THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Spectral properties of quantum mechanical operators with magnetic field

Mikael Persson

CHALMERS | UNIVERSITY OF GOTHENBURG

Department of Mathematical Sciences
Chalmers University of Technology
and Gothenburg University
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Mikael Persson
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Department of Mathematical Sciences
Division of Mathematics
Chalmers University of Technology and University of Gothenburg
SE-412 96 Göteborg
Sweden
Telephone +46 (0)31-772 1000

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Abstract: This thesis contains four papers on spectral theory of magnetic quantum operators in even dimensions.

In the first two papers we consider the interaction of a charged particle with spin 1/2, such as the electron, and a very singular magnetic field in two dimensions. In quantum mechanics this situation is described by the Pauli Hamiltonian. The singularities of the magnetic field enables several different self-adjoint Pauli Hamiltonians to exist, each of them describing the situation differently. It is the different ways the particles interact with the singularities of the field that gives different realizations.

We discuss some natural physical properties that the Pauli Hamiltonian should satisfy, and compare some of the extensions. The result is that no realization studied satisfies all the wanted properties. Along the way we show how many bound states the different extensions have, giving some variants of the classical Aharonov-Casher theorem.

In the third paper, we study the Pauli operator corresponding to a regular magnetic field in higher even-dimensional Euclidean space. We try to correct a mistake in a paper from 1993 about the number of bound states, and succeed partially. The main result in the third paper is that zero is not an eigenvalue if the magnetic field decays faster than quadratically at infinity. However, if the magnetic field decays quadratically, then zero might be an eigenvalue, and we give a lower bound for its multiplicity. The methods are based on complex analysis which restricts the types of magnetic fields studied.

In the fourth paper we consider perturbations of the Landau Hamiltonian in even-dimensional Euclidean space. We perturb by introducing a compact obstacle, imposing magnetic Neumann conditions at the boundary. Several different perturbations of the Landau Hamiltonian have been studied lately, such as perturbing by electric field, magnetic field and by an obstacle with Dirichlet boundary conditions. For weak perturbations the rate of accumulation of the eigenvalues are the same for the different perturbations.

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Papers in this thesis

Paper I: M. Persson. On the Aharonov-Casher formula for different self-adjoint extensions of the Pauli operator with singular magnetic field. *Electron. J. Differential Equations*, 2005(55):1–16 (electronic), 2005.

Paper II: M. Persson. On the Dirac and Pauli operators with several Aharonov-Bohm solenoids. *Lett. Math. Phys.*, 78 (2006), no. 2, 139–156.

Paper III: M. Persson. Zero modes for the magnetic Pauli operator in even-dimensional Euclidean space. *Submitted to Lett. Math. Phys.*

Paper IV: M. Persson. Eigenvalue asymptotics of the even-dimensional exterior Landau-Neumann Hamiltonian. *To be submitted.*

Introduction

The influence on a spin-free particle of charge q from a magnetic field $\vec{B}$ and a potential field $\vec{E}$, both constant in time, in three-dimensional Euclidean space is in classical Newtonian physics given by the so-called Lorentz force

$$\vec{F} = q\left(\vec{v} \times \vec{B} + \vec{E}\right).$$

Here $\vec{v}$ is the velocity of the particle. According to Newton's second law, this should equal $m\dot{\vec{v}}$, where m is the mass of the particle. The energy of the system is described by the Hamiltonian H, which is given by the sum of kinetic and potential energy

$$H = \frac{1}{2m}\left(\vec{p} - q\vec{a}\right)^2 + V. \tag{1.1}$$

Here $\vec{p}$ is the momentum of the particle, $\vec{p} = m\vec{v} + q\vec{a}$, $\vec{a}$ is the magnetic vector potential, $\vec{B} = \operatorname{curl}\vec{a}$, and V is the potential energy, $\nabla V = -q\vec{E}$. These two last conditions imply that $\vec{B}$ is divergence-free and $\vec{E}$ is rotation-free,

$$\begin{cases} \operatorname{div}\vec{B} = 0; \\ \operatorname{curl}\vec{E} = 0. \end{cases} \tag{1.2}$$

These equations are called the Maxwell's equations. In Section 1 we transfer the concept of magnetic fields from $\mathbb{R}^3$ to $\mathbb{R}^n$, $n \geq 2$, and introduce different kind of magnetic potentials.

According to the correspondence principle, quantities in classical physics should correspond to operators in quantum physics. The momentum operator $\vec{p}$ is given by $\vec{p} = -i\hbar\nabla$, where $\hbar$ denotes the Planck constant divided by 2π. The energy operator H in (1.1) becomes

$$H = \frac{1}{2m}\left(-i\hbar\nabla - q\vec{a}\right)^2 + V.$$

In this thesis we work without potential field $\vec{E}$, and we choose dimensionless units $\hbar = 1$, $q = 1$ and $m = 1/2$, so the Hamiltonian H reads

$$H = \left(-i\nabla - \vec{a}\right)^2.$$

To define H in a satisfactory way, we have to tell on what kind of functions it should operate. This is in general a nontrivial task, which we will discuss more in Section 2. We will return to the Hamiltonians and the quantum mechanics in Section 3. In Sections 4–6 we discuss the four papers in this thesis and put them into their context. We end this introduction by discussing some open problems and suggestions for further research in this topic in Section 7.

1 Magnetic fields

In this section we discuss the notion of magnetic fields in $\mathbb{R}^n$ in general, magnetic vector and scalar potentials and the Aharonov-Bohm magnetic field in $\mathbb{R}^2$.

1.1 Magnetic fields in Euclidean space

A common object we study in all papers in this thesis is the magnetic field, which by definition is a field that satisfies Maxwell's equations, which in three dimensions, as we just saw, says that the field $\vec{B}$ should be divergence-free. If we write $\vec{B} = (B_1, B_2, B_3)$, then this means that

$$\frac{\partial B_1}{\partial x^1} + \frac{\partial B_2}{\partial x^2} + \frac{\partial B_3}{\partial x^3} = 0 \tag{1.3}$$

Here and elsewhere we use the standard coordinates $x = (x^1, \ldots, x^n)$ in $\mathbb{R}^n$. Let us introduce the 2-form

$$B(x) = B_3(x)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 + B_1(x)\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 + B_2(x)\,\mathrm{d}x^3 \wedge \mathrm{d}x^1.$$

Then (1.3) is equivalent to the equation

$$\mathrm{d}B = 0, \tag{1.4}$$

where d denotes the exterior derivative. It turns out that (1.4) is a form of Maxwell's equations for a magnetic field that is easily generalized to n dimensions. Thus, by a magnetic field B in $\mathbb{R}^n$ we mean a real 2-form B satisfying $\mathrm{d}B = 0$, i.e. a real closed 2-form. Any magnetic field B can be written as

$$B(x) = \sum_{j<k} b_{j,k}(x)\,\mathrm{d}x^j \wedge \mathrm{d}x^k,$$

where $b_{j,k}$ are real-valued functions. With this notation equation (1.4) becomes

$$0 = \mathrm{d}B = \sum_{j<k<l} \left(\frac{\partial b_{j,k}}{\partial x^l} - \frac{\partial b_{j,l}}{\partial x^k} + \frac{\partial b_{k,l}}{\partial x^j} \right) \mathrm{d}x^j \wedge \mathrm{d}x^k \wedge \mathrm{d}x^l.$$

In this thesis we only work in even dimensions. In the following example we see how the two-dimensional case fits into $\mathbb{R}^3$.

Example 1.1 A magnetic field in $\mathbb{R}^2$ is just a real two-form

$$B = b_{1,2}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2.$$

All such forms are automatically closed, so equation (1.4) is superfluous. We see that we have the same situation if we introduce a third coordinate x^3 but let the magnetic field be the same, not depending on x^3. Then, as we saw in the previous example, it is identified with the magnetic field

$$\vec{B} = \big(0, 0, b_{1,2}(x^1, x^2)\big).$$

Hence, in two dimensions we can interpret the magnetic field as something two-dimensional or we can introduce a third coordinate x^3 and think of the magnetic field as a vector field orthogonal to the plane $x^3 = 0$. For simplicity we identify the magnetic field B with the function $b_{1,2}$.

1.2 Magnetic vector potentials

Given a magnetic field B there exists by the Poincaré lemma a one-form

$$a(x) = \sum_{j=1}^{n} a_j(x)\,\mathrm{d}x^j \tag{1.5}$$

such that

$$B = \mathrm{d}a = \sum_{j<k} \left(\frac{\partial a_k}{\partial x^j} - \frac{\partial a_j}{\partial x^k} \right) \mathrm{d}x^j \wedge \mathrm{d}x^k.$$

The one-form a is usually called the magnetic one-form or the magnetic vector potential. The latter term is motivated by the identification of the one-form a in (1.5) and the vector

$$\vec{a} = (a_1, \dots, a_n).$$

The solution a to the equation $B = \mathrm{d}a$ is not unique. In fact, given one such solution a, another one is given by $\tilde{a} = a + \mathrm{d}f$, where f is a not too singular function. This follows easily by the fact that $\mathrm{d}^2 = 0$,

$$\mathrm{d}\tilde{a} = \mathrm{d}(a + \mathrm{d}f) = \mathrm{d}a + \mathrm{d}^2 f = B.$$

The choice of magnetic one-form is referred to as the choice of gauge.

Example 1.2 To the constant magnetic field of strength B_0 in $\mathbb{R}^2$,

$$B = B_0 \, \mathrm{d}x^1 \wedge \mathrm{d}x^2,$$

one usually works in the *symmetric gauge*

$$a(x) = \frac{B_0}{2} \left(-x^2 \, \mathrm{d}x^1 + x^1 \, \mathrm{d}x^2\right).$$

Another choice is the *Landau gauge*, given by $\tilde{a}(x) = -B_0 x^2 \, \mathrm{d}x^1$.

1.3 Scalar potentials

In some situations it is also possible to associate a scalar potential W to a magnetic field B. A scalar potential W in two dimensions is by definition a function $W\colon \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$-\Delta W \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 = B,$$

or just

$$-\Delta W = B,$$

since we often identify the magnetic field and its coefficient function in two dimensions. We note that the scalar potential is unique only up to addition of harmonic functions. Since $-\Delta \log|x| = 2\pi\delta_0$ in the sense of distributions, one choice of W is

$$W(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} B(y) \log|x - y| \, \mathrm{d}m(y), \tag{1.6}$$

at least when the magnetic field B is sufficient regular so the integral makes sense. Here, and elsewhere, $\mathrm{d}m$ denotes the Lebesgue measure.

In higher dimensions it is not clear in general what one should mean by a scalar potential. However, in even dimensions it is possible to introduce a suitable notion for special types of magnetic fields. Let

$$B(x) = \sum_{j<k} b_{j,k}(x) \, \mathrm{d}x^j \wedge \mathrm{d}x^k \tag{1.7}$$

be a magnetic field in $\mathbb{R}^{2d}$. We identify the point $x = (x^1, \ldots, x^{2d})$ in $\mathbb{R}^{2d}$ with $z = (z^1, \ldots, z^d)$ in $\mathbb{C}^d$, where $z^j = x^{2j-1} + i x^{2j}$, and define tangent and cotangent vectors by

$$\begin{aligned}
\frac{\partial}{\partial z^j} &= \frac{1}{2}\left(\frac{\partial}{\partial x^{2j-1}} - i\frac{\partial}{\partial x^{2j}}\right), \\
\frac{\partial}{\partial \bar{z}^j} &= \frac{1}{2}\left(\frac{\partial}{\partial x^{2j-1}} + i\frac{\partial}{\partial x^{2j}}\right), \\
\mathrm{d}z^j &= \mathrm{d}x^{2j-1} + i\,\mathrm{d}x^{2j}, \quad \text{and} \\
\mathrm{d}\bar{z}^j &= \mathrm{d}x^{2j-1} - i\,\mathrm{d}x^{2j}.
\end{aligned}$$

In the case $d = 1$ we write $z = x^1 + i x^2$.

Written in complex terms, the magnetic field (1.7) can be written as a sum of (1, 1), (2, 0), and (0, 2) type forms as

$$B(z) = \sum_{j,k=1}^{d} \mathfrak{b}_{j,k}(z)\,\mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k + \sum_{j,k=1}^{d} \mathfrak{b}'_{j,k}(z)\,\mathrm{d}z^j \wedge \mathrm{d}z^k + \sum_{j,k=1}^{d} \mathfrak{b}''_{j,k}(z)\,\mathrm{d}\bar{z}^j \wedge \mathrm{d}\bar{z}^k.$$

The magnetic field B is of type (1, 1) if $\mathfrak{b}'_{j,k} = \mathfrak{b}''_{j,k} = 0$ for all j and k, i.e. if it can be written in the form

$$B(z) = \sum_{j,k=1}^{d} \mathfrak{b}_{j,k}(z)\,\mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k. \tag{1.8}$$

To such a (1, 1)-type magnetic field one can associate a scalar potential in the sense that

$$B = 2i\bar{\partial}\partial W = -2i\sum_{j,k=1}^{d} \frac{\partial^2 W}{\partial z^j \partial \bar{z}^k}\,\mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k. \tag{1.9}$$

In $\mathbb{R}^2$ all magnetic fields are of type (1, 1), and we see that the latter definition of scalar potential agrees with the first one,

$$B = -2i\frac{\partial^2 W}{\partial z \partial \bar{z}}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} = -\Delta W\,\mathrm{d}x^1 \wedge \mathrm{d}x^2.$$

Proposition 1.1 *Let B be a magnetic field of type (1, 1) in $\mathbb{R}^{2d}$ with smooth coefficient functions. Then there exists a smooth function W that is a scalar potential to B in the sense of (1.9).*

Proof See Lemma II.2.15 in [Wel80]. □

1.4 Aharonov-Bohm magnetic fields

Above, we have not given any regularity restrictions on the coefficient functions $b_{j,k}$ of the magnetic field. In the first two papers we work with very singular magnetic fields in two dimensions. In the paper [AB59] a magnetic field having support in only one point was introduced. Mathematically, the coefficient function of such a magnetic field is the Dirac delta distribution, so the magnetic two-form becomes a current. Such a magnetic field is called an Aharonov-Bohm (AB) solenoid. An experiment was proposed in [AB59] where one has a source sending electrons along two different paths, γ_1 and γ_2, running on different sides of the support of the AB solenoid. On the other side an observation screen is receiving the electrons, see Figure 1.1. They proposed that even though the electrons did not travel in any magnetic field, there would be a phase shift in the wave functions. This is called the AB effect, and has later been verified in experiments, see [Cha60]. It seems that a similar effect was proposed in [Lon48]. More information about the AB effect can be found in [PT89], and the references therein.

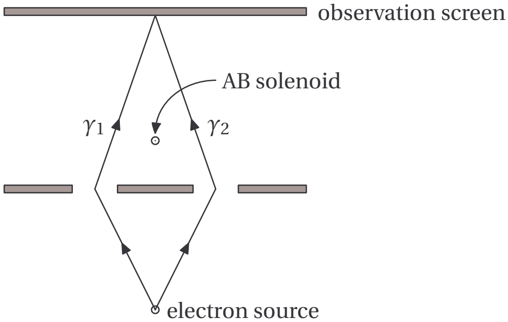


Figure 1.1 The experiment proposed by Aharonov and Bohm.

Even though the magnetic field is zero outside the singular point, the magnetic one-form is not. This suggests that the magnetic one-form is not just a mathematical fiction, but actually something with a physical meaning. Let us write the formulas for the AB field and its potentials. We introduce an AB solenoid at the origin. Then

$$B(x) = 2\pi\alpha\delta(x)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2,$$

where α is called the AB intensity (we use the term intensity for the flux divided by 2π) of the solenoid. It is easy to see that the one-form

$$a(x) = \frac{\alpha}{|x|^2}\left(-x^2\,\mathrm{d}x^1 + x^1\,\mathrm{d}x^2\right)$$

satisfies $\mathrm{d}a = B$ and that the function

$$W(x) = \alpha \log|x|$$

satisfies $-\Delta W\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 = B$ *in the sense of distributions*.

2 Self-adjoint operators

In this section we take a closer look at self-adjoint operators. General references for this subject are [Kat76, Wei80, BS87, AG93], and the classical series of books [RS80, RS75, RS79, RS78]. We recall that a Hilbert space $\mathcal{H}$ is a vector space that has an inner product $\langle\cdot,\cdot\rangle$ such that the metric defined via the norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ in the natural way is complete.

A (linear) operator in $\mathcal{H}$ is a mapping $T : \mathrm{Dom}(T) \subset \mathcal{H} \to \mathcal{H}$ such that

$$T(\alpha u + \beta v) = \alpha T u + \beta T v \quad \text{for all } u, v \in \mathrm{Dom}(T) \text{ and } \alpha, \beta \in \mathbb{C}.$$

$\mathrm{Dom}(T)$ is called the *domain* of T. If $\mathrm{Dom}(T) = \mathcal{H}$ then one usually say that T is an operator *on* $\mathcal{H}$. The *range* of T is defined as $\mathrm{Ran}(T) = T(\mathrm{Dom}(T))$.

2.1 Unbounded operators

A linear operator T in $\mathcal{H}$ is *bounded* if there exists a constant C such that

$$\|Tu\| \le C\|u\| \quad \text{for all } u \in \mathrm{Dom}(T). \tag{2.1}$$

Otherwise T is *unbounded*. An operator is *densely defined* if its domain is a dense subset of $\mathcal{H}$, i.e. $\overline{\mathrm{Dom}(T)} = \mathcal{H}$.

Example 2.1 Let $\mathcal{H} = L_2([0,1])$ with the inner product

$$\langle u, v\rangle = \int_0^1 u(x)\overline{v(x)}\,\mathrm{d}m(x).$$

We define the operator T_0 in $\mathcal{H}$ as

$$\mathrm{Dom}(T_0) = \left\{ u \in C^1([0,1]) \;\middle|\; u(0) = 0 = u(1) \right\};$$
$$T_0 u = -iu', \quad u \in \mathrm{Dom}(T_0).$$

The operator T_0 is densely defined, since $\mathrm{Dom}(T_0)$ is dense in $\mathcal{H}$. To see that T_0 is unbounded, let $u_j(x) = \frac{1}{\sqrt{2}}\sin(j\pi x)$, $j = 1, 2, \ldots$. Then $u_j \in \mathrm{Dom}(T_0)$ and $\|u_j\| = 1$ for all j. However $\|T_0 u_j\| = \pi j$, so there exists no C such that (2.1) holds for all elements in the $\mathrm{Dom}(T_0)$.

For a densely defined operator T in $\mathcal{H}$ one can define the *adjoint operator* T^*. The domain of T^* consists of all elements $v \in \mathcal{H}$ such that the linear functional

$$l(u) = \langle Tu, v\rangle$$

is a continuous on $\mathrm{Dom}(T)$. Then, according to the Riesz representation theorem, there exists a unique element $\tilde{v}$ (depending on v) such that

$$\langle Tu, v\rangle = \langle u, \tilde{v}\rangle$$

for all $u \in \mathrm{Dom}(T)$. The adjoint T^* is defined as $T^*v = \tilde{v}$. One easily checks that the operator T has to be densely defined in order to obtain a unique adjoint T^*.

The operator S is said to be an *extension* of the operator T, written $T \subset S$, if $\mathrm{Dom}(T) \subset \mathrm{Dom}(S)$ and $Tu = Su$ for all $u \in \mathrm{Dom}(T)$.

An operator T is *symmetric* if it is densely defined and $T \subset T^*$. This is equivalent to the condition that

$$\langle Tu, v\rangle = \langle u, Tv\rangle, \quad \text{for all } u, v \in \mathrm{Dom}(T).$$

Example 2.1 *(continued)* We show that T_0 is symmetric, using integration by parts. For $u, v \in \mathrm{Dom}(T_0)$,

$$\begin{aligned}\langle T_0 u, v\rangle &= \int_0^1 -iu'(x)\overline{v(x)}\,\mathrm{d}x \\ &= -iu(1)\overline{v(1)} + iu(0)\overline{v(0)} + \int_0^1 u(x)\cdot\overline{-iv'(x)}\,\mathrm{d}x \\ &= \langle u, T_0 v\rangle.\end{aligned}$$

Here we used the property that the elements in $\mathrm{Dom}(T_0)$ vanishes at the endpoints.

An operator T is *closed* if $u_j \in \mathrm{Dom}(T)$, $u_j \to u$ and $Tu_j \to v$ implies that $u \in \mathrm{Dom}(T)$ and $Tu = v$. It is *closable* if it has a closed extension. T is closable if and only if $u_j \in \mathrm{Dom}(T)$, $u_j \to 0$ and $Tu_j \to v$ implies that $v = 0$. If T is closable it has a minimal closed extension $\overline{T}$, called the *closure* of T.

Proposition 2.1 *If an operator is symmetric then it is closable.*

Proof Let T be a symmetric operator. Then $T \subset T^*$ and T^* is closed. □

Example 2.1 *(continued)* We show that T_0 is not closed. Let $\{u_j\}$ be the sequence

$$u_j(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2} - \frac{1}{j}; \\ \frac{j}{2}x(1-x) - \frac{(j-2)^2}{8j}, & \frac{1}{2} - \frac{1}{j} < x < \frac{1}{2} + \frac{1}{j}; \\ 1-x, & \frac{1}{2} + \frac{1}{j} \le x \le 1. \end{cases}$$

In Figure 2.1 we see the graphs of some of the functions u_j together with the graph of the function

$$u(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2}; \\ 1-x, & \frac{1}{2} < x \le 1. \end{cases}$$

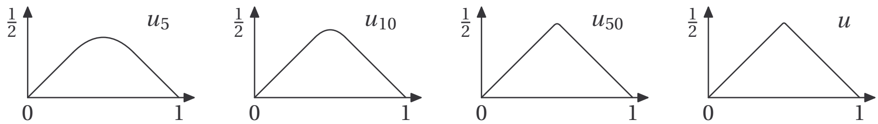


Figure 2.1

It is clear that $u_j \in \mathrm{Dom}(T_0)$ for all j and that $u_j \to u$ in $\mathcal{H}$. Moreover, $T_0 u_j$ is given by

$$T_0 u_j(x) = \begin{cases} -i, & 0 \le x \le \frac{1}{2} - \frac{1}{j}; \\ \frac{ij}{2}(2x-1), & \frac{1}{2} - \frac{1}{j} < x < \frac{1}{2} + \frac{1}{j}; \\ i, & \frac{1}{2} + \frac{1}{j} \le x \le 1; \end{cases}$$

so $T_0 u_j \to v$ in $\mathcal{H}$, where

$$v(x) = \begin{cases} -i, & 0 \le x < \frac{1}{2}; \\ 0, & x = \frac{1}{2}; \\ i, & \frac{1}{2} < x \le 1. \end{cases}$$

Since u is not differentiable at the point $x = \frac{1}{2}$, it does not belong to $\mathrm{Dom}(T_0)$, so the operator T_0 is not closed. However, it seems natural to include functions like u in the domain, since if we take the derivative of u in the sense of distributions we get an element in $\mathcal{H}$. This is done by taking the closure of T_0. Indeed, one can show that

$$\operatorname{Dom}(\overline{T}_0) = \{ u \in \mathcal{H} \mid -iu' \in \mathcal{H} \text{ and } u(0) = 0 = u(1) \};$$
$$\overline{T}_0 u = -iu', \quad u \in \operatorname{Dom}(\overline{T}_0).$$

Here the derivative is the distribution derivative.

