both equal to zero.*

Then T preserves real-rootedness. Moreover, $T(f)$ has simple zeros if f has simple zeros.

If the leading coefficient of Q_0 in (3) is positive, then one can prove that T must be monotone in order to preserve real-rootedness, see [5]. In [5] we generalize the theory developed in [8, 9].

In the above setting we may view $f \Diamond g$ as $T(f)$ where T has symbol

$$\sum_{k\geq 0} \frac{g^{(k)}(x)}{k!k!} x^k(1+x)^k z^k.$$

In [8] we show that if g has all its zeros in the interval $[-1,0]$ then the above differential operator T does indeed preserve real-rootedness. This amounts to the following generalization of Theorem 7.

Theorem 9. *Suppose that all zeros of g are in the interval $[-1,0]$ and all zeros of f are real. Then all zeros of $f \Diamond g$ are real.*

In [8] we also use Theorem 9 to exhibit explicit interlacing properties for the (P,ω)-Eulerian polynomials for classes of labeled posets for which the Neggers-Stanley conjecture is known to hold.

In [9] we also study the linear transformation $\mathcal{E} : \mathbb{R}[x] \to \mathbb{R}[x]$ which takes $\binom{x}{k}$ to x^k for $k \in \mathbb{N}$. Positivity properties for this linear operator have been studied in [10, 26]. The importance of this operator stems from the fact that it maps the order-polynomial $\Omega(P,\omega;x)$ to $E(P,\omega;x)$. In [9] we prove the following theorem.

Theorem 10. *Let f be a polynomial of degree d. Suppose that f can be written as*

$$f(x) = \sum_{i=0}^{d} a_i x^i (1+x)^{d-i},$$

where $a_i \geq 0$. Then all zeros of $\mathcal{E}(f)$ are real, simple and located in the interval $[-1,0]$.

Let the f- and h-polynomial of a simplicial complex Δ be $f_\Delta(x) = \sum_{i=0}^{d} f_{i-1}(\Delta)x^i$ and $h_\Delta(x) = \sum_{i=0}^{d} h_i(\Delta)x^i$, respectively. The following striking consequence of Theorem 10 was observed by Brenti and Welker [12]. The linear transformation $\mathcal{E}$ maps the f-polynomial of a simplicial complex Δ to the f-polynomial of the barycentric subdivision, $\mathrm{sd}(\Delta)$, of Δ. That the f-polynomial can be written as in Theorem 10 is by (1) equivalent to saying that the h-polynomial of Δ has nonnegative coefficients. In other words if the h-polynomial of a simplicial complex

Δ has nonnegative coefficients (for example if Δ is Cohen-Macaulay), then the f-polynomial and h-polynomial of the barycentric subdivision $\mathrm{sd}(\Delta)$ have real zeros only.

In [9] we also use the theory developed there on linear transformations preserving real-rootedness to settle some conjectures and open problems in combinatorics raised by Bóna, Brenti and Reiner-Welker.

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SIGN-GRADED POSETS, UNIMODALITY OF W-POLYNOMIALS AND THE CHARNEY-DAVIS CONJECTURE

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Dedicated to Richard Stanley on the occasion of his 60th birthday

Abstract. We generalize the notion of graded posets to what we call sign-graded (labeled) posets. We prove that the W-polynomial of a sign-graded poset is symmetric and unimodal. This extends a recent result of Reiner and Welker who proved it for graded posets by associating a simplicial polytopal sphere to each graded poset. By proving that the W-polynomials of sign-graded posets has the right sign at -1, we are able to prove the Charney-Davis conjecture for these spheres (whenever they are flag).

1. Introduction and preliminaries

Recently Reiner and Welker [10] proved that the W-polynomial of a graded poset (partially ordered set) P has unimodal coefficients. They proved this by associating to P a simplicial polytopal sphere, $\Delta_{eq}(P)$, whose h-polynomial is the W-polynomial of P, and invoking the g-theorem for simplicial polytopes (see [15, 16]). Whenever this sphere is flag, i.e., its minimal non-faces all have cardinality two, they noted that the Neggers-Stanley conjecture implies the Charney-Davis conjecture for $\Delta_{eq}(P)$. In this paper we give a different proof of the unimodality of W-polynomials of graded posets, and we also prove the Charney-Davis conjecture for $\Delta_{eq}(P)$ (whenever it is flag). We prove it by studying a class of labeled posets, which we call sign-graded posets, of which the class of graded naturally labeled posets is a sub-class.

In this paper all posets will be finite and non-empty. For undefined terminology on posets we refer the reader to [13]. We denote the cardinality of a poset P with the letter p. Let P be a poset and let $\omega : P \to \{1, 2, \ldots, p\}$ be a bijection. The pair (P, ω) is called a *labeled poset*. If ω is order-preserving then (P, ω) is said to be *naturally labeled*. A (P, ω)-*partition* is a map $\sigma : P \to \{1, 2, 3, \ldots\}$ such that

- σ is order reversing, that is, if $x \leq y$ then $\sigma(x) \geq \sigma(y)$,
- if $x < y$ and $\omega(x) > \omega(y)$ then $\sigma(x) > \sigma(y)$.

The theory of (P, ω)-partitions was developed by Stanley in [14]. The number of (P, ω)-partitions σ with largest part at most n is a polynomial of degree p in n called the *order polynomial* of (P, ω) and is denoted $\Omega(P, \omega; n)$. The W-polynomial

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of (P,ω) is defined by

$$\sum_{n\geq 0} \Omega(P,\omega;n+1)t^n = \frac{W(P,\omega;t)}{(1-t)^{p+1}}. \tag{1.1}$$

The set, $\mathcal{L}(P,\omega)$, of permutations $\omega(x_1),\omega(x_2),\ldots,\omega(x_p)$ where $x_1,x_2,\ldots,x_p$ is a linear extension of P is called the *Jordan-Hölder set* of (P,ω). A *descent* in a permutation $\pi=\pi_1\pi_2\cdots\pi_p$ is an index $1\leq i\leq p-1$ such that $\pi_i>\pi_{i+1}$. The number of descents in π is denoted $\mathrm{des}(\pi)$. A fundamental result in the theory of (P,ω)-partitions, see [14], is that the W-polynomial can be written as

$$W(P,\omega;t) = \sum_{\pi\in\mathcal{L}(P,\omega)} t^{\mathrm{des}(\pi)}.$$

The Neggers-Stanley conjecture is the following:

Conjecture 1.1 (Neggers-Stanley)**.** *Let (P,ω) be a labeled poset. Then $W(P,\omega;t)$ has real zeros only.*

This was first conjectured by Neggers [8] in 1978 for natural labelings and by Stanley in 1986 for arbitrary labelings. The conjecture has been proved for some special cases, see [1, 2, 10, 17] for the state of the art. If a polynomial has only real non-positive zeros then its coefficients form a unimodal sequence. For the W-polynomials of graded posets unimodality was first proved by Gasharov [7] whenever the rank is at most 2, and as mentioned by Reiner and Welker [10] for all graded posets.

For the relevant definitions concerning the topology behind the Charney-Davis conjecture we refer the reader to [3, 10, 16].

Conjecture 1.2 (Charney-Davis, [3])**.** *Let Δ be a flag simplicial homology $(d-1)$-sphere, where d is even. Then the h-vector, $h(\Delta,t)$, of Δ satisfies*

$$(-1)^{d/2}h(\Delta,-1)\geq 0.$$

Recall that the nth *Eulerian polynomial*, $A_n(x)$, is the W-polynomial of an anti-chain of n elements. The Eulerian polynomials can be written as

$$A_n(x) = \sum_{i=0}^{\lfloor (n-1)/2\rfloor} a_{n,i}x^i(1+x)^{n-1-2i},$$

where $a_{n,i}$ is a nonnegative integer for all i, see [5, 11]. From this expansion we see immediately that $A_n(x)$ is symmetric and that the coefficients in the standard basis are unimodal. It also follows that $(-1)^{(n-1)/2}A_n(-1)\geq 0$.

We will in Section 2 define a class of labeled poset whose members we call sign-graded posets. This class includes the class of naturally labeled graded posets. In Section 4 we show that the W-polynomial of a sign-graded poset (P,ω) of rank r can be expanded, just as the Eulerian polynomials, as

$$W(P,\omega;t) = \sum_{i=0}^{\lfloor (p-r-1)/2\rfloor} a_i(P,\omega)t^i(1+t)^{p-r-1-2i}, \tag{1.2}$$

where $a_i(P,\omega)$ are nonnegative integers. Hence, symmetry and unimodality follow, and $W(P,\omega;t)$ has the right sign at -1. Consequently, whenever the associated sphere $\Delta_{eq}(P)$ of a graded poset P is flag the Charney-Davis conjecture holds for

$\Delta_{eq}(P)$. We also note that all symmetric polynomials with non-positive zeros only, admit an expansion such as (1.2). Hence, that $W(P, \omega; t)$ has such an expansion can be seen as further evidence for the Neggers-Stanley conjecture. After the completion of the first version of this paper we were informed that S. Gal [6] has conjectured that if Δ is flag simplicial homology $(d-1)$-sphere, then its h-vector admits an expansion

$$h(\Delta, t) = \sum_{i=0}^{\lfloor d/2 \rfloor} a_i(\Delta) t^i (1+t)^{d-2i},$$

where $a_i(\Delta)$ are nonnegative integers. This would imply the Charney-Davis conjecture and (1.2) can be seen as further evidence for Gal's conjecture.

In [9] the Charney-Davis quantity of a graded naturally labeled poset (P, ω) of rank r was defined to be $(-1)^{(p-1-r)/2} W(P, \omega; -1)$. In Section 5 we give a combinatorial interpretation of the Charney-Davis quantity as counting certain reverse alternating permutations. Finally in Section 7 we characterize sign-graded posets in terms of properties of order polynomials.

2. Sign-graded posets

Recall that a poset P is *graded* if all maximal chains in P have the same length. If P is graded one may associate a *rank function* $\rho : P \to \mathbb{N}$ by letting $\rho(x)$ be the length of any saturated chain from a minimal element to x. The *rank* of a graded poset P is defined as the length of any maximal chain in P. In this section we will generalize the notion of graded posets to labeled posets.

Let (P, ω) be a labeled poset. An element y *covers* x, written $x \prec y$, if $x < y$ and $x < z < y$ for no $z \in P$. Let $E = E(P) = \{(x, y) \in P \times P : x \prec y\}$ be the covering relations of P. We associate a labeling $\epsilon : E \to \{-1, 1\}$ of the covering relations defined by

$$\epsilon(x, y) = \begin{cases} 1 \text{ if } & \omega(x) < \omega(y), \\ -1 \text{ if } & \omega(x) > \omega(y). \end{cases}$$

If two labelings ω and λ of P give rise to the same labeling of $E(P)$ then it is easy to see that the set of (P, ω)-partitions and the set of (P, λ)-partitions are the same. In what follows we will often refer to ϵ as the labeling and write (P, ϵ).

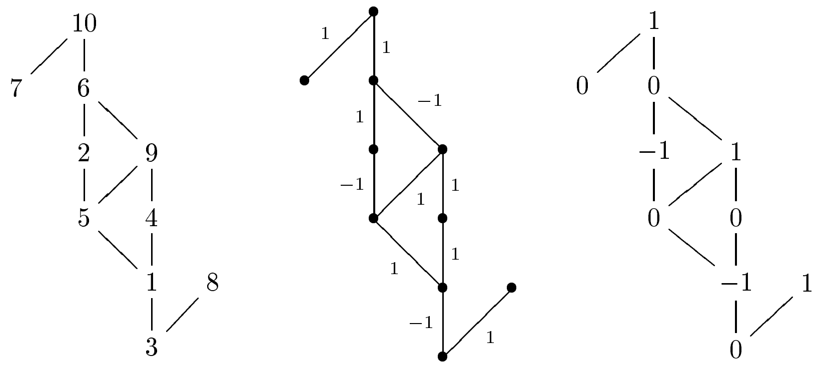
Definition 2.1. Let (P, ω) be a labeled poset and let ϵ be the corresponding labeling of $E(P)$. We say that (P, ω) is *sign-graded*, and that P is *ϵ-graded* (and *ω-graded*) if for every maximal chain $x_0 \prec x_1 \prec \cdots \prec x_n$ the sum

$$\sum_{i=1}^{n} \epsilon(x_{i-1}, x_i)$$

is the same. The common value of the above sum is called the *rank* of (P, ω) and is denoted $r(\epsilon)$.

We say that the poset P is *ϵ-consistent* (and ω-consistent) if for every $y \in P$ the principal order ideal $\Lambda_y = \{x \in P : x \leq y\}$ is ϵ_y-graded, where ϵ_y is ϵ restricted to $E(\Lambda_y)$. The *rank function* $\rho : P \to \mathbb{Z}$ of an ϵ-consistent poset P is defined by $\rho(x) = r(\epsilon_x)$. Hence, an ϵ-consistent poset P is ϵ-graded if and only if ρ is constant on the set of maximal elements.

FIGURE 1. A sign-graded poset, its two labelings and the corresponding rank function.



See Fig. 1 for an example of a sign-graded poset. Note that if ϵ is identically equal to 1, i.e., if (P, ω) is naturally labeled, then a sign-graded poset with respect to ϵ is just a graded poset. Note also that if P is ϵ-graded then P is also $-\epsilon$-graded, where $-\epsilon$ is defined by $(-\epsilon)(x, y) = -\epsilon(x, y)$. Up to a shift, the order polynomial of a sign-graded labeled poset only depends on the underlying poset:

Theorem 2.2. *Let P be ϵ-graded and μ-graded. Then*

$$\Omega(P, \epsilon; t - \frac{r(\epsilon)}{2}) = \Omega(P, \mu; t - \frac{r(\mu)}{2}).$$

Proof. Let ρ_ϵ and ρ_μ denote the rank functions of (P, ϵ) and (P, μ) respectively, and let $\mathcal{A}(\epsilon)$ denote the set of (P, ϵ)-partitions. Define a function $\xi : \mathcal{A}(\epsilon) \to \mathbb{Q}^P$ by $\xi\sigma(x) = \sigma(x) + \Delta(x)$, where

$$\Delta(x) = \frac{r(\epsilon) - \rho_\epsilon(x)}{2} - \frac{r(\mu) - \rho_\mu(x)}{2}.$$

TABLE 1

$\epsilon(x,y)$	$\mu(x,y)$	σ	Δ	$\xi\sigma$
1	1	$\sigma(x) \geq \sigma(y)$	$\Delta(x) = \Delta(y)$	$\xi\sigma(x) \geq \xi\sigma(y)$
1	-1	$\sigma(x) \geq \sigma(y)$	$\Delta(x) = \Delta(y) + 1$	$\xi\sigma(x) > \xi\sigma(y)$
-1	1	$\sigma(x) > \sigma(y)$	$\Delta(x) = \Delta(y) - 1$	$\xi\sigma(x) \geq \xi\sigma(y)$
-1	-1	$\sigma(x) > \sigma(y)$	$\Delta(x) = \Delta(y)$	$\xi\sigma(x) > \xi\sigma(y)$

The four possible combinations of labelings of a covering-relation $(x, y) \in E$ are given in Table 1.

According to the table $\xi\sigma$ is a (P, μ)-partition provided that $\xi\sigma(x) > 0$ for all $x \in P$. But $\xi\sigma$ is order-reversing so it attains its minima on maximal elements and if z is a maximal element we have $\xi\sigma(z) = \sigma(z)$. Hence $\xi : \mathcal{A}(\epsilon) \to \mathcal{A}(\mu)$. By symmetry we also have a map $\eta : \mathcal{A}(\mu) \to \mathcal{A}(\epsilon)$ defined by

$$\eta\sigma(x) = \sigma(x) + \frac{r(\mu) - \rho_\mu(x)}{2} - \frac{r(\epsilon) - \rho_\epsilon(x)}{2}.$$

Hence, $\eta = \xi^{-1}$ and ξ is a bijection.

Since σ and $\xi\sigma$ are order-reversing they attain their maxima on minimal elements. But if z is a minimal element then $\xi\sigma(z) = \sigma(z) + \frac{r(\epsilon)-r(\mu)}{2}$, which gives

$$\Omega(P,\mu;n) = \Omega(P,\epsilon;n + \frac{r(\mu)-r(\epsilon)}{2}),$$

for all nonnegative integers n and the theorem follows. □

Theorem 2.3. *Let P be ϵ-graded. Then*

$$\Omega(P,\epsilon;t) = (-1)^p\Omega(P,\epsilon;-t-r(\epsilon)).$$

Proof. We have the following reciprocity for order polynomials, see [14]:

$$\Omega(P,-\epsilon;t) = (-1)^p\Omega(P,\epsilon;-t). \tag{2.1}$$

Note that $r(-\epsilon) = -r(\epsilon)$, so by Theorem 2.2 we have:

$$\Omega(P,-\epsilon;t) = \Omega(P,\epsilon;t-r(\epsilon)),$$

which, combined with (2.1), gives the desired result. □

Corollary 2.4. *Let P be an ϵ-graded poset. Then $W(P,\epsilon;t)$ is symmetric with center of symmetry $(p-r(\epsilon)-1)/2$. If P is also μ-graded then*

$$W(P,\mu;t) = t^{(r(\epsilon)-r(\mu))/2}W(P,\epsilon;t).$$

Proof. Suppose that $W(P,\epsilon;t) = \sum_{i\geq 0} w_i(P,\epsilon)t^i$. From (1.1) it follows that $\Omega(P,\epsilon;t) = \sum_{i\geq 0} w_i(P,\epsilon)\binom{t+p-1-i}{p}$. Let $r = r(\epsilon)$. Theorem 2.3 gives:

$$\begin{aligned}
\Omega(P,\epsilon;t) &= \sum_{i\geq 0} w_i(P,\epsilon)(-1)^p\binom{-t-r+p-1-i}{p} \\
&= \sum_{i\geq 0} w_i(P,\epsilon)\binom{t+r+i}{p} \\
&= \sum_{i\geq 0} w_{p-r-1-i}(P,\epsilon)\binom{t+p-1-i}{p},
\end{aligned}$$

so $w_i(P,\epsilon) = w_{p-r-1-i}(P,\epsilon)$ for all i, and the symmetry follows. The relationship between the W-polynomials of (P,ϵ) and (P,μ) follows from Theorem 2.2 and the expansion of order-polynomials in the basis $\binom{t+p-1-i}{p}$. □

We say that a poset P is *parity graded* if the size of all maximal chains in P have the same parity. Also, a poset is P is *parity consistent* if for all $x \in P$ the order ideal Λ_x is parity graded. These classes of posets were studied in [12] in a different context. The following theorem tells us that the class of sign-graded posets is considerably greater than the class of graded posets.