Even though we only consider closable/closed operators in this thesis, we give here an example of an operator S that is not closable.

Example 2.2 Let $\mathcal{H} = L_2([0,1])$, and define S as

$$\operatorname{Dom}(S) = C([0,1]);$$
$$(Su)(x) = u(0), \quad u \in \operatorname{Dom}(S).$$

The functions $u_j(x) = (1-x)^j$, $j = 1, 2, \ldots$, belong to $\operatorname{Dom}(S)$, and $u_j \to 0$ in $\mathcal{H}$. However $Su_j = 1$ (the constant function one) for all $j = 1, 2, \ldots$, so the operator S is not closable.

An operator T in $\mathcal{H}$ is *self-adjoint* if $T = T^*$, and *essentially self-adjoint* if $\overline{T}$ is self-adjoint.

Example 2.1 *(continued)* The operator T_0 is not essentially self-adjoint. The reason for this is that the domain of $\overline{T}_0$ contains too few elements. One can show that

$$\operatorname{Dom}(\overline{T}_0^*) = \{ u \in \mathcal{H} \mid -iu' \in \mathcal{H} \};$$
$$\overline{T}_0^* u = -iu', \quad u \in \operatorname{Dom}(\overline{T}_0^*);$$

i.e. no restrictions are made on the boundary values of elements in $\operatorname{Dom}(\overline{T}_0^*)$.

One question that naturally arises is if T_0 has any self-adjoint extensions? The answer is yes, it has uncountably many. The way to obtain them is to balance the boundary conditions. This is generally done by a method of Kreĭn and von Neumann, which we look at below. □

2.2 Self-adjoint extensions of symmetric operators

Let T be a closed symmetric operator and let $\lambda \in \mathbb{C}$. The point λ is said to be a *quasi-regular point* if $T - \lambda I$ has a continuous inverse on $\operatorname{Ran}(T - \lambda I)$, i.e. there exists a constant $C > 0$ such that

$$\|(T-\lambda I)u\| \geq C\|u\|, \quad u \in \mathrm{Dom}(T).$$

The set of quasi-regular points of T is denoted by $\hat{\rho}(T)$. It is an open set in $\mathbb{C}$. The *defect number* of T at λ, denoted $d_T(\lambda)$, is defined as

$$d_T(\lambda) = \mathrm{codim}\left(\overline{\mathrm{Ran}(T-\lambda I)}\right) = \dim\left(\mathcal{H} \ominus \overline{\mathrm{Ran}(T-\lambda I)}\right).$$

The Hilbert space $\mathcal{H}$ can be written as $\mathcal{H} = \overline{\mathrm{Ran}(T-\lambda I)} \oplus \mathrm{Ker}(T^* - \bar{\lambda} I)$, so the defect number can be written as

$$d_T(\lambda) = \dim \mathrm{Ker}(T^* - \bar{\lambda} I).$$

The defect number is stable under small perturbations. It follows that it is constant on each connected component of $\hat{\rho}(T)$.

Proposition 2.2 *Let T be closed and symmetric. The half planes* $\mathrm{Im}\,\lambda > 0$ *and* $\mathrm{Im}\,\lambda < 0$ *are contained in* $\hat{\rho}(T)$.

Proof Let $\mathrm{Im}\,\lambda \neq 0$. Then, for $u \in \mathrm{Dom}(T)$,

$$\|(T-\lambda I)u\|^2 = \|(T - \mathrm{Re}\,\lambda I)u\|^2 + \|(\mathrm{Im}\,\lambda)u\|^2 \geq |\mathrm{Im}\,\lambda|^2 \|u\|^2,$$

so $\|(T-\lambda)u\| \geq |\mathrm{Im}\,\lambda| \|u\|$. □

We define the *deficiency numbers* $n_\pm$ of T as

$$n_\pm(T) = d_T(\pm i) = \dim \ker(T^* \pm iI).$$

We also denote by $N_\pm$ the *deficiency spaces* $N_\pm = \ker(T^* \pm iI)$.

Example 2.1 *(continued)* The symmetric operator $\overline{T}_0$ has deficiency indices $(n_+(T), n_-(T)) = (1,1)$. Indeed, the deficiency space N_+ consists of all functions $u \in \mathrm{Dom}(T^*)$ such that $-iu' + iu = 0$. This space is one-dimensional and is spanned by the function $u_+(x) = e^x$. Similarly, one easily checks that N_- is spanned by $u_-(x) = e^{1-x}$.

2.2.1 The method of Kreĭn and von Neumann

Theorem 2.3 *A symmetric operator T is self-adjoint if and only if its deficiency indices equal zero. It has self-adjoint extensions if and only if the deficiency indices are equal.*

Let T be a closed symmetric operator in $\mathcal{H}$ with equal deficiency indices $n_+(T) = n_-(T) > 0$. Also, let $U : N_+ \to N_-$ be a unitary operator and define the operator T_U as follows

$$\mathrm{Dom}(T_U) = \mathrm{Dom}(T) + \left\{ u_1 + Uu_1 \;\middle|\; u_1 \in N_+ \right\};$$
$$T_U(u_0 + u_1 + Uu_1) = Tu_0 + iu_1 - iUu_1; \quad u_0 \in \mathrm{Dom}(T),\ u_1 \in N_+.$$

Theorem 2.4 *The operator T_U is self-adjoint. Moreover, all self-adjoint extensions of T are obtained in this way.*

We refer to [Wei80] for the proofs of these theorems, but let us explain the extension procedure. The problem of extending a symmetric operator to a self-adjoint operator is reduced to extending an isometric operator to a unitary operator via the Cayley transform. We introduce the Cayley transform V of T as

$$\mathrm{Dom}(V) = \mathrm{Ran}(T + i);$$
$$V = (T - i)(T + i)^{-1}.$$

It is clear that V maps $\mathrm{Ran}(T + i)$ onto $\mathrm{Ran}(T - i)$ and that V is isometric. Moreover, the symmetric operator T can be recovered from T,

$$\mathrm{Dom}(T) = \mathrm{Ran}(I + V);$$
$$T = i(I + V)(I - V)^{-1}.$$

If V_1 and V_2 are the Cayley transforms of the symmetric operators T_1 and T_2 then V_2 is an extension of V_1 if and only if T_2 is an extension of T_1. The operator T is self-adjoint if and only if its Cayley transform V is unitary. It follows from the closedness of T that the sets $\mathrm{Dom}(V) = \mathrm{Ran}(T + i)$ and $\mathrm{Ran}(V) = \mathrm{Ran}(T - i)$ are closed in $\mathcal{H}$. Since the orthogonal complement of these spaces are the deficiency spaces N_+ and N_-, the extending procedure is trivial, and the result is given in the theorem.

Example 2.1 *(continued)* Using Theorem 2.4, we are now able to describe all self-adjoint extensions of $\overline{T}_0$. We know that the deficiency indices of $\overline{T}_0$ are $(1, 1)$, so the extensions are parameterized by a one-dimensional unitary transform $U : N_+ \to N_-$. Any such operator U maps u_+ to $e^{i\theta}u_-$ for some $\theta \in [0, 2\pi)$. Thus, all self-adjoint extensions T_θ of $\overline{T}_0$ are (we use θ as a parameter instead of U)

$$\operatorname{Dom}(T_\theta) = \operatorname{Dom}\left(\overline{T}_0\right) + \operatorname{span}\left\{e^x + e^{i\theta}e^{1-x}\right\};$$

$$T_\theta\left(u + c(e^x + e^{i\theta}e^{1-x})\right) = -iu' + c(ie^x - ie^{i\theta}e^{1-x}); \quad u \in \operatorname{Dom}\left(\overline{T}_0\right),\ c \in \mathbb{C}.$$

It is often convenient to describe the domains of the self-adjoint extensions via the boundary-values instead. Indeed, for all elements u in $\operatorname{Dom}(T_\theta)$ we have

$$\frac{u(1)}{u(0)} = \frac{e + e^{i\theta}}{1 + e^{i\theta+1}} =: \Theta(\theta)$$

It is easily seen that Θ is a bijective map from $[0, 2\pi)$ onto the unit circle. The result is that we can write the self-adjoint extensions as

$$\operatorname{Dom}(T_\theta) = \left\{u \in \mathcal{H} \,\middle|\, -iu' \in \mathcal{H} \text{ and } u(1) = \Theta(\theta)u(1)\right\};$$

$$T_\theta u = -iu', \quad u \in \operatorname{Dom}(T_\theta).$$

Example 2.3 Let $\mathcal{H} = L_2([0,\infty))$, and define T as

$$\operatorname{Dom}(T) = C_0^1([0,\infty));$$

$$Tu = -iu', \quad u \in \operatorname{Dom}(T).$$

It holds that $\operatorname{Dom}(T)$ is dense in $\mathcal{H}$ and that T is symmetric. The closure of T is given by

$$\operatorname{Dom}\left(\overline{T}\right) = \left\{u \in \mathcal{H} \,\middle|\, -iu' \in \mathcal{H} \text{ and } u(0) = 0\right\};$$

$$\overline{T}u = -iu', \quad u \in \operatorname{Dom}\left(\overline{T}\right);$$

and its adjoint is given by

$$\operatorname{Dom}\left(\overline{T}^*\right) = \left\{u \in \mathcal{H} \,\middle|\, -iu' \in \mathcal{H}\right\};$$

$$\overline{T}^* u = -iu', \quad u \in \operatorname{Dom}\left(\overline{T}^*\right).$$

It is easily seen that the deficiency space $N_+ = \{0\}$ and that N_- is spanned by the function e^{-x}, so the deficiency indices of $\overline{T}$ are $(0,1)$. The operator T is an example of a symmetric operator that has no self-adjoint extension.

2.2.2 Extensions of semi-bounded symmetric operators

A symmetric operator T is *bounded below* if there exists a constant $m > -\infty$ such that

$$\langle Tu, u\rangle \geq m\|u\|^2, \quad u \in \mathrm{Dom}(T).$$

The maximal m for which the inequality above is still valid is called the *greatest lower bound* of T and we denote it by m_T.

Proposition 2.5 *Let T be bounded below with bound m_T. Then*

$$(-\infty, m_T) \subset \hat{\rho}(T),$$

and so $n_+(T) = n_-(T)$.

Proof Let $\lambda < m_T$. Then

$$\|(T - \lambda I)u\|\|u\| \geq \langle (T - \lambda I)u, u\rangle \geq (m_T - \lambda)\|u\|^2, \quad u \in \mathrm{Dom}(T),$$

so $\|(T - \lambda I)u\| \geq (m_T - \lambda)\|u\|$, i.e. $\lambda \in \hat{\rho}(T)$. □

Since the deficiency indices of a semi-bounded operator are equal, it has self-adjoint extensions according to Theorem 2.3. One of them is called the Friedrichs extension, and we describe below how to obtain it.

We need some terminology of sesquilinear forms. A sesquilinear form t in a Hilbert space $\mathcal{H}$ is a mapping

$$t : \mathrm{Dom}(t) \times \mathrm{Dom}(t) \subset \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

which is linear in the first argument and anti-linear in the second. One also speaks about the corresponding quadratic form $t(u, u)$. A sesqui-linear form t is semi-bounded (from below) if there exists a constant m such that $t(u, u) \geq m\|u\|^2$ for all $u \in \mathrm{Dom}(t)$. We denote by m_t the maximal lower bound of t. In this case one can define a new inner product $\langle\cdot,\cdot\rangle_t$ on $\mathrm{Dom}(t)$ as

$$\langle u, v\rangle_t = (1 - m_t)\langle u, v\rangle + t(u, v)$$

with the property $\|u\|_t \geq \|u\|$ for all $u \in \mathrm{Dom}(t)$. Here $\|\cdot\|_t$ is the norm equipped with the inner product $\langle\cdot,\cdot\rangle_t$. The form t is said to be *closable* if for every sequence $u_j \in \mathrm{Dom}(t)$ that is Cauchy in $\|\cdot\|_t$ and $\|u_j\| \to 0$ it holds that $\|u_j\|_t \to 0$.

We also define a new Hilbert space $\mathcal{H}_t$ as the $\|\cdot\|_t$-completion of $\mathrm{Dom}(t)$. The inclusion map $i : \mathrm{Dom}(T) \to \mathcal{H}$ is bounded with norm not greater than one, and thus extends to $\mathcal{H}_t$ with the same norm. The closedness of t implies that the inclusion map is injective, so $\mathcal{H}_t$ can be thought of as a subspace of $\mathcal{H}$. The sesquilinear form

$$\mathrm{Dom}(\bar{t}) = \mathcal{H}_t;$$
$$\bar{t}(u,v) = \langle u,v\rangle_t - (1-m)\langle u,v\rangle, \quad u,v \in \mathrm{Dom}(\bar{t});$$

is called the *closure* of t in $\mathcal{H}$.

Now let T be a closed symmetric operator with lower bound m_T. We define a sesquilinear form t as

$$\mathrm{Dom}(t) = \mathrm{Dom}(T);$$
$$t(u,v) = \langle Tu, v\rangle, \quad u,v \in \mathrm{Dom}(t).$$

Then clearly t is symmetric, in the sense that $t(u,v) = t(v,u)$ for all $u, v \in \mathrm{Dom}(t)$. The symmetry of T implies that t is closable. Moreover, the closure $\bar{t}$ has the same lower bound $m_{\bar{t}} = m_T$ as T. Define the operator $\widetilde{T}$ as

$$\mathrm{Dom}\big(\widetilde{T}\big) = \Big\{ u \in \mathcal{H}_t \;\Big|\; \text{there exists a } v \in \mathcal{H} \text{ such that } \bar{t}(u,v) = \langle u,v\rangle \text{ for all } u \in \mathrm{Dom}(T) \Big\},$$
$$\widetilde{T}u = v, \quad u \in \mathrm{Dom}\big(\widetilde{T}\big).$$

Theorem 2.6 *$\widetilde{T}$ is a self-adjoint extension of T with lower bound m_T. The domain of $\widetilde{T}$ coincides with the set $\mathrm{Dom}(T^*) \cap \mathcal{H}_t$, and $\widetilde{T}$ is the only self-adjoint extension of T whose domain is contained in $\mathcal{H}_t$.*

Proof See [Wei80]. □

Example 2.4 Let $\mathcal{H} = L_2([0,\infty))$ and define T by

$$\mathrm{Dom}(T) = C_0^2([0,\infty));$$
$$Tu = -u'', \quad u \in \mathrm{Dom}(T).$$

Then T is a densely defined symmetric operator. Since

$$\langle Tu, u\rangle = \int_0^\infty |u'(x)|^2 \,\mathrm{d}m(x) \geq 0, \quad u \in \mathrm{Dom}(T),$$

we see that T is bounded below by zero, so it possesses self-adjoint extensions. The deficiency indices of T are $(1,1)$.

2.3 Fredholm operators

We explain how a self-adjoint operator can be obtained via some Fredholm theory. This theory is treated in for example [Kat76]. Assume that T is a closed densely defined operator in $\mathcal{H}$. It is said to be *Fredholm* is there exist an operator (called the parametrix) R in $\mathcal{H}$ such that the operators

$$RT - I \quad \text{and} \quad TR - I$$

are compact. If this is the case, then one can define the index of T as

$$\operatorname{ind}(T) = \dim\ker T - \dim\ker T^*.$$

We note that the parametrix R also is Fredholm and that $\operatorname{ind}(R) = -\operatorname{ind}(T)$.

If S is a self-adjoint operator, and $0 \neq t \in \mathbb{R}$, then the operator $S + itI$ is Fredholm with index zero. Indeed, the resolvent $R_t = (S + itI)^{-1}$ is a parametrix and the index is zero since the deficiency indices of a self-adjoint operator are $(0, 0)$.

The property of being Fredholm as well as the index are invariant under compact perturbations.

The following proposition is useful when showing that a specific operator is self-adjoint.

Proposition 2.7 *Assume that $0 \neq t \in \mathbb{R}$, and that S is a self-adjoint operator in $\mathcal{H}$ and that T is a symmetric operator such that the operator $T + itI$ (considered as an operator from the graph space of T) is a Fredholm operator with parametrix R_t. If $R_t - (S + itI)^{-1}$ is compact, then T is self-adjoint.*

Proof Firstly, the Fredholm properties give

$$\operatorname{ind}(T + itI) = -\operatorname{ind}(R_t) = -\operatorname{ind}((S + itI)^{-1}) = \operatorname{ind}(S + itI) = 0.$$

Secondly, the symmetry gives

$$\|(T + itI)u\|^2 = \|Tu\|^2 + t^2\|u\|^2 \geq s^2\|u\|^2.$$

These two facts imply that

$$\dim\ker(T^* - itI) = \dim\ker(T + itI) - \operatorname{ind}(T + itI) = 0.$$

Choosing t positive and negative implies that the deficiency indices of T is $(0, 0)$, so T is self-adjoint. □

2.4 The spectrum of self-adjoint operators

Let T be a closed operator in a Hilbert space $\mathcal{H}$. The number $\lambda \in \mathbb{C}$ is an *eigenvalue* of T if there exists an $u \in \mathcal{H}$, $u \neq 0$ such that $Tu = \lambda u$. The subspace $\ker(T - \lambda I)$ is called the *eigenspace* of λ and the dimension of $\ker(T - \lambda I)$ is called the *multiplicity* of the eigenvalue.

The set

$$\rho(T) = \left\{ \lambda \in \mathbb{C} \;\middle|\; T - \lambda I \text{ is bijective} \right\}$$

is called the *resolvent set* of T. The *spectrum* of T, $\sigma(T)$, is the complement of the resolvent set,

$$\sigma(T) = \mathbb{C} \setminus \rho(T).$$

Proposition 2.8 *The set $\rho(T)$ is open in $\mathbb{C}$, and hence the set $\sigma(T)$ is closed.*

Proof See [Wei80].

The set of eigenvalues, denoted by $\sigma_p(T)$, is called the *point spectrum* of T, and is clearly a subset of $\sigma(T)$.

Now assume that T is self-adjoint. Then the spectrum is a subset of the real line, $\sigma(T) \subset \mathbb{R}$. The *essential spectrum* of T, denoted $\sigma_e(T)$ is the subset of $\sigma(T)$ that consists of eigenvalues of infinite multiplicity and accumulation points of $\sigma(T)$.

There are several characterizations of these different types of spectral points. We give one, and refer again to [Wei80] for the proof.

Proposition 2.9 *The number λ belongs to $\sigma_p(T)$ if and only if there exists a Cauchy sequence $\{u_j\} \subset \mathrm{Dom}(T)$ such that $\lim \|u_j\| > 0$ and*

$$\lim_{j\to\infty} (T - \lambda I)u_j = 0.$$

The number λ belongs to $\sigma_e(T)$ if and only if there exists a sequence $\{u_j\} \subset \mathrm{Dom}(T)$ converging weakly to 0, while $\liminf \|u_j\| > 0$ and

$$\lim_{j\to\infty} (T - \lambda)u_j = 0.$$

Example 2.1 *(continued)* We calculate the spectrum of T_θ, by solving the equation

$$\begin{cases} -iu'(x) = \lambda u(x), & 0 < x < 1; \\ u(1) = \Theta(\theta)u(0). \end{cases}$$

The eigenvalues are given by $\lambda_j = \lambda_0 + 2\pi j$, where $j \in \mathbb{Z}$. Here λ_0 is the unique solution in $[0, 2\pi)$ to the equation $e^{i\lambda} = \Theta(\theta)$. The corresponding normalized eigenfunctions are given by $u_j(x) = e^{i\lambda_j x}$.

3 Quantum mechanics in $\mathbb{R}^n$

In this section we only discuss the parts from quantum mechanics needed to motivate our work in defining self-adjoint operators. We refer to the books [LL58, Jau68] for more background and theory of quantum mechanics.

A particle living in $\mathbb{R}^n$, is in quantum mechanics described by a *wave function* $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$ such that

$$\int_{\mathbb{R}^n} |u(x,t)|^2 \, \mathrm{d}m(x) = 1$$

for all $t \in \mathbb{R}$. Here x is thought of as a point in space and t denotes time. Quantities that can be observed, such as position and momentum, are called *observables*. It is not possible to measure the exact position of the particle at time t, but the probability to find the particle in a domain $\Omega \subset \mathbb{R}^n$ is given by $\int_\Omega |u(x,t)|^2 \, \mathrm{d}x$.

In general quantum mechanics takes place in a *configuration space*, which should be a Hilbert space $\mathcal{H}$. A state is an element in $\mathcal{H}$ of norm one. Assume that the Quantum mechanical system is in the state $u(0)$ (we omit the space variable x) at time $t = 0$. Then there should be a unique state $u(t)$ that describes the system at time $t \in \mathbb{R}$. Let us write

$$u(t) = U(t)u(0), \quad t \in \mathbb{R}.$$

This process is in general non-linear, but in quantum mechanics the linear approximation is studied, so $U(t)$ is assumed to be linear. Moreover, $u(t)$ should also be a state, so $\|U(t)u(0)\| = \|u(0)\|$. $U(t)$ should also be defined for all states and it should be possible to reach all states. In mathematical terms, for fixed t, such an operator $U(t)$ is called a unitary operator. Since the state $u(t)$ should be uniquely determined, it follows that

$$\begin{aligned} U(t+s) &= U(t)U(s), \quad t,s \in \mathbb{R}; \\ U(0) &= I. \end{aligned}$$

Such a family $U(t)$ of operators is called a *one-parameter unitary group*. It is natural to assume that this family should satisfy some kind of continuity. One says that it is *strongly continuous* if

$$\lim_{t \to t_0} U(t)u = U(t_0)u, \quad u \in \mathcal{H}.$$

If the group is strongly continuous, it also has an *infinitesimal generator* T, defined as

$$\operatorname{Dom}(T) = \left\{ u \in \mathcal{H} \;\middle|\; \lim_{t\to 0} \frac{1}{t}(U(t) - I)u \quad \text{exists} \right\},$$

$$Tu = -i \lim_{t\to 0} \frac{1}{t}(U(t) - I)u, \quad u \in \operatorname{Dom}(T).$$

The infinitesimal generator T is an (often unbounded) linear operator in $\mathcal{H}$. It is called the *Schrödinger operator* or the *Hamiltonian* of the system. If we have $u(0) \in \operatorname{Dom}(T)$ then $u(t) \in \operatorname{Dom}(T)$ for all $t \in \mathbb{R}$ and $u(t)$ satisfies the *Schrödinger equation*

$$-i \frac{\mathrm{d}}{\mathrm{d}t} u(t) = Tu(t).$$

The expression $\langle Tu, u \rangle$ corresponds to the *energy* of the state u, and should be real. From this, it follows that T should be a selfadjoint operator in $\mathcal{H}$. Luckily, this is the case. It is a theorem by Stone, that says that if we have a strongly continuous one-parameter unitary group, then the operator T defined as above is self-adjoint. On the other hand, if one starts with a self-adjoint operator T, then it is possible to define a strongly continuous one-parameter unitary group $U(t)$ as

$$U(t) = e^{itT}, \quad t \in \mathbb{R}.$$

The operator e^{itT} is defined via the spectral theorem for self-adjoint operators. We refer to [Wei80] for the details.

Example 2.1 *(continued)* Motivated by an example in [RS75], we return once more to the operator T_0, and the self-adjoint extensions of it. Let u be a state vector of the system, and assume that it is differentiable and has compact support in $[0, 1]$. For such u we define $U(t)$ as the right translation of u,

$$U(t)u = u(\cdot - t),$$

see Figure 3.1. Given a state u, we only define $U(t)$ for such small values of t such that $u(x - t)$ has its support in $(0, 1)$.

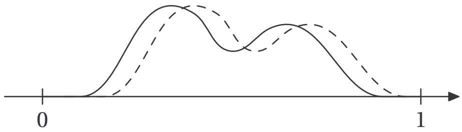


Figure 3.1 The solid curve shows the modulus of a state $u(x)$. The dashed curve shows the modulus of the translated state $u(x-t)$, for small t.

Then we can calculate the infinitesimal generator,

$$-i \lim_{t\to 0} \frac{1}{t}(U(t) - I)u(x) = -i \lim_{t\to 0} \frac{1}{t}(u(x-t) - u(x)) = -iu'(x).$$

This procedure gives an operator that coincides with T_0. Hence U, defined as above, is isometric, but not unitary. To extend it to a unitary operator, we have to tell how it acts on functions that are not compactly supported, that is, we have to tell what happens at the boundary. For $U(t)$ to be unitary, the wave going out at the right boundary must enter at the left boundary again (perhaps shifted in phase), and we must have

$$\int_0^1 |u(x-t)|^2 \, \mathrm{d}m(x) = \int_0^1 |u(x)|^2 \, \mathrm{d}m(x),$$

where $(x-t)$ is calculated mod 1, see Figure 3.2. This requires that the states u all have the same modulus at the endpoints, i.e. $|u(1)| = |u(0)|$. We can parameterize this by the function $\Theta(\theta)$ as before.

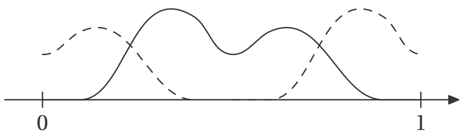


Figure 3.2 The solid curve shows the modulus of a state $u(x)$. The dashed curve shows the modulus of the translated state $u(x-t)$, note that it has the same modulus at both endpoints.

3.1 Magnetic Quantum operators

Quantum operators with magnetic fields have been studied by physicists in the twentieth century. However, defining these operators in a mathematically stringent way for quite general magnetic fields, has not been possible until some fundamental papers appeared in the sixties and seventies. We mention [IK62] which allows smooth magnetic potentials not decaying to zero at infinity, and the series of four papers by Avron, Herbst and Simon, starting with [AHS78]. Other fundamental papers that relax the smoothness properties of the magnetic fields are [Sim79b, LS81], see the details below. For very singular magnetic fields, such as the fields the AB solenoids generate, the question of defining a good self-adjoint operator gets more intricate. The quantum operators, first defined on smooth functions with support not touching the singular set of the field, are not essentially self-adjoint. The result is that there are several self-adjoint realizations describing the situation. They describe different physics, in the sense that different boundary conditions are imposed on the singular set, i.e. the particle interacts with the magnetic field in different ways for different realizations. We mention the papers [AT98, DŠ98, EŠV02, GŠ04a, GŠ04b, Tam03] for more details.

We refer to [MR94, RM05] for good reviews over the theory of Schrödinger operators with magnetic fields.

3.1.1 The Schrödinger operator in $\mathbb{R}^n$

The dynamics of a spinless quantum particle in $\mathbb{R}^n$ is described by the Schrödinger operator. Let $\vec{a}$ be a given magnetic vector potential. The Schrödinger operator $H : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$ is formally defined as

$$H = (-i\nabla - \vec{a})^2.$$

It is an unbounded operator, so one should be more specific about the domain $\operatorname{Dom}(H)$. To be able to define H on $C_0^\infty(\mathbb{R}^n)$ it is sufficient that

$$\vec{a} \in L_{4,\mathrm{loc}}(\mathbb{R}^n) \otimes \mathbb{R}^n, \quad \operatorname{div} \vec{a} \in L_{2,\mathrm{loc}}(\mathbb{R}^n). \tag{3.1}$$

This follows by expanding H as

$$H = -\Delta + i \operatorname{div} \vec{a} + 2i\vec{a} \cdot \nabla + \vec{a} \cdot \vec{a}.$$

It was conjectured in [Sim79b] and proved in [LS81] that H, first defined on $C_0^\infty(\mathbb{R}^n)$, is essentially self-adjoint if (3.1) is satisfied.

As we have seen, one can also obtain self-adjoint operators via quadratic forms. The quadratic form corresponding to the Schrödinger operator is given by

$$h(u,u) = \int_{\mathbb{R}^n} \left|(-i\nabla - \vec{a})u\right|^2 \, \mathrm{d}m(x). \tag{3.2}$$

Assuming that $\vec{a} \in L_{2,\mathrm{loc}}(\mathbb{R}^n) \otimes \mathbb{R}^n$, we can define two forms $h_{\min}$ and $h_{\max}$ as

$$\begin{aligned}\operatorname{Dom}(h_{\min}) &= C_0^\infty(\mathbb{R}^n);\\ h_{\min}(u,u) &= h(u,u), \quad u \in \operatorname{Dom}(h_{\min});\end{aligned}$$

and

$$\begin{aligned}\operatorname{Dom}(h_{\max}) &= \left\{ u \in L_2(\mathbb{R}^n) \;\middle|\; h(u,u) < \infty \right\}\\ h_{\max}(u,u) &= h(u,u), \quad u \in \operatorname{Dom}(h_{\max}).\end{aligned}$$

It is proved in [Sim79b] that the form $h_{\min}$ is closable and that its closure $\bar{h}_{\min}$ coincides with $h_{\max}$. Thus a self-adjoint operator can be defined via these forms.

As we mentioned in Section 1.2 the magnetic vector potential is not unique. Indeed, given one potential $\vec{a}_1$ we can find another one by adding the gradient of a gauge function f, $\vec{a}_2 = \vec{a}_1 + \nabla f$. Formally we get unitarily equivalent Schrödinger operators,

$$e^{if}(-i\nabla - \vec{a}_1)^2 e^{-if} = (-i\nabla - \vec{a}_2)^2.$$

We refer to [Lei83] for conditions on the vector potentials and the gauge function to make this unitarily equivalence rigorous.

3.1.2 The Schrödinger operator in $\mathbb{R}^2$ with AB solenoids

We turn to the plane and consider one AB solenoid placed at the origin, i.e. we let the magnetic field be given by

$$B(x) = 2\pi\alpha\delta(x)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2.$$

Then, as we saw in Section 1, a magnetic vector potential is given by

$$\vec{a}(x) = \frac{\alpha}{|x|^2}\left(-x^2, x^1\right).$$

We note that the singularity at the origin is so strong that a does not belong to $L_{2,\mathrm{loc}}(\mathbb{R}^2) \otimes \mathbb{R}^2$, so defining a self-adjoint Schrödinger Hamiltonian via the machinery from last section does not work. Indeed, let $u \in C_0^\infty(\mathbb{R}^2)$ be a function that is constant equal to 1 in a neighborhood of the unit disk. Then

$$\begin{aligned} h(u,u) &= \int_{\mathbb{R}^2} |(-i\nabla - \vec{a})u|^2 \,\mathrm{d}m(x) \geq \int_{|x|<1} |(-i\nabla - \vec{a})u|^2 \,\mathrm{d}m(x)\\ &= \int_{|x|<1} |\vec{a}|^2 \,\mathrm{d}m(x) = \int_{|x|<1} \frac{\alpha^2}{|x|^2} \,\mathrm{d}m(x) = +\infty\end{aligned}$$

However, it is possible to define the operator as a symmetric operator on $C_0^\infty(\mathbb{R}^2 \setminus \{0\})$. That operator has deficiency indices $(2,2)$. The self-adjoint extensions can be parameterized by boundary conditions at the singular point, as we have seen. We write $z = re^{i\theta}$ and assume that the AB intensity α is normalized to the interval $(0,1)$. Then functions in the domain of the self-adjoint extensions has the asymptotics

$$u(re^{i\theta}) \sim c_{-\alpha} r^{-\alpha} + c_{\alpha-1} r^{\alpha-1} e^{-i\theta} + c_{\alpha} r^{\alpha} + c_{1-\alpha} r^{1-\alpha} e^{-i\theta}, \quad r \searrow 0,$$

and it is the coefficients (functionals on the definition space) $c_{\pm\alpha}$ and $c_{\pm(1-\alpha)}$ that determine the extension. This result was obtained in [AT98] and [DŠ98] independently.

The Schrödinger operator with several, including infinitely many, AB solenoids together with a uniform field is studied in [Min05]. It is proved that with n solenoids, the operator, first defined on functions with compact support not touching the solenoids, has deficiency indices $(2n, 2n)$. This is done by a certain gluing process of the operators corresponding to one AB solenoid. Some spectral results are also obtained, see the discussion in Section 6.3.

3.1.3 The Dirac operator in $\mathbb{R}^{2d}$

In dimension $2d$ there are $2d + 1$ Dirac matrices $\{\gamma_d^j\}_{j=0}^{2d}$ that generate a Clifford algebra, i.e. they satisfy

$$\left(\gamma_d^j\right)^* = \gamma_d^j, \quad \text{and} \quad \gamma_d^j \gamma_d^k + \gamma_d^k \gamma_d^j = 2\delta^{j,k} I_{2^d}, \quad j,k = 0,1,\ldots,2d. \tag{3.3}$$

They may be defined as follows. For $d = 1$ the Dirac matrices are given by

$$\gamma_1^0 = \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_1^1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \gamma_1^2 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The matrices σ_0, σ_1 and σ_2 are called the Pauli matrices. If $\gamma_d^0, \gamma_d^1, \ldots, \gamma_d^{2d}$ are given Dirac matrices for dimension $2d$, then the Dirac matrices for dimension $2d + 2$ are given by

$$\gamma_{d+1}^0 = \begin{pmatrix} I_{2^{d-1}} & 0 \\ 0 & -I_{2^{d-1}} \end{pmatrix}, \ \gamma_{d+1}^1 = \begin{pmatrix} 0 & \gamma_d^1 \\ \gamma_d^1 & 0 \end{pmatrix}, \ \ldots, \ \gamma_{d+1}^{2d} = \begin{pmatrix} 0 & \gamma_d^{2d} \\ \gamma_d^{2d} & 0 \end{pmatrix},$$

$$\gamma_{d+1}^{2d+1} = \begin{pmatrix} 0 & \gamma_d^0 \\ \gamma_d^0 & 0 \end{pmatrix}, \quad \text{and} \quad \gamma_{d+1}^{2d+2} = \begin{pmatrix} 0 & -iI_{2^{d-1}} \\ iI_{2^{d-1}} & 0 \end{pmatrix}.$$

When there is no doubt about the dimension, we omit the subscripts. The Dirac operator $\mathfrak{D}$ in $L_2(\mathbb{R}^{2d}) \otimes \mathbb{C}^{2^d}$ is formally defined by

$$\mathfrak{D} = \sum_{j=1}^{2d} \gamma^j \Big(-i \frac{\partial}{\partial x^j} - a_j(x) \Big).$$

If the magnetic field is regular, then $\mathfrak{D}$, first defined on $C_0^\infty(\mathbb{R}^{2d}) \otimes \mathbb{C}^{2^d}$, is essentially self-adjoint, see [Che73]. We refer to [Tha92] for more information about the Dirac operator, and to Section 4 for a discussion of how to define the Dirac operator with AB solenoids.

3.1.4 The Pauli operator in $\mathbb{R}^2$

A charged spin 1/2 particle is described by the Pauli Hamiltonian, which acts in $L_2(\mathbb{R}^2) \otimes \mathbb{C}^2$, and is formally defined as

$$\mathfrak{P} = \begin{pmatrix} H - \frac{g}{2}B & 0 \\ 0 & H + \frac{g}{2}B \end{pmatrix}. \tag{3.4}$$

Here H is the two-dimensional Schrödinger Hamiltonian $H = (-i\nabla - \vec{a})^2$, B is the magnetic field (In two dimensions we identify the two-form and the coefficient function), and g is the *gyromagnetic ratio*. We identify the real point (x^1, x^2) with the complex number $z = x^1 + ix^2$, and denote a scalar potential of B by W,

$$-\Delta W = B.$$

We set $\Pi_j = -i\frac{\partial}{\partial x^j} - a_j$ and

$$\mathfrak{Q} = \Pi_1 - i\Pi_2, \quad \mathfrak{Q}^* = \Pi_1 + i\Pi_2,$$

and note that

$$\mathfrak{Q}\mathfrak{Q}^* = \mathfrak{Q}^*\mathfrak{Q} + 2B, \quad H = \mathfrak{Q}^*\mathfrak{Q} + B = \mathfrak{Q}\mathfrak{Q}^* - B. \tag{3.5}$$

From (3.4) and (3.5) we get

$$\mathfrak{P} = \begin{pmatrix} \mathfrak{Q}^*\mathfrak{Q} - \frac{g-2}{2}B & 0 \\ 0 & \mathfrak{Q}\mathfrak{Q}^* + \frac{g-2}{2}B \end{pmatrix}. \tag{3.6}$$

The number $\frac{g-2}{2}$ is called the *anomaly factor* of the magnetic moment. Experiments give an anomaly factor of 0.00159 for the electron [BV93]. We assume that $g = 2$, which is the simplest case. Thus, the Pauli Hamiltonian we study in this thesis is formally defined by

$$\mathfrak{P} = \begin{pmatrix} \mathfrak{Q}^*\mathfrak{Q} & 0 \\ 0 & \mathfrak{Q}\mathfrak{Q}^* \end{pmatrix}. \tag{3.7}$$

The Pauli operator can be written as the square of the Dirac operator

$$\mathfrak{P} = \mathfrak{D}^2 = \Big(\sum_{j=1}^{2} \sigma_j\big(-i\frac{\partial}{\partial x^j} - a_j\big)\Big)^2 = \begin{pmatrix} 0 & \mathfrak{Q}^* \\ \mathfrak{Q} & 0 \end{pmatrix}^2 \tag{3.8}$$

from which it follows that it is a non-negative operator. Now let us be more precise about the domains. As in the case of the Schrödinger Hamiltonian there is a problem in defining the Pauli Hamiltonian if the magnetic field is too singular. The quadratic form corresponding to $\mathfrak{P}$ is given by

$$\mathfrak{p}(\psi,\psi) = \int_{\mathbb{R}^2} \Big|\sum_{j=1}^{2} \sigma_j\big(-i\frac{\partial}{\partial x^j} - a_j\big)\psi\Big|^2 \,\mathrm{d}m(x). \tag{3.9}$$

If $\vec{a} \in L_{2,\mathrm{loc}}(\mathbb{R}^2) \otimes \mathbb{R}^2$ then $\mathfrak{p}(\psi,\psi)$ makes sense for $\psi \in C_0^\infty(\mathbb{R}^2) \otimes \mathbb{C}^2$. We define the *minimal* Pauli form $\mathfrak{p}_{\min}$ as

$$\begin{aligned} \mathrm{Dom}(\mathfrak{p}_{\min}) &= C_0^\infty(\mathbb{R}^2) \otimes \mathbb{C}^2; \\ \mathfrak{p}_{\min}(\psi,\psi) &= \mathfrak{p}(\psi,\psi), \quad \psi \in \mathrm{Dom}(\mathfrak{p}_{\min}). \end{aligned}$$

It is closable and thus a self-adjoint operator $\mathfrak{P}_{\min}$ can be defined. We also define the *maximal* Pauli form $\mathfrak{p}_{\max}$ as

$$\begin{aligned} \mathrm{Dom}(\mathfrak{p}_{\max}) &= \big\{\psi \in L_2(\mathbb{R}^2) \otimes \mathbb{C}^2 \;\big|\; \mathfrak{p}(\psi,\psi) < \infty\big\}; \qquad (3.10) \\ \mathfrak{p}_{\max}(\psi,\psi) &= \mathfrak{p}(\psi,\psi), \quad \psi \in \mathrm{Dom}(\mathfrak{p}_{\max}). \end{aligned}$$

In the presence of AB solenoids, $\vec{a}$ does not belong to $L_{2,\mathrm{loc}}(\mathbb{R}^2) \otimes \mathbb{R}^2$. It was proved in [Sob96] that the Pauli form can not be defined on smooth compactly supported ψ via (3.9) in this case. The way out of this is to redefine the Pauli form $\mathfrak{p}$ by an expression that makes sense even in this more singular case. This is done in [EV02] by writing the operators $\mathfrak{Q}$ and $\mathfrak{Q}^*$ as

$$\mathfrak{Q} = -2ie^{W}\frac{\partial}{\partial \bar{z}}e^{-W} \quad \text{and} \quad \mathfrak{Q}^* = -2ie^{-W}\frac{\partial}{\partial z}e^{W}, \tag{3.11}$$

and noting that the quadratic form

$$\mathfrak{p}(\psi,\psi) = 4\int_{\mathbb{R}^2} \left|\frac{\partial}{\partial \bar{z}}\left(e^{-W}\psi_+\right)\right|^2 e^{2W} + \left|\frac{\partial}{\partial z}\left(e^{W}\psi_-\right)\right|^2 e^{-2W}\,\mathrm{d}m(x), \tag{3.12}$$

$\psi = (\psi_+,\psi_-)^t$ makes sense even with this more singular field. If $\mathfrak{p}$ is defined on a maximal domain in the same way as in (3.10), it yields a self-adjoint operator even with this singular field, usually called the *maximal* Pauli operator. The forms in (3.9) and (3.12) coincides for more regular fields.

3.2 The Aharonov-Casher theorem

We will go through the original proof of the Aharonov-Casher (AC) theorem, see [AC79], when the magnetic field B is smooth and has compact support. The regularity properties on B for the AC theorem to hold have been relaxed considerably, see [Mil82, CFKS87, Ara93, HO01, MOR04] and especially [EV02] which deals with measure-valued magnetic fields.

For a real positive number s we denote by $\{s\}$ the lower integer part, i.e. the largest integer strictly less than s.

Theorem 3.1 *Assume that* $B \in C_0^1(\mathbb{R}^2)$*, and let* $\Phi = \frac{1}{2\pi}\int_{\mathbb{R}^2} B(x)\,\mathrm{d}m(x)$*. Then*

$$\dim\ker\mathfrak{P} = \{|\Phi|\}.$$

Proof The magnetic field B has a potential W defined by (1.6). It satisfies

$$W(z) = \Phi\log|z|(1+o(1)), \quad |z|\to\infty. \tag{3.13}$$

It follows from (3.8) that $\psi = (\psi_+,\psi_-)^t$ is an element of the kernel of $\mathfrak{P}$ if and only if

$$\mathfrak{Q}\psi_+ = 0, \quad \text{and} \quad \mathfrak{Q}^*\psi_- = 0.$$

By (3.11) this means that the function $f_+ = \psi_+e^{-W}$ is holomorphic and $f_- = \psi_-e^{W}$ is antiholomorphic in $\mathbb{C}$. By (3.13) we see that

$$e^{\pm W(z)} = |z|^{\pm\Phi}(1+o(1)), \quad |z|\to\infty.$$

Assume that $\Phi \geq 0$. Then, since $\psi_+ \in L_2(\mathbb{C})$, the function f_+ is a holomorphic function that decays to zero at infinity. By Liouville's theorem it must be zero. The function f_-, on the other hand is allowed to grow no faster than a polynomial at infinity for ψ_- to be in $L_2(\mathbb{C})$ and the degree of the polynomial must be strictly less than $\Phi - 1$. This means that it can be a polynomial in $\bar{z}$ of degree at most $\{\Phi\} - 1$. Since there are $\{\Phi\}$ linearly independent such polynomials the result follows. If $\Phi < 0$ the same argument shows that the spin-down component $\psi_- = 0$ and that the spin-up component ψ_+ is given by a polynomial of degree at most $\{-\Phi\} - 1$ times e^W. □

For constant magnetic field the kernel of $\mathfrak{P}$ coincides with the lowest Landau level, giving an example infinite dimensional kernel. It was conjectured in [Mil82] that the dimension of the kernel of $\mathfrak{P}$ should be at least $\lfloor \Phi \rfloor$ for a magnetic field $B \geq 0$ with total flux $2\pi\Phi$.

This result was proved for finite fluxes in [EV02]. In [GG02] it was shown that if one has a system of AB solenoids placed at the points of an infinite lattice with equal intensities at every vortex, then the kernel of the maximal Pauli operator is infinite-dimensional. This was also the first example studied with non-trivial spin-up and the spin-down null-spaces (in fact, both infinitely dimensional). This result was extended by means of a certain perturbation of the lattice in [GŠ04b].