Theorem 2.5. *Let P be a poset. Then*

- *there exists a labeling $\epsilon : E \to \{-1,1\}$ such that P is ϵ-consistent if and only if P is parity consistent,*
- *there exists a labeling $\epsilon : E \to \{-1,1\}$ such that P is ϵ-graded if and only if P is parity graded.*

Moreover, the labeling ϵ can be chosen so that the corresponding rank function has values in $\{0,1\}$.

Proof. It suffices to prove the equivalence regarding parity graded posets. It is clear that if P is ϵ-graded then P is parity graded.

Let P be parity graded. Then, for any $x \in P$, all saturated chains from a minimal element to x have the same length modulo 2. Hence, we may define a labeling $\epsilon : E(P) \to \{-1, 1\}$ by $\epsilon(x, y) = (-1)^{\ell(x)}$, where $\ell(x)$ is the length of any saturated chain starting at a minimal element and ending at x. It follows that P is ϵ-graded and that its rank function has values in $\{0, 1\}$. □

We say that $\omega : P \to \{1, 2, \dots, p\}$ is *canonical* if (P, ω) has a rank-function ρ with values in $\{0, 1\}$, and $\rho(x) < \rho(y)$ implies $\omega(x) < \omega(y)$. By Theorem 2.5 we know that P admits a canonical labeling if P is ϵ-consistent for some ϵ.

3. The Jordan-Hölder set of an ϵ-consistent poset

Let P be ω-consistent. We may assume that $\omega(x) < \omega(y)$ whenever $\rho(x) < \rho(y)$. This is because any labeling ω' of P for which $\rho(x) < \rho(y)$ implies $\omega'(x) < \omega'(y)$ will give rise to the same labeling of $E(P)$ as (P, ω).

Suppose that $x, y \in P$ are incomparable and that $\rho(y) = \rho(x) + 1$. Then the Jordan-Hölder set of (P, ω) can be partitioned into two sets: One where in all permutations $\omega(x)$ comes before $\omega(y)$ and one where $\omega(y)$ always comes before $\omega(x)$. This means that $\mathcal{L}(P, \omega)$ is the disjoint union

$$\mathcal{L}(P, \omega) = \mathcal{L}(P', \omega) \sqcup \mathcal{L}(P'', \omega), \tag{3.1}$$

where P' is the transitive closure of $E \cup \{x \prec y\}$, and P'' is the transitive closure of $E \cup \{y \prec x\}$.

Lemma 3.1. *With definitions as above P' and P'' are ω-consistent with the same rank-function as (P, ω).*

Proof. Let $c : z_0 \prec z_1 \prec \cdots \prec z_k = z$ be a saturated chain in P'', where z_0 is a minimal element of P''. Of course z_0 is also a minimal element of P. We have to prove that

$$\rho(z) = \sum_{i=0}^{k-1} \epsilon''(z_i, z_{i+1}),$$

where ϵ'' is the labeling of $E(P'')$ and ρ is the rank-function of (P, ω).

All covering relations in P'', except $y \prec x$, are also covering relations in P. If y and x do not appear in c, then c is a saturated chain in P and there is nothing to prove. Otherwise

$$c : y_0 \prec \cdots \prec y_i = y \prec x = x_{i+1} \prec x_{i+2} \prec \cdots \prec x_k = z.$$

Note that if $s_0 \prec s_1 \prec \cdots \prec s_\ell$ is any saturated chain in P then $\sum_{i=0}^{\ell-1} \epsilon(s_i, s_{i+1}) = \rho(s_\ell) - \rho(s_0)$. Since $y_0 \prec \cdots \prec y_i = y$ and $x = x_{i+1} \prec x_{i+2} \prec \cdots \prec x_k = z$ are saturated chains in P we have

$$\begin{aligned} \sum_{i=0}^{k-1} \epsilon''(z_i, z_{i+1}) &= \rho(y) + \epsilon''(y, x) + \rho(z) - \rho(x) \\ &= \rho(y) - 1 - \rho(x) + \rho(z) \\ &= \rho(z), \end{aligned}$$

as was to be proved. The statement for (P', ω) follows similarly. □

We say that a ω-consistent poset P is *saturated* if for all $x, y \in P$ we have that x and y are comparable whenever $|\rho(y) - \rho(x)| = 1$. Let P and Q be posets on the same set. Then Q *extends* P if $x <_Q y$ whenever $x <_P y$.

Theorem 3.2. *Let P be a ω-consistent poset. Then the Jordan-Hölder set of (P, ω) is uniquely decomposed as the disjoint union*

$$\mathcal{L}(P, \omega) = \bigsqcup_Q \mathcal{L}(Q, \omega),$$

where the union is over all saturated ω-consistent posets Q that extend P and have the same rank-function as (P, ω).

Proof. That the union exhausts $\mathcal{L}(P, \omega)$ follows from (3.1) and Lemma 3.1. Let Q_1 and Q_2 be two different saturated ω-consistent posets that extend P and have the same rank-function as (P, ω). We may assume that Q_2 does not extend Q_1. Then there exists a covering relation $x \prec y$ in Q_1 such that $x \not< y$ in Q_2. Since $|\rho(x) - \rho(y)| = 1$ we must have $y < x$ in Q_2. Thus $\omega(x)$ precedes $\omega(y)$ in any permutation in $\mathcal{L}(Q_1, \omega)$, and $\omega(y)$ precedes $\omega(x)$ in any permutation in $\mathcal{L}(Q_2, \omega)$. Hence, the union is disjoint and unique. □

We need two operations on labeled posets: Let (P, ϵ) and (Q, μ) be two labeled posets. The *ordinal sum*, $P \oplus Q$, of P and Q is the poset with the disjoint union of P and Q as underlying set and with partial order defined by $x \le y$ if $x \le_P y$ or $x \le_Q y$, or $x \in P, y \in Q$. Define two labelings of $E(P \oplus Q)$ by

$$\begin{aligned}
(\epsilon \oplus_1 \mu)(x, y) &= \epsilon(x, y) \text{ if } (x, y) \in E(P), \\
(\epsilon \oplus_1 \mu)(x, y) &= \mu(x, y) \text{ if } (x, y) \in E(Q) \text{ and} \\
(\epsilon \oplus_1 \mu)(x, y) &= 1 \text{ otherwise.} \\
(\epsilon \oplus_{-1} \mu)(x, y) &= \epsilon(x, y) \text{ if } (x, y) \in E(P), \\
(\epsilon \oplus_{-1} \mu)(x, y) &= \mu(x, y) \text{ if } (x, y) \in E(Q) \text{ and} \\
(\epsilon \oplus_{-1} \mu)(x, y) &= -1 \text{ otherwise.}
\end{aligned}$$

With a slight abuse of notation we write $P \oplus_{\pm 1} Q$ when the labelings of P and Q are understood from the context. Note that ordinal sums are associative, i.e., $(P \oplus_{\pm 1} Q) \oplus_{\pm 1} R = P \oplus_{\pm 1} (Q \oplus_{\pm 1} R)$, and preserve the property of being sign-graded. The following result is easily obtained by combinatorial reasoning, see [2, 17]:

Proposition 3.3. *Let (P, ω) and (Q, ν) be two labeled posets. Then*

$$W(P \oplus Q, \omega \oplus_1 \nu; t) = W(P, \omega; t) W(Q, \nu; t)$$

and

$$W(P \oplus Q, \omega \oplus_{-1} \nu; t) = t W(P, \omega; t) W(Q, \nu; t).$$

Proposition 3.4. *Suppose that (P, ω) is a saturated canonically labeled ω-consistent poset. Then (P, ω) is the direct sum*

$$(P, \omega) = A_0 \oplus_1 A_1 \oplus_{-1} A_2 \oplus_1 A_3 \oplus_{-1} \cdots \oplus_{\pm 1} A_k,$$

where the A_is are anti-chains.

Proof. Let $\pi \in \mathcal{L}(P, \omega)$. Then we may write π as $\pi = w_0 w_1 \cdots w_k$ where the w_is are maximal words with respect to the property: If a and b are letters of w_i then $\rho(\omega^{-1}(a)) = \rho(\omega^{-1}(b))$. Hence $\pi \in \mathcal{L}(Q, \omega)$ where

$$(Q, \omega) = A_0 \oplus_1 A_1 \oplus_{-1} A_2 \oplus_1 A_3 \oplus_{-1} \cdots \oplus_{\pm 1} A_k,$$

and A_i is the anti-chain consisting of the elements $\omega^{-1}(a)$, where a is a letter of w_i (A_i is an anti-chain, since if $x < y$ where $x, y \in A_i$ there would be a letter in π between $\omega(x)$ and $\omega(y)$ whose rank was different than that of x, y). Now, (Q, ω) is saturated so $P = Q$. $\square$

Note that the argument in the above proof also can be used to give a simpler proof of Theorem 3.2 when ω is canonical.

4. The W-polynomial of a sign-graded poset

The space S^d of symmetric polynomials in $\mathbb{R}[t]$ with center of symmetry $d/2$ has a basis

$$B_d = \{t^i(1+t)^{d-2i}\}_{i=0}^{\lfloor d/2 \rfloor}.$$

If $h \in S^d$ has nonnegative coefficients in this basis it follows immediately that the coefficients of h in the standard basis are unimodal. Let S_+^d be the nonnegative span of B_d. Thus S_+^d is a cone. Another property of S_+^d is that if $h \in S_+^d$ then it has the correct sign at -1 i.e.,

$$(-1)^{d/2}h(-1) \geq 0.$$

Lemma 4.1. *Let $c, d \in \mathbb{N}$. Then*

$$\begin{aligned} S^c S^d &\subset S^{c+d} \\ S_+^c S_+^d &\subset S_+^{c+d}. \end{aligned}$$

Suppose further that $h \in S^d$ has positive leading coefficient and that all zeros of h are real and non-positive. Then $h \in S_+^d$.

Proof. The inclusions are obvious. Since $t \in S_+^2$ and $(1+t) \in S_+^1$ we may assume that none of them divides h. But then we may collect the zeros of h in pairs $\{\theta, \theta^{-1}\}$. Let $A_\theta = -\theta - \theta^{-1}$. Then

$$h = C \prod_{\theta < -1} (t^2 + A_\theta t + 1),$$

where $C > 0$. Since $A_\theta > 2$ we have

$$t^2 + A_\theta t + 1 = (t+1)^2 + (A_\theta - 2)t \in S_+^2,$$

and the lemma follows. $\square$

We can now prove our main theorem.

Theorem 4.2. *Suppose that (P, ω) is a sign-graded poset of rank r. Then $W(P, \omega; t) \in S_+^{p-r-1}$.*

Proof. By Corollary 2.4 and Lemma 2.5 we may assume that (P, ω) is canonically labeled. If Q extends P then the maximal elements of Q are also maximal elements of P. By Theorem 3.2 we know that

$$W(P, \omega; t) = \sum_Q W(Q, \omega; t),$$

where (Q, ω) is saturated and sign-graded with the same rank function and rank as (P, ω). The W-polynomials of anti-chains are the Eulerian polynomials, which have real nonnegative zeros only. By Propositions 3.3 and 3.4 the polynomial $W(Q, \omega; t)$ has only real non-positive zeros so by Lemma 4.1 and Corollary 2.4 we have $W(Q, \omega; t) \in S_+^{p-r-1}$. The theorem now follows since S_+^{p-r-1} is a cone. $\square$

Corollary 4.3. *Let (P,ω) be sign-graded of rank r. Then $W(P,\omega;t)$ is symmetric and its coefficients are unimodal. Moreover, $W(P,\omega;t)$ has the correct sign at -1, i.e.,*

$$(-1)^{(p-1-r)/2}W(P,\omega;-1) \geq 0.$$

Corollary 4.4. *Let P be a graded poset. Suppose that $\Delta_{eq}(P)$ is flag. Then the Charney-Davis conjecture holds for $\Delta_{eq}(P)$.*

Theorem 4.5. *Suppose that P is an ω-consistent poset and that $|\rho(x)-\rho(y)| \leq 1$ for all maximal elements $x, y \in P$. Then $W(P,\omega;t)$ has unimodal coefficients.*

Proof. Suppose that the ranks of maximal elements are contained in $\{r, r+1\}$. If Q is any saturated poset that extends P and has the same rank function as (P,ω) then Q is ω-graded of rank r or $r+1$. By Theorems 3.2 and 4.2 we know that

$$W(P,\omega;t) = \sum_{Q} W(Q,\omega;t),$$

where $W(Q,\omega;t)$ is symmetric and unimodal with center of symmetry at $(p-1-r)/2$ or $(p-2-r)/2$. The sum of such polynomials is again unimodal. □

5. The Charney-Davis quantity

In [9] Reiner, Stanton and Welker defined the *Charney-Davis quantity* of a graded naturally labeled poset (P,ω) of rank r to be

$$CD(P,\omega) = (-1)^{(p-1-r)/2}W(P,\omega;-1).$$

We define it in the exact same way for sign-graded posets. Since, by Corollary 2.4, the particular labeling does not matter we write $CD(P)$. Let $\pi = \pi_1\pi_2\cdots\pi_n$ be any permutation. We say that π is *alternating* if $\pi_1 > \pi_2 < \pi_3 > \cdots$ and *reverse alternating* if $\pi_1 < \pi_2 > \pi_3 < \cdots$. Let (P,ω) be a canonically labeled sign-graded poset. If $\pi \in \mathcal{L}(P,\omega)$ then we may write π as $\pi = w_0w_1\cdots w_k$ where w_i are maximal words with respect to the property: If a and b are letters of w_i then $\rho(\omega^{-1}(a)) = \rho(\omega^{-1}(b))$. The words w_i are called the *components* of π. The following theorem is well known, see for example [5, 11, 13], and gives the Charney-Davis quantity of an anti-chain.

Proposition 5.1. *Let $n \geq 0$ be an integer. Then $(-1)^{(n-1)/2}A_n(-1)$ is equal to 0 if n is even and equal to the number of (reverse) alternating permutations of the set $\{1,2,\dots,n\}$ if n is odd.*

Theorem 5.2. *Let (P,ω) be a canonically labeled sign-graded poset. Then the Charney-Davis quantity, $CD(P)$, is equal to the number of reverse alternating permutations in $\mathcal{L}(P,\omega)$ such that all components have an odd number of letters.*

Proof. It suffices to prove the theorem when (P,ω) is saturated. By Proposition 3.4 we know that

$$(P,\omega) = A_0 \oplus_1 A_1 \oplus_{-1} A_2 \oplus_1 A_3 \oplus_{-1} \cdots \oplus_{\pm 1} A_k,$$

where the A_is are anti-chains. Thus $CD(P) = CD(A_0)CD(A_1)\cdots CD(A_k)$. Let $\pi = w_0w_1\cdots w_k \in \mathcal{L}(P,\omega)$ where w_i is a permutation of $\omega(A_i)$. Then π is a reverse alternating permutation such that all components have an odd number of letters if and only if, for all i, w_i is reverse alternating if i is even and alternating if i is odd. Hence, by Proposition 5.1, the number of such permutations is indeed $CD(A_0)CD(A_1)\cdots CD(A_k)$. □

If $h(t)$ is any polynomial with integer coefficients and $h(t) \in S^d$, it follows that $h(t)$ has integer coefficients in the basis $t^i(1+t)^{d-2i}$. Thus we know that if (P,ω) is sign-graded of rank r, then

$$W(P,\omega;t) = \sum_{i=0}^{\lfloor (p-r-1)/2 \rfloor} a_i(P,\omega) t^i (1+t)^{p-r-1-2i},$$

where $a_i(P,\omega)$ are nonnegative integers. By Theorem 5.2 we have a combinatorial interpretation of the $a_{(p-r-1)/2}(P,\omega)$. A similar but more complicated interpretation of $a_i(P,\omega)$, $i = 0,1,\dots,\lfloor (p-r-1)/2 \rfloor$ can be deduced from Proposition 3.4 and the work in [5, 11]. We omit this.

6. The right mode

Let $f(x) = a_0 + a_1 x + \cdots + a_d x^d$ be a polynomial with real coefficients. The *mode*, $\mathrm{mode}(f)$, of f is the average value of the indices i such that $a_i = \max\{a_j\}_{j=0}^d$. One can easily compute the mode of a polynomial with real non-positive zeros only:

Theorem 6.1. [4] *Let f be a polynomial with real non-positive zeros only and with positive leading coefficient. Then*

$$\left| \frac{f'(1)}{f(1)} - \mathrm{mode}(f) \right| < 1.$$

It is known, see [2, 14, 17], that

$$W(P,\omega;x) = \sum_{i=1}^{p} e_i(P,\omega) x^{i-1}(1-x)^{p-i},$$

where $e_i(P,\omega)$ is the number of surjective (P,ω)-partitions $\sigma : P \to \{1,2,\dots,i\}$. A simple calculation gives

$$\frac{W'(P,\omega;1)}{W(P,\omega;1)} = p - 1 - \frac{e_{p-1}(P,\omega)}{e_p(P,\omega)}. \tag{6.1}$$

If P is ω-graded of rank r we know by Theorem 4.2 that $\mathrm{mode}(W(P,\omega;x)) = (p-r-1)/2$. The Neggers-Stanley conjecture, Theorem 6.1 and (6.1) suggest that $2e_{p-1}(P,\omega) = (p+r-1)e_p(P,\omega)$. Stanley [14] proved this for graded posets and it generalizes to sign-graded posets:

Proposition 6.2. *Let P be ω-graded of rank r. Then*

$$2e_{p-1}(P,\omega) = (p+r-1)e_p(P,\omega).$$

Proof. The identity follows when expanding $\Omega(P,\omega;t)$ in powers of t using Theorem 2.3. See [14, Corollary 19.4] for details. □

7. A characterization of sign-graded posets

Here we give a characterization of sign-graded posets along the lines of the characterization of graded posets given by Stanley in [14]. Let (P,ϵ) be a labeled poset. Define a function $\delta = \delta_\epsilon : P \to \mathbb{Z}$ by

$$\delta(x) = \max\{\sum_{i=1}^{\ell} \epsilon(x_{i-1}, x_i)\},$$

where $x = x_0 \prec x_1 \prec \cdots \prec x_\ell$ is any saturated chain starting at x and ending at a maximal element x_ℓ. Define a map $\Phi = \Phi_\epsilon : \mathcal{A}(\epsilon) \to \mathbb{Z}^P$ by

$$\Phi\sigma = \sigma + \delta.$$

We have

$$\delta(x) \geq \delta(y) + \epsilon(x, y). \tag{7.1}$$

This means that $\Phi\sigma(x) > \Phi\sigma(y)$ if $\epsilon(x,y) = 1$ and $\Phi\sigma(x) \geq \Phi\sigma(y)$ if $\epsilon(x,y) = -1$. Thus $\Phi\sigma$ is a $(P, -\epsilon)$-partition provided that $\Phi\sigma(x) > 0$ for all $x \in P$. But $\Phi\sigma$ is order reversing so it attains its minimum at maximal elements and for maximal elements, z, we have $\Phi\sigma(z) = \sigma(z)$. This shows that $\Phi : \mathcal{A}(\epsilon) \to \mathcal{A}(-\epsilon)$ is an injection.

The *dual*, (P^*, ϵ^*), of a labeled poset (P, ϵ) is defined by $x <_{P^*} y$ if and only if $y <_{P^*} x$, with labeling defined by $\epsilon^*(y, x) = -\epsilon(x, y)$. We say that P is *dual ϵ-consistent* if P^* is ϵ^*-consistent.

Proposition 7.1. *Let (P, ϵ) be labeled poset. Then $\Phi_\epsilon : \mathcal{A}(\epsilon) \to \mathcal{A}(-\epsilon)$ is a bijection if and only if P is dual ϵ-consistent.*

Proof. If P is dual ϵ-consistent then P is also dual $-\epsilon$-consistent and $\delta_{-\epsilon}(x) = -\delta_\epsilon(x)$ for all $x \in P$. Thus the if part follows since the inverse of Φ_ϵ is $\Phi_{-\epsilon}$.

For the only if direction note that P is dual ϵ-consistent if and only if for all $(x, y) \in E$ we have

$$\delta(x) = \delta(y) + \epsilon(x, y)$$

Hence, if P is not dual ϵ-consistent then by (7.1), there is a covering relation $(x_0, y_0) \in E$ such that either $\epsilon(x_0, y_0) = 1$ and $\delta(x_0) \geq \delta(y_0) + 2$ or $\epsilon(x_0, y_0) = -1$ and $\delta(x_0) \geq \delta(y_0)$.

Suppose that $\epsilon(x_0, y_0) = 1$. It is clear that there is a $\sigma \in \mathcal{A}(-\epsilon)$ such that $\sigma(x_0) = \sigma(y_0) + 1$. But then

$$\sigma(x_0) - \delta(x_0) \leq \sigma(y_0) - \delta(y_0) - 1,$$

so $\sigma - \delta \notin \mathcal{A}(\epsilon)$.

Similarly, if $\epsilon(x_0, y_0) = -1$ then we can find a partition $\sigma \in \mathcal{A}(-\epsilon)$ with $\sigma(x_0) = \sigma(y_0)$, and then

$$\sigma(x_0) - \delta(x_0) \leq \sigma(y_0) - \delta(y_0),$$

so $\sigma - \delta \notin \mathcal{A}(\epsilon)$. $\square$

Let (P, ϵ) be a labeled poset. Define $r(\epsilon)$ by

$$r(\epsilon) = \max\{\sum_{i=1}^{\ell} \epsilon(x_{i-1}, x_i) : x_0 \prec x_1 \prec \cdots \prec x_\ell \text{ is maximal}\}.$$

We then have:

$$\begin{aligned} \max\{\Phi\sigma(x) : x \in P\} &= \max\{\sigma(x) + \delta_\epsilon(x) : x \text{ is minimal}\} \\ &\leq \max\{\sigma(x) : x \in P\} + r(\epsilon). \end{aligned}$$

So if we let $\mathcal{A}_n(\epsilon)$ be the (P, ϵ)-partitions with largest part at most n we have that $\Phi_\epsilon : \mathcal{A}_n(\epsilon) \to \mathcal{A}_{n+r(\epsilon)}(-\epsilon)$ is an injection. A labeling ϵ of P is said to satisfy the *λ-chain condition* if for every $x \in P$ there is a maximal chain $c : x_0 \prec x_1 \prec \cdots \prec x_\ell$ containing x such that $\sum_{i=1}^{\ell} \epsilon(x_{i-1}, x_i) = r(\epsilon)$.

Lemma 7.2. *Suppose that n is a nonnegative integer such that $\Omega(P,\epsilon;n) \neq 0$. If*

$$\Omega(P,-\epsilon;n+r(\epsilon)) = \Omega(P,\epsilon;n)$$

then ϵ satisfies the λ-chain condition.

Proof. Define $\delta^* : P \to \mathbb{Z}$ by

$$\delta^*(x) = \max\{\sum_{i=1}^{\ell} \epsilon(x_{i-1},x_i) : x_0 \prec x_1 \prec \cdots \prec x_\ell = x\},$$

where the maximum is taken over all maximal chains starting at a minimal element and ending at x. Then

$$\delta(x) + \delta^*(x) \leq r(\epsilon) \tag{7.2}$$

for all x, and ϵ satisfies the λ-chain condition if and only if we have equality in (7.2) for all $x \in P$. It is easy to see that the map $\Phi^* : \mathcal{A}_n(\epsilon) \to \mathcal{A}_{n+r(\epsilon)}(-\epsilon)$ defined by

$$\Phi^*\sigma(x) = \sigma(x) + r(\epsilon) - \delta^*(x),$$

is well-defined and is an injection. By (7.2) we have $\Phi\sigma(x) \leq \Phi^*\sigma(x)$ for all σ and all $x \in P$, with equality if and only if x is in a maximal chain of maximal weight. This means that in order for $\Phi : \mathcal{A}_n(\epsilon) \to \mathcal{A}_{n+r(\epsilon)}(-\epsilon)$ to be a bijection it is necessary for ϵ to satisfy the λ-chain condition. $\square$

Theorem 7.3. *Let ϵ be a labeling of P. Then*

$$\Omega(P,\epsilon;t) = (-1)^p\Omega(P,\epsilon;-t-r(\epsilon))$$

if and only if P is ϵ-graded of rank $r(\epsilon)$.

Proof. The "if" part is Theorem 2.3, so suppose that the equality of the theorem holds. By reciprocity we have

$$(-1)^p\Omega(P,\epsilon;-t-r(\epsilon)) = \Omega(P,-\epsilon;t+r(\epsilon)),$$

and since $\Phi_\epsilon : \mathcal{A}_n(\epsilon) \to \mathcal{A}_{n+r(\epsilon)}(-\epsilon)$ is an injection it is also a bijection. By Proposition 7.1 we have that P is dual ϵ-consistent and by Lemma 7.2, we have that all minimal elements are members of maximal chains of maximal weight. In other words P is ϵ-graded. $\square$

It should be noted that it is not necessary for P to be ϵ-graded in order for $W(P,\epsilon;t)$ to be symmetric. For example, if (P,ϵ) is any labeled poset then the W-polynomial of the disjoint union of (P,ϵ) and $(P,-\epsilon)$ is easily seen to be symmetric. However, we have the following:

Corollary 7.4. *Suppose that*

$$\Omega(P,\epsilon;t) = \Omega(P,-\epsilon;t+s),$$

for some $s \in \mathbb{Z}$. Then $-r(-\epsilon) \leq s \leq r(\epsilon)$, with equality if and only if P is ϵ-graded.

Proof. We have an injection $\Phi_\epsilon : \mathcal{A}_n(\epsilon) \to \mathcal{A}_{n+r(\epsilon)}(-\epsilon)$. This means that $s \leq r(\epsilon)$. The lower bound follows from the injection $\Phi_{-\epsilon}$, and the statement of equality follows from Theorem 7.3. $\square$

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ON OPERATORS ON POLYNOMIALS PRESERVING REAL-ROOTEDNESS AND THE NEGGERS-STANLEY CONJECTURE

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Abstract. We refine a technique used in a paper by Schur on real-rooted polynomials. This amounts to an extension of a theorem of Wagner on Hadamard products of Pólya frequency sequences. We also apply our results to polynomials for which the Neggers-Stanley Conjecture is known to hold. More precisely, we settle interlacing properties for E-polynomials of series-parallel posets and column-strict labelled Ferrers posets.

1. Introduction

Several polynomials associated to combinatorial structures are known to have real zeros. Often one can say more about the location of the zeros than just that they are on the real axis. The *matching polynomial* of a graph is not only real-rooted, but it is known that the matching polynomial of the graph obtained by deleting a vertex of G interlaces that of G [5]. The same is true for the *characteristic polynomial* of graph (see e.g., [3]). If A is a nonnegative matrix and A' is the matrix obtained by either deleting a row or a column, then the *rook polynomial* of A' interlaces that of A (see [5, 8]).

The Neggers-Stanley Conjecture asserts that certain polynomials associated to posets, see Section 3, have real zeros; see [1, 10, 14] for the state of the art. For classes of posets for which the conjecture is known to hold we will exhibit explicit interlacing relationships.

The first part of this paper is concerned with operators on polynomials which preserve real-rootedness. The following classical theorem is due to Schur [11]:

Theorem 1 (Schur). *Let $f = a_0 + a_1x + \cdots + a_nx^n$ and $g = b_0 + b_1x + \cdots + b_mx^m$ be polynomials in $\mathbb{R}[x]$. Suppose that f and g have only real zeros and that the zeros of g are all of the same sign. Then the polynomial*

$$f \odot g := \sum_k k!a_kb_kx^k,$$

has only real zeros. If $a_0b_0 \neq 0$ then all the zeros of $f \odot g$ are distinct.

In this paper we will refine the technique used in Schur's proof of the theorem to extend a theorem of Wagner [15, Theorem 0.3]. The *diamond product* of two polynomials f and g is the polynomial

$$f\diamond g = \sum_{n\geq 0} \frac{f^{(n)}(x)}{n!}\frac{g^{(n)}(x)}{n!}x^n(x+1)^n.$$

Here $f^{(n)}(x)$ denotes the nth derivative of $f(x)$. Brenti [1] conjectured an equivalent form of Theorem 2 and Wagner proved it in [15, Theorem 0.3].

Theorem 2 (Wagner). *If $f, g \in \mathbb{R}[x]$ have all their zeros in the interval $[-1, 0]$ then so does $f\diamond g$.*

This theorem has important consequences in combinatorics [14], and it also has implications to the theory of total positivity [15]. Namely, that if $\{f(i)\}_{i=0}^{\infty}$ and $\{g(i)\}_{i=0}^{\infty}$ are Pólya frequency sequences where f and g are polynomials, then the sequence $\{f(i)g(i)\}_{i=0}^{\infty}$ is also a Pólya frequency sequence. This is not true when the requirement that f and g should be polynomials is dropped.

In this paper we will refine the technique used in Schur's proof of Theorem 1 to extend Theorem 2 as follows:

Theorem 3. *Let h be $[-1,0]$-rooted and let f be real-rooted.*

(a) *Then $f\diamondsuit h$ is real-rooted, and if $g \preceq f$ then*
$$g\diamondsuit h \preceq f\diamondsuit h.$$

(b) *If h is $(-1,0)$- and simple-rooted and f is simple-rooted then $f\diamondsuit h$ is simple-rooted and*
$$g\diamondsuit h \prec f\diamondsuit h,$$
for all $g \prec f$.

Here, the symbols $\preceq$ and $\prec$ denotes the interlacing- and the strict interlacing property, respectively (see Section 2 for the precise definition). Theorem 2 thus follows from part (a) of Theorem 14 since the hypotheses is weaker (we don't require both polynomials to be $[-1,0]$-rooted) and the conclusion stronger.

In the second part of the paper we settle interlacing properties for E-polynomials of series-parallel posets and column-strict labelled Ferrers posets.

We will implicitly use the fact that the zeros of a polynomial are continuous functions of the coefficients of the polynomial. In particular, the limit of real-rooted polynomials will again be real-rooted. For a treatment of these matters we refer the reader to [7].

2. Sturm sequences and linear operators preserving real-rootedness

Let f and g be real polynomials. We say that f and g *alternate* if f and g are real-rooted and either of the following conditions hold:

(A) $\deg(g) = \deg(f) = d$ and
$$\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \cdots \leq \beta_{d-1} \leq \alpha_d \leq \beta_d,$$
where $\alpha_1 \leq \cdots \leq \alpha_d$ and $\beta_1 \leq \cdots \leq \beta_d$ are the zeros of f and g respectively

(B) $\deg(f) = \deg(g) + 1 = d$ and
$$\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \cdots \leq \beta_{d-1} \leq \alpha_d$$
where $\alpha_1 \leq \cdots \leq \alpha_d$ and $\beta_1 \leq \cdots \leq \beta_{d-1}$ are the zeros of f and g respectively.

If all the inequalities above are strict then f and g are said to *strictly alternate.* Moreover, if f and g are as in (B) then we say that g *interlaces* f, denoted $g \preceq f$. In the strict case we write $g \prec f$. If the leading coefficient of f is positive we say that f is *standard.*

For $z \in \mathbb{R}$ let $T_z : \mathbb{R}[x] \to \mathbb{R}[x]$ be the translation operator defined by $T_z(f(x)) = f(x+z)$. For any linear operator $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ we define a linear transform $\mathcal{L}_\phi : \mathbb{R}[x] \to \mathbb{R}[x,z]$ by
$$\begin{aligned}\mathcal{L}_\phi(f) &:= \phi(T_z(f)) \\ &= \sum_n \phi(f^{(n)})(x)\frac{z^n}{n!} \qquad (1) \\ &= \sum_n \frac{\phi(x^n)}{n!} f^{(n)}(z).\end{aligned}$$

Definition 4. Let $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. Define a function $d_\phi : \mathbb{R}[x] \to \mathbb{N} \cup \{-\infty\}$ by the following. If $\phi(f^{(n)}) = 0$ for all $n \in \mathbb{N}$, let $d_\phi(f) = -\infty$. Otherwise, let $d_\phi(f)$ be the smallest integer d such that $\phi(f^{(n)}) = 0$ for all $n > d$. Note that $d_\phi(f) \leq \deg(f)$ for all $f \in \mathbb{R}[x]$.

The set $\mathscr{A}^+(\phi) \subset \mathbb{R}[x]$ is defined as follows: If $d_\phi(f) = -\infty$, or $d_\phi(f) = 0$ and $\phi(f)$ is standard real- and simple-rooted, then $f \in \mathscr{A}^+(\phi)$. Moreover, $f \in \mathscr{A}^+(\phi)$ if $d = d_\phi(f) \geq 1$ and all of the following conditions are satisfied:

(i) $\phi(f^{(i)})$ is standard for all i and $\deg(\phi(f^{(i-1)})) = \deg(\phi(f^{(i)})) + 1$ for $1 \leq i \leq d$,
(ii) $\phi(f)$ and $\phi(f')$ have no common real zero,
(iii) $\phi(f^{(d)}) \prec \phi(f^{(d-1)})$,
(iv) for all $\xi \in \mathbb{R}$ the polynomial $\mathcal{L}_\phi(f)(\xi, z)$ is real-rooted.

Let $\mathscr{A}^-(\phi) := \{-f : f \in \mathscr{A}^+(\phi)\}$ and $\mathscr{A}(\phi) := \mathscr{A}^-(\phi) \cup \mathscr{A}^+(\phi)$.

Example 5. If ϕ is the identity operator then $\mathscr{A}(\phi)$ is just the set of polynomials in $\mathbb{R}[x]$ with real and simple zeros only.