The conjecture in [Mil82] was finally proved in [RS06], where also some more general results concerning magnetic fields with varying sign of infinite flux were obtained.

4 Paper I and II

In the first two papers we compare different self-adjoint realizations of the Pauli operator in two dimensions corresponding to the magnetic field given by a regular part with compact support and a singular part consisting of a finite number of AB solenoids, i.e. the magnetic field is

$$B(z) = B_0(z) + \sum_{j=1}^{n} 2\pi\alpha_j \delta_{z_j}, \tag{4.1}$$

where $B_0 \in C_0^\infty(\mathbb{R}^2)$. We denote by Λ the set $\{z_j\}_{j=1}^n \subset \mathbb{C}$ of singular points.

We consider the two-dimensional Pauli operator corresponding to this magnetic field, which models a spin 1/2 quantum particle confined to a plane. Such a plane could consist of a very thin film made of for example gold, see Figure 4.1.

Figure 4.1 A quantum particle moving in a thin film of gold, orthogonal to an external magnetic field.

The singular field we study can be obtained when a superconductor of type II is put on top on a two-dimensional electron gas, see Figure 4.2 and the discussion in [Mor96b]. Another situation that leads to this type of magnetic fields are certain impurities in crystal structures.

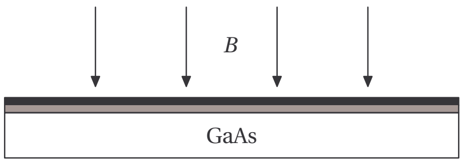


Figure 4.2 A superconductor of type II (black) put on top of a two-dimensional electron gas (gray). If the magnetic field is strong enough so-called Abrikosov vortices will appear. In the level of the electron gas, they will appear as flux tubes with negligible radius, i.e. as AB solenoids.

The Pauli operators considered are all self-adjoint extensions of the symmetric Pauli operator, originally defined on $C_0^\infty(\mathbb{R}^2 \setminus \Lambda)$.

In [EV02] the Pauli operator is defined for a more general set of magnetic fields via the maximal quadratic form (3.12). However, the distribution derivatives in (3.9) are considered in the distribution space $\mathcal{D}'(\mathbb{R}^2)$. The derivatives "feels" the singular points at Λ, which forces the AB intensities α_j to be bounded, $-1 < \alpha_j < 1$. Since it is possible to add integer intensities by means of gauge transformations, the Pauli operator in [EV02] is defined by normalizing the AB intensities to any interval of length one in $(-1, 1)$, the interval $[-1/2, 1/2)$ is chosen, by means of these gauge transformations. We denote the resulting, normalized operator by $\mathfrak{P}_{\mathrm{EV}}$.

In [GG02] a Pauli Hamiltonian is defined via the same expression of the maximal quadratic form, (3.12), but with the fundamental difference that the derivatives are considered in the distribution space $\mathcal{D}'(\mathbb{R}^2 \setminus \Lambda)$, i.e. the derivatives does not feel the singular points of the magnetic field. This allowes arbitrary intensities α. In Paper I we follow [GG02] and the resulting operator is denoted by $\mathfrak{P}_{\max}$.

The purpose of Paper II is to study all self-adjoint Pauli operators that are realized as the square of the Dirac operator. In Paper I we used the semi-boundedness of the Pauli operator to define a certain self-adjoint realization. Since the Dirac operator lacks semiboundedness, we can not use this method in Paper II, so defining self-adjoint extensions becomes more complicated. We start by defining the Dirac operator for one AB solenoid. The deficiency indices for the operator first defined on functions with support not touching the AB solenoid are $(1,1)$, so there is a number $\tau \in [0,2\pi)$ that parametrizes the different self-adjoint extensions. They are also characterized by the boundary values they obey at the AB solenoid.

To define the Dirac operator with magnetic field (4.1), inspired by [AR04], we use a certain gluing process and the Fredholm theory from Section 2.3 to show that the resulting operator is self-adjoint. The result is a Dirac operator $\mathfrak{D}_{\tilde{\tau}}$, where $\tilde{\tau} = (\tau_1, \ldots, \tau_n)$ denotes the boundary value parameters for each AB solenoid. Since different values of the parameters give different physics it is natural to assume that all τ_j are equal. The Pauli operator we study in Paper II is the one defined as the square of this Dirac operator, $\mathfrak{P}_{\tilde{\tau}} = \mathfrak{D}_{\tilde{\tau}}^2$.

Some Dirac extensions have been studied before, see [Ara93, AH05, dSG89, HO01, Tam03].

Below we discuss some natural properties one expect from a Pauli operator. No extension studied in the papers fulfill all these properties. We leave it for the physicists to decide which one of the extensions that is describing the real physical situation in the best way.

Property 1 *The Pauli operator $\mathfrak{P}$ is gauge invariant, i.e. changing the AB intensity by an integer multiple results in unitarily equivalent operators.*

This property is fulfilled by all extensions studied. Note that $\mathfrak{P}_{\mathrm{EV}}$ satisfies it by definition.

Property 2 *The physical situation is the same if the magnetic field change sign. Mathematically, the two Pauli operators corresponding to B and $-B$ are (anti)-unitarily equivalent.*

We show that $\mathfrak{P}_{\max}$ satisfies this property. For $\mathfrak{P}_{\tilde{\tau}}$ it depend on the value of the parameter. There is a coupling between the spin-up and spin-down components of the functions in the domain. If $\tau_j = 0$ or π for all j, then there is only singularities in one of the components, so the resulting operator is quite asymmetric and do not satisfy Property 2. We show that for all values of τ_j but $\pi/2$ and $3\pi/2$ this coupling asymmetry is present, and for these two exceptional values Property 2 is fulfilled.

For the operator $\mathfrak{P}_{\mathrm{EV}}$ it is noted that if $0 < \alpha_j < 1/2$ then the boundary condition is the same as for the square of the Dirac operator when $\tau_j = \pi$ and for $-1/2 \le \alpha_j \le 0$ it matches with $\tau_j = 0$. In both cases singularities are only allowed in one component, so the asymmetry in the domain is present. The operator does not satisfy Property 2.

Property 3 *The Pauli operator is the square of a self-adjoint Dirac operator.*

The operator $\mathfrak{P}_{\tilde{\tau}}$ satisfies this by definition. From the discussion in the last paragraph we also see that $\mathfrak{P}_{\mathrm{EV}}$ satisfy this property, however, depending on the AB intensities, the boundary conditions might be different at different solenoids.

The maximal operator, $\mathfrak{P}_{\max}$, is not the square of a Dirac operator. This can be seen from the Aharonov-Casher theorem, or by noting that singular terms are included in both spin components, without coupling.

Property 4 *The Pauli operator satisfies Theorem 3.1, the Aharonov-Casher theorem.*

It is not that important that the Pauli operator satisfies the original Aharonov-Casher formula, but physicists are interested in the number of zero-modes (see for example the discussion in [BRF+02] and its references). We calculate the number of zero-modes for the different extensions, and use it as a tool to easily distinguish between different extensions.

The only extension we study that satisfy the original Aharonov-Casher theorem is $\mathfrak{P}_{\mathrm{EV}}$. The extension $\mathfrak{P}_{\max}$ accepts more singular elements in the domain than the other extensions, and thus it has the biggest kernel. For the operators $\mathfrak{P}_{\tilde{\tau}}$ there are only two values of the parameter that give non-trivial kernels; the ones where the spin components are uncoupled.

Property 5 *The boundary values at the AB solenoids do not depend on the boundary values at the other AB solenoids.*

All extensions studied satisfy this property.

Property 6 *The Pauli operator $\mathfrak{P}$ can be approximated by Pauli operators corresponding to more regular magnetic fields.*

This question is studied in [Tam03, BV93]. The result is expressed in terms of boundary conditions. We compare them and condluce that $\mathfrak{P}_{\mathrm{EV}}$ can be approximated as a Hamiltonian, i.e. both spin-up and spin-down components are approximated simultaneously, while the components in $\mathfrak{P}_{\max}$ can be approximated, but in different ways. The operators $\mathfrak{P}_{\tilde{\tau}}$ can all be approximated component-wise, and the extensions where the components are not coupled can be approximated as a Hamiltonians.

We note that new ways of approximating the operators might appear, leading to new results.

5 Paper III

When the first two papers were completed it was a natural step to investigate the possibility to generalize the Aharonov-Casher theorem to higher dimensions. This question was raised in [Shi91] and further studied in [Ogu93].

We consider a charged spin-1/2 particle in $\mathbb{R}^{2d}$, $d > 1$, influenced by a smooth magnetic field B. This is described by the Pauli Hamiltonian

$$\mathfrak{P} = \mathfrak{D}^2 = \left(\sum_{j=1}^{2d} \gamma^j \left(-i \frac{\partial}{\partial x^j} - a_j(x) \right) \right)^2$$

acting in $L_2(\mathbb{R}^{2d}) \otimes \mathbb{C}^{2^d}$. In [Shi91] methods from complex analysis in several variables were successfully used under the assumption that B is a $(1,1)$ type form

$$B = \sum_{j,k=1}^{d} \mathfrak{b}_{j,k}(z)\, \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k.$$

Let $\mu(z)$ be the eigenvalue of $\{\mathfrak{b}_{j,k}(z)\}$ that has the smallest absolute value. It was proved that if $\lim |\mu(z) z^2| \to \infty$ as $|z| \to \infty$ then the kernel of $\mathfrak{P}$ is infinite-dimensional.

In two dimensions the Aharonov-Casher theorem, Theorem 3.1 says that if $\Phi = \int_{\mathbb{R}^2} B(x)\, \mathrm{d}m(x)$ then the dimension of the kernel of $\mathfrak{P}$ is $\{|\Phi|\}$. In the path of the proof we used the asymptotic expansion

$$W(z) = \Phi \log |z| (1 + o(1)), \quad |z| \to \infty \tag{5.1}$$

for the scalar potential. In [Ogu93] the scalar potential is the starting point. Assuming that W satisfies (5.1) (now in $\mathbb{R}^{2d}$) the main result is that

$$\dim \ker \mathfrak{P} = N_d(\Phi) = \binom{\{|\Phi|\}}{d}. \tag{5.2}$$

Here $N_d(\Phi)$ is the number of all monomials in d variables of degree less than $|\Phi| - d$. The method in the proof is similar to the original Aharonov-Casher theorem, i.e. to reduce the equation $\mathfrak{P}\psi = 0$ to the Cauchy-Riemann equations, and deduce that the zero-modes must be polynomials times a weight $e^{\pm W}$. This approach works well for two of the 2^d components of ψ, but the calculation for the other ones is not correct. The two components that are calculated correctly are the ones where the zero-modes occur. Thus, what is really proved in [Ogu93] is the inequality

$$\dim \ker \mathfrak{P} \geq \binom{\{|\Phi|\}}{d} \tag{5.3}$$

instead of (5.2).

The aim of Paper III is to correct this mistake. We manage to prove (5.2) for $|\Phi| < d$, which means that the right hand side is zero.

The tools we use come mainly from complex analysis in several variables. We do the same transformation of the problem as is done in [Shi91]. This leads to a question of estimations of solutions to a weighted $\bar{\partial}$-equation. We write our solution with help of the Bochner-Martinelli-Koppelmann formula, see [Ran86]. To estimate the solution we use a weighted version of the Hardy-Littlewood-Sobolev theorem, see [Ste70] for the original theorem and [DL98] for the weighted version we use. The estimation requires that the weight, which is $e^{\pm 2W}$, belongs to a certain Muckenhoupt weight class. We show that it does if $|\Phi| < d$.

In terms of the magnetic field, we show that the kernel is trivial if B is a $(1,1)$ type magnetic field and there exist constants $C > 0$ and $\rho > 2$ such that

$$|B(x)| \leq \frac{C}{(1+|x|)^{\rho}}, \quad \text{for all } x \in \mathbb{R}^{2d}.$$

6 Paper IV

The last paper in this thesis is about spectral asymptotics of a certain perturbation of the Landau Hamiltonian, i.e. the quantum operator in two (or even) dimensions with constant magnetic field. We start by discussing the dynamics of a particle in classical Newtonian physics in this setting.

6.1 Classical dynamics

As we saw in the introduction, in classical physics, a particle of mass m and charge q, moving with speed $\vec{v}$ in a magnetic field $\vec{B}$ in $\mathbb{R}^3$ is influenced by a force $\vec{F}$, given by

$$\vec{F} = q\vec{v} \times \vec{B}.$$

Using Newton's second law of motion, $\vec{F} = m\dot{\vec{v}}$, we get the differential equation

$$\dot{\vec{v}} = \frac{q}{m}\vec{v} \times \vec{B}. \tag{6.1}$$

If we restrict the particle to the plane $x^3 = 0$, and let the magnetic field $\vec{B}$ be the constant magnetic field $\vec{B} = (0, 0, B_0)$ orthogonal to the plane, the differential equation (6.1) is easily solved. The particle travels along a given circle of radius $m|\vec{v}|/|B_0 q|$, see Figure 6.1, left side. If an obstacle is introduced not much will happen. If the original orbit (the circle) of the particle does not intersect the obstacle, then the particle will go on as if there were no obstacle. However, if the orbit intersects the obstacle, then the particle will bounce as in the right side of Figure 6.1.

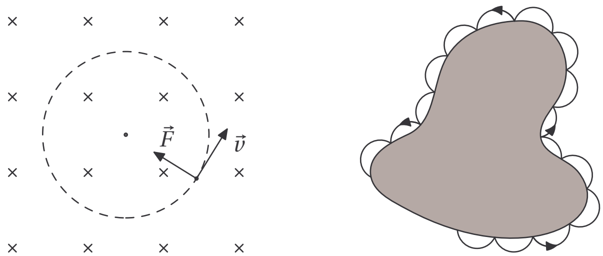


Figure 6.1 *Left*: The classical picture of a charged particle moving with speed $\vec{v}$ in a plane with an orthogonal constant magnetic field acting on it. Here the magnetic field is directed into the paper. The orbit of the particle is the dashed circle. *Right*: If an obstacle is introduced then the particle will bounce against it.

6.2 The Landau Hamiltonian in $\mathbb{R}^2$

We now turn to a quantum mechanical particle. Even though we consider the general even-dimensional case $\mathbb{R}^{2d}$, $d \geq 1$ in the paper, let us for clarity work in $\mathbb{R}^2$, which we often consider as the x^1x^2-plane embedded in $\mathbb{R}^3$.

We let the magnetic field be uniform and orthogonal to the plane $x^3 = 0$,

$$\vec{B} = (0,0,B_0), \quad B_0 > 0.$$

We work in the symmetric gauge

$$\vec{a}(x) = \frac{B_0}{2}(-x^2, x^1, 0).$$

The Landau Hamiltonian L is defined in $\mathcal{H} = L_2(\mathbb{R}^2)$ as the closure of

$$\begin{aligned} \mathrm{Dom}(L_0) &= C_0^2(\mathbb{R}^2); \\ L_0 u &= (-i\nabla - \vec{a})^2 u, \quad u \in \mathrm{Dom}(L_0). \end{aligned}$$

This is one of the first explicitly solvable Hamiltonians studied, see [Foc28, Lan30]. To determine the spectra of L, it is convenient to introduce the creation operator $\mathfrak{Q}^*$ and the annihilator operator $\mathfrak{Q}$ as in Section 3.1.4. We identify the real point (x^1, x^2) with the complex number $z = x^1 + ix^2$, and denote a scalar potential of B by W,

$$W(z) = -\frac{B}{4}|z|^2.$$

We define the annihilation operator $\mathfrak{Q}$ and its adjoint, the creation operator $\mathfrak{Q}^*$, as

$$\mathfrak{Q} = -2ie^W \frac{\partial}{\partial \bar{z}} e^{-W}, \qquad \mathfrak{Q}^* = -2ie^{-W} \frac{\partial}{\partial z} e^W,$$

and note that

$$\mathfrak{Q}\mathfrak{Q}^* = \mathfrak{Q}^*\mathfrak{Q} + 2B, \quad L = \mathfrak{Q}^*\mathfrak{Q} + B = \mathfrak{Q}\mathfrak{Q}^* - B. \tag{6.2}$$

It is clear that $\mathfrak{Q}\mathfrak{Q}^* \geq 0$ and $\mathfrak{Q}^*\mathfrak{Q} \geq 0$ from which it follows that $L \geq B$. Moreover, except for zero, the spectrum of $\mathfrak{Q}\mathfrak{Q}^*$ and $\mathfrak{Q}^*\mathfrak{Q}$ coincide. Zero is an eigenvalue of $\mathfrak{Q}^*\mathfrak{Q}$ of infinite dimension. Indeed, $\mathfrak{Q}^*\mathfrak{Q}u = 0$ implies that $\mathfrak{Q}u = 0$. If we let $v = e^{-W}u$ this implies that v satisfies the Cauchy-Riemann equations. Hence the function v should be an element of the Fock space

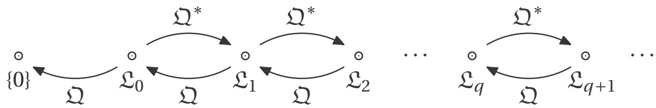
$$\mathfrak{F}_B^2 = \Big\{ v \;\Big|\; v \text{ entire, and } \int_{\mathbb{C}} |v|^2 e^{-\frac{B_0}{2}|z|^2}\, \mathrm{d}m(z) < \infty \Big\}.$$

The monomials $v_j(z) = z^j$, $j = 0, 1, \ldots$, form a basis for $\mathfrak{F}_B^2$, so the functions $u_j(z) = e^W v_j$, $j = 0, 1, \ldots$ form a basis for $\ker(\mathfrak{Q}^* \mathfrak{Q})$. It follows from (3.5) that the spectrum of $\mathfrak{Q}^* \mathfrak{Q}$ consists of infinitely degenerate eigenvalues $2Bq$, $q = 0, 1, \ldots$, and hence that the spectrum of L consists of infinitely degenerate eigenvalues

$$\Lambda_q = B(2q + 1), \quad q = 0, 1, \ldots.$$

with corresponding eigenspaces $\mathfrak{L}_q$. The creation and annihilation operators act between the eigenspaces as

$$\mathfrak{Q}^* \mathfrak{L}_q = \mathfrak{L}_{q+1}, \quad \mathfrak{Q}\mathfrak{L}_{q+1} = \mathfrak{L}_q, \quad \mathfrak{Q}\mathfrak{L}_0 = \{0\}, \quad q = 0, 1, \ldots.$$



In Figure 6.2 we see the graphs of the modulus of some states in $\mathfrak{L}_0$. We refer to [Ave76] for a more extensive theory of creation and annihilation operators.

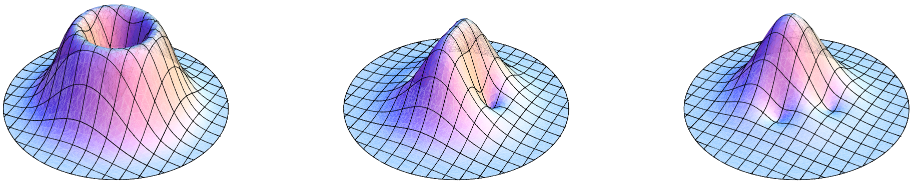


Figure 6.2 Plots of $|u|$ for three different elements in $\mathfrak{L}_0$ of the Landau Hamiltonian. This indicates that the dynamics of the quantum particle is much more complex than for the classical particle. *Left*: An example of a state where the particle moves in a circle, similar to the classical situation. *Middle, right*: States that differ from the classical movement.

6.3 Perturbations of the Landau Hamiltonian

It is possible to perturb the Landau Hamiltonian in several different ways, and this has been a very active field in the last years. The expectation is that the Landau levels Λ_q split into clusters of eigenvalues. We describe below some different perturbations. We don't pretend to give a complete list of perturbations studied in the literature. In some cases, to keep the presentation as simple as possible, we don't give the most general result obtained in the referred paper. We denote by $N(c_1, c_2, T)$ the number of eigenvalues (including multiplicity) of the operator T in the interval (c_1, c_2) (or (c_2, c_1) if $c_2 < c_1$). It is convenient to denote the Schrödinger operator with magnetic potential $\vec{a}$ and electric potential V by $H(\vec{a}, V)$, i.e.

$$H(\vec{a}, V) = (-i\nabla - \vec{a})^2 + V.$$

We also denote the magnetic vector potential of the constant magnetic field by $\vec{a}_0$.

The first example we give is also one of the starting papers of the perturbations. In [Raĭ90] the operator $H(\vec{a}_0, V)$ is considered, where the potential V obeys

$$V(x) \sim c(1 + |x|^2)^{-\alpha/2}, \quad |x| \to \infty, \quad \alpha \in (2, \infty).$$

The result is the quasi-classical power-like asymptotics

$$N\big(\Lambda_q \pm \lambda, \mu_\pm, H(\vec{a}_0, \pm V)\big) \sim \lambda^{-2/\alpha}, \quad \lambda \searrow 0,$$

where $\mu_- \in (\Lambda_{q-1}, \Lambda_q)$ and $\mu_+ \in (\Lambda_q, \Lambda_{q+1})$, $q = 0, 1, \ldots$.

Showing asymptotic formulas for fast decaying, or compactly supported, potentials turned out to be harder since the microlocal analysis arguments used in [Raĭ90] no longer applies, see the discussion in the introduction of [MR03]. However, using Toeplitz-type operators and theory of complex polynomials, it was proved in [RW02a, RW02b, MR03] that if the potential $V \geq 0$ decays sufficiently fast, or even has compact support, and is positive on an open set, then

$$N\big(\Lambda_q \pm \lambda, \mu_\pm, H(\vec{a}_0, \pm V)\big) \sim \frac{|\log \lambda|}{\log|\log \lambda|}, \quad \lambda \searrow 0,$$

where $\mu_\pm$ are as above.

The border between quasi-classical and non-classical asymptotics is considered in [RW02b]. A bit simplified, the result is that if V decays as $e^{-|x|^2}$ or faster at infinity then the asymptotics is non-classical. In [MR03] the case of compact V is considered also in higher-dimensional spaces. Moreover, also the Dirac operator is considered.