The following theorem is the basis for our analysis:

Theorem 6. *Let $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. If $f \in \mathscr{A}(\phi)$ then $\phi(f)$ is real- and simple-rooted and if $d_\phi(f) \geq 1$ we have*

$$\phi(f^{(d)}) \prec \phi(f^{(d-1)}) \prec \cdots \prec \phi(f') \prec \phi(f).$$

Before we give a proof of Theorem 6 we will need a couple of lemmas. Note that $\frac{\partial}{\partial z}\mathcal{L}_\phi(f) = \mathcal{L}_\phi(f')$ so by Rolle's Theorem we know that $\mathcal{L}_\phi(f')$ is real-rooted (in z) if $\mathcal{L}_\phi(f)$ is. By Theorem 6 it follows that $\mathscr{A}(\phi)$ is closed under differentiation. A (generalised) *Sturm sequence* is a sequence $f_0, f_1, \dots, f_n$ of standard polynomials such that $\deg(f_i) = i$ for $0 \leq i \leq n$ and

$$f_{i-1}(\theta) f_{i+1}(\theta) < 0, \tag{2}$$

whenever $f_i(\theta) = 0$ and $1 \leq i \leq n-1$. If f is a standard polynomial with real simple zeros, we know from Rolle's Theorem that the sequence $\{f^{(i)}\}_i$ is a Sturm sequence. The following lemma is folklore.

Lemma 7. *Let $f_0, f_1, \dots, f_n$ be a sequence of standard polynomials with $\deg(f_i) = i$ for $0 \leq i \leq n$. Then the following statements are equivalent:*

(i) *$f_0, f_1, \dots, f_n$ is a Sturm sequence,*
(ii) *$f_0 \prec f_1 \prec \cdots \prec f_n$.*

The next lemma is of interest for real-rooted polynomials encountered in combinatorics.

Lemma 8. *Let $a_m x^m + a_{m+1}x^{m+1} + \cdots + a_n x^n \in \mathbb{R}[x]$ be real-rooted with $a_m a_n \neq 0$. Then the sequence a_i is strictly log-concave, i.e.,*

$$a_i^2 > a_{i-1}a_{i+1}, \qquad (m+1 \leq i \leq n-1).$$

Proof. See Lemma 3 on page 337 of [6]. □

Proof of Theorem 6. Let $f \in \mathscr{A}^+(\phi)$. Clearly we may assume that $d = d_\phi(f) > 1$. We claim that for $1 \leq n \leq d-1$:

$$\phi(f^{(n)})(\theta) = 0 \implies \phi(f^{(n-1)})(\theta)\phi(f^{(n+1)})(\theta) < 0. \tag{3}$$

If $1 \leq n \leq d-1$ and $\phi(f^{(n)})(\theta) = 0$, then by condition (ii) and (iii) of Definition 4 we have that there are integers $0 \leq \ell < n < k \leq d$ with $\phi(f^{(\ell)})(\theta)\phi(f^{(k)})(\theta) \neq 0$. By Lemma 8 and the real-rootedness of $\mathcal{L}_\phi(f)(\theta, z)$ this verifies (3).

If $\phi(f^{(d)})$ is a constant then $\{\phi(f^{(n)})\}_n$ is a Sturm sequence. Otherwise let $g = \phi(f^{(d)})$. Then, since $g' \prec g \prec \phi(f^{(d-1)})$, we have that (2) is satisfied everywhere in the sequence $\{g^{(n)}\}_n \cup \{\phi(f^{(n)})\}_n$. This proves the theorem by Lemma 7. $\square$

In order to make use of Theorem 6 we will need further results on real-rootedness and interlacings of polynomials. There is a characterisation of alternating polynomials due to Obreschkoff and Dedieu. Obreschkoff proved the case of strictly alternating polynomials, see [9, Satz 5.2], and Dedieu [2] generalised it in the case $\deg(f) = \deg(g)$. But his proof also covers this slightly more general theorem:

Theorem 9. *Let f and g be real polynomials. Then f and g alternate (strictly alternate) if and only if all polynomials in the space*

$$\{\alpha f + \beta g : \alpha, \beta \in \mathbb{R}\}$$

are real-rooted (real- and simple-rooted).

A direct consequence of Theorem 9 is the following theorem.

Theorem 10. *If $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ is a linear operator preserving real-rootedness, then $\phi(f)$ and $\phi(g)$ alternate if f and g alternate. Moreover, if ϕ preserves real- and simple-rootedness then $\phi(f)$ and $\phi(g)$ strictly alternate if f and g strictly alternate.*

Proof. The theorem is an immediate consequence of Theorem 9 since the concept of alternating zeros is translated into a linear condition. $\square$

Lemma 11. *Let $0 \neq h, f, g \in \mathbb{R}[x]$ be standard and real-rooted. If $h \prec f$ and $h \prec g$, then $h \prec \alpha f + \beta g$ for all $\alpha, \beta \geq 0$ not both equal to zero.*

Note that Lemma 11 also holds (by continuity arguments) when all instances of $\prec$ are replaced by $\preceq$ in Lemma 11.

Proof. If θ is a zero of h then clearly $\alpha f + \beta g$ has the same sign as f and g at θ. Since $\{h^{(i)}\}_i \cup \{f\}$ is a Sturm sequence by Lemma 7, so is $\{h^{(i)}\}_i \cup \{\alpha f + \beta g\}$. By Lemma 7 again the proof follows. $\square$

We will need two classical theorems on real-rootedness. The first theorem is essentially due to Hermite and Poulain and the second is due to Laguerre.

Theorem 12 (Hermite, Poulain)**.** *Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ and g be real-rooted. Then the polynomial*

$$f(\frac{d}{dx})g := a_0 g(x) + a_1 g'(x) + \cdots + a_n g^{(n)}(x)$$

is real-rooted. Moreover, if $x^N \nmid f$ and $\deg(g) \geq N-1$ then any multiple zero of $f(\frac{d}{dx})g$ is a multiple zero of g.

Proof. The case $N = 1$ is the Hermite-Poulain theorem. A proof can be found in any of the references [6, 9, 11]. For the general result it will suffice to prove that if $\deg(g) \neq 0$ then any multiple zero of g' is a multiple zero of g. Let

$$g = c_0 + c_1(x-\theta) + \cdots + c_M(x-\theta)^M,$$

where $c_M \neq 0$, $M > 0$ and $(x-\theta)^2 | g'$. Then $c_1 = c_2 = 0$ and $M > 2$. If $c_0 = 0$ we are done and if $c_0 \neq 0$ we have by Lemma 8 that $0 = c_1^2 > c_0 c_2 = 0$, which is a contradiction. $\square$

Theorem 13 (Laguerre)**.** *If $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is real-rooted then so is*

$$a_0 + a_1 x + \frac{a_2}{2!}x^2 + \cdots + \frac{a_n}{n!}x^n.$$

For a proof of Theorem 13, see [6, 11]. We are now in a position to extend Theorem 2.

Theorem 14. *Let h be $[-1,0]$-rooted and let f be real-rooted.*

(a) *Then $f\diamond h$ is real-rooted, and if $g \preceq f$ then*

$$g\diamond h \preceq f\diamond h.$$

(b) *If h is $(-1,0)$- and simple-rooted and f is simple-rooted then $f\diamond h$ is simple-rooted and*

$$g\diamond h \prec f\diamond h,$$

for all $g \prec f$.

Proof. First we assume that $\deg(h) > 0$ and that h is standard, $(-1,0)$-rooted and has simple zeros. Let $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ be the linear operator defined by $\phi(f) = f\diamond h$.

We will show that $f \in \mathscr{A}^+(\phi)$ if f is standard real- and simple-rooted. Clearly we may assume that $\deg(f) = d \geq 1$. Condition (i) of Definition 4 follows immediately from the definition of the diamond product. Now, $f^{(d-1)} = ax + b$, where $a, b \in \mathbb{R}$ and $a > 0$ so

$$\begin{aligned} \phi(f^{(d)}) &= ah \quad \text{and} \\ \phi(f^{(d-1)}) &= (ax+b)h + ax(x+1)h', \end{aligned}$$

and since $h \preceq (ax+b)h$ and $h \preceq x(x+1)h'$ we have by the discussion following Lemma 11 that $h \preceq \phi(f^{(d-1)})$. If θ is a common zero of h and $\phi(f^{(d-1)})$, then $\theta(\theta+1)h'(\theta) = 0$, which is impossible since $\theta \in (-1,0)$ and $h'(\theta) \neq 0$. Thus $\phi(f^{(d)}) \prec \phi(f^{(d-1)})$, which verifies condition (iii) of Definition 4. Given $\xi \in \mathbb{R}$ we have

$$\begin{aligned} \mathcal{L}_\phi(f)(\xi, z) &= \sum_n \frac{h^{(n)}(\xi)}{n!n!}\xi^n(\xi+1)^n \frac{d^n f(\xi+z)}{dz^n} \\ &= H_\xi(\frac{d}{dz})f(\xi+z), \end{aligned}$$

where

$$H_\xi(x) = \sum_n \frac{h^{(n)}(\xi)}{n!n!}\{\xi(\xi+1)x\}^n.$$

By Theorem 13 H_ξ is real-rooted, which by Theorem 12 verifies condition (iv).

Suppose that ξ is a common zero of $\phi(f')$ and $\phi(f)$. From the definition of the diamond product it follows that $\xi \notin \{0,-1\}$, so $x^2 \nmid H_\xi(x)$. Since ξ is supposed to be a common zero of $\phi(f')$ and $\phi(f)$ we have, by (1), that 0 is a multiple zero of $\mathcal{L}_\phi(f)(\xi,z)$. It follows from Theorem 12 that 0 is a multiple zero of $f(z+\xi)$, that is, ξ is a multiple zero of f, contrary to assumption that f is simple-rooted. This verifies condition (ii), and we can conclude that $f \in \mathscr{A}^+(\phi)$. Part (b) of the theorem now follows from Theorem 10.

If h is merely $[-1,0]$-rooted and f is real-rooted then we can find polynomials h_n and f_n whose limits are h and f respectively, such that h_n and f_n are real- and simple-rooted and h_n is $(-1,0)$-rooted. Now, $f_n\diamond h_n$ is real-rooted by the above and, by continuity, so is $f\diamond g$. The proof now follows from Theorem 10. $\square$

There are many products on polynomials for which a similar proof applies. With minor changes in the above proof, Theorem 14 also holds for the product

$$(f,g) \to \sum_{n\geq 0} \frac{f^{(n)}(x)g^{(n)}(x)}{n!}x^n(x+1)^n.$$

3. Interlacing zeros and the Neggers-Stanley Conjecture

Let P be any finite poset of cardinality p. An injective function $\omega : P \to \mathbb{N}$ is called a *labelling* of P and (P, ω) is a called a *labelled poset.* A (P, ω)-partition with largest part $\leq n$ is a map $\sigma : P \to [n]$ such that

- σ is order reversing, that is, if $x \leq y$ then $\sigma(x) \geq \sigma(y)$,
- if $x < y$ and $\omega(x) > \omega(y)$ then $\sigma(x) > \sigma(y)$.

The number of (P, ω)-partitions with largest part $\leq n$ is denoted $\Omega(P, \omega, n)$ and is easily seen to be a polynomial in n. Indeed, if we let $e_k(P, \omega)$ be the number of surjective (P, ω)-partitions $\sigma : P \to [k]$, then by a simple counting argument we have:

$$\Omega(P, \omega, x) = \sum_{k=1}^{|P|} e_k(P, \omega) \binom{x}{k}. \tag{4}$$

The polynomial $\Omega(P, \omega, x)$ is called the *order polynomial* of (P, ω). The *E-polynomial* of (P, ω) is the polynomial

$$E(P, \omega) = \sum_{k=1}^{p} e_k(P, \omega) x^k,$$

so $E(P, \omega)$ is the image of $\Omega(P, \omega, x)$ under the invertible linear operator $\mathcal{E} : \mathbb{R}[x] \to \mathbb{R}[x]$ which takes $\binom{x}{k}$ to x^k.

The Neggers-Stanley Conjecture asserts that the polynomial $E(P, \omega)$ is real-rooted for all choices of P and ω. The conjecture has been verified for series-parallel posets [14], column-strict labelled Ferrers posets and Gaussian posets [1] and for all labelled posets having at most seven elements.

There are two operations on labelled posets under which E-polynomials behave well. The first operation is the *ordinal sum*:

Let (P, ω) and (Q, ν) be two labelled posets. The *ordinal sum*, $P \oplus Q$, of P and Q is the poset with the disjoint union of P and Q as underlying set and with partial order defined by $x \leq y$ if either $x \leq_P y$, $x \leq_Q y$, or $x \in P, y \in Q$. For $i = 0, 1$ let $\omega \oplus_i \nu$ be any labellings of $P \oplus Q$ such that

- $(\omega \oplus_0 \nu)(x) < (\omega \oplus_0 \nu)(y)$ if $\omega(x) < \omega(y)$, $\nu(x) < \nu(y)$ or $x \in P, y \in Q$.
- $(\omega \oplus_1 \nu)(x) < (\omega \oplus_1 \nu)(y)$ if $\omega(x) < \omega(y)$, $\nu(x) < \nu(y)$ or $x \in Q, y \in P$.

The following result follows easily by combinatorial reasoning:

Proposition 15. *Let (P, ω) and (Q, ν) be as above. Then*

$$E(P \oplus Q, \omega \oplus_1 \nu) = E(P, \omega) E(Q, \nu)$$

and

$$xE(P \oplus Q, \omega \oplus_0 \nu) = (x + 1) E(P, \omega) E(Q, \nu),$$

if P and Q are nonempty.

Proof. See [1, 14]. □

The *disjoint union*, $P \sqcup Q$, of P and Q is the poset on the disjoint union with $x < y$ in $P \sqcup Q$ if and only if $x <_P y$ or $x <_Q y$. Let $\omega \sqcup \nu$ be any labelling of $P \sqcup Q$ such that

$$(\omega \sqcup \nu)(x) < (\omega \sqcup \nu)(y),$$

if $\omega(x) < \omega(y)$ or $\nu(x) < \nu(y)$. It is immediate by construction that

$$\Omega(P \sqcup Q, \omega \sqcup \nu) = \Omega(P, \omega) \Omega(Q, \nu)$$

Here is where the diamond product comes in. Wagner [14] showed that the diamond product satisfies

$$f \diamondsuit g = \mathcal{E}(\mathcal{E}^{-1}(f) \mathcal{E}^{-1}(g)), \tag{5}$$

which implies:

$$E(P \sqcup Q, \omega \sqcup \nu) = E(P, \omega) \diamond E(Q, \nu), \tag{6}$$

for all pairs of labelled posets (P, ω) and (Q, ν).

If P is nonempty and $x \in P$ we let $P \setminus x$ be the poset on $P \setminus \{x\}$ with the order inherited by P. If (P, ω) is labelled then $P \setminus x$ is labelled with the restriction of ω to $P \setminus x$. By a slight abuse of notation we will write $(P \setminus x, \omega)$ for this labelled poset.

A series-parallel labelled poset (S, μ) is either the empty poset, a one element poset or

(a) $(S, \mu) = (P \oplus Q, \omega \oplus_0 \nu)$,
(b) $(S, \mu) = (P \oplus Q, \omega \oplus_1 \nu)$ or
(c) $(S, \mu) = (P \sqcup Q, \omega \sqcup \nu)$

where (P, ω) and (Q, ν) are series-parallel. Note that if (S, μ) is series-parallel then so is $(S \setminus x, \mu)$ for all $x \in S$. Let $\mathscr{I}$ denote the class of finite labelled posets (S, μ) such that $E(S, \mu)$ is real-rooted and

$$E(S \setminus x, \mu) \preceq E(S, \mu),$$

for all $x \in S$. Note that the empty poset and the singleton posets are members of $\mathscr{I}$ which by the following theorem gives that series-parallel posets are in $\mathscr{I}$.

Theorem 16. *The class $\mathscr{I}$ is closed under ordinal sum and disjoint union.*

Proof. Suppose that $(P, \omega), (Q, \nu) \in \mathscr{I}$.
(a): Let $(S, \mu) = (P \oplus Q, \omega \oplus_0 \nu)$. Now, if $y \in P$ we have

$$(S \setminus y, \mu) = (P \setminus y \oplus Q, \omega \oplus_0 \nu).$$

If $|P| = 1$ then by Proposition 15 we have $E(S \setminus y, \mu) = E(Q, \nu)$ and $E(S, \mu) = (x+1)E(Q, \nu)$ so $E(S \setminus y, \mu) \preceq E(S, \mu)$. If $|P| > 1$ then

$$\begin{aligned} xE(S \setminus y, \mu) &= (x+1)E(P \setminus y, \omega)E(Q, \nu) \\ &\preceq (x+1)E(P, \omega)E(Q, \nu) \\ &= xE(S, \mu), \end{aligned}$$

which gives $E(S \setminus y, \mu) \preceq E(S, \mu)$. A similar argument applies to the case $y \in Q$.
(b): The case $(S, \mu) = (P \oplus Q, \omega \oplus_0 \nu)$ follows as in (a).
(c): $(S, \mu) = (P \sqcup Q, \omega \sqcup \nu)$. If $y \in P$ we have by (6) and Theorem 14:

$$\begin{aligned} E(S \setminus y, \mu) &= E(P \setminus y \sqcup Q, \omega \sqcup \nu) \\ &= E(P \setminus y, \omega) \diamond E(Q, \nu) \\ &\preceq E(P, \omega) \diamond E(Q, \nu) \\ &= E(S, \mu). \end{aligned}$$

This proves the theorem. $\square$

In [12] Simion proved a special case of the following corollary. Namely the case when S is a disjoint union of chains and μ is order-preserving.

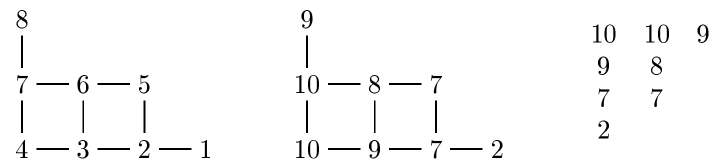
Corollary 17. *If (S, μ) is series-parallel and $x \in S$ then*

$$E(S \setminus x, \mu) \preceq E(S, \mu).$$

Next we will analyse interlacings of E-polynomials of Ferrers posets. For undefined terminology in what follows we refer the reader to [13, Chapter 7]. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ be a partition. The *Ferrers poset* P_λ is the poset

$$P_\lambda = \{(i, j) \in \mathbb{P} \times \mathbb{P} : 1 \leq i \leq \ell, 1 \leq j \leq \lambda_i\},$$

FIGURE 1. From left to right: A column-strict labelling ω of P_λ with $\lambda = (3,2,2,1)$, a (P_λ, ω)-partition and the corresponding reverse SSYT.



ordered by the standard product ordering. A labelling ω of P_λ is *column strict* if $\omega(i,j) > \omega(i+1,j)$ and $\omega(i,j) < \omega(i,j+1)$ for all $(i,j) \in P_\lambda$. If ω is a column strict labelling then any (P_λ, ω)-partition must necessarily be strictly decreasing in the x-direction and weakly decreasing in the y-direction.

It follows that the (P_λ, ω)-partitions are in a one-to-one correspondence with with the reverse SSYT's of shape λ (see Figure 1). The number of reverse SSYT's of shape λ with largest part $\leq n$ is by the combinatorial definition of the Schur function equal to $s_\lambda(1^n)$ which by the hook-content formula [13, Corollary 7.21.4] gives us.

$$\Omega(P_\lambda, \omega, z) = \prod_{u \in P_\lambda} \frac{z + c_\lambda(u)}{h_\lambda(u)}, \tag{7}$$

where for $u = (x,y) \in P_\lambda$

$$h_\lambda(u) := |\{(x,j) \in \lambda : j \geq y\}| + |\{(i,y) \in \lambda : i \geq x\}| - 1$$

and $c_\lambda(u) := y - x$ are the *hook length* respectively *content* at u. In [1] Brenti showed that the E-polynomials of column strict labelled Ferrers posets are real-rooted. In the next theorem we refine this result. If $x < y$ in a poset P and $x < z < y$ for no $z \in P$ we say that y *covers* x. If we remove an element from P_λ the resulting poset will not necessarily be a Ferrers poset. But if we remove a maximal element m from P_λ we will have $P_\lambda \setminus m = P_\mu$ for a partition μ covered by λ in the Young's lattice.

Theorem 18. *Let (P_λ, ω) be labelled column strict. Then $E(P_\lambda, \omega)$ is real-rooted. Moreover, if λ covers μ in the Young's lattice, then*

$$E(P_\mu, \omega) \preceq E(P_\lambda, \omega).$$

Proof. The proof is by induction over n, where $\lambda \vdash n$. It is trivially true for $n = 1$. If $\lambda \vdash n+1$ and λ covers μ we have that $P_\lambda = P_\mu \cup \{m\}$ for some maximal element $m \in P_\lambda$. By definition $c_\mu(u) = c_\lambda(u)$ for all $u \in P_\mu$, so by (7) we have that for some $C > 0$:

$$\Omega(P_\lambda, \omega, x) = C(x + c_\lambda(m))\Omega(P_\mu, \omega, x),$$

and by (5):

$$E(P_\lambda, \omega) = C(x + c_\lambda(m)) \Diamond E(P_\mu, \omega).$$

Wagner [14] showed that all real zeros of E-polynomials are necessarily in $[-1, 0]$, so by induction we have that $E(P_\mu, \omega)$ is $[-1,0]$-rooted. By Theorem 14 this suffices to prove the theorem. $\square$

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ON LINEAR TRANSFORMATIONS PRESERVING THE PÓLYA FREQUENCY PROPERTY

PETTER BRÄNDÉN

ABSTRACT. We prove that certain linear operators preserve the Pólya frequency property and real-rootedness. The results attained are applied to settle some conjectures and open problems in combinatorics proposed by Bóna, Brenti and Reiner-Welker.

1. INTRODUCTION

Many sequences encountered in various areas of mathematics, statistics and computer science are known or conjectured to be unimodal or log-concave, see [7, 30, 32]. A sufficient condition for a sequence to enjoy these properties is that it is a Pólya frequency (PF for short) sequence, or equivalently for finite sequences, that its generating function has only real and non-positive zeros. It is often the case that the generating function of a finite PF-sequence has more transparent properties when expanded in a basis other than the standard basis $\{x^i\}_{i\geq 0}$ of $\mathbb{R}[x]$. Therefore it is natural to investigate how PF-sequences translate when expressed in various basis. This amounts to studying properties of the linear operator that maps one basis to another. A systematic study of this was first pursued by Brenti in [6]. This is also the theme of this paper.

In Section 3 we will study linear operators of the type

$$\phi_F = \sum_{k=0}^{n} Q_k(x)\frac{d^k}{dx^k},$$

where $F(x,z) = \sum_{k=0}^{n} Q_k(x)z^k \in \mathbb{R}[x,z]$. Here we will give sufficient conditions on F for ϕ_F to preserve the PF-property. The results attained generalizes and unifies theorems of Hermite, Poulain, Pólya and Schur. We will also in this section give a sufficient condition for a family of natural $\mathbb{R}$-bilinear forms to preserve the PF-property in both arguments. This generalizes results of Wagner [10, 35, 36].

An important linear operator in combinatorics is the operator defined by $\mathcal{E}(\binom{x}{i}) = x^i$, for all $i \in \mathbb{N}$. In Section 4 we will prove that whenever a polynomial f of degree d has nonnegative coefficients when expanded in the basis $\{x^i(x+1)^{d-i}\}_{i=0}^{d}$ the polynomial $\mathcal{E}(f)$ will have only real, non-positive and simple zeros.

In the remainder of the paper we use the theory developed to settle some conjectures and open problems raised in combinatorics. In Section 5 we prove that the numbers $\{W_t(n,k)\}_{k=0}^{n-1}$ of t-stack sortable permutations in $\mathcal{S}_n$ with k descents

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form PF-sequences when $t = 2, n-2$, and thereby settling two new cases of an open problem proposed by Bóna [2, 3].

In Section 6 we prove that the q-Eulerian polynomials, $A_n(x;q)$, defined by Foata and Schützenberger [16] and further studied by Brenti in [9] have only real zeros for all integers q. This settles a conjecture raised by Brenti. Here we also continue the study of the W-Eulerian polynomials, defined for any finite Coxeter group W and the q-analog $B_n(x;q)$, initiated by Brenti in [8].

In Section 7 we prove that the h-vectors of a family simplicial complexes associated to finite Weyl groups defined by Fomin and Zelevinski [17] are PF, thus settling an open problem raised by Reiner and Welker [28].

2. Notation and preliminaries

In this section we collect definitions, notation and results that will be used frequently in the rest of the paper. Let $\{a_i\}_{i=0}^\infty$ be a sequence of real numbers. It is *unimodal* if there is a number p such that $a_0 \leq a_1 \leq \cdots \leq a_p \geq a_{p+1} \geq \cdots$, and *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $i > 0$.

An infinite matrix $A = (a_{ij})_{i,j\geq 0}$ of real numbers is *totally positive*, TP, if all minors of A are nonnegative. An infinite sequence $\{a_i\}_{i=0}^\infty$ of real numbers is a *Pólya frequency sequence*, PF-sequence, if the matrix $(a_{i-j})_{i,j\geq 0}$ is TP. Thus a PF-sequence is by definition log-concave and therefore also unimodal. A finite sequence $a_0, a_1, a_2, \dots, a_n$ is said to be PF if the infinite sequence $a_0, a_1, a_2, \dots, a_n, 0, 0, \dots$ is PF. A sequence $\{a_i\}_{i=0}^\infty$ is said to be PF_r if all minors of size at most r of $(a_{i-j})_{i,j\geq 0}$ are nonnegative. If the polynomials $\{b_i(x)\}_{i=0}^d$ are linearly independent over $\mathbb{R}$ and $r \in \mathbb{N}$ we define the set $PF_r[\{b_i(x)\}_{i=0}^d]$ to be

$$PF_r[\{b_i(x)\}_{i=0}^d] = \{\sum_{i=0}^d \lambda_i b_i(x) : \{\lambda_i\}_{i=0}^\infty \text{ is } PF_r\},$$

and $PF[\{b_i(x)\}_{i=0}^d] = \bigcap_{r=1}^\infty PF_r[\{b_i(x)\}_{i=0}^d]$.

The following theorem characterizes PF-sequences. It was conjectured by Schoenberg and proved by Edrei [15], see also [22].

Theorem 2.1. *Let $\{a_i\}_{i=0}^\infty$ be a sequence of real numbers with $a_0 = 1$. Then it is a PF-sequence if and only if the generating function can be expanded, in a neighborhood of the origin, as*

$$\sum_{i\geq 0} a_i z^i = e^{\gamma z} \frac{\prod_{i\geq 0}(1+\alpha_i z)}{\prod_{i\geq 0}(1-\beta_i z)},$$

where $\gamma \geq 0$, $\alpha_i, \beta_i > 0$ and $\sum_{i\geq 0}(\alpha_i + \beta_i) < \infty$.

A consequence of this theorem is that a finite sequence is PF if and only if its generating function is a polynomial with only real non positive zeros.

Let $f, g \in \mathbb{R}[x]$ be real-rooted with zeros: $\alpha_1 \leq \cdots \leq \alpha_i$ and $\beta_1 \leq \cdots \leq \beta_j$, respectively. We say that f *interlaces* g, denoted $f \preceq g$, if $j = i+1$ and

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \cdots \leq \beta_{j-1} \leq \alpha_{j-1} \leq \beta_j.$$

We say that f *alternates left of* g, denoted $f \ll g$, if $i = j$ and

$$\alpha_1 \leq \beta_1 \leq \cdots \leq \beta_{i-1} \leq \alpha_i \leq \beta_i.$$

If in addition f and g have no common zero then we say that f *strictly interlaces* g and f *strictly alternates left of* g, respectively. We also say that two polynomials f and g *alternate* if one of the polynomials alternates left of or interlaces the other. We will need two simple lemmata concerning these concepts. A polynomial is said to be *standard* if its leading coefficient is positive.

Lemma 2.2. *Let g and $\{f_i\}_{i=1}^n$ be real-rooted standard polynomials.*

(i) *If for each $1 \leq i \leq n$ we have either $g \ll f_i$ or $g \preceq f_i$. Then the sum $F = f_1 + f_2 + \cdots + f_n$ is real-rooted with $g \preceq F$ or $g \ll F$, depending on the degree of F.*

(ii) *If for each $1 \leq i \leq n$ we have either $f_i \ll g$ or $f_i \preceq g$. Then the sum $F = f_1 + f_2 + \cdots + f_n$ is real-rooted with $F \preceq g$ or $F \ll g$, depending on the degree of F.*

Proof. The lemma follows easily by counting the sign-changes of F at the zeros of g, see e.g., [37, Prop. 3.5]. □

The next lemma is obvious:

Lemma 2.3. *If $f_0, f_1, \ldots, f_n$ are real-rooted polynomials with $f_0 \ll f_n$ and $f_{i-1} \ll f_i$ for all $1 \leq i \leq n$, then $f_i \ll f_j$ for all $0 \leq i \leq j \leq n$.*

The following theorem is a characterization of alternating polynomials due to Obreschkoff [24] and Dedieu [13]:

Theorem 2.4. *Let $f, g \in \mathbb{R}[x]$. Then f and g alternate (strictly alternate) if and only if all polynomials in the space*

$$\{\alpha f + \beta g : \alpha, \beta \in \mathbb{R}\},$$

have only real (real and simple) zeros.

An immediate but non-trivial consequence of this theorem is:

Corollary 2.5. *Let $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. Then ϕ preserves the real-rootedness property (real- and simple-rootedness property) only if ϕ preserves the alternating property (strictly alternating property).*

We denote by $\mathbb{N}$ the set of natural numbers $\{0, 1, 2, \ldots\}$. The symmetric group of bijections $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is denoted by $\mathcal{S}_n$. A *descent* in a permutation $\pi \in \mathcal{S}_n$ is an index $1 \leq i \leq n-1$ such that $\pi(i) > \pi(i+1)$. Let $\mathrm{des}(\pi)$ denote the number of descents in π. The *Eulerian polynomials*, $A_n(x)$, are defined by $A_n(x) = \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{des}(\pi)+1}$ and satisfies, see e.g. [11]

$$\sum_{k \geq 0} k^n x^k = \frac{A_n(x)}{(1-x)^{n+1}}.$$

The binomial polynomials are defined by $\binom{x}{0} = 1$ and $\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}$ for $k \geq 1$.

In several proofs we will implicitly use the fact that the zeros of a polynomial are continuous functions of the coefficients of the polynomial. In particular the limit of a sequence of real-rooted polynomials is again real-rooted. For a treatment of these matters we refer the reader to [23].

3. A class of linear operators preserving the PF-property

For any polynomial $F(x,z) = \sum_{k=0}^{n} Q_k(x)z^k \in \mathbb{R}[x,z]$ we define a linear operator $\phi_F : \mathbb{R}[x] \to \mathbb{R}[x]$ by,

$$\phi_F(f) := \sum_{k=0}^{n} Q_k(x)\frac{d^k}{dx^k}f(x).$$

In this section we will investigate for which $F \in \mathbb{R}[x,z]$ the linear operator ϕ_F preserves real-rootedness- and the PF-property .

We will need some terminology and a theorem from [5]. For $\xi \in \mathbb{R}$ let $T_\xi : \mathbb{R}[x] \to \mathbb{R}[x]$ be the translation operator defined by $T_\xi(f(x)) = f(x+\xi)$. For any linear operator $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ we define a linear transform $\mathcal{L}_\phi : \mathbb{R}[x] \to \mathbb{R}[x,z]$ by

$$\begin{aligned} \mathcal{L}_\phi(f) &:= \phi(T_z(f)) \\ &= \sum_n \phi(f^{(n)})(x)\frac{z^n}{n!} \\ &= \sum_n \frac{\phi(x^n)}{n!}f^{(n)}(z). \end{aligned} \tag{3.1}$$

Definition 3.1. Let $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. Define a function $d_\phi : \mathbb{R}[x] \to \mathbb{N} \cup \{-\infty\}$ by: If $\phi(f^{(n)}) = 0$ for all $n \in \mathbb{N}$, we let $d_\phi(f) = -\infty$. Otherwise let $d_\phi(f)$ be the smallest integer d such that $\phi(f^{(n)}) = 0$ for all $n > d$. Hence $d_\phi(f) \leq \deg f$ for all $f \in \mathbb{R}[x]$.

The set $\mathscr{A}(\phi)$ is defined as follows: If $d_\phi(f) \in \{-\infty, 0\}$ and $\phi(f)$ is standard real- and simple-rooted, then $f \in \mathscr{A}(\phi)$. Moreover, $f \in \mathscr{A}(\phi)$ if $d = d_\phi(f) \geq 1$ and all of the following conditions are satisfied:

(i) $\phi(f^{(i)})$ all have leading coefficients of the same sign and $\deg(\phi(f^{(i-1)})) = \deg(\phi(f^{(i)})) + 1$ for $1 \leq i \leq d$,
(ii) $\phi(f)$ and $\phi(f')$ have no common real zero,
(iii) $\phi(f^{(d)})$ strictly interlaces $\phi(f^{(d-1)})$,
(iv) for all $\xi \in \mathbb{R}$ the polynomial $\mathcal{L}_\phi(f)(\xi, z)$ is real-rooted.

The following theorem is proved in [5]:

Theorem 3.2. *Let $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. If $f \in \mathscr{A}(\phi)$ then $\phi(f)$ is real- and simple-rooted.*

We will also need the following classical theorem of Hermite and Poulain. For a proof see [24].

Theorem 3.3. *Let $f = a_0 + a_1x + \cdots + a_nx^n$ and g be real-rooted polynomials. Then the polynomial*

$$f(\frac{d}{dx})g := a_0g(x) + a_1g'(x) + \cdots + a_ng^{(n)}(x)$$

is real-rooted. Moreover, if $f(\frac{d}{dx})g \neq 0$ then any multiple zero of $f(\frac{d}{dx})g$ is a multiple zero of g.

The following theorem gives a sufficient condition for a polynomial to be mapped onto a real-rooted polynomial.

Theorem 3.4. *Let $F = \sum_{k=0}^{n} Q_k(x)z^k$ be such that $Q_0 \neq 0$ and*

(I) *For all $\xi \in \mathbb{R}$, $F(\xi, z)$ is real-rooted,*

(II) Q_0 *strictly interlaces or strictly alternates left of* Q_1, *and* $\deg Q_0 = 0$ *or* Q_0 *and* Q_1 *have leading coefficients of the same sign.*

Suppose that

(III) f *is real- and simple-rooted and that for* $0 \leq k \leq \deg f$ *the polynomials* $\phi_F(f^{(k)})$ *have their leading term of the same sign with*

$$\deg \phi_F(f^{(k)}) = \deg Q_0 + \deg f - k.$$

Then $\phi_F(f)$ *is real- and simple-rooted.*

Proof. We will show that the set of real- and simple-rooted polynomials satisfying (III) is a subset of $\mathscr{A}(\phi_F)$ by verifying conditions (i)-(iv) of Definition 3.1. Condition (i) follows immediately from (III). For condition (iv) note that

$$\mathcal{L}_\phi(f)(\xi, z) = \sum_{k=0}^{n} Q_k(\xi) f^{(k)}(\xi + z),$$

so by the Theorem 3.3 condition (iv) is satisfied. Suppose that η is a common zero of $\phi_F(f)$ and $\phi_F(f')$. By (3.1) we have that 0 is a multiple zero of $\mathcal{L}_\phi(f)(\eta, z)$. Moreover, since $\mathcal{L}_\phi(f)(\eta, z)$ is not identically equal to zero, by (II), Theorem 3.3 tells us that 0 is a multiple zero of $f(\eta + z)$. This means that η is a multiple zero of f contrary to the assumption that f is simple-rooted, and verifies condition (ii).