When these asymptotic formulas were proved, new types of questions were asked. We mention [RS08], dealing with expanding electric fields. Let V be a potential that decays fast at infinity (typically with compact support). The number of eigenvalues in a given interval of the t-dependent operator $H^t = H^t(\vec{a}_0, V(x/t))$ is studied, as $t \to \infty$. More precisely, for a given interval $(\mu_1, \mu_2) \subset (\Lambda_q, \Lambda_{q+1})$ the asymptotics of $N(\mu_1, \mu_2, H^t)$ as $t \to \infty$ is given in terms of the measure of level sets of the perturbation of V. See [RS08] for exact details.

Results concering perturbed magnetic fields are generally harder, since even a perturbation by a compactly supported field, the perturbation magnetic vector potential will not in general have compact support, and it is the vector potential and not the magnetic field that appears in the Hamiltonian $H(\vec{a}, V)$. Thus small perturbations of the magnetic field does not necessarily give small perturbations of the operator. However, if the magnetic field is asymptotically constant, then no new points appear in the essential spectrum, see [Iwa83]. This result was generalized to higher dimensions in [Shi91].

In [Rai03] the spectral asymptotics of the lowest Landau level was completed in the case of perturbation by an oscillating magnetic field. The result is similar as in the non-classical perturbation with an electric field.

In [FP06], using theory of orthogonal polynomials, a more precise asymptotic formula was given. Let

$$\lambda_{1,q}^{+} \geq \lambda_{2,q}^{+} \geq \cdots, \qquad \lambda_{1,q}^{-} \leq \lambda_{2,q}^{-} \leq \cdots$$

be the eigenvalues of $H(\vec{a}, \pm V)$ in $(\Lambda_q, \Lambda_{q+1})$ (for +) and $(\Lambda_{q-1}, \Lambda_q)$ (for −), and let $\mathrm{Cap}(K)$ be the logarithmic capacity of the set K, see [Lan72]. Then, if the magnetic scalar potential can be written as $W = -\frac{B_0}{4}|z|^2 + W_\infty$, where W_∞ is bounded, and V has support in a compact K, $V \geq c > 0$ on K, it holds that

$$\lim_{j\to\infty} \left(\pm j!(\lambda_{j,0}^{\pm} - \Lambda_0)\right)^{1/j} = \frac{B_0}{2}\,\mathrm{Cap}(K)^2.$$

Moreover, if $W_\infty = 0$, then they are able to prove a similar formula for the higher Landau levels,

$$\lim_{j\to\infty} \left(\pm j!(\lambda_{j,q}^{\pm} - \Lambda_q\right)^{1/j} = \frac{B_0}{2}\,\mathrm{Cap}(K)^2, \quad q = 0, 1, \ldots.$$

The asymptotics with perturbed magnetic field for higher Landau levels was given in [RT08], using approximate creation and annihilation operators.

Perturbations of a constant magnetic field by AB solenoids are considered in [EŠV02] for one AB solenoid and [Min05] for arbitrary many (uniformly separated) AB solenoids. We explain the results of the latter paper. The presense of AB solenoids introduces several self-adjoint extensions. Assume that there is a finite number n of AB solenoids. We denote an arbitrary self-adjoint extension by H. Then the essential spectrum does not change, i.e.

the Landau levels Λ_q are still infinitely degenerate eigenvalues. Moreover, it is proved that

$$N(-\infty, \Lambda_0, H) \leq 2n,$$
$$N(\Lambda_q, \Lambda_{q+1}, H) \leq qn, \quad q = 0, 1, \ldots.$$

Moreover the Aharonov-Bohm hamiltonian (the extension obtained as the Friedrich's extension) has no eigenvalues below Λ_0.

Another type of problem studied for the magnetic perturbation, which is close to the type of problems we mentioned above, is the behavior of eigenvalues in spectral gaps of the operator $H(\vec{a}_0 + t\vec{a}_1, V)$, as the so-called coupling constant t tends to infinity. Here $\vec{a}_0$ corresponds to the constant magnetic field, $\vec{a}_1$ corresponds to a magnetic field that decays fast at infinity, or even with compact support, and V is a weak perturbation. These type of operators are well-studied, we refer to [HL98, Hem99, Bes00, HB03], and the references therein. A typical result is that infinitely many eigenvalues will cross each gap. However, it seems that the eigenvalues do not move monotonically with the parameter t, for small values of t.

Finally, we turn to perturbations of the domain. In [PR07] a compact obstacle K is introduced in the plane, and the question is again how the eigenvalues split. The operator studied is the one with Dirichlet boundary conditions at the boundary of K. The method is again to do certain reduction steps to Toeplitz-type operators as in [FP06]. The result is that the eigenvalue clusters converge very fast to the Landau levels. Moreover, they can only accumulate from above. The asymptotic formula obtained is

$$\lim_{j\to\infty} \left(j!(\lambda^+_{j,q} - \Lambda_q)\right)^{1/j} = \frac{B_0}{2} \operatorname{Cap}(K)^2, \quad q = 0, 1, \ldots.$$

The result is obtained for quite singular domains K. However, it turns out that the reduction step does not work if one instead impose magnetic Neumann conditions at the boundary. This is what we do in Paper IV. The result is similar, but the eigenvalues accumulate from below instead.

$$\lim_{j\to\infty} \left(j!(\Lambda_q - \lambda^-_{j,q})\right)^{1/j} = \frac{B_0}{2} \operatorname{Cap}(K)^2, \quad q = 0, 1, \ldots.$$

We reduce the situation to a certain Toeplitz-type operator. We do this by a certain reduction to an elliptic Pseudodifferential operator on the boundary of K.

We mention also the fundamental paper [HS02], where the influence of a compact obstacle onto the spectrum of Landau and an impact for more detailed analysis is made. Both Dirichlet and Neumann boundary conditions were considered. They focus more on the properties of the states, especially the property of a state being a bulk state or an edge state. In classical theory a bulk state corresponds to the circular orbit as we saw before, i.e. no interaction with the obstacle is seen. If the particle hits the boundary then the "state" is called an edge state, see Figure 6.1. In quantum mechanics the states usually "live" away from the boundary or close to the boundary. However, since the functions living far away from the boundary are not in general zero close to the boundary, they in fact interact weakly with the boundary, so no pure bulk states exists. The presence of edge-states which are important in many physical situations is discussed, and also some connections between the interior and the exterior problem is given.

7 Open problems

Problem 7.1 If the gyromagnetic ratio g is not equal to 2 then the supersymmetry of the Pauli operator breaks. According to experiments, this is the case for the electron. Thus, it is admirable to define a Pauli operator as in (3.6),

$$\mathfrak{P} = \begin{pmatrix} \mathfrak{Q}^*\mathfrak{Q} & 0 \\ 0 & \mathfrak{Q}\mathfrak{Q}^* \end{pmatrix} + \begin{pmatrix} -\frac{g-2}{2}B & 0 \\ 0 & \frac{g-2}{2}B \end{pmatrix}, \tag{7.1}$$

for $g \neq 2$. This works well for example if the magnetic field is regular and decays fast to zero at infinity. Then one can view it as a perturbation of the $g = 2$ Pauli Hamiltonian. Observe that the Pauli operator in (7.1) is no longer non-negative. Instead of finding formulas for the dimension of the kernel it is now interesting to determine the number of negative eigenvalues. The perturbation works as an electric potential and the theory for it is quite extensive, we refer to [RM05].

The situation gets more intricate if one tries to define a $g \neq 2$ Pauli operator for singular magnetic fields. For example, if an AB solenoid is located at the origin, i.e. $B = 2\pi\alpha\delta$. If the operator is first considered on smooth functions with compact support not touching the singular point, then the operator agrees with the $g = 2$ Pauli operator. It is unclear if any of the self-adjoint extensions of this operator reflects the additional singular term.

This kind of point interactions has been studied in [AGHKH05], and the $g \neq 2$ situation is discussed in several papers. We refer to [BV93, Mor95, BP03] and the references therein.

Problem 7.2 (This problem was proposed to me by Prof. Ari Laptev in private communication) Study Schrödinger operators with singular potentials in higher dimensions. For example, one could try to describe the self-adjoint extensions of the Schrödinger operator in $\mathbb{R}^{2d}$ with magnetic one-form

$$a(x) = \sum_{j \neq k} \Phi_{j,k} \frac{x^{2k} - x^{2j}}{|z^j - z^k|^2} \, \mathrm{d}x^{2j-1} + \Phi_{j,k} \frac{x^{2j-1} - x^{2k-1}}{|z^j - z^k|^2} \, \mathrm{d}x^{2j}$$

initially defined on $C_0^\infty\big(\mathbb{R}^{2d} \setminus \{z^j = z^k\}_{j \neq k}\big)$.

The main difficulty is that the vector potential is singular, not just in one point as in two dimensions, but in all hyperplanes $z^j = z^k$. In two dimensions the extensions were described by certain singular boundary terms at the singular points.

Problem 7.3 Does the asymptotics $W \sim \Phi \log|z|$, $|z| \to \infty$, imply that the dimension of the kernel of the Pauli operator with scalar potential W is given by $\binom{\{|\Phi|\}}{d}$?

Even more interesting, what is the dimension of the kernel of the Pauli operator in $\mathbb{R}^{2d}$ if the magnetic field is not a $(1,1)$ form? The reason to study this problem is that we know no physically reason that a magnetic field should be of type $(1,1)$. Showing some results for general types of magnetic fields will probably include some new ideas or tools, since the techniques of complex analysis used effectively so far no longer applies.

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Paper I

On the Aharonov-Casher formula for different self-adjoint extensions of the Pauli operator with singular magnetic field

Mikael Persson

Abstract: Two different self-adjoint Pauli extensions describing a spin-1/2 two-dimensional quantum system with singular magnetic field are studied. An Aharonov-Casher type formula is proved for the maximal Pauli extension and the possibility of approximation of the two different self-adjoint extensions by operators with regular magnetic fields is investigated.

1 Introduction

Two-dimensional spin-1/2 quantum systems involving magnetic fields are described by the self-adjoint Pauli operator. One interesting question about such systems is the appearance of zero modes (eigenfunctions with eigenvalue zero). Aharonov and Casher proved in [AC79] that if the magnetic field is bounded and compactly supported, then zero modes can arise, and the number of zero modes is simply connected to the total flux of the magnetic field. Since then, Aharonov-Casher type formulas have been proved for more and more singular magnetic fields in different settings, see [CFKS87, GG02, LL58, Mil82]. Recently they were proved for measure-valued magnetic fields in [EV02] by Erdős and Vougalter.

We are interested in the Pauli operator when the magnetic field consists of a regular part with compact support and a singular part with a finite number of Aharonov-Bohm (AB) solenoids [AB59]. The Pauli operator for such singular magnetic fields, defined initially on smooth functions with support not touching the singularities, is not essentially self-adjoint. Thus there are several ways of defining the self-adjoint Pauli extension, depending on what boundary conditions one sets at the AB solenoids, see [AT98, DŠ98, EŠV02, GŠ04a, GŠ04b]. Different extensions describe different physics, and there is a discussion going on about which extensions describe the real physical situation.

There are two possible approaches to making the choice of the extension: trying to describe boundary conditions at the singularities by means of modelling actual interaction of the particle with an AB solenoid, or considering approximations of singular fields by regular ones, see [BP03, Tam03]. We are going to study the maximal extension introduced in [GG02], called the Maximal Pauli operator, and compare it with the extension defined in [EV02], that we will call the EV Pauli operator. These two extensions were recently studied in [RS06] in the presence of infinite number of AB solenoids, and it was proved that a magnetic field with infinite flux gives an infinite-dimensional space of zero modes for both extensions.

When studying the Pauli operator in the presence of AB solenoids one must always keep in mind the possibility to reduce the intensities of solenoids by arbitrary integers by means of singular gauge transformations. In Section 2 we define both extensions via quadratic forms. The Maximal Pauli operator can be defined directly for arbitrary strength of the AB fluxes, while the EV Pauli operator is defined via gauge transformations if the AB intensities do not belong to the interval $[-1/2, 1/2)$.

The EV Pauli operator has the advantage that the Aharonov-Casher type formula in its original form holds even for singular AB magnetic fields. However, as we show, it does not satisfy another natural requirement, that the number of zero modes is invariant under the change of sign of the magnetic field. This absence of invariance exhibits itself only if both singular and regular parts of the field are present. This justifies our attempt to study the Maximal Pauli operator which lacks the latter disadvantage. The price we have to pay for this is that our Aharonov-Casher type formula has certain extra terms.

For the Dirac operators with strongly singular magnetic field the question on the number of zero modes was considered in [HO01]. The definition of the self-adjoint operator considered there is close to the one in Erdős-Vougalter, however it is not gauge invariant, therefore the Aharonov Casher-type formula obtained in [HO01] depends on intensity of each AB solenoid separately.

In Section 3 we establish that the Maximal Pauli operator is gauge invariant and that changing the sign of the magnetic field leads to anti-unitarily equivalence. Our main result is the Aharonov-Casher type formula for the Maximal Pauli operator. An interesting fact is that this operator can have both spin-up and spin-down zero modes, in contrary to the EV Pauli operator and the Pauli operator for less singular magnetic fields, which have either spin-up or spin-down zero modes, but not both. In [GG02] a setting with an infinite lattice of AB solenoids with equal AB flux at each solenoid is studied, having both spin-up and spin-down zero modes, both with infinite multiplicity.

In Section 4 we discuss the approximation by more regular fields in the sense of Borg and Pulé, see [BP03]. It turns out that both the Maximal Pauli operator and the EV Pauli operator can be approximated in this way. However, the EV Pauli operator can be approximated as a Pauli Hamiltonian, while the Maximal Pauli operator can only be approximated one component at a time. Since different ways of approximating the magnetic field may lead to different results, see [BV93, Tam03], we leave the question if the Maximal Pauli operator can be approximated as Pauli Hamiltonian open.

2 Definition of the Pauli operators

The Pauli operator is formally defined in the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^2) \otimes \mathbb{C}^2$ as

$$\mathfrak{P} = \big(\sigma \cdot (-i\nabla - \vec{a})\big)^2 = \big(-i\nabla - \vec{a}\big)^2 - \sigma_0 B.$$

Here $\sigma = (\sigma_1, \sigma_2)$, where σ_0, σ_1 and σ_2 are the Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$\vec{a} = (a_1, a_2)$ is the real magnetic vector potential and $B = \frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2}$ is the magnetic field. This definition does not work if the magnetic field B is too singular, see the discussion in [EV02, Sob96]. If $\vec{a} \in L_{2,\mathrm{loc}}(\mathbb{R}^2) \otimes \mathbb{R}^2$, using the notation $\Pi_j = -i\partial_j - a_j$, for $j = 1, 2$,

$$\mathfrak{Q} = \Pi_1 - i\Pi_2, \quad \mathfrak{Q}^* = \Pi_1 + i\Pi_2,$$

and $\mathrm{d}m$ for the Lebesgue measure, the Pauli operator can be defined via the quadratic form

$$\mathfrak{p}[\psi] = \|\mathfrak{Q}\psi_+\|^2 + \|\mathfrak{Q}^*\psi_-\|^2 = \int_{\mathbb{R}^2} |\sigma \cdot (-i\nabla - \vec{a})\psi|^2 \,\mathrm{d}m(x), \tag{2.1}$$

the domain being the closure in the sense of the metrics $\mathfrak{p}[\psi]$ of the core consisting of smooth compactly supported functions. With this notation, we can write the Pauli operator $\mathfrak{P}$ as

$$\mathfrak{P} = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix} = \begin{pmatrix} \mathfrak{Q}^*\mathfrak{Q} & 0 \\ 0 & \mathfrak{Q}\mathfrak{Q}^* \end{pmatrix}. \tag{2.2}$$

However, defining the Pauli operator via the quadratic form $\mathfrak{p}[\psi]$ in (2.1) requires that the vector potential $\vec{a}$ belongs to $L_{2,\mathrm{loc}}(\mathbb{R}^2) \otimes \mathbb{R}^2$, otherwise $\mathfrak{p}[\psi]$ can be infinite for nice functions ψ, see [Sob96]. If the magnetic field consists of only one AB solenoid located at the origin with intensity (flux divided by 2π) α, then a magnetic vector potential $\vec{a}$ is given by

$$\vec{a}(x^1, x^2) = \frac{\alpha}{|x|^2}\left(-x^2, x^1\right)$$

which is not in $L_{2,\mathrm{loc}}(\mathbb{R}^2) \otimes \mathbb{R}^2$. Here, and elsewhere we identify a point $x = (x^1, x^2)$ in the two-dimensional space $\mathbb{R}^2$ with $z = x^1 + ix^2$ in the complex plan $\mathbb{C}$.

Following [EV02], we will define the Pauli operator via another quadratic form, that agrees with $\mathfrak{p}[\psi]$ for less singular magnetic fields. We start by describing the magnetic field.

Even though the Pauli operator can be defined for more general magnetic fields, in order to demonstrate the main features of the study, without extra technicalities, we restrict ourself to a magnetic field consisting of a sum of two parts, the first being a smooth function with compact support, the second consisting of finitely many AB solenoids. Let

$$\Lambda = \left\{z_j\right\}_{j=1}^{n}$$

be a set of distinct points in $\mathbb{C}$ and let $\alpha_j \in \mathbb{R} \setminus \mathbb{Z}$. The magnetic field we will study in this paper has the form

$$B(z) = B_0(z) + \sum_{j=1}^{n} 2\pi\alpha_j\delta_{z_j}, \tag{2.3}$$

where $B_0 \in C_0^1(\mathbb{R}^2)$. In [EV02] the magnetic field is given by a signed real regular Borel measure μ on $\mathbb{R}^2$ with locally finite total variation. It is clear that $\mathrm{d}\mu = B_0(z)\,\mathrm{d}m(z) + \sum_{j=1}^{n} 2\pi\alpha_j\delta_{z_j}$ is such a measure.

The function W_0 given by

$$W_0(z) = \frac{1}{2\pi}\int_{\mathbb{C}} \log|z-z'|B_0(z')\,\mathrm{d}m(z')$$

satisfies $-\Delta W_0 = B_0$ since $B_0 \in C_0^1(\mathbb{R}^2)$ and $-\Delta \log|z-z_j| = 2\pi\delta_{z_j}$ in the sense of distributions. The function

$$W(z) = W_0(z) + \sum_{j=1}^{n} \alpha_j \log|z-z_j|$$

satisfies $-\Delta W = B$. It is easily seen that $W_0(z) \sim \Phi_0 \log|z|$ as $|z| \to \infty$, and thus the asymptotics of $e^{\pm W(z)}$ is

$$e^{\pm W(z)} \sim \begin{cases} |z|^{\pm\Phi}, & |z| \to \infty; \\ |z-z_j|^{\pm\alpha_j}, & z \to z_j; \end{cases}$$

where $\Phi_0 = \frac{1}{2\pi}\int_{\mathbb{C}} B_0(z)\,\mathrm{d}m(z)$ and $\Phi = \frac{1}{2\pi}\int_{\mathbb{C}} B(z)\,\mathrm{d}m(z) = \Phi_0 + \sum_{j=1}^{n}\alpha_j$.

We are now ready to define the two self-adjoint Pauli operators. The decisive difference between them is the sense in which we are taking derivatives. This leads to different domains, and, as we will see in later sections, to different properties of the operators. Let us introduce notation for taking derivatives in the different spaces of distributions. Remember that $\Lambda = \{z_j\}_{j=1}^{n}$ is a finite set of distinct points in $\mathbb{C}$. We let the derivatives in $\mathcal{D}'(\mathbb{R}^2)$ be denoted by ∂ and the derivatives in $\mathcal{D}'(\mathbb{R}^2 \setminus \Lambda)$ be denoted by ∂ with a tilde over it, that is $\tilde{\partial}$. Thus, for example, by ∂_z we mean $\frac{\partial}{\partial z}$ in the space $\mathcal{D}'(\mathbb{R}^2)$ and by $\tilde{\partial}_z$ we mean $\frac{\partial}{\partial z}$ in the space $\mathcal{D}'(\mathbb{R}^2 \setminus \Lambda)$.

2.1 The EV Pauli operator

We follow [EV02] and define the sesquilinear forms $\mathfrak{p}_{\mathrm{EV}}^{+}$ and $\mathfrak{p}_{\mathrm{EV}}^{-}$ by

$$\mathfrak{p}_{\mathrm{EV}}^{+}(\psi_+,\xi_+) = 4\int_{\mathbb{C}} \partial_{\bar z}\left(e^{-W}\psi_+\right)\overline{\partial_{\bar z}\left(e^{-W}\xi_+\right)}e^{2W}\,\mathrm{d}m(z);$$
$$\mathrm{Dom}(\mathfrak{p}_{\mathrm{EV}}^{+}) = \left\{\psi_+ \in L_2(\mathbb{R}^2) \;\middle|\; \mathfrak{p}_{\mathrm{EV}}^{+}(\psi_+,\psi_+) < \infty\right\};$$

and

$$\mathfrak{p}_{\mathrm{EV}}^{-}(\psi_-,\xi_-) = 4\int_{\mathbb{C}} \partial_{z}\left(e^{W}\psi_-\right)\overline{\partial_{z}\left(e^{W}\xi_-\right)}e^{-2W}\,\mathrm{d}m(z);$$
$$\mathrm{Dom}(\mathfrak{p}_{\mathrm{EV}}^{-}) = \left\{\psi_- \in L_2(\mathbb{R}^2) \;\middle|\; \mathfrak{p}_{\mathrm{EV}}^{-}(\psi_-,\psi_-) < \infty\right\}.$$

Set

$$\mathfrak{p}_{\mathrm{EV}}(\psi,\xi)=\mathfrak{p}^{+}_{\mathrm{EV}}(\psi_+,\xi_+)+\mathfrak{p}^{-}_{\mathrm{EV}}(\psi_-,\xi_-);$$
$$\mathrm{Dom}(\mathfrak{p}_{\mathrm{EV}})=\left\{\psi=\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}\in\mathcal{H}\;\middle|\;\mathfrak{p}_{\mathrm{EV}}(\psi,\psi)<\infty\right\}.$$

Let us make more accurate the description of the domains of the forms $\mathfrak{p}^{+}_{\mathrm{EV}}$, $\mathfrak{p}^{-}_{\mathrm{EV}}$ and $\mathfrak{p}_{\mathrm{EV}}$. For example, what is required of a function ψ_+ to be in $\mathrm{Dom}(\mathfrak{p}^{+}_{\mathrm{EV}})$? It should belong to $L_2(\mathbb{R}^2)$, and the expression

$$\mathfrak{p}^{+}_{\mathrm{EV}}(\psi_+,\psi_+)=4\int_{\mathbb{C}}\left|\partial_{\bar z}\left(e^{-W}\psi_+\right)\right|^2 e^{2W}\,\mathrm{d}m(z)$$

should have a meaning and be finite. This means that the distribution $\partial_{\bar z}\left(e^{-W}\psi_+\right)$ actually must be a function and its modulus multiplied with e^{W} must belong to $L_2(\mathbb{R}^2)$, that is $|\partial_{\bar z}\left(e^{-W}\psi_+\right)|e^{W}\in L_2(\mathbb{R}^2)$. This forces all the intensities α_j to be in the interval $(-1,1)$, see [EV02].