For condition (iii) we have to show that for all $\alpha \in \mathbb{R}$ such that $x+\alpha$ satisfies (III) the polynomial $\phi_F(1) = Q_0$ strictly interlaces $f(x) := \phi_F(x+\alpha) = (x+\alpha)Q_0 + Q_1$. This follows from (II) when analyzing the sign of $f(x) := \phi_F(x+\alpha)$ at the zeros of Q_0: Let $\alpha_k < \alpha_{k-1} < \cdots < \alpha_1$ be the zeros of Q_0 ordered by size. Suppose that Q_0 and Q_1 are standard and that Q_0 strictly interlaces or strictly alternates left of Q_1. Then $\operatorname{sgn} f(\alpha_i) = \operatorname{sgn} Q_1(\alpha_i) = (-1)^i$ for $1 \leq i \leq k$. By Rolle's theorem we know that f has a zero in each interval (α_i, α_{i+1}). This accounts for $k-1$ real zeros of f. Since Q_0 has positive sign, so does f by condition (III). Now, because $f(\alpha_1) < 0$ and f is standard, f must have a zero to the right of α_1. We now know that f has k zeros real. The signs at α_i forces the remaining zero to be in the interval $(-\infty, \alpha_k)$. Thus Q_0 strictly interlaces f as was to be shown.

Now, if $Q_0 = A \in \mathbb{R}$ then $\deg Q_1 \leq 1$. Suppose that $Q_1 = B \in \mathbb{R}$. Then clearly A strictly interlaces $(x+\alpha)A + B$. If $Q_0 = A$ and $Q_1 = Cx + D$ where $A, C, D \in \mathbb{R}$, then $f = (A+C)x + A\alpha + D$, so by (III) we have that Q_0 strictly interlaces f. This concludes the proof. $\square$

In some cases it may be convenient to have sharper hypothesis. Therefore we state the following form of the theorem.

Corollary 3.5. *Let* $d \in \mathbb{N}$ *be given and let* $F = \sum_{k=0}^{n} Q_k(x) z^k$ *be such that* $Q_0 \neq 0$ *and*

(i) *For all* $\xi \in \mathbb{R}$, $F(\xi, z)$ *is real-rooted,*
(ii) Q_0 *strictly interlaces or strictly alternates left of* Q_1, *and* $\deg Q_0 = 0$ *or* Q_0 *and* Q_1 *have leading coefficients of the same sign.*
(iii) *The polynomials* $\phi_F(x^k)$, $0 \leq k \leq d$ *have the same sign and*

$$\deg \phi_F(x^k) = \deg Q_0 + k.$$

Then $\phi_F(f)$ *is real-rooted (real- and simple-rooted) if* f *is real-rooted (real and simple-rooted) and* $\deg(f) \leq d$.

Proof. The case of real- and simple-rooted f follows immediately from Theorem 3.4 since (iii) implies (III). If f is a real-rooted polynomial of degree at most d, then f is the limit of a sequence $\{f_k\}_{k=0}^{\infty}$ of real- and simple-rooted polynomials of degree at most d. It follows that $\phi_F(f)$ is the limit of $\phi_F(f_k)$, and the thesis follows by continuity. □

In the language of PF-sequences we have:

Theorem 3.6. *Let $d \in \mathbb{N}$ be given and let $F = \sum_{k=0}^{n} Q_k(x)z^k \in \mathbb{R}[x,z]$ be such that $Q_0 \neq 0$ and*

(i) *For all $\xi \in \mathbb{R}$, $F(\xi, z)$ is real-rooted,*
(ii) *$\phi_F(1)$ strictly interlaces $\phi_F(x)$.*
(iii) *For all $0 \le k \le d$*
$$\deg \phi_F(x^k) = \deg Q_0 + k,$$
and $\phi_F(x^k) \in PF_1$.

Then $PF[\{\phi_F(x^i)\}_{i=0}^{d}] \subseteq PF[x^i]$.

Several old results can be derived from these last few theorems. In [25, p. 163] Pólya gave a theorem which he states probably was the most general theorem on real-rootedness known at the time. "Dieser Satz gehört wohl zu den allgemeinsten bekannten Sätzen über Wurzelrealität.":

Theorem 3.7. *Let $f(x)$ be a real-rooted polynomial of degree n, and let*
$$b_0 + b_1x + \cdots + b_{n+m}x^{n+m}, \quad (m \ge 0)$$
be a real-rooted polynomial such that $b_i > 0$ for $0 \le i \le n$. Then the equation
$$G(x,y) := b_0 f(y) + b_1 x f'(y) + b_2 x^2 f''(y) + \cdots + b_n x^n f^{(n)}(y) = 0,$$
has n real intersection points, (counted with multiplicity), with the line
$$sx - ty + u = 0,$$
provided that $s,t \ge 0$, $s+t > 0$ and $u \in \mathbb{R}$.

Proof. We may assume that $s,t > 0$ since the other cases follows by continuity when s and/or t tends to zero. Thus we may write the equation as
$$a_0 g(x) + a_1 x g'(x) + a_2 x^2 g''(x) + \cdots + a_n x^n g^{(n)}(x) = 0,$$
where $g(x) = f(st^{-1}x + ut^{-1})$ and $a_i = s^i t^{-i} b_i$. Now, we see that all hypothesis of Corollary 3.5 are satisfied for
$$F(x,z) = a_0 + a_1 xz + a_2 x^2 z^2 + \cdots + a_{n+m} x^{n+m} z^{n+m},$$
when $d = n$. □

We will later need one famous consequence of this theorem, $t = 1, s = u = 0$, due to Schur [29].

Theorem 3.8. *Let $f = \sum_{k=0}^{n} a_k x^k$ and $g = \sum_{k=0}^{m} b_k x^k$ be two real-rooted polynomials such that g has all zeros of the same sign. Then the polynomial*
$$(fSg)(x) = \sum_{k\ge 0}^{M} k! a_k b_k x^k,$$
where $M = \min(m,n)$ has only real zeros.

3.1. **Multiplier-sequences.** A multiplier-sequence is a sequence $T = \{\gamma_i\}_{i=0}^{\infty}$ of real numbers such that if a polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ has only real zeros, then the polynomial

$$T[f(x)] := a_0\gamma_0 + a_1\gamma_1 x + \cdots + a_n\gamma_n x^n,$$

also has only real zeros. There is a characterization of multiplier-sequences due to Pólya and Schur [25, p. 100-124]:

Theorem 3.9. *Let $T = \{\gamma_i\}_{i=0}^{\infty}$ be a sequence of non-negative real numbers and let $\phi(x) = T[e^x] = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!}$ be its exponential generating function. Then T is a multiplier-sequence if and only if ϕ is a real entire function which can be written as*

$$\phi(x) = cx^n e^{\beta x} \prod_{k=1}^{\infty} (1 + \delta_k x),$$

where $c > 0$, $\beta \geq 0$, $\delta_k \geq 0$, $n \in \mathbb{N}$ and $\sum_{k=1}^{\infty} \delta_k < \infty$.

The following lemma is well-known but elementary, so we give a proof here.

Lemma 3.10. *A multiplier-sequence is strictly log-concave. In particular, a non-negative multiplier-sequence has no internal zeros.*

Proof. If $f(x) = a_m x^m + a_{m+1}x^{m+1} + \cdots + a_n x^n$ is real-rooted with $a_m a_n \neq 0$, then the coefficients satisfy (see [20, p. 52]):

$$\frac{a_i^2}{\binom{n}{i}^2} > \frac{a_{i-1}}{\binom{n}{i-1}} \frac{a_{i+1}}{\binom{n}{i+1}} \quad (m < i < n).$$

Now, if $\Gamma = \{\gamma_i\}_{i=0}^{\infty}$ is a multiplier-sequence, then $\Gamma[(x+1)^n]$ is real-rooted for all $n \in \mathbb{N}$, which implies

$$\gamma_i^2 > \gamma_{i-1}\gamma_{i+1},$$

for all i such that there are integers $m < i < n$ with $\gamma_m \gamma_n \neq 0$.

□

Theorem 3.11. *Let $\{\lambda_k\}_{k=0}^{\infty}$ be a non-negative multiplier-sequence, and let $\alpha < \beta \in \mathbb{R}$ be given. Define two $\mathbb{R}$-bilinear forms $\mathbb{R}[x] \times \mathbb{R}[x] \to \mathbb{R}[x]$ by*

$$\begin{aligned} f \cdot g &:= \sum_{k \geq 0} \frac{\lambda_k}{k!} f^{(k)}(x) g^{(k)}(x) (x-\alpha)^k (x-\beta)^k, \\ f \circ g &:= \sum_{k \geq 0} \frac{\lambda_k}{k!} f^{(k)}(x) g^{(k)}(x) (x-\alpha)^k, \end{aligned}$$

If f is real-rooted and g is $[\alpha, \beta]$-rooted, then $f \cdot g$ is real-rooted. If f is real-rooted and g is $[-\infty, \alpha]$-rooted, then $f \circ g$ is real-rooted.

Proof. We prove the statement for $\cdot$ since the case of $\circ$ is similar. We may assume that $\lambda_0 > 0$. Clearly the theorem is true if $\lambda_i = 0$ for all $i > 0$, so by Lemma 3.10 we may assume that $\lambda_1 > 0$. Let g have all zeros simple and in the interval (α, β), and let ϕ be the linear operator defined by $\phi(f) = f \cdot g$. Then $\phi = \phi_F$, where

$$F(x, z) = \sum_{k \geq 0} \lambda_k \frac{g^{(k)}(x)}{k!} (x-\alpha)^k (x-\beta)^k z^k.$$

Since $\{\lambda_k\}_{k \geq 0}$ is a multiplier sequence $F(\xi, z)$ is real-rooted for all real choices of ξ. Now, $Q_0 = \lambda_0 g(x)$ and $Q_1 = \lambda_1 (x-\alpha)(x-\beta) g'(x)$, so Q_0 strictly interlaces Q_1.

Moreover, $\deg \phi(x^k) = \deg Q_0 + k$ for all k, so all the hypothesis of Corollary 3.5 are fulfilled. Since any $[\alpha, \beta]$-rooted polynomial is the limit of polynomials which are (α, β)- and simple-rooted the thesis follows by continuity. □

There are a few bilinear forms on polynomials that occur frequently in combinatorics. Let $\# : \mathbb{R}[x] \times \mathbb{R}[x] \to \mathbb{R}[x]$ be defined by

$$(f\#g)(x) := \sum_{k\geq 0} f^{(k)}(x) g^{(k)}(x) \frac{x^k}{k!}.$$

This product is important when analyzing how the the zeros of σ-polynomials behave under disjoint union of graphs, see [10].

Theorem 3.12. *Let f be real-rooted and let g have only real zeros of the same sign. Then $f\#g$ is real-rooted.*

Proof. The theorem follows from Theorem 3.11, since $\{1\}_{k=0}^{\infty}$ is trivially a multiplier-sequence. □

This generalizes a result of Wagner, who proved that $f\#g$ is real-rooted whenever f and g have only non-negative zeros, see [10, 35].

The *diamond product* of two polynomials f and g is given by

$$(f \diamond g)(x) = \sum_{k\geq 0} \frac{f^{(k)}(x)}{k!} \frac{g^{(k)}(x)}{k!} x^k (x+1)^k. \tag{3.2}$$

This product is important in the theory of (P, ω)-partitions and the Neggers-Stanley conjecture and was first studied by Wagner in [36, 37], see also Section 4 of this paper. Applying Theorem 3.11 with the multiplier-sequence $\{\frac{1}{k!}\}_{k\geq 0}$ we get:

Theorem 3.13. *Let f be real-rooted and let g have all zeros in the interval $[-1, 0]$. Then $f \diamond g$ is real-rooted.*

This was first proved by Wagner [37] under the additional hypothesis that f has all zeros in $[-1, 0]$, and generalized by the present author in [5].

A sequence of real numbers $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ is called a *multiplier n-sequence* if for any real-rooted polynomial $f = a_0 + a_1 x + \cdots + a_n x^n$ of degree at most n the polynomial $\Gamma[f] := a_0\gamma_0 + a_1\gamma_1 x + \cdots + a_n\gamma_n x^n$ is real-rooted. There is a simple algebraic characterization of multiplier n-sequences [12]:

Theorem 3.14. *Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers. Then Γ is a multiplier n-sequence if and only if $\Gamma[(x+1)^n]$ is real-rooted with all its zeros of the same sign.*

Recall the definition of the *hypergeometric function* ${}_2F_1$:

$${}_2F_1\,(a, b; c; z) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m z^m}{(c)_m m!},$$

where $(\alpha)_0 = 1$ and $(\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1)$ when $m \geq 1$. The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ can be expressed as follows [26, p. 254]:

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1\left(-n, 1+\alpha+\beta+n; 1+\alpha; \frac{1-x}{2}\right), \tag{3.3}$$

We need the following lemma:

Lemma 3.15. *Let n be a positive integer and r a non-negative real number. Then $\Gamma = \{\binom{-n-r}{k}\}_{k=0}^{\infty}$ is a multiplier n-sequence.*

Proof. Let $r > 0$. Then

$$\begin{aligned}\Gamma[(x+1)^n] &= \sum_{k=0}^{n} \binom{-n-r}{k}\binom{n}{k} x^k \\ &= {}_2F_1\left(-n, n+r; 1; x\right) \\ &= P_n^{(0,r-1)}(1-2x),\end{aligned}$$

where the last equality follows from (3.3). Since the Jacobi polynomials are known, see [26], to have all their zeros in $[-1,1]$ when $\alpha, \beta > -1$, we have that $\Gamma[(x+1)^n]$ has all its zeros in $[0,1]$. The case $r = 0$ follows by continuity when we let r tend to zero from above. □

For any real number q let $\Gamma_q := \{q+k\}_{k=0}^{\infty}$. The following Corollary was known already to Laguerre:

Corollary 3.16. *Let $n > 1$ be a positive integer. Then Γ_q is a multiplier n-sequence if and only if $q \notin (-n, 0)$.*

Proof. Let $q \in \mathbb{R}$ be given. We have to determine for which $n > 1$ the zeros of $\Gamma_q[(x+1)^n]$ are all real and of the same sign. Now,

$$\Gamma_q[(x+1)^n] = (x+1)^{n-1}\{(n+q)x + q\},$$

and the theorem follows. □

4. The E-transformation

The *E-transformation* is the invertible linear operator, $\mathcal{E} : \mathbb{R}[x] \to \mathbb{R}[x]$, defined by

$$\mathcal{E}(\binom{x}{i}) = x^i,$$

for all $i \in \mathbb{N}$. The PF-preserving properties of this linear operator was first studied in [6] and later in [36, 37] and [5]. It is important in the theory of (P,ω)-partitions since it maps the order-polynomial of a labeled poset to the E-polynomial of the same labeled poset, see [6, 36]. In, [6] Brenti proved the following theorem. Let $\lambda(f)$ and $\Lambda(f)$ denote the smallest and the largest real zero of the polynomial f, respectively.

Theorem 4.1. *Suppose that $f \in \mathbb{R}[x]$ has only real zeros and that $f(n) = 0$ for all $n \in ([\lambda(f), -1] \cup [0, \Lambda(f)]) \cap \mathbb{Z}$. Then $\mathcal{E}(f)$ has all zeros real and non-positive.*

In this section we will prove the following theorem:

Theorem 4.2. *For all $n \in \mathbb{N}$ we have*

$$PF_1[\{x^i(x+1)^{n-i}\}_{i=0}^{n}] \subseteq PF[\binom{x}{i}]$$

Moreover if $f \in PF_1[\{x^i(x+1)^{n-i}\}_{i=0}^{n}]$ then $\mathcal{E}(f)$ has simple zeros and

$$\mathcal{E}((x+1)^d) \ll \mathcal{E}(f) \ll \mathcal{E}(x^d).$$

The diamond product (3.2) is intimately connected with the E-transformation. By the Vandermonde identity

$$\binom{x}{i}\binom{x}{j} = \sum_{k\geq 0}\binom{k}{k-i,i+j-k,k-j}\binom{x}{k},$$

it follows, see [37], that

$$(f \diamond g)(x) := \mathcal{E}(\mathcal{E}^{-1}(f)\mathcal{E}^{-1}(g)). \tag{4.1}$$

We will later need a symmetry property of $\mathcal{E}$. Let $\mathcal{R} : \mathbb{R}[x] \to \mathbb{R}[x]$ be the algebra automorphism defined by $\mathcal{R}(x) = -1 - x$.

Lemma 4.3.

$$\mathcal{R}\mathcal{E} = \mathcal{E}\mathcal{R}$$

Proof. Let n be a nonnegative integer. Using the identity

$$\binom{x+n}{n} = \sum_{k=0}^{n}\binom{n}{k}\binom{x}{k},$$

and the fact that $\binom{-x-1}{n} = (-1)^n\binom{x+n}{n}$ we get

$$\begin{aligned}\mathcal{E}\mathcal{R}\binom{x}{n} &= (-1)^n\mathcal{E}\binom{x+n}{n}\\ &= (-1)^n\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}\binom{x}{k}\\ &= (-1-x)^n\\ &= \mathcal{R}\mathcal{E}\binom{x}{n},\end{aligned}$$

and the lemma follows. □

Lemma 4.4. *Let $\alpha \in [-1,0]$ and let f be a polynomial such that $\mathcal{E}(f)$ is $[-1,0]$-rooted. Then $\mathcal{E}((x-\alpha)f)$ is $[-1,0]$-rooted and $\mathcal{E}(f)$ interlaces $\mathcal{E}((x-\alpha)f)$. If $\mathcal{E}(f)$ in addition only has simple zeros, then so does $\mathcal{E}((x-\alpha)f)$.*

Proof. Let $g = \mathcal{E}(f)$ and let $\alpha \in [-1,0]$. By (3.2) and (4.1) we have that

$$\mathcal{E}((x-\alpha)f) = (x-\alpha)g + x(x+1)g'. \tag{4.2}$$

Since g interlaces $(x-\alpha)g$ and $x(x+1)g'$ it also interlaces the sum, by Lemma 2.2. Also, if $x \notin [-1,0]$ then the summands have the same sign so $\mathcal{E}((x-\alpha)f)$ cannot have any zeros outside $[-1,0]$. Suppose that g has only simple zeros. Then by (4.2) the only possible common zeros of g and $\mathcal{E}((x-\alpha)f)$ are 0 and -1. If $\deg(f) \geq 1$ it also follows from (4.2) that the multiplicities of 0 and -1 of $\mathcal{E}((x-\alpha)f)$ are the same as those of g. Hence the (simple) zeros of g separate the zeros of $\mathcal{E}((x-\alpha)f)$ except possibly at $0,-1$, and we conclude that $\mathcal{E}((x-\alpha)f)$ has only simple zeros. □

Lemma 4.5. *For all integers $n \geq 1$ we have*

$$(x+1)\mathcal{E}(x^n) = x\mathcal{E}((x+1)^n).$$

Proof. We may write

$$x^n = \sum_{k=1}^{n} a_k \binom{x}{k},$$

where $a_k \in \mathbb{R}$. Thus

$$\begin{aligned} \mathcal{E}((x+1)^n) &= \sum_{k=1}^{n} a_k \mathcal{E}[\binom{x}{k} + \binom{x}{k-1}] \\ &= \sum_{k=1}^{n} a_k (x^k + x^{k-1}) \\ &= (x+1)x^{-1}\mathcal{E}(x^n). \end{aligned}$$

□

Let f and g be standard real-rooted polynomials of degree n and let the zeros of f and g be $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_n$ and $\beta_1 \le \beta_2 \le \cdots \le \beta_n$ respectively. We write $f \le g$, if $\alpha_i \le \beta_i$ for $1 \le i \le n$.

Theorem 4.6. *Suppose that f and g are $[-1,0]$-rooted with $f \le g$. Then $\mathcal{E}(f)$ and $\mathcal{E}(g)$ are $[-1,0]$- and simple-rooted, with $\mathcal{E}(f) \ll \mathcal{E}(g)$.*

Proof. By Lemma 4.4 and induction we only have to show that $\mathcal{E}(f) \ll \mathcal{E}(g)$. If f and g have the same zeros except for one, i.e., $f = (x-\alpha)h$ and $g = (x-\beta)h$, where $\alpha < \beta$, then

$$\mathcal{E}(g) = \mathcal{E}(f) - (\beta - \alpha)\mathcal{E}(h),$$

and since $\mathcal{E}(h)$ interlaces $\mathcal{E}(f)$ we have $\mathcal{E}(f) \ll \mathcal{E}(g)$ by Lemma 2.2.

Now, suppose that f and g are $[-1,0]$-rooted polynomials of degree n such that $f \le g$. Then there are $[-1,0]$-rooted polynomials $\{h_i\}_{i=0}^M$ with

$$(x+1)^n = h_0 \le h_1 \le \cdots \le h_M = x^n,$$

such that $f, g \in \{h_i\}_{i=0}^M$ and h_{i-1} and h_i only differ in one zero for $1 \le i \le n$. We therefore have

$$\mathcal{E}(h_0) \ll \mathcal{E}(h_1) \ll \cdots \ll \mathcal{E}(h_M),$$

and since $\mathcal{E}(h_0) \ll \mathcal{E}(h_M)$, by Lemma 4.5, the theorem follows from Lemma 2.3. □

A consequence of Theorem 4.6 is that if $\{f_i\}_{i=1}^m$ is a sequence of standard $[-1,0]$-rooted polynomials of the same degree d, then by Lemma 2.2 and Theorem 4.6, the image under $\mathcal{E}$ of any non-negative sum $F = \sum_{i=1}^m \mu_i f_i$ will be $[-1,0]$-rooted with

$$\mathcal{E}((x+1)^d) \ll \mathcal{E}(F) \ll \mathcal{E}(x^d).$$

It is easy to see that a standard polynomial f of degree d is $[-1,0]$-rooted if and only if f can be written as

$$f(x) = (x+1)^d g(\frac{x}{x+1}),$$

where g is a standard and $(-\infty,0)$-rooted. On the other hand, since $x^i(x+1)^{d-i}$ is $[-1,0]$-rooted we have that F can be written as a non-negative sum of standard $[-1,0]$-rooted polynomials of degree d if and only if

$$F(x) = \sum_{i=0}^{d} a_i x^i (x+1)^{d-i},$$

where $a_i \ge 0$. This proves Theorem 4.2.

5. t-STACK SORTABLE PERMUTATIONS

For relevant definitions regarding t-stack sortable permutations we refer the reader to [2]. Let $W_t(n,k)$ be the number of t-stack sortable permutations in the symmetric group, $\mathcal{S}_n$, with k descents, and let

$$W_{n,k}(x) = \sum_{k=0}^{n-1} W_t(n,k)x^k.$$

Recently, Bóna [1, 3] showed that for fixed n and t the numbers $\{W_t(n,k)\}_{k=0}^{n-1}$ form a unimodal sequence. When $t = n-1$ and $t = 1$ we get the Eulerian and the Narayana numbers (see [34] and [31, Exercise 6.36]), respectively. These are known to be PF-sequences and Bóna [2, 3] has raised the question if this is true for general t. Here we will settle the problem to the affirmative for $t = 2$ and $t = n-2$.

The numbers $W_2(n,k)$ are surprisingly hard to determine despite their compact and simple form. It was recently shown that

$$W_2(n,k) = \frac{(n+k)!(2n-k-1)!}{(k+1)!(n-k)!(2k+1)!(2n-2k-1)!}.$$

See [4, 14, 19, 21] for proofs and more information on 2-stack sortable permutations.

From the case $r = 0$ in Lemma 3.15 and the identity

$$\sum_{k=0}^{n} \binom{2n-k-1}{n-1}\binom{n}{k}x^k = (-1)^n \sum_{k=0}^{n} \binom{-n}{k}\binom{n}{k}(-x)^{n-k},$$

it follows that $\binom{2n-k-1}{n-1}$ is an n-sequence.

Theorem 5.1. *For all $n \geq 0$ the sequence $\{W_2(n,k)\}_{k=0}^{n-1}$, which records 2-stack sortable permutations by descents, is PF.*

Proof. We may write $W_2(n,k)$ as

$$W_2(n,k) = \frac{\binom{2n-k-1}{n-1}\binom{n+k}{n-1}\binom{2n}{2k+1}}{n^2\binom{2n}{n}}.$$

A simple consequence of the notion of PF-sequences reads as follows: If $\{a_i\}_{i\geq 0}$ is PF then so is $\{a_{ki}\}_{i\geq 0}$, where k is any positive integer. Applying this to the polynomial $x(1+x)^{2n}$ we see that $\sum_k \binom{2n}{2k+1}x^k$ is real-rooted. Therefore the polynomial,

$$\sum_{k=0}^{n-1} \binom{n+k}{n-1}\binom{2n}{2k+1}x^k = \sum_{k=0}^{n-1} \binom{2n-k-1}{n-1}\binom{2n}{2k+1}x^{n-1-k},$$

is real-rooted. Another application of Lemma 3.15 gives that $W_{n,2}(x)$ is real-rooted. $\square$

It is easy to see that a permutation $\pi \in \mathcal{S}_n$ is $(n-2)$-stack sortable if and only if it is not of the form $\sigma n1$. Thus the generating function satisfies

$$xW_{n,n-2}(x) = A_n(x) - xA_{n-2}(x),$$

where $A_n(x)$ is the nth Eulerian polynomial.

Theorem 5.2. *For all real numbers $t > -2$ and integers $n > 2$, the polynomial*

$$A_n(t, x) = A_n(x) + txA_{n-2}(x),$$

is real- and simple-rooted. Moreover, $A_n(t, x)/x$ strictly interlaces $A_{n+1}(t, x)/x$ for $-2 < t \leq 3$.

Corollary 5.3. *For all $n \geq 2$ we have that $\{W_{n-2}(n, k)\}_{k=0}^{n-1}$ is PF. Moreover, $W_{n,n-2}(x)$ strictly interlaces $W_{n+1,n-1}(x)$.*

Proof of Theorem 5.2. It is well known that $A_{n-1}(x) \ll xA_{n-2}(x)$ and $A_{n-1}(x) \preceq A_n(x)$. So by Lemma 2.2 we have that $A_n(t, x)$ is real- and simple-rooted for $t \geq 0$. However, when $t < 0$ a similar argument does not apply.

Let $E_n(t, x) = A_n(t, \frac{x}{1+x})$. Then

$$E_n(t, x) = E_n(x) + tx(1 + x)E_{n-2}(x),$$

where the coefficient to x^k in $E_n(x)$ counts the number of surjections $\sigma : [n] \to [k]$, see [6, 36]. These polynomials satisfy the recursion:

$$E_n(x) = x\frac{d}{dx}((1 + x)E_{n-1}(x)),$$

with initial condition $E_1(x) = x$. Thus, if we let $G_n(x) = E_{n+1}(x)/x$ we have the following recursion:

$$G_n(x) = \frac{d}{dx}(x(1 + x)G_{n-1}(x)), \tag{5.1}$$

with $G_0(x) = 1$. Obviously $G_n(x)$ is real- and simple-rooted. If we apply (5.1) two times we get the equation:

$$G_n(x) = (1 + 6x + 6x^2)G_{n-2}(x) + 3x(1 + 2x)(1 + x)G'_{n-2}(x)+ \\ x^2(1 + x)^2G''_{n-2}(x),$$

and for $G_n(t, x) := G_n(x) + tx(1 + x)G_{n-2}(x)$ we have

$$G_n(t, x) = (1 + (6 + t)x + (6 + t)x^2)G_{n-2}(x) + 3x(1 + 2x)(1 + x)G'_{n-2}(x)+ \\ x^2(1 + x)^2G''_{n-2}(x).$$

To apply Theorem 3.4 we need show that for all $\xi \in \mathbb{R}$ and $-2 < t < 0$ the polynomial

$$F(\xi, z) := (1 + (6 + t)\xi + (6 + t)\xi^2) + 3\xi(1 + 2\xi)(1 + \xi)z + \xi^2(1 + \xi)^2z^2$$

is real-rooted. The discriminant of $F(\xi, z)$,

$$\Delta(F(\xi, z)) = \xi^2(1 + \xi)^2(2 + t + (3 - t)(1 + 2\xi)^2),$$

is non-negative when $-2 \leq t \leq 3$, so $F(\xi, z)$ real-rooted for these t. Since all the Q_ks are standard it is easy to see that condition (III) in the statement of Theorem 3.4 is satisfied. Moreover, $1+(6+t)x+(6+t)x^2$ strictly interlaces $3x(1+2x)(1+x)$ when $t > -2$ so Theorem 3.4 applies. Since G_n strictly interlaces G_{n+1} we have by Theorem 3.4 and Corollary 2.5 that $\phi_F(G_n)$ strictly interlaces $\phi_F(G_{n+1})$. Thus $A_n(t, x)$ strictly interlaces $A_{n+1}(t, x)$. □

6. q-Eulerian and W-Eulerian polynomials

A q-analog of the Eulerian polynomials was introduced and studied in [16] and further studied in [9]. It is defined by

$$A_n(x;q) := \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{exc}(\pi)} q^{c(\pi)},$$

where $\mathrm{c}(\pi)$ and $\mathrm{exc}(\pi)$ denotes the number of *cycles* and *excedances* in π respectively. These polynomials satisfy the recursion

$$A_{n+1}(x;q) = (nx+q)A_n(x;q) - x(x-1)\frac{\partial}{\partial x}A_n(x;q),$$

with initial condition $A_0(x;q) := 1$. See [9] for a proof. The following theorem appears in [9].

Theorem 6.1. *Let $q \in \mathbb{R}$, $q > 0$. Then the polynomials $A_n(x,q)$ have only real non-positive simple zeros.*

Brenti also makes the following conjecture:

Conjecture 6.2. *Let $n, m \in \mathbb{N}$. Then $A_n(x;-m)$ has only real zeros.*

In what follows we will prove this conjecture using multiplier n-sequences. For $n \in \mathbb{N}$ define the polynomials $E_n(x;q)$ by:

$$E_n(x;q) := (1+x)^n A_n(\frac{x}{1+x};q).$$

It is clear that $E_n(x;q)$ is real-rooted if and only if $A_n(x;q)$ is real-rooted. These polynomials satisfy a somewhat easier recursion. Namely,

$$E_{n+1}(x;q) = (1+x)\{qE_n(x;q) + x\frac{\partial}{\partial x}E_n(x;q)\}, \tag{6.1}$$

with initial condition $E_0(x;q) = 1$. Now, for $q \in \mathbb{R}$ let $\Gamma_q : \mathbb{R}[x] \to \mathbb{R}[x]$ be the linear operator defined by $\Gamma_q(f(x)) = qf(x) + xf'(x)$. Since $\Gamma_q(x^n) = (q+n)x^n$ we may apply Corollary 3.16.

Theorem 6.3. *Let $q \in \mathbb{R}$ and $n \in \mathbb{N}$. If $q \geq 0$, $n \leq -q$ or $q \in \mathbb{Z}$ then $E_n(x;q)$ has only real zeros.*

Proof. We may write (6.1) as

$$E_{n+1}(x;q) = (x+1)\Gamma_q[E_n(x;q)].$$

The cases $q \geq 0$ and $n \leq -q$ follow from Corollary 3.16 by induction. We may therefore assume that $q = -m$ is a negative integer. We claim that $\deg E_n(x;q) = n$ if $n \leq m$ and $\deg E_n(x;q) = m$ if $n \geq m$. From this the real-rootedness follows by Corollary 3.16 and induction. The case $n \leq m$ is clear since $\Gamma_q[x^{n-1}] = -(m - n + 1) < 0$. The case $n > m$ also follows by induction. Suppose that $n \geq m$ and that $\deg E_n(x;q) = m$. Then by the recursion we have that $\deg E_{n+1}(x;q) \leq m+1$. Moreover, since $\Gamma_q[x^m] = 0$ we have that $\deg E_{n+1}(x;q) \leq m$. Let $a \neq 0$ be the coefficient to x^m of $E_n(x;q)$. Then the coefficient to x^m of $E_{n+1}(x;q)$ is $a\Gamma_q[x^{m-1}] = -a$, so $\deg E_{n+1}(x;q) = m$, and the thesis follows. □

The Eulerian polynomial, $P(W,x)$, of a finite Coxeter group W is the polynomial,

$$P(W,x) = \sum_{\sigma \in W} x^{d_W(\sigma)},$$

where $d_W(\sigma)$ is the number of W-descents of σ, see [8]. This polynomial is also the generating function for the h-vector of the Coxeter complex associated to (W, S). For Coxeter groups of type A_n we have that $P(A_n, x) = A_n(x)/x$, the shifted Eulerian polynomial. Also, for Coxeter groups of type B_n it is known, see [8], that $P(B_n, x)$, has only real zeros. It is easy to see that $P(W_1 \times W_2, x) = P(W_1, x)P(W_2, x)$ for finite Coxeter groups W_1 and W_2. Also, the real-rootedness can be checked ad hoc for the exceptional groups. Thus, by the classification of finite irreducible Coxeter groups, to prove that $P(W, x)$ has only real zeros for all finite Coxeter groups it suffices to prove that $P(D_n, x)$ is real-rooted for Coxeter groups of type D_n. The real-rootedness of $P(D_n, x)$ is conjectured by Brenti in [8]. It is known that the Eulerian polynomials of type A_n, B_n and D_n are related by, see [8, 27, 33]:

$$P(D_n, x) = P(B_n, x) - n2^{n-1}xP(A_{n-1}, x).$$

This relationship was first noticed by Stembridge [33]. One step towards proving the real-rootedness of $P(D_n, x)$ is to learn more about the relationships between the zeros of $P(B_n, x)$ and $P(A_n, x)$.

Brenti [8] introduced a q-analog of $P(B_n, x)$

$$B_n(x; q) = \sum_{\sigma \in B_n} q^{N(\sigma)} x^{d_B(\sigma)}, \tag{6.2}$$

where $d_B(\sigma)$ is the number of B_n-descents of σ and $N(\sigma)$ is the number of negative entries of σ, see [8]. He proved that

$$\sum_{i \geq 0} ((1+q)i + 1)^n x^i = \frac{B_n(x; q)}{(1-x)^{n+1}}, \tag{6.3}$$

and that $B_n(x; q)$ is real- and simple-rooted for all $q \geq 0$. Suppose that $f(i)$ is a polynomial in i of degree d, then the polynomial $W(f)$ is defined by

$$\sum_{i \geq 0} f(i)x^i = \frac{W(f)(x)}{(1-x)^{d+1}},$$

One can show, see [6], that $\mathcal{E}(f)$ and $W(f)$ are related by:

$$\mathcal{E}(f)(x) = (1+x)^{\deg(f)} W(f)(\frac{x}{1+x}). \tag{6.4}$$

It follows that $W(f)$ has only real non-positive roots if and only if $\mathcal{E}(f)$ is $[-1, 0]$-rooted.