Next we define the norm by

$$|||\psi|||^2_{\mathfrak{p}_{\mathrm{EV}}}=|||\psi_+|||^2_{\mathfrak{p}^{+}_{\mathrm{EV}}}+|||\psi_-|||^2_{\mathfrak{p}^{-}_{\mathrm{EV}}},$$

where

$$|||\psi_+|||^2_{\mathfrak{p}^{+}_{\mathrm{EV}}}=\|\psi_+\|^2+\left\|\partial_{\bar z}\big(e^{-W}\psi_+\big)e^{W}\right\|^2$$

and

$$|||\psi_-|||^2_{\mathfrak{p}^{-}_{\mathrm{EV}}}=\|\psi_-\|^2+\left\|\partial_{z}\big(e^{W}\psi_-\big)e^{-W}\right\|^2.$$

The form $\mathfrak{p}_{\mathrm{EV}}$ is symmetric, nonnegative and closed with respect to $\|\cdot\|$, again see [EV02], and hence it defines a unique self-adjoint operator $\widetilde{\mathfrak{P}}_{\mathrm{EV}}$ via

$$\mathrm{Dom}(\widetilde{\mathfrak{P}}_{\mathrm{EV}})=\left\{\psi\in\mathrm{Dom}(\mathfrak{p}_{\mathrm{EV}})\;\middle|\;\mathfrak{p}_{\mathrm{EV}}(\psi,\cdot)\in\mathcal{H}'\right\};\tag{2.4}$$
$$\left\langle\widetilde{\mathfrak{P}}_{\mathrm{EV}}\psi,\xi\right\rangle=\mathfrak{p}_{\mathrm{EV}}(\psi,\xi),\quad \psi\in\mathrm{Dom}(\widetilde{\mathfrak{P}}_{\mathrm{EV}}),\ \xi\in\mathrm{Dom}(\mathfrak{p}_{\mathrm{EV}}).\tag{2.5}$$

We call this operator $\widetilde{\mathfrak{P}}_{\mathrm{EV}}$ the *non-reduced EV Pauli operator*.

If some of the intensities α_j belong to $\mathbb{R}\setminus[-1/2,1/2)$, we let α_j^* be the real number in $[-1/2,1/2)$ such that α_j and α_j^* differ by an integer, that is $\alpha_j^*-\alpha_j=m_j\in\mathbb{Z}$. We define the *reduced EV Pauli operator* (or just the *EV Pauli operator*), $\mathfrak{P}_{\mathrm{EV}}$, to be

$$\mathfrak{P}_{\mathrm{EV}}=\exp(i\varphi)\widetilde{\mathfrak{P}}_{\mathrm{EV}}\exp(-i\varphi),\tag{2.6}$$

where $\varphi(z) = \sum_{j=1}^{n} m_j \arg(z - z_j)$. Hence, if there are some α_j outside the interval $(-1, 1)$ only the reduced EV Pauli operator is well-defined. If all the intensities α_j belong to the interval $[-1/2, 1/2)$ then we do not have to perform the reduction and hence there is only one definition. However, if there are intensities α_j inside the interval $(-1, 1)$ but outside the interval $[-1/2, 1/2)$ then we have two different definitions of the EV Pauli operator, the direct one and the one obtained by reduction. In Section 3 we will show that these two operators are not the same.

2.2 The Maximal Pauli operator

Let $\alpha_j \in \mathbb{R} \setminus \mathbb{Z}$. We define the forms

$$\mathfrak{p}^+_{\max}(\psi_+, \xi_+) = 4 \int_{\mathbb{C}} \tilde{\partial}_{\bar{z}} \left(e^{-W} \psi_+\right) \overline{\tilde{\partial}_{\bar{z}} \left(e^{-W} \xi_+\right)} e^{2W} \, \mathrm{d}m(z);$$
$$\mathrm{Dom}(\mathfrak{p}^+_{\max}) = \left\{ \psi_+ \in L_2(\mathbb{R}^2) \,\middle|\, \mathfrak{p}^+_{\max}(\psi_+, \psi_+) < \infty \right\},$$

and

$$\mathfrak{p}^-_{\max}(\psi_-, \xi_-) = 4 \int_{\mathbb{C}} \tilde{\partial}_{z} \left(e^{W} \psi_-\right) \overline{\tilde{\partial}_{z} \left(e^{W} \xi_-\right)} e^{-2W} \, \mathrm{d}m(z);$$
$$\mathrm{Dom}(\mathfrak{p}^-_{\max}) = \left\{ \psi_- \in L_2(\mathbb{R}^2) \,\middle|\, \mathfrak{p}^-_{\max}(\psi_-, \psi_-) < \infty \right\}.$$

Set

$$\mathfrak{p}_{\max}(\psi, \xi) = \mathfrak{p}^+_{\max}(\psi_+, \xi_+) + \mathfrak{p}^-_{\max}(\psi_-, \xi_-);$$
$$\mathrm{Dom}(\mathfrak{p}_{\max}) = \left\{ \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \in \mathcal{H} \,\middle|\, \mathfrak{p}_{\max}(\psi, \psi) < \infty \right\}.$$

Again, let us make clear about the domains of the forms. For a function ψ_+ to be in $\mathrm{Dom}(\mathfrak{p}^+_{\max})$ it is required that $\psi_+ \in L_2(\mathbb{R}^2)$ and that $\tilde{\partial}_z(e^{-W}\psi_+)$ is a function. After taking the modulus of this derivative and multiplying by e^W we should get into $L_2(\mathbb{R}^2 \setminus \Lambda)$, that is $|\tilde{\partial}_{\bar{z}}(e^{-W}\psi_+)|e^W \in L_2(\mathbb{R}^2 \setminus \Lambda)$. Note that the form $\mathfrak{p}_{\max}$ does not feel the AB fluxes at Λ since the derivatives are taken in the space $\mathcal{D}'(\mathbb{R}^2 \setminus \Lambda)$, and integration does not feel Λ either since Λ has Lebesgue measure zero. This enable the AB solenoids to have intensities that lie outside $(-1, 1)$.

Also, define the norm

$$|||\psi|||^2_{\mathfrak{p}_{\max}} = |||\psi_+|||^2_{\mathfrak{p}^+_{\max}} + |||\psi_-|||^2_{\mathfrak{p}^-_{\max}},$$

where

$$|||\psi_+|||^2_{\mathfrak{p}^+_{\max}} = \|\psi_+\|^2 + \left\|\tilde{\partial}_{\bar{z}}\left(e^{-W}\psi_+\right)e^{W}\right\|^2$$

and

$$|||\psi_-|||^2_{\mathfrak{p}^-_{\max}} = \|\psi_-\|^2 + \left\|\tilde{\partial}_{z}\left(e^{W}\psi_-\right)e^{-W}\right\|^2.$$

Proposition 2.1 *The form $\mathfrak{p}_{max}$ defined above is symmetric, nonnegative and closed with respect to $\|\cdot\|$.*

Proof It is clear that $\mathfrak{p}_{\max}$ is symmetric and nonnegative. Assume that $\psi_j = (\psi_{j,+}, \psi_{j,-})$ is a Cauchy sequence in the norm $|||\cdot|||_{\mathfrak{p}_{\max}}$. This implies that

$$\begin{aligned}
\psi_{j,\pm} \to \psi_\pm \quad &\text{in} \quad L_2\big(\mathbb{R}^2, \, \mathrm{d}m(z)\big);\\
\tilde{\partial}_{\bar{z}}\big(e^{-W}\psi_{j,+}\big) \to u_+ \quad &\text{in} \quad L_2\big(\mathbb{R}^2, e^{2W}\, \mathrm{d}m(z)\big);\\
\tilde{\partial}_{z}\big(e^{W}\psi_{j,-}\big) \to u_- \quad &\text{in} \quad L_2\big(\mathbb{R}^2, e^{-2W}\, \mathrm{d}m(z)\big).
\end{aligned}$$

We shall show that

$$\begin{aligned}
u_+ &= \tilde{\partial}_{\bar{z}}\big(e^{-W}\psi_+\big), \quad \text{and}\\
u_- &= \tilde{\partial}_{z}\big(e^{W}\psi_-\big).
\end{aligned}$$

For any function $\varphi \in C_0^\infty(\mathbb{R}^2 \setminus \Lambda)$,

$$\begin{aligned}
\left|\left\langle \left(u_+ - \tilde{\partial}_{\bar{z}}\left(e^{-W}\psi_+\right)\right), \varphi\right\rangle\right| &= \left|\int_{\mathbb{C}} \left(u_+ - \tilde{\partial}_{\bar{z}}\left(e^{-W}\psi_+\right)\right)\overline{\varphi}\,\mathrm{d}m(z)\right|\\
&\leq \left|\int_{\mathbb{C}} \left(u_+ - \tilde{\partial}_{\bar{z}}\left(e^{-W}\psi_{j,+}\right)\right)\overline{\varphi}\,\mathrm{d}m(z)\right|\\
&\quad + \left|\int_{\mathbb{C}} \left(\psi_+ - \psi_{j,+}\right)\tilde{\partial}_{\bar{z}}(\overline{\varphi})e^{-W}\,\mathrm{d}m(z)\right|\\
&\leq \left\|u_+ - \tilde{\partial}_{\bar{z}}\left(e^{-W}\psi_{j,+}\right)\right\|_{L_2(e^{2W}\,\mathrm{d}m(z))} \cdot \|\overline{\varphi}e^{-W}\|\\
&\quad + \|\psi_+ - \psi_{j,+}\| \cdot \left\|\tilde{\partial}_{\bar{z}}(\overline{\varphi})e^{-W}\right\|.
\end{aligned}$$

The above expression tends to zero as $j \to \infty$, since the first factor in each term tend to zero, and the second factor is bounded (thanks to φ). The proof is the same for the spin down component. This shows that the form $\mathfrak{p}_{\max}$ is closed. □

Hence $\mathfrak{p}_{\max}$ defines a unique self-adjoint operator $\mathfrak{P}_{\max}$ via

$$\operatorname{Dom}(\mathfrak{P}_{\max}) = \left\{ \psi \in \operatorname{Dom}(\mathfrak{p}_{\max}) \,\middle|\, \mathfrak{p}_{\max}(\psi, \cdot) \in \mathcal{H}' \right\}; \tag{2.7}$$

$$\left\langle \mathfrak{P}_{\max}\psi, \xi \right\rangle = \mathfrak{p}_{\max}(\psi, \xi), \quad \psi \in \operatorname{Dom}(\mathfrak{P}_{\max}),\ \xi \in \operatorname{Dom}(\mathfrak{p}_{\max}). \tag{2.8}$$

We call this operator $\mathfrak{P}_{\max}$ the *Maximal Pauli operator*.

3 Properties of the Pauli operators

In this section we will compare some properties of the two Pauli operators $\mathfrak{P}_{EV}$ and $\mathfrak{P}_{\max}$ defined in the previous section. We start by showing that $\mathfrak{P}_{\max}$ is gauge invariant while the non-reduced EV Pauli operator $\widetilde{\mathfrak{P}}_{EV}$ is not.

3.1 Gauge transformations

Let $B(z) = B_0(z) + \sum_{j=1}^{n} 2\pi\alpha_j\delta_{z_j}$ be the same magnetic field as before and let $\widehat{B}(z)$ be another magnetic field that differs from $B(z)$ only by multiples of the delta functions, that is

$$\widehat{B}(z) = B(z) + \sum_{j=1}^{n} 2\pi m_j \delta_{z_j},$$

where m_j are integers, not all equal to zero. Then the corresponding scalar potentials $\widehat{W}(z)$ and $W(z)$ differ only by the corresponding logarithms

$$\widehat{W}(z) = W(z) + \sum_{j=1}^{n} m_j \log|z - z_j|.$$

With

$$\varphi(z) = \sum_{j=1}^{n} m_j \arg(z - z_j)$$

we get

$$\widehat{W}(z) + i\varphi(z) = W(z) + \sum_{j=1}^{n} m_j \log(z - z_j).$$

The function $\widehat{W}$ is multivalued, but, since m_j, $j = 1, 2, \ldots, n$ are integers, we have

$$\partial_{\bar z}\left(\widehat{W}(z)+i\varphi(z)\right) = \partial_{\bar z}W(z)+\sum_{j=1}^{n} m_j\partial_{\bar z}\log(z-z_j); \tag{3.1}$$

$$\tilde\partial_{\bar z}\left(\widehat{W}(z)+i\varphi(z)\right) = \tilde\partial_{\bar z}W(z); \tag{3.2}$$

$$e^{\widehat{W}+i\varphi} = e^{W}\prod_{j=1}^{m}(z-z_j)^{m_j}. \tag{3.3}$$

Let us check what happens with $\mathfrak{p}_{\max}$ when we do gauge transforms. Let $\psi = (\psi_+,\psi_-)^t \in \mathrm{Dom}(\mathfrak{p}_{\max}(W))$. Here, and in the continuation, we add the scalar potential to the notation of the forms and operators when necessary to avoid confusion. We should check that $\widehat{\psi} = e^{-i\varphi}\psi$ belongs to $\mathrm{Dom}(\mathfrak{p}_{\max}(\widehat{W}))$, where $\varphi(z) = \sum_{j=1}^{n} m_j \arg(z-z_j)$ is the harmonic conjugate to $\widehat{W}(z)-W(z)$. We do this for $\mathfrak{p}^+_{\max}(\widehat{W})$. A calculation using (3.2) and (3.3) gives

$$\begin{aligned}\left|\tilde\partial_{\bar z}\big(e^{-\widehat{W}}\widehat{\psi}_+\big)\right|e^{\widehat{W}} &= \left|\tilde\partial_{\bar z}\big(e^{-\widehat{W}-i\varphi}\psi_+(z)\big)\right|e^{\widehat{W}} \\ &= \left|\tilde\partial_{\bar z}\Big(e^{-W}\prod_{j=1}^{n}(z-z_j)^{-m_j}\psi_+(z)\Big)\right|e^{W}\prod_{j=1}^{n}|z-z_j|^{m_j} \\ &= \left|\tilde\partial_{\bar z}\big(e^{-W}\psi_+\big)\right|e^{W}.\end{aligned} \tag{3.4}$$

Hence $\psi_+ \in \mathrm{Dom}(\mathfrak{p}^+_{\max}(W))$ implies that $\widehat{\psi}_+ = e^{-i\varphi}\psi_+ \in \mathrm{Dom}(\mathfrak{p}^+_{\max}(\widehat{W}))$. In the same way it follows that $\widehat{\psi}_- = e^{-i\varphi}\psi_- \in \mathrm{Dom}(\mathfrak{p}^-_{\max}(\widehat{W}))$ if $\psi_- \in \mathrm{Dom}(\mathfrak{p}^-_{\max}(W))$. Thus $e^{-i\varphi}\,\mathrm{Dom}(\mathfrak{p}_{\max}(W)) \subset \mathrm{Dom}(\mathfrak{p}_{\max}(\widehat{W}))$. In the same way we can show that $e^{i\varphi}\,\mathrm{Dom}(\mathfrak{p}_{\max}(\widehat{W})) \subset \mathrm{Dom}(\mathfrak{p}_{\max}(W))$, and thus we can conclude that $e^{-i\varphi}\,\mathrm{Dom}(\mathfrak{p}_{\max}(W)) = \mathrm{Dom}(\mathfrak{p}_{\max}(\widehat{W}))$. From the calculation in (3.4) and a similar calculation for the spin-down component ψ_- it also follows that

$$\mathfrak{p}_{\max}(\widehat{W})\left(e^{-i\varphi}\psi, e^{-i\varphi}\psi\right) = \mathfrak{p}_{\max}(W)(\psi,\psi).$$

Hence we can conclude that if $\psi \in \mathrm{Dom}(\mathfrak{P}_{\max}(W))$ and $\xi \in \mathrm{Dom}(\mathfrak{p}_{\max}(W))$ then $e^{-i\varphi}\psi \in \mathrm{Dom}(\mathfrak{P}_{\max}(\widehat{W}))$ and $e^{-i\varphi}\xi \in \mathrm{Dom}(\mathfrak{p}_{\max}(\widehat{W}))$. If we denote by U_φ the unitary operator of multiplication by $e^{i\varphi}$, then we get

$$\begin{aligned}\left\langle \mathfrak{P}_{\max}(W)\psi,\xi\right\rangle &= \mathfrak{p}_{\max}(W)(\psi,\xi) = \mathfrak{p}_{\max}(\widehat{W})(U_\varphi^*\psi, U_\varphi^*\xi) \\ &= \left\langle \mathfrak{P}_{\max}(\widehat{W})U_\varphi^*\psi, U_\varphi^*\xi\right\rangle = \left\langle U_\varphi\mathfrak{P}_{\max}(\widehat{W})U_\varphi^*\psi,\xi\right\rangle,\end{aligned}$$

and hence $\mathfrak{P}_{\max}(W)$ and $\mathfrak{P}_{\max}(\widehat{W})$ are unitarily equivalent. We have proved the following proposition.

Proposition 3.1 *Let B and $\widehat{B}$ be two singular magnetic fields as in (2.3), with difference $\widehat{B} - B = \sum_{j=1}^{n} 2\pi m_j \delta_{z_j}$, where m_j are integers, not all equal to zero. Then their corresponding Maximal Pauli operators defined by (2.7) and (2.8) are unitarily equivalent.*

To see that $\widetilde{\mathfrak{P}}_{\mathrm{EV}}$ is not gauge invariant it is enough to look at an example. Note that this operator is defined only for intensities belonging to the interval $(-1, 1)$. Let $n = 1$, $z_1 = 0$, $\alpha_1 = -1/2$ and $m_1 = 1$, so the two magnetic fields are $B(z) = B_0(z) - \pi\delta_0$ and $\widehat{B}(z) = B_0(z) + \pi\delta_0$. The scalar potentials are given by $W(z) = W_0(z) - \frac{1}{2}\log|z|$ and $\widehat{W}(z) = W_0(z) + \frac{1}{2}\log|z|$ respectively, where $W_0(z)$ is a smooth function with asymptotics $\Phi_0 \log|z|$ as $|z| \to \infty$. We should show that $\mathrm{Dom}(\mathfrak{p}_{\mathrm{EV}}(\widehat{W}))$ is not given by $e^{-i\varphi}\,\mathrm{Dom}(\mathfrak{p}_{\mathrm{EV}}(W))$, where $\varphi(z) = \arg(z)$. Then it follows that $\mathfrak{p}_{\mathrm{EV}}(W)$ and $\mathfrak{p}_{\mathrm{EV}}(\widehat{W})$ do not define unitarily equivalent operators.

Let $\psi_+ \in \mathrm{Dom}(\mathfrak{p}^+_{\mathrm{EV}}(W))$. This means, in particular, that $\partial_{\bar z}(\psi_+ e^{-W})$ belongs to the space $L_{1,\mathrm{loc}}(\mathbb{R}^2)$. Now let $\widehat{\psi}_+ = e^{-i\varphi}\psi_+$. Then, according to (3.3) we get

$$\partial_{\bar z}(\widehat{\psi}_+ e^{-\widehat{W}}) = \partial_{\bar z}(\psi_+ e^{-\widehat{W} - i\varphi}) = \partial_{\bar z}\left(\frac{\psi_+ e^{-W}}{z}\right) = \partial_{\bar z}(\psi_+ e^{-W})\frac{1}{z} + \psi_+ e^{-W}\pi\delta_0$$

which is not in $L_{1,\mathrm{loc}}(\mathbb{R}^2)$ since it is a distribution involving δ_0 (for non-smooth ψ_+ it is not even well-defined). Thus the sets $\mathrm{Dom}(\mathfrak{p}^+_{\mathrm{EV}}(\widehat{W}))$ and $e^{-i\varphi}\,\mathrm{Dom}(\mathfrak{p}^+_{\mathrm{EV}}(W))$ are not equal, and hence the sets $\mathrm{Dom}(\mathfrak{p}_{\mathrm{EV}}(\widehat{W}))$ and $e^{-i\varphi}\,\mathrm{Dom}(\mathfrak{p}_{\mathrm{EV}}(W))$ also differ, so the forms $\mathfrak{p}_{\mathrm{EV}}(W)$ and $\mathfrak{p}_{\mathrm{EV}}(\widehat{W})$ are not defining unitarily equivalent operators $\widetilde{\mathfrak{P}}_{\mathrm{EV}}(W)$ and $\widetilde{\mathfrak{P}}_{\mathrm{EV}}(\widehat{W})$.

3.2 Zero modes

When studying spectral properties of the operator $\mathfrak{P}_{\max}$ it is sufficient to consider AB intensities α_j that belong to the interval $(0, 1)$, since the operator is gauge invariant. See the discussion after the proof of Theorem 3.3 for more details about what happens when we do gauge transformations.

Lemma 3.2 *Let $c_j \in \mathbb{C}$ and $z_j \in \mathbb{C}$, $j = 1, \ldots, n$, where $z_j \neq z_i$ if $j \neq i$ and not all c_j are equal to zero. Then*

$$\sum_{j=1}^{n} \frac{c_j}{z - z_j} \sim \Big(\sum_{j=1}^{n} c_j z_j^l\Big)|z|^{-l-1} + O(|z|^{-l-2}), \quad |z| \to \infty, \tag{3.5}$$

where l is the smallest nonnegative integer such that $\sum_{j=1}^{n} c_j z_j^l \neq 0$.