Since $((1+q)i+1)^n$ is a $[-1, 0]$-rooted polynomial in i for any $q \geq 0$ it follows from e.g. Theorem 4.2 that $B_n(x; q)$ is real-rooted in x for any fixed $q \geq 0$. It is natural to generalize $B_n(x; q)$ to have $n+1$ parameters as $B_n(x; \mathbf{q}) := W\big(\prod_{i=0}^n \big((1+q_i)x+1\big)\big)$. This polynomial has a nice combinatorial interpretation:

Theorem 6.4. *For all $n \in \mathbb{N}$ we have:*

$$B_n(x, \mathbf{q}) = \sum_{\sigma \in B_n} q_1^{\chi_1(\sigma)} q_2^{\chi_2(\sigma)} \cdots q_n^{\chi_n(\sigma)} t^{d_B(\sigma)},$$

where

$$\chi_i(\sigma) = \begin{cases} 1 & \text{if } \sigma_i < 0, \\ 0 & \text{if } \sigma_i > 0. \end{cases}$$

Proof. The proof is an obvious generalization of the proof of Theorem 3.4 of [8]. $\square$

Note that this theorem gives a semi-combinatorial interpretation of the W-transform of any $[-1,0)$-rooted polynomial.

Corollary 6.5. *Let $n \in \mathbb{N}$ and let $q_1, q_2, \dots, q_n$ be non-negative real numbers. Then $B_n(x;\mathbf{q})$ has only real and simple zeros.*

We need the following lemma on the degree of $W(f)$.

Lemma 6.6. *Let $f \in \mathbb{R}[x]$. Then*

$$\deg W(f) = \deg f - \mathrm{mult}(-1, \mathcal{E}(f)).$$

Moreover, $\mathrm{mult}(-1,\mathcal{E}(f))$ *is equal to the maximal integer k such that $(x+1)(x+2)\cdots(x+k)$ divides f.*

Proof. Since $\deg \mathcal{E}(f) = \deg f$ for all f we have by (6.4) that $\deg W(f) = \deg f - \mathrm{mult}(-1,\mathcal{E}(f))$. If we expand f in the basis $\{\binom{-x-1}{i}\}$ as:

$$\begin{aligned} f(x) &= \sum_{i\geq 0}(-1)^i a_i \binom{-x-1}{i}, \\ &= \sum_{i \geq 0} \frac{a_i}{i!}(x+1)\cdots(x+i), \end{aligned}$$

we have by Lemma 4.3 that

$$\mathcal{E}(f)(x) = \sum_{i\geq 0} a_i (x+1)^i,$$

and the lemma follows. □

We now have more precise knowledge of the location of the zeros of $B_n(x;q)$ for any given $q \geq 0$.

Theorem 6.7. *Let $0 < q < t \in \mathbb{R}$ and $n > 0$ be an integer. Then*

$$B_n(x;0) \preceq B_n(x;t) \ll B_n(x;q) \ll xB_n(x;0),$$

where the three first polynomials have no common zeros.

Proof. Let $0 < r < s < 1$. Then by the proof of Lemma 4.4 we have

$$\begin{aligned} \mathcal{E}(x^n) \ll \mathcal{E}(x(x+r)^{n-1}) \ll_{strict} \mathcal{E}((x+r)^n) \ll_{strict} \mathcal{E}((x+r)^{n-1}(x+s)) \ll \\ \mathcal{E}((x+s)^n) \ll_{strict} \mathcal{E}((x+s)^{n-1}(x+1)) \ll \mathcal{E}((x+1)^n), \end{aligned}$$

where $\ll_{strict}$ means strictly alternating left of. Since $(x+1)\mathcal{E}(x^n) = x\mathcal{E}((x+1)^n)$ this implies

$$\mathcal{E}(x^n) \ll_{strict} \mathcal{E}((x+r)^n) \ll_{strict} \mathcal{E}((x+s)^n) \ll_{strict} \mathcal{E}((x+1)^n).$$

Now since

$$B_n(x;q) = (q+1)^n W((x+\frac{1}{1+q})^n) = (q+1)^n(1-x)^n\mathcal{E}((x+\frac{1}{1+q})^n)(\frac{x}{1-x}),$$

we see by Lemma 6.6 that $\deg B_n(x;0) = n-1$ and $\deg B_n(x;q) = n$ if $q \neq 0$. Moreover, the alternating property is preserved under the operation (6.4) and the theorem follows. □

It follows from (6.2) that $P(B_n, x) = B_n(x;1)$ and $P(A_n,x) = B_n(x;0)$.

Corollary 6.8. *For all integers $n \geq 1$ we have that $P(A_n, x)$ strictly interlaces $P(B_n, x)$.*

Since $P(A_n, x) \ll xP(A_{n-1}, x)$ and $P(A_n, x) \preceq P(B_n, x)$, we have by Lemma 2.2 that for all $t \geq 0$ the polynomial $P(B_n, x) + txP(A_{n-1}, x)$ is real-rooted. Unfortunately a similar argument does not apply when $t < 0$.

One can extract more from (6.3). Brenti [8] proved that the polynomial

$$\sum_{\sigma \in B_n, N(\sigma) \in \{k, n-k\}} x^{d_B(\sigma)},$$

is real-rooted for all choices of $0 \leq k \leq n$. Using Theorem 4.6 we can extend this result to:

Corollary 6.9. *Let S be any subset of $[0, n]$. Then the polynomial*

$$P(B_n, S; x) := \sum_{\sigma \in B_n, N(\sigma) \in S} x^{d_B(\sigma)},$$

has only real and simple zeros.

Proof. Comparing the coefficient of q^i in both sides of (6.3) we see that $P(B_n, S; x) = W(f_n(S; x))$ where

$$f_n(S; x) = \sum_{s \in S} \binom{n}{s} x^s (x+1)^{n-s}.$$

So the theorem follows from Theorem 4.2. □

One instance of Theorem 6.9 is particularly interesting. Recall that a Coxeter group of type D_n is isomorphic to the subgroup

$$D_n = \{\sigma \in B_n : \ \ 2 \mid N(\sigma)\}$$

Hence, we have the following corollary

Corollary 6.10. *For all $n \in \mathbb{N}$ the polynomial*

$$\sum_{\sigma \in D_n} x^{d_B(\sigma)}$$

has only real and simple zeros.

Note that the above polynomial is not $P(D_n, x)$, since B_n-descents and D_n-descents are not the same.

7. The h-vector of a family of simplicial complexes defined by Fomin and Zelevinsky

Fomin and Zelevinsky [17] recently associated to any finite Weyl group W a simplicial complex $\Delta_{FZ}(W)$. For the classical Weyl groups the corresponding h-polynomials are given by

$$\begin{aligned}
h(\Delta_{FZ}(A_{n-1}), x) &= \frac{1}{n} \sum_{k=0}^{n-1} \binom{n}{k} \binom{n}{k+1} x^k, \\
h(\Delta_{FZ}(B_n), x) &= \sum_{k=0}^{n} \binom{n}{k} \binom{n}{k} x^k, \\
h(\Delta_{FZ}(D_n), x) &= h(\Delta_{FZ}(B_n), x) - nxh(\Delta_{FZ}(A_{n-2}), x).
\end{aligned}$$

It is known that the h-polynomials corresponding to A_n and B_n have only real zeros. We will here show that so has $h(\Delta_{FZ}(D_n), x)$.

Theorem 7.1. *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha \geq 0, 2\alpha + \beta > 0$ and let $n \geq 2$ be an integer. Then the polynomial*

$$F_n(\alpha, \beta) := \alpha h(\Delta_{FZ}(B_n), x) + \beta n x h(\Delta_{FZ}(A_{n-2}), x),$$

is real- and simple-rooted. Moreover, $h(\Delta_{FZ}(B_{n-1}), x)$ strictly interlaces $F_n(\alpha, \beta)$ if $\alpha > 0$ and strictly alternates left of $F_n(\alpha, \beta)$ if $\alpha = 0$.

Corollary 7.2. *Let W be a finite Weyl group. Then $h(\Delta_{FZ}(W), x)$ has only real and simple zeros.*

Proof. For the exceptional Weyl group one can check the real-rootedness ad hoc, see [28]. The other cases follows from Theorem 7.1. $\square$

The *Hadamard product* of two polynomials

$$\begin{aligned} p(x) &= a_0 + a_1 x + \cdots + a_m x^m \\ q(x) &= b_0 + b_1 x + \cdots + b_n x^n \end{aligned}$$

is the polynomial

$$(p \star q)(x) = a_0 b_0 + a_1 b_1 x + \cdots + a_N b_N x^N,$$

where $N = \min(m, n)$. Málo proved that if the zeros of p are real and the zeros of q are real and of the same sign then the zeros of $p \star q$ are real as well. This also follows from Theorem 3.8 since $p \star q = \Gamma[pSq]$ where Γ is the multiplier sequence $\{\frac{1}{k!}\}_{k=0}^{\infty}$. It is known, see e.g. [18], that if f has only real zeros then all zeros of $\Gamma[f]$ are real and simple except for possibly at the origin.

Proof of Theorem 7.1. We may write $F_n(\alpha, \beta)$ as

$$F_n(\alpha, \beta) = \alpha(x+1)f + (2\alpha + \beta)g,$$

where $f = (x+1)^{n-1} \star (x+1)^{n-1}$ and $g = \big(x(x+1)^{n-1}\big) \star (x+1)^{n-1}$.

By the discussion before this proof we have that for all real choices of $\gamma, \delta \in \mathbb{R}$ the polynomial

$$\gamma f + \delta g = \big((\gamma + \delta x)(x+1)^{n-1}\big) \star (x+1)^{n-1},$$

is real- and simple-rooted. By the Obreschkoff theorem we infer that f strictly alternates left of g. Now, since $f \preceq (x+1)f$ and $f \ll g$ we know by Lemma 2.2 that f either interlaces or alternates left of $F_n(\alpha, \beta)$ for all $\alpha, \beta \in \mathbb{R}$ such that $\operatorname{sgn}(\alpha) = \operatorname{sgn}(2\alpha + \beta)$. Moreover, since g and f have no common zeros nor does $F_n(\alpha, \beta)$ and f (provided that $2\alpha + \beta \neq 0$). $\square$

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COUNTEREXAMPLES TO THE NEGGERS-STANLEY CONJECTURE

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ABSTRACT. The Neggers-Stanley conjecture asserts that the polynomial counting the linear extensions of a labeled finite partially ordered set by the number of descents has real zeros only. We provide counterexamples to this conjecture.

A finite partially ordered set (*poset*) P of cardinality p is said to be *labeled* if its elements are identified with the integers $1, 2, \ldots, p$. We will use the symbol $\prec$ to denote the partial order on P and $<$ to denote the usual order on the integers. The *Jordan-Hölder set* $\mathcal{L}(P)$ is the set of permutations $\pi = (\pi_1, \ldots, \pi_p)$ of $[p] \stackrel{\text{def}}{=} \{1, 2, \ldots, p\}$ which encode the linear extensions of P. More precisely, $\pi \in \mathcal{L}(P)$ if $\pi_i \prec \pi_j$ implies $i < j$.

A *descent* in a permutation π is an index i such that $\pi_i > \pi_{i+1}$. Let $\operatorname{des}(\pi)$ denote the number of descents in π. The *W-polynomial* of a labeled poset P is defined by

$$W(P, t) = \sum_{\pi \in \mathcal{L}(P)} t^{\operatorname{des}(\pi)}.$$

W-polynomials appear naturally in many combinatorial contexts [2, 7, 8], and are connected to Hilbert series of the Stanley-Reisner rings of simplicial complexes [10, Section III.7] and algebras with straightening laws [9, Theorem 5.2.].

Example 1. Let $P_{2,2}$ be the labeled poset shown in Figure 1. Then

$$\mathcal{L}(P_{2,2}) = \{(1,3,2,4), (1,3,4,2), (3,1,2,4), (3,1,4,2), (3,4,1,2)\},$$

so $W(P_{2,2}, t) = 4t + t^2$.

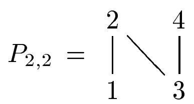

FIGURE 1. The poset $P_{2,2}$.

When P is a p-element antichain, then $\mathcal{L}(P)$ consists of all permutations of $[p]$, and $W(P, t)$ is the pth *Eulerian polynomial*. The Eulerian polynomials are known [3] to have only real zeros. In this instance, the *Neggers-Stanley conjecture* holds:

Conjecture 1 (Neggers-Stanley). *For any finite labeled poset P, all zeros of the polynomial $W(P, t)$ are real.*

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A poset P is *naturally labeled* if $i \prec j$ implies $i < j$. Conjecture 1 was made by J. Neggers [4] in 1978 for naturally labeled posets, and extended by R. P. Stanley in 1986 to arbitrary labelings. It has been proved in some special cases (see [2, 11]). A weaker *unimodality* property of W-polynomials was recently proved [6] (see also [1]) for *graded* naturally labeled posets.

In this note, we construct counterexamples to Conjecture 1 utilizing the following construction. Let $\mathbf{m} \sqcup \mathbf{n}$ denote the disjoint union of the chains $1 \prec 2 \prec \cdots \prec m$ and $m+1 \prec m+2 \prec \cdots \prec m+n$. Let $P_{m,n}$ be the labeled poset obtained by adding the relation $m+1 \prec m$ to the relations in $\mathbf{m} \sqcup \mathbf{n}$; see Figure 2.

$P_{3,4} =$

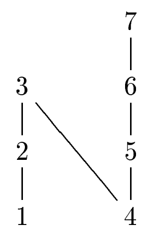

FIGURE 2. The poset $P_{3,4}$.

Theorem 1. *Let M be a positive integer. The polynomial $W(P_{m,n}, t)$ has more than M non-real zeros provided $\min(m, n)$ is sufficiently large.*

The posets $P_{m,n}$ are not naturally labeled, so the original conjecture of Neggers remains open.

The rest of the paper is devoted to the proof of Theorem 1. At the end, we discuss specific (minimal) counterexamples obtained from Theorem 1.

Lemma 1. $W(P_{m,n}, t) = \sum_{k=1}^{\min(n,m)} \binom{m}{k}\binom{n}{k} t^k$.

Proof. Let $\pi = (\pi_1, \pi_2, \ldots)$ be a permutation. If i is a descent in π, we say that π_i is a *descent top* and π_{i+1} is a *descent bottom*. Any $\pi \in \mathcal{L}(\mathbf{m} \sqcup \mathbf{n})$ is uniquely determined by its descent tops (which are necessarily elements of $[m+n] \setminus [m]$) and descent bottoms (which are elements of $[m]$). It follows that the number of permutations in $\mathcal{L}(\mathbf{m} \sqcup \mathbf{n})$ with exactly k descents is $\binom{m}{k}\binom{n}{k}$, implying that $W(\mathbf{m} \sqcup \mathbf{n}, t) = \sum_{k=0}^{\min(n,m)} \binom{m}{k}\binom{n}{k} t^k$. Since the only element of $\mathcal{L}(\mathbf{m} \sqcup \mathbf{n}) \setminus \mathcal{L}(P_{m,n})$ is $(1, 2, \ldots, m+n)$, we have $W(\mathbf{m} \sqcup \mathbf{n}, t) = 1 + W(P_{m,n}, t)$, and Lemma 1 follows. $\square$

We note that all zeros of $W(\mathbf{m} \sqcup \mathbf{n}, t)$ are real and simple (R. Simion [7]).

Proof of Theorem 1. Recall that the *Bessel function* of order 0 is given by

$$J_0(z) = \frac{2}{\pi} \int_0^1 \frac{\cos(zt)}{\sqrt{1-t^2}} dt = \sum_{k=0}^{\infty} \frac{1}{k!k!} \left(\frac{-z^2}{2} \right)^k. \tag{2}$$

It is known that $J_0(z)$ has infinitely many zeros, all of which are real and simple. It follows from (2) that $|J_0(\theta)| \leq 1$ for all real θ, with equality only if $\theta = 0$. Hence the function

$$F(z) = \sum_{k=0}^{\infty} \frac{1}{k!k!} z^k$$

has infinitely many zeros, all of them negative and simple. Also, $|F(\theta)| < 1$ for $\theta < 0$.

Let $f_{m,n}(z) = W(P_{m,n}, z/mn)$. Then

$$f_{m,n}(z) = \sum_{k=1}^{\min(m,n)} \frac{(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k-1}{n})}{k!} \frac{(1-\frac{1}{m})(1-\frac{2}{m})\cdots(1-\frac{k-1}{m})}{k!} z^k.$$

Let $m_1, n_1, m_2, n_2, \dots$ be positive integers such that $\lim_{j\to\infty} \min(m_j, n_j) = \infty$. Then

$$\lim_{j\to\infty} f_{m_j,n_j}(z) + 1 = F(z),$$

where the convergence is uniform on any compact subset of $\mathbb{C}$. Let $(-a, 0)$ be an interval containing more than M zeros of $F(z)$. It follows from Hurwitz's theorem [5, Theorem 1.3.8] that the polynomial $f_{m_j,n_j}(z) + 1$ has more than M zeros in $(-a, 0)$ for sufficiently large j. By continuity we also have $|f_{m_j,n_j}(z) + 1| < 1$ for $z \in (-a, 0)$ and j large. Thus by subtracting 1 from $f_{m_j,n_j}(z) + 1$, we will lose at least M real zeros. □

By applying Sturm's Theorem [5, Section 10.5], one can find specific counterexamples. The polynomial $W(P_{11,11}, t)$ has two non-real zeros which are approximately

$$z = -0.10902 \pm 0.01308i.$$

A counterexample with a polynomial of lower degree is

$$W(P_{36,6}, t) = 216t + 9450t^2 + 142800t^3 + 883575t^4 + 2261952t^5 + 1947792t^6.$$

This polynomial has two non-real zeros.

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