Proof If $|z|$ is large in comparison with all $|z_j|$ we have

$$\begin{aligned} \sum_{j=1}^{n} \frac{c_j}{z - z_j} &= \frac{1}{z}\sum_{j=1}^{n} \frac{c_j}{1 - z_j/z} \\ &= \sum_{k=0}^{\infty}\Big(\sum_{j=1}^{n} c_j z_j^k\Big)\frac{1}{z^{k+1}} \\ &= \Big(\sum_{j=1}^{n} c_j z_j^l\Big)\frac{1}{z^{l+1}} + O(|z|^{-l-2}) \end{aligned}$$

and thus $\sum_{j=1}^{n} \frac{c_j}{z-z_j} \sim |z|^{-l-1}$ as $|z| \to \infty$. □

Remark 3.1 We note that l in Lemma 3.2 may never be greater than $n-1$. Indeed, if $l \geq n$ then we would have the linear system of equations $\left\{\sum_{j=1}^{n} c_j z_j^k = 0\right\}_{k=0}^{n-1}$. But the determinant of this system is $\prod_{i>j}(z_i - z_j) \neq 0$, and this would force all c_j to be zero.

Note also that for $l < n$ the linear system of l equations

$$\Big\{\sum_{j=1}^{n} c_j z_j^k = 0\Big\}_{k=0}^{l-1}$$

with n unknowns c_j appear, and that the $l \times n$ matrix $\left\{z_j^k\right\}$ has rank l.

Theorem 3.3 *Let $B(z)$ be the magnetic field (2.3) with all $\alpha_j \in (0,1)$, and let $\mathfrak{P}_{max}$ be the Pauli operator defined by (2.7) and (2.8) in Section 2 corresponding to $B(z)$. Then*

$$\dim\ker \mathfrak{P}_{max} = \{n - \Phi\} + \{\Phi\},$$

where $\Phi = \frac{1}{2\pi}\int_{\mathbb{C}} B(z)\,\mathrm{d}m(z)$, and $\{x\}$ denotes the largest integer strictly less than x if $x > 1$ and 0 if $x \leq 1$. Using the notation $\mathfrak{Q}$ and $\mathfrak{Q}^*$ introduced in Section 2, we also have

$$\dim\ker \mathfrak{Q} = \{n - \Phi\} \quad \textit{and} \quad \dim\ker \mathfrak{Q}^* = \{\Phi\}.$$

Proof We follow the reasoning originating in [AC79], with necessary modifications. First we note that $(\psi_+, \psi_-)^t$ belongs to $\ker \mathfrak{P}_{\max}$ if and only if ψ_+ belongs to $\ker \mathfrak{Q}$ and ψ_- belongs to $\ker \mathfrak{Q}^*$, which is equivalent to

$$\tilde{\partial}_{\bar{z}} \left(e^{-W} \psi_+ \right) = 0 \quad \text{and} \quad \tilde{\partial}_z \left(e^{W} \psi_- \right) = 0.$$

This means exactly that $f_\pm(z) = e^{\mp W} \psi_\pm(z)$ are holomorphic $(+)$ and anti-holomorphic $(-)$ functions in $z \in \mathbb{R}^2 \setminus \Lambda$. It is the change in the domain where the functions are holomorphic that influences the result.

Let us start with the spin-up component ψ_+. The function f_+ is allowed to have poles of order at most one at z_j, $j = 1, \dots, n$, and no others, since $e^h \sim |z - z_j|^{\alpha_j}$ as $z \to z_j$ and $\psi_+ = f_+ e^W$ should belong to $L_2(\mathbb{R}^2)$. Hence there exist constants c_j such that the function $f_+(z) - \sum_{j=1}^n \frac{c_j}{z - z_j}$ is entire. From the asymptotics $e^W \sim |z|^{\Phi}$, $|z| \to \infty$, it follows that $f_+ - \sum_{j=1}^n \frac{c_j}{z - z_j}$ may only be a polynomial of degree at most $N = -\Phi - 2$. Hence

$$f_+(z) = \sum_{j=1}^n \frac{c_j}{z - z_j} + a_0 + a_1 z + \dots a_N z^N,$$

where we let the polynomial part disappear if $N < 0$. Now, the asymptotics for ψ_+ is

$$\psi_+(z) \sim |z|^{-l-1+\Phi} + |z|^{N+\Phi}, \quad |z| \to \infty,$$

where l is the smallest nonnegative integer such that $\sum_{j=1}^n c_j z_j^l \neq 0$. To have ψ_+ in $L_2(\mathbb{R}^2)$ we take l to be the smallest nonnegative integer strictly greater than Φ. Remember also from the remark after Lemma 3.2 that $l \leq n - 1$. We get three cases. If $\Phi < -1$, then all complex numbers c_j can be chosen freely, and a polynomial of degree $\{-\Phi\} - 1$ may be added which results $\{n - \Phi\}$ degrees of freedom. If $-1 \leq \Phi < n - 1$ we have no contribution from the polynomial, and we have to choose the coefficients c_j such that $\sum_{j=1}^n c_j z_j^k = 0$ for $k = 0, 1, \dots, l - 1$. The dimension of the null-space of the matrix $\{z_j^k\}$ is $n - l = \{n - \Phi\}$. If $\Phi \geq n - 1$ then we must have all coefficients c_j equal to zero and we get no contribution from the polynomial. Hence, in all three cases we have $\{n - \Phi\}$ spin-up zero modes.

Let us now focus on the spin-down component ψ_-. The function f_- may not have any singularities, since the asymptotics of e^{-W} is $|z - z_j|^{-\alpha_j}$ as $z \to z_j$. Hence f_- must be entire. Moreover, f_- may grow no faster than a polynomial of degree $\Phi - 1$ for ψ_- to be in $L_2(\mathbb{R}^2)$. Thus f_- has to be a polynomial of degree at most $\{\Phi\} - 1$, which gives us $\{\Phi\}$ spin-down zero modes. □

The number of zero modes for $\mathfrak{P}_{\max}$ and $\mathfrak{P}_{\mathrm{EV}}$ are not the same. The Aharonov-Casher theorem for the EV Pauli operator (Theorem 3.1 in [EV02]) states for the field under consideration:

Theorem 3.4 *Let $B(z)$ be as in (2.3) and let $\widehat{B}(z)$ be the unique magnetic field where all AB intensities α_j are reduced to the interval $[-1/2, 1/2)$, that is $\widehat{B}(z) = B(z) + \sum_{j=1}^{n} 2\pi m_j \delta_{z_j}$, where $\alpha_j + m_j \in [-1/2, 1/2)$. Moreover, let $\Phi = \frac{1}{2\pi}\int_{\mathbb{C}} \widehat{B}(z)\, \mathrm{d}m(z)$. Then the dimension of the kernel of the EV Pauli operator $\mathfrak{P}_{\mathrm{EV}}$ is given by $\{|\Phi|\}$. All zero modes belong only to the spin-up or only to the spin-down component (depending on the sign of Φ).*

Below we explain by some concrete examples how the spectral properties of the two Pauli operators $\mathfrak{P}_{\max}$ and $\mathfrak{P}_{\mathrm{EV}}$ differ.

Example 3.1 Since $\mathfrak{P}_{\mathrm{EV}}$ is not gauge invariant we must not expect that the number of zero modes of $\mathfrak{P}_{\mathrm{EV}}$ is invariant under gauge transforms. To see that this property in fact can fail, let us look at the Pauli operators $\mathfrak{P}_{\mathrm{EV}}(W_1)$ and $\mathfrak{P}_{\mathrm{EV}}(W_2)$ induced by the magnetic fields

$$B_1(z) = B_0(z) + \pi\delta_0,$$
$$B_2(z) = B_0(z) - \pi\delta_0$$

respectively, where B_0 has compact support and $\Phi_0 = \frac{1}{2\pi}\int_{\mathbb{C}} B_0(z)\, \mathrm{d}m(z) = \frac{3}{4}$. Then B_2 is reduced (that is, its AB intensity belong to $[-1/2, 1/2)$) but B_1 has to be reduced. Due to Theorem 3.4, the EV Pauli operators $\mathfrak{P}_{\mathrm{EV}}(W_1)$ and $\mathfrak{P}_{\mathrm{EV}}(W_2)$ corresponding to B_1 and B_2 have no zero modes. However, a direct computation for the non-reduced EV Pauli operator $\widetilde{\mathfrak{P}}_{\mathrm{EV}}(W_1)$ corresponding to B_1 shows that it actually has one zero mode. The situation is getting more interesting when we introduce the operator that should correspond to

$$B_3 = B_0(z) + 3\pi\delta_0.$$

The AB intensity for B_3 is too strong so we have to make a reduction. In [EV02] the reduction is made to the interval $[-1/2, 1/2)$, and we have followed this convention, but physically there is nothing that says that this is the natural choice. Reducing the AB intensity of B_3 to $-1/2$ gives an operator with no zero modes and reducing it to $1/2$ gives an operator with one zero mode.

The Maximal Pauli operators $\mathfrak{P}_{\max}(W_1)$, $\mathfrak{P}_{\max}(W_2)$ and $\mathfrak{P}_{\max}(W_3)$ for these three magnetic fields all have one zero mode. This is easily seen by an application of Theorem 3.3 to $\mathfrak{P}_{\max}(W_1)$, and then using the fact that the operators are unitarily equivalent.

However, more understanding is achieved when looking more closely at how the eigenfunctions for these three Maximal Pauli operators look like. Let W_k be the scalar potential for B_k, $k = 1,2,3$. Then, as we have seen before

$$\begin{aligned} W_1(z) &= W_0(z) + \frac{1}{2}\log|z|, \\ W_2(z) &= W_0(z) - \frac{1}{2}\log|z|, \quad \text{and} \\ W_3(z) &= W_0(z) + \frac{3}{2}\log|z|, \end{aligned}$$

where $W_0(z)$ corresponds to $B_0(z)$. Following the reasoning from the proof of Theorem 3.3 we see that the solution space to $\mathfrak{P}_{\max}(W_1)\psi = 0$ is spanned by $\psi = (0, e^{-W_1})^t$.

Next, we see what the solutions to $\mathfrak{P}_{\max}(W_2)\psi = 0$ look like. The flux is this time

$$\Phi_2 = \frac{1}{2\pi}\int_{\mathbb{C}} B_2(z)\,\mathrm{d}m(z) = 1/4 > 0.$$

Let us begin with the spin-up component ψ_+. This time, the holomorphic $f_+ = e^{-W_2}\psi_+$ may not have any poles since then ψ_+ would not belong to $L_2(\mathbb{R}^2)$, and $f_+(z) = e^{-W_2}\psi_+(z) \to 0$ as $|z| \to \infty$, so we must have $f_+ \equiv 0$, and thus $\psi_+ \equiv 0$. For $\psi_-(z)$ to be in $L_2(\mathbb{R}^2)$ it is possible for f_- to have a pole of order 1 at the origin. Hence there exist a constant c such that $f_-(z) - c/\bar{z}$ is antiholomorphic in the whole plane. The function $f_-(z) \to 0$ as $|z| \to \infty$ since the total intensity $\Phi_2 > 0$. This implies, by Liouville's theorem, that $f_-(z) \equiv c/\bar{z}$, so the solution space to $\mathfrak{P}_{\max}(W_2)\psi = 0$ is spanned by $\psi(z) = (0, e^{-W_2}/\bar{z})$.

Finally, we determine the solutions to $\mathfrak{P}_{\max}(W_3)\psi = 0$. For this field, the flux is given by

$$\Phi_3 = \frac{1}{2\pi}\int_{\mathbb{C}} B_3(z)\,\mathrm{d}m(z) = 9/4.$$

Consider the spin-up part ψ_+. For ψ_+ to be in $L_2(\mathbb{R}^2)$ our function f_+ may have a pole of order no more than two at the origin. As before, there exist constants c_1 and c_2 such that $f_+(z) - c_1/z - c_2/z^2$ is entire and its limit is zero as $|z| \to \infty$, and thus $f_+(z) \equiv c_1/z + c_2/z^2$. Again, both c_1 and c_2 must vanish for ψ_+ to be in $L_2(\mathbb{R}^2)$ (otherwise we would not stay in L_2 at infinity). Thus $\psi_+ \equiv 0$. On the other hand, the function f_- may not have any poles (these poles would push ψ_- out of $L_2(\mathbb{R}^2)$), so it is antiholomorphic in the whole plane. It also may grow no faster than $|z|^{5/4}$ as $|z| \to \infty$, and thus f_- has to be a first order polynomial in $\bar{z}$, that is $f_-(z) = c_0 + c_1\bar{z}$. Moreover for ψ_- to be in $L_2(\mathbb{R}^2)$ it must have a zero of order 1 at the origin, and thus $f_-(z) = c_1\bar{z}$. We conclude that the solutions to $\mathfrak{P}_{\max}(W_3)\psi = 0$ are spanned by $(0, \bar{z}e^{-W_3})^t$.

A natural property one should expect of a reasonably defined Pauli operator is that its spectral properties are invariant under the reversing the direction of the magnetic field: $B \mapsto -B$. The corresponding operators are formally anti-unitary equivalent under the transformation $\psi \mapsto \overline{\psi}$ and interchanging of ψ_+ and ψ_-.

Example 3.2 The number of zero modes for $\mathfrak{P}_{\mathrm{EV}}$ is not invariant under $B(z) \mapsto -B(z)$, which we should not expect since the interval $[-1/2, 1/2)$ is not symmetric. We check this by showing that the number of zero modes are not the same. To see this, let $B(z) = B_0(z) + \pi\delta_0$, where B_0 has compact support and $\Phi_0 = \frac{1}{2\pi}\int_{\mathbb{C}} B_0(z)\,\mathrm{d}m(z) = \frac{3}{4}$. Then B has to be reduced since the AB intensity at zero is $1/2 \notin [-1/2, 1/2)$. After reduction we get the magnetic field $\widehat{B}(z) = B_0(z) - \pi\delta_0$, and we can apply Theorem 3.4. We put $\widehat{\Phi} = \frac{1}{2\pi}\int_{\mathbb{C}} \widehat{B}\,\mathrm{d}m(z) = \frac{1}{4}$. Thus the number of zero modes for $\mathfrak{P}_{\mathrm{EV}}(W)$ is 0. Next, we consider the Pauli operator $\mathfrak{P}_{\mathrm{EV}}(-W)$ defined by the magnetic field $B_-(z) = -B(z) = -B_0(z) - \pi\delta_0$. This magnetic field is reduced and thus we can apply Theorem 3.4 directly. The total intensity of this field is $\Phi_- = \frac{1}{2\pi}\int_{\mathbb{C}} -B(z)\,\mathrm{d}m(z) = -\frac{5}{4}$, so the number of zero modes for $\mathfrak{P}_{\mathrm{EV}}(-W)$ is 1. If B has several AB fluxes then the difference in the number of zero modes of $\mathfrak{P}_{\mathrm{EV}}(W)$ and $\mathfrak{P}_{\mathrm{EV}}(-W)$ can be made arbitrarily large.

Remark 3.2 If there are only AB solenoids then the $\mathfrak{P}_{\mathrm{EV}}$ preserves the number of zero modes under $B \mapsto -B$, so the absence of signflip invariance can be noticed only in the presence of both AB and nice part.

Example 3.3 The number of zero modes for $\mathfrak{P}_{\max}$ is invariant when we flip the magnetic field, $B(z) \mapsto -B(z)$. Since it is clear that the number of zero modes is invariant under $z \mapsto \bar{z}$ we look instead at how the Pauli operators change when we do $B(z) \mapsto \widehat{B}(z) = -B(\bar{z})$. If we set $\zeta = \bar{z}$ we get $\widehat{B}(\zeta) = -B(z)$ and the scalar potentials satisfy $\widehat{W}(\zeta) = -W(z)$. Assume that $\psi = (\psi_+(z), \psi_-(z))^t \in \mathrm{Dom}(\mathfrak{p}_{\max}(W))$, and denote by S is the isometric spin-flip operator $S((\psi_+, \psi_-)^t) = (\psi_-, \psi_+)^t$. Then

$$\begin{aligned}
\mathfrak{p}_{\max}(W(z))(\psi,\psi) &= 4\int_{\mathbb{C}} \left|\tilde{\partial}_{\bar z}(\psi_+(z)e^{-W(z)})\right|^2 e^{2W(z)} \\
&\quad + \left|\tilde{\partial}_z(\psi_-(z)e^{W(z)})\right|^2 e^{-2W(z)}\,\mathrm{d}m(z) \\
&= 4\int_{\mathbb{C}} \left|\tilde{\partial}_{\zeta}(\psi_+(\bar\zeta)e^{\widehat{W}(\bar\zeta)})\right|^2 e^{-2\widehat{W}(\zeta)} \\
&\quad + \left|\tilde{\partial}_{\bar\zeta}(\psi_-(\bar\zeta)e^{-\widehat{W}(\zeta))}\right|^2 e^{2\widehat{W}(\zeta)}\,\mathrm{d}m(\zeta) \\
&= \mathfrak{p}_{\max}(\widehat{W}(\bar z))(S\psi, S\psi)
\end{aligned}$$

We see that $(\psi_+, \psi_-)^t$ belongs to $\mathrm{Dom}(\mathfrak{P}_{\max}(W(z)))$ if and only if $(\psi_-, \psi_+)^t$ belongs to $\mathrm{Dom}\big(\mathfrak{P}_{\max}(\widehat{W}(\bar z))\big)$ and then

$$\mathfrak{P}_{\max}\big(\widehat{W}(\bar z)\big) = \mathfrak{P}_{\max}(W(z))S$$

Hence it is clear that $\mathfrak{P}_{\max}\big(\widehat{W}(\bar z)\big)$ and $\mathfrak{P}_{\max}(W(z))$ have the same number of zero modes.

Example 3.4 In the previous example we saw that changing the sign of the magnetic field results in unitarily equivalent Maximal Pauli operators. This implies that the number of zero modes for the Maximal Pauli operators corresponding to B and $-B$ are the same. This, however, can be seen directly from the Aharonov-Casher formula in Theorem 3.3. To be able to apply the theorem to $-B = -B_0 - \sum_{j=1}^n 2\pi\alpha_j\delta_j$ we have to do gauge transformations, adding 1 to all the AB intensities, resulting in $\widehat{B} = -B_0 + \sum_{j=1}^n 2\pi(1-\alpha_j)\delta_j$. Now according to Theorem 3.3 the number of zero modes of $\mathfrak{P}_{\max}(-W)$ is equal to

$$\dim\ker\mathfrak{P}_{\max}(-W) = \{\widehat{\Phi}\} + \{n - \widehat{\Phi}\} = \{n - \Phi\} + \{\Phi\} = \dim\ker\mathfrak{P}_{\max}(W),$$

where we have used that $\widehat{\Phi} = \frac{1}{2\pi}\int_{\mathbb{C}} \widehat{B}\,\mathrm{d}m(z) = n - \Phi$.

4 Approximation by regular fields

We have mentioned that the different Pauli extensions depend on which boundary conditions are induced at the AB fluxes. Let us now make this more precise. Since the self-adjoint extension only depends on the boundary condition at the AB solenoids it is enough to study the case of one such solenoid and no smooth field. For simplicity, let the solenoid be located at the origin, with intensity $\alpha \in (0,1)$, that is, let the magnetic field be given by $B = 2\pi\alpha\delta_0$. We consider self-adjoint extensions of the Pauli operator $\mathfrak{P}$ that can be written in the form

$$\mathfrak{P} = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix} = \begin{pmatrix} \mathfrak{Q}^*\mathfrak{Q} & 0 \\ 0 & \mathfrak{Q}\mathfrak{Q}^* \end{pmatrix},$$

with some explicitly chosen domains that ensures closedness of $\mathfrak{Q}^*$ and $\mathfrak{Q}$. It is exactly such extensions $\mathfrak{P}$ that can be defined by the quadratic form (2.1). A function ψ_+ belongs to $\mathrm{Dom}(P_+)$ if and only if ψ_+ belongs to $\mathrm{Dom}(\mathfrak{Q})$ and $\mathfrak{Q}\psi_+$ belongs to $\mathrm{Dom}(\mathfrak{Q}^*)$, and similarly for P_-.

With each self-adjoint extension $P_+ = \mathfrak{Q}^*\mathfrak{Q}$ and $P_- = \mathfrak{Q}\mathfrak{Q}^*$ one can associate (see [DŠ98, EŠV02, GŠ04a, Tam03]) functionals $c^\pm_{-\alpha}$, $c^\pm_{\alpha}$, $c^\pm_{\alpha-1}$ and $c^\pm_{1-\alpha}$, by

$$\begin{aligned}
c^\pm_{-\alpha}(\psi) &= \lim_{r\to 0} r^{\alpha}\frac{1}{2\pi}\int_0^{2\pi}\psi_\pm d\theta,\\
c^\pm_{\alpha}(\psi) &= \lim_{r\to 0} r^{-\alpha}\left(\frac{1}{2\pi}\int_0^{2\pi}\psi_\pm d\theta - r^{-\alpha}c^\pm_{\alpha}(\psi)\right),\\
c^\pm_{\alpha-1}(\psi) &= \lim_{r\to 0} r^{1-\alpha}\frac{1}{2\pi}\int_0^{2\pi}\psi_\pm e^{i\theta} d\theta,\\
c^\pm_{1-\alpha}(\psi) &= \lim_{r\to 0} r^{\alpha-1}\left(\frac{1}{2\pi}\int_0^{2\pi}\psi_\pm e^{i\theta} d\theta - r^{\alpha-1}c^\pm_{1-\alpha}(\psi)\right).
\end{aligned}$$

such that $\psi_\pm \in \mathrm{Dom}(P_\pm)$ if and only if

$$\psi_\pm \sim c^\pm_{-\alpha}r^{-\alpha} + c^\pm_{\alpha}r^{\alpha} + c^\pm_{\alpha-1}r^{\alpha-1}e^{-i\theta} + c^\pm_{1-\alpha}r^{1-\alpha}e^{-i\theta} + O(r^\gamma) \qquad (4.1)$$

as $r \to 0$, where $\gamma = \min(1+\alpha, 2-\alpha)$ and $z = re^{i\theta}$.

Any two nontrivial independent linear relations between these functionals determine a self-adjoint extension. In order that the operator be rotation-invariant, none of these relations may involve both α and $1-\alpha$ terms simultaneously. Accordingly, the parameters $\nu^\pm_0 = c^\pm_{\alpha}/c^\pm_{-\alpha}$ and $\nu^\pm_1 = c^\pm_{1-\alpha}/c^\pm_{\alpha-1}$, with possible values in $(-\infty,\infty]$, are introduced in [BP03], and it is proved that the operators $P_\pm$ can be approximated by operators with regularized magnetic fields in the norm resolvent sense if and only if $\nu_0^\pm = \infty$ and $\nu_1^\pm \in (-\infty, \infty]$ or if $\nu_0^\pm \in (-\infty, \infty]$ and $\nu_1^\pm = \infty$.

Before we check what parameters the Maximal and EV Pauli operators correspond to, let us in a few words discuss how the approximation in [BP03] works.

The vector magnetic potential $\vec{a}$ is approximated with the vector potential

$$\vec{a}_R(z) = \begin{cases} \vec{a}(z) & |z| > R \\ 0 & |z| < R \end{cases}$$

avoiding the singularity in the origin. The corresponding Hamiltonian H_R, formally defined as

$$H_R = (-i\nabla - \vec{a}_R)^2 + \frac{\beta}{R}\delta(r - R),$$

where $\beta = \beta(\alpha, R)$, is studied. It is decomposed into angular momentum operators $h_{m,R}$. Only the operators $h_{m,R}$ where $m = 0$ or $m = 1$ have non-trivial deficiency space. Let $h_{m,R}^\beta$ be self-adjoint extensions of $h_{m,R}$ and let $H_R^\beta = \bigoplus_{m=-\infty}^{\infty} h_{m,R}^\beta$. Theorem 1 in [BP03] says (here we use the notation ν_0 and ν_1 for what could be $\nu_0^\pm$ and $\nu_1^\pm$ respectively):

(I) If $(\beta(\alpha, R) + \alpha)R^{-2\alpha} \to 2\alpha\nu_0$ as $R \to 0$, then H_R^β converges in the norm resolvent sense to one component of the Pauli Hamiltonian corresponding to $\nu_1 = \infty$.

(II) If $(\beta(\alpha, R) - \alpha + 2)R^{2(\alpha-1)} \to 2(1-\alpha)\nu_1$ as $R \to 0$, then H_R^β converges in the norm resolvent sense to one component of the Pauli Hamiltonian corresponding to $\nu_0 = \infty$.

We are now going to check what parameters the Maximal and EV Pauli operators corresponds to. Generally, for the function ψ_+ to be in Dom(P_+), it must belong to Dom$(\mathfrak{Q})$ and $\mathfrak{Q}\psi_+$ must belong to Dom$(\mathfrak{Q}^*)$. We will find out what is required for a function g to be in Dom$(\mathfrak{Q}^*)$. Take any $\varphi_+ \in$ Dom$(\mathfrak{Q})$, then the integration by parts on the domain $\varepsilon < |z|$ gives

$$\begin{aligned}\langle g, \mathfrak{Q}\varphi_+\rangle &= \lim_{\varepsilon\to 0}\int_{|z|>\varepsilon} g(z)\overline{\left(-2i\frac{\partial}{\partial\bar z}(e^{-W}\varphi_+(z))e^{W}\right)}\,\mathrm{d}m(z)\\ &= \lim_{\varepsilon\to 0}\int_{|z|>\varepsilon} -2i\frac{\partial}{\partial z}(g(z)e^{W})e^{-W}\overline{\varphi_+(z)}\,\mathrm{d}m(z)\\ &\quad+\lim_{\varepsilon\to 0}\varepsilon\int_0^{2\pi} g(\varepsilon e^{i\theta})\overline{\varphi_+(\varepsilon e^{i\theta})}e^{-i\theta}d\theta\\ &= \langle\mathfrak{Q}^* g,\varphi_+\rangle + \lim_{\varepsilon\to 0}\frac{\varepsilon}{2}\int_0^{2\pi} g(\varepsilon e^{i\theta})\overline{\varphi_+(\varepsilon e^{i\theta})}e^{-i\theta}d\theta\end{aligned}$$

Hence, for g to belong to $\mathrm{Dom}(\mathfrak{Q}^*)$ it is necessary and sufficient that

$$\lim_{\varepsilon\to 0}\varepsilon\int_0^{2\pi} g(\varepsilon e^{i\theta})\overline{\varphi_+(\varepsilon e^{i\theta})}e^{-i\theta}d\theta = 0$$

for all $\varphi_+ \in \mathrm{Dom}(\mathfrak{p}^+)$, and thus for $\mathfrak{Q}\psi_+$ to belong to $\mathrm{Dom}(\mathfrak{Q}^*)$ it is necessary and sufficient that

$$\lim_{\varepsilon\to 0}\varepsilon\int_0^{2\pi}\left(\frac{\partial}{\partial\bar z}(e^{-W}\psi_+)e^{W}\right)\Big|_{z=\varepsilon e^{i\theta}}\overline{\varphi_+(\varepsilon e^{i\theta})}e^{-i\theta}d\theta = 0$$

for all $\varphi_+ \in \mathrm{Dom}(\mathfrak{p}^+)$. We know that ψ_+ has asymptotics as in (4.1) and that $\frac{\partial}{\partial\bar z} = \frac{e^{i\theta}}{2}\left(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\theta}\right)$ in polar coordinates. A calculation gives

$$\varepsilon\frac{\partial}{\partial\bar z}(e^{-W}\psi_+)e^{W}e^{-i\theta}\Big|_{z=\varepsilon e^{i\theta}} \sim -2\alpha c^+_{-\alpha}\varepsilon^{-\alpha} + 2(1-\alpha)c^+_{1-\alpha}\varepsilon^{1-\alpha}e^{-i\theta} + O(r^{\gamma}),$$

hence we must have

$$\lim_{\varepsilon\to 0}\int_0^{2\pi}\left(-2\alpha c^+_{-\alpha}\varepsilon^{-\alpha} + 2(1-\alpha)c^+_{1-\alpha}\varepsilon^{1-\alpha}e^{-i\theta}\right)\overline{\varphi_+(\varepsilon e^{i\theta})}\,\mathrm{d}\theta = 0 \tag{4.2}$$

for all $\varphi_+ \in \mathrm{Dom}(\mathfrak{p}^+)$. A similar calculation for the spin-down component yields

$$\lim_{\varepsilon\to 0}\int_0^{2\pi}\left(2\alpha c^-_{\alpha}\varepsilon^{\alpha} + 2(\alpha-1)c^-_{\alpha-1}\varepsilon^{\alpha-1}e^{i\theta}\right)\overline{\varphi_-(\varepsilon e^{i\theta})}\,\mathrm{d}\theta = 0. \tag{4.3}$$

To calculate what parameters $\nu_0^\pm$ and $\nu_1^\pm$ the Maximal and EV Pauli extensions correspond to, it is enough to study the asymptotics of the functions in the form core.

Let us first consider the Maximal Pauli extension. Functions on the form $(\varphi_0^+ c/z)e^W$ constitute a form core for $\mathfrak{p}^+_{\max}$, where φ_0 is smooth. Hence there are elements in $\mathrm{Dom}(\mathfrak{p}^+_{\max})$ that asymptotically behave as r^α and also elements with asymptotics $r^{\alpha-1}e^{-i\theta}$. According to (4.2) this means that $c^+_{-\alpha}$ and $c^+_{1-\alpha}$ must be zero. Similarly, the elements that behave like $r^{-\alpha}$ and elements that behave like $r^{1-\alpha}e^{i\theta}$ constitute a form core for $\mathfrak{p}^-_{\max}$, which by (4.3) forces c^-_α and $c^-_{\alpha-1}$ to be zero. The parameters $\nu^\pm_0$ and $\nu^\pm_1$ are given by $\nu^+_0 = c^+_\alpha/c^+_{-\alpha} = \infty$, $\nu^+_1 = c^+_{1-\alpha}/c^+_{\alpha-1} = 0$, $\nu^-_0 = c^-_\alpha/c^-_{-\alpha} = 0$ and $\nu^-_1 = c^-_{1-\alpha}/c^-_{\alpha-1} = \infty$. We see that the spin-up component can be approximated as in (II), while the spin-down component can be approximated as in (I).

Let us now consider the EV Pauli extension, and study the case when $\alpha \in (0, 1/2)$. The case $\alpha < 0$ follows in a a similar way. A form core for $\mathfrak{p}^+_{\mathrm{EV}}$ is given by $e^W\varphi_0$ where φ_0 is smooth, see [EV02]. These functions have asymptotic behavior r^α. From (4.2) follows that $c^+_{-\alpha}$ must vanish. However, ψ_+ belonging to $\mathrm{Dom}(\mathfrak{Q})$ must also belong to $\mathrm{Dom}(\mathfrak{p}^+_{\mathrm{EV}})$ and since the functions in the form core for $\mathfrak{p}^+_{\mathrm{EV}}$ behave as r^α or nicer, we see that the term $c^+_{\alpha-1}r^{\alpha-1}e^{-i\theta}$ gets too singular to be in $\mathrm{Dom}(\mathfrak{Q})$ if $c^+_{\alpha-1} \neq 0$, and hence $c^+_{\alpha-1}$ must be zero.

Similarly, a form core for $\mathfrak{p}^-_{\mathrm{EV}}$ is given by $e^{-W}\varphi_0$, with φ_0 smooth. Functions in this form core have asymptotic behavior $r^{-\alpha}$ or $r^{-\alpha+1}e^{i\theta}$ which forces c^-_α and $c^-_{\alpha-1}$ to be zero.

Hence the parameters $\nu^\pm_0$ and $\nu^\pm_1$ are given by $\nu^+_0 = c^+_\alpha/c^+_{-\alpha} = \infty$, $\nu^+_1 = c^+_{1-\alpha}/c^+_{\alpha-1} = \infty$, $\nu^-_0 = c^-_\alpha/c^-_{-\alpha} = 0$ and $\nu^-_1 = c^-_{1-\alpha}/c^-_{\alpha-1} = \infty$.

We conclude that the spin-up part of the EV Pauli operator can be approximated in either of the ways (I) or (II), while the spin-down part can be approximated in way (I).

Remark 4.1 From the calculations above it follows that the EV Pauli operator can be approximated as a Pauli Hamiltonian in the sense of [BP03], while the Maximal Pauli operator cannot be approximated as a Pauli Hamiltonian, since the spin-up and spin-down components are approximated in different ways.

Since AB is defined up to a singular gauge transformation and regular fields can not be transformed in this way it is unclear which additional physical requirements or principles can decide on which way of approximation is the most physically reasonable.

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Paper II

On the Dirac and Pauli operators with several Aharonov-Bohm solenoids

Mikael Persson

Abstract: We study the self-adjoint Pauli operators that can be realized as the square of a self-adjoint Dirac operator and correspond to a magnetic field consisting of a finite number of Aharonov-Bohm solenoids and a regular part, and prove an Aharonov-Casher type formula for the number of zero-modes for these operators. We also see that essentially only one of the Pauli operators are spin-flip invariant, and this operator does not have any zero-modes.

1 Introduction

The paper is devoted to the study of self-adjoint realizations of Dirac and Pauli operators involving strongly singular magnetic fields, in particular, to the analysis of admissibility of such realizations.

A basic principle of quantum mechanics requires that a system with conserved energy must be described by a self-adjoint Hamiltonian. For a vast majority of situations, this requirement does not cause any trouble, a naturally defined operator proves to be essentially self-adjoint, so only one self-adjoint realization exists. Complications arise for operators involving singular fields. Here quite often the energy operator, defined on smooth functions with support not touching the singularity may admit many self-adjoint extensions. Physically, such extensions differ by the way how the particle interacts with the singularity, mathematically a kind of boundary conditions at singularity must be imposed; anyway, different choices of the self-adjoint extension describe different physics. Sometimes it is possible to describe all self-adjoint realizations explicitly, we mention here especially the paper [AT98], one of the starting points of our study. In other cases only some of such extensions can be found. However the question remains, which of the extensions may correspond to actual physical situations, and which surely are just a mathematical fiction, irrelevant to the reality. In the present paper we consider the Pauli and Dirac operators with singular magnetic fields and attempt to perform the above selection, using as a criterion several intrinsic physical principles which the operators must obey. We find out that, in fact, very few of the rich set of self-adjoint extensions follow all of these principles.

Two-dimensional spin-$\frac{1}{2}$ non-relativistic quantum systems with magnetic fields are described by the Pauli operator. For regular magnetic fields the Pauli operator is usually defined as the square of the Dirac operator. However, for more singular magnetic fields, such as the delta field, an Aharonov-Bohm (AB) solenoid, generates (see [AB59]), there are many self-adjoint realizations of both the Dirac and the Pauli operator. We consider the magnetic field consisting of finitely many AB solenoids and a smooth field with compact support. Up to now only two Pauli extensions have been studied for this type of magnetic field (see [EV02, Per05]), both defined via a quadratic form. Since the Pauli operator classically is the square of the Dirac operator it is natural to study those self-adjoint Pauli extensions that can be obtained in this way.

Another natural property to expect from the Pauli operator is that it transforms in an (anti)-unitary way when the sign (direction) of the magnetic field is changed to the opposite one and the spin-up and spin-down components are switched. This property is usually called spin-flip invariance, and we want to answer the question of which Pauli operators defined in different ways satisfy it.

One more natural property to expect is the possibility to approximate our operator by ones with regular magnetic fields. For one AB solenoid such Pauli extensions were described in [BP03] and the conditions were expressed in the terms of the asymptotics at the singular point of the functions in the domain of the operator. We extend these results to the case of several solenoids.

These and some other principles, explicitly formulated in the paper, leave rather few of all possible self-adjoint operators. One of the important features to be studied for such operators is the dimension of the space of zero modes, given in the regular case by the Aharonov-Casher formula (see [AC79]). This formula and its modifications have been proved in different settings, see [CFKS87, GG02, Mil82]. Recently this formula was also proved for one of the extensions for a very singular magnetic field (containing the case with AB solenoids) in [EV02]. Another extension was introduced in [GG02], and in [Per05] an Aharonov-Casher type formula was established for that extension. We find out how the (admissible) choice of the self-adjoint extension influences the dimension of the zero subspace.

Although the Pauli operator is the main object of our study, much attention is given to the Dirac operator with strongly singular magnetic field, for which we address the same kind of questions.

So, in Section 2 we investigate systematically the Dirac operator. For some special configurations of singular magnetic fields such operators have been studied before in [Ara93, AH05, dSG89, HO01, Tam03]. In order to be able to treat the general case, we need first to repeat in details the description of all self-adjoint extensions corresponding to only one AB solenoid, given in [Tam03]. To construct the self-adjoint operators in the case of several solenoids, we use the glue-together procedure, proposed in [AR04]. After that we check which extensions are spin-flip invariant and finally we prove a formula for the dimension of the kernel of the Dirac extensions. In [HO01] a formula for the dimension of the kernel of the Dirac operator was proved for two different asymmetric self-adjoint extensions (i.e. those with different behavior of spin-up and spin-down components), and it was demonstrated that, in fact, this dimension may differ for quite natural self-adjoint realizations. These extensions are closely related to the ones introduced in [Ara93]. In both these articles the magnetic field is the same as the one we consider (the one in [Ara93] does not have the regular part), with the addition of even more singular terms containing derivatives of the delta distributions (although by means of proper gauge transformations one can dispose of these derivatives.)

In Section 3 we consider the Pauli operators that are the square of some self-adjoint Dirac operator defined in Section 2. We show exactly which Pauli extensions are obtained in this way, in terms of the asymptotics of functions in the domain of the Pauli operator at the points where the singular AB solenoids are located. We also find an Aharonov-Casher type formula for these Pauli operators. It turns out that there are only two of them that have zero-modes. These two extensions are very asymmetric though, admitting singularities in one component only, which looks rather non-physical. All the other extensions have singularities in both the spin-up and spin-down components, and they are coupled.

It turns out that the Pauli operator studied in [EV02] is a sort of mixture of these two asymmetric extensions, admitting different interaction with the singularity of the field at different AB solenoids. In the end of the article we present a discussion of the properties of the self-adjoint Pauli extensions with respect to different ways of normalization of AB intensities when choosing a representative in the gauge equivalence class.

Not pretending to give the final answer to the question which are correct self-adjoint extensions of Pauli and Dirac operators in the presence of AB solenoids, we hope that the results of the paper may lead to a certain enlightening in this problem.

2 The Dirac operator with singular magnetic field

The goal in this section is to describe the self-adjoint Dirac operators corresponding to a magnetic field consisting of several (but finitely many) AB solenoids together with a smooth field, and to find an Aharonov-Casher type formula for the dimension of the kernel of these self-adjoint operators. Let us introduce some notation that will be used throughout the article. As usual we identify the point $x = (x^1, x^2)$ in $\mathbb{R}^2$ with the complex number $z = x^1 + ix^2$, and we will often write z in polar coordinates, $z = re^{i\theta}$. Sometimes it will be convenient to use the polar coordinates $r_j e^{i\theta_j}$ with z_j as the origin. The magnetic field will consist of a regular part $B_0 \in C_0^1(\mathbb{R}^2)$ and a singular part consisting of n AB solenoids located at the points $\Lambda = \{z_j\}_1^n$, so that the magnetic field B has the form

$$B(z) = B_0(z) + \sum_{j=1}^{n} 2\pi\alpha_j \delta_{z_j}. \tag{2.1}$$

Owing to gauge equivalence (see [Tam03]) we can assume that all the AB intensities α_j (fluxes divided by 2π) belong to the interval $(0,1)$. All derivatives will be considered in the distribution space $\mathcal{D}'(\mathbb{R}^2 \setminus \Lambda)$. We will denote by W a magnetic scalar potential satisfying $-\Delta W = B$. The magnetic scalar potential is uniquely defined modulo addition of a harmonic function. We will use the scalar potential

$$W(z) = \frac{1}{2\pi}\int_{\mathbb{C}} B_0(\zeta)\log|z-\zeta|\,\mathrm{d}m(\zeta) + \sum_{j=1}^{n}\alpha_j \log|z - z_j| = W_0(z) + \sum_{j=1}^{n} W_j(z),$$

where $\mathrm{d}m$ is the Lebesgue measure. The *actions* $\mathfrak{Q}$ and its formal adjoint $\mathfrak{Q}^*$, which will be used to describe how the Dirac operator acts, are defined by

$$\mathfrak{Q}u = -2ie^{W}\frac{\partial}{\partial\bar{z}}\left(e^{-W}u\right) \quad \text{and} \quad \mathfrak{Q}^*u = -2ie^{-W}\frac{\partial}{\partial z}\left(e^{W}u\right).$$

These actions $\mathfrak{Q}$ and $\mathfrak{Q}^*$ are usually called the spin-up and spin-down actions, respectively. The Dirac action is given by

$$\eth = \begin{pmatrix} 0 & \mathfrak{Q}^* \\ \mathfrak{Q} & 0 \end{pmatrix}$$

To be able to describe the self-adjoint Dirac operators with several AB solenoids we first study the self-adjoint extensions of the Dirac operator with one AB solenoid, originally defined on smooth functions with compact support not touching the singular point. The Hilbert space we are working in is

$$\mathcal{H} = L_2(\mathbb{R}^2) \otimes \mathbb{C}^2.$$

We will also denote by $\mathcal{H}^1$ the Sobolev space $H^1(\mathbb{R}^2) \otimes \mathbb{C}^2$.

2.1 The Dirac operator with one AB solenoid

The case of one AB solenoid has been studied before (see [dSG89, Tam03]), and we just sketch the way it was done since we need the detailed information about these extensions for our further analysis. We let the AB solenoid have intensity $\alpha = \alpha_1 \in (0,1)$ and be located at the origin. We will describe all self-adjoint extensions of the Dirac operator originally defined on $C_0^\infty(\mathbb{R}^2 \setminus \{0\}) \otimes \mathbb{C}^2$.

The minimal Dirac operator $\mathfrak{D}_{\min}$, obviously symmetric, is defined by

$$\begin{aligned} \mathrm{Dom}(\mathfrak{D}_{\min}) &= C_0^\infty(\mathbb{R}^2 \setminus \{0\}) \otimes \mathbb{C}^2; \\ \mathfrak{D}_{\min}\psi &= \eth\psi, \quad \psi \in \mathrm{Dom}(\mathfrak{D}_{\min}). \end{aligned}$$

It can be seen that $\overline{\mathfrak{D}}_{\min}$ has deficiency index $(1,1)$, and the deficiency spaces $N_\pm = \ker\big(\overline{\mathfrak{D}}^*_{\min} \pm i\big)$ are spanned by

$$\xi_\pm(re^{i\theta}) = \begin{pmatrix} K_{1-\alpha}(r)e^{-i\theta} \\ \mp K_\alpha(r) \end{pmatrix}.$$

Denote by U any unitary operator from N_+ to N_-. Then U takes ξ_+ to $e^{i\tau}\xi_-$ for some $\tau \in [0, 2\pi)$. According to the theorem of Kreĭn and von Neumann, described in [AG93], all self-adjoint extensions can be parametrized by τ as

$$\begin{aligned} \mathrm{Dom}(\mathfrak{D}_\tau) &= \Big\{ \psi = \psi_0 + \mu(\xi_+ + e^{i\tau}\xi_-) \;\Big|\; \psi_0 \in \mathrm{Dom}\big(\overline{\mathfrak{D}}_{\min}\big),\ \mu \in \mathbb{C} \Big\}, \\ \mathfrak{D}_\tau\psi &= \eth\psi_0 + i\mu(\xi_+ - e^{i\tau}\xi_-), \quad \psi \in \mathrm{Dom}(\mathfrak{D}_\tau). \end{aligned} \tag{2.2}$$

It is also possible to describe the self-adjoint extensions by studying the asymptotic behavior of the functions in the domain at the origin. To see this, let us define the linear functionals $c^{\pm}_{-\alpha}$ and $c^{\pm}_{\alpha-1}$ on $\mathrm{Dom}(\mathfrak{D}_\tau)$ as

$$c^{\pm}_{-\alpha}(\psi) = \lim_{r\to 0} r^{\alpha}\frac{1}{2\pi}\int_0^{2\pi}\psi_{\pm}(re^{i\theta})d\theta, \text{ and}$$
$$c^{\pm}_{\alpha-1}(\psi) = \lim_{r\to 0} r^{1-\alpha}\frac{1}{2\pi}\int_0^{2\pi}\psi_{\pm}(re^{i\theta})e^{i\theta}d\theta.$$

For $\psi = \psi_0 + \mu(\xi_+ + e^{i\tau}\xi_-)$ in $\mathrm{Dom}(\mathfrak{D}_\tau)$, where $\psi_0 \in \mathrm{Dom}\big(\overline{\mathfrak{D}}_{\min}\big)$, applying these functionals gives no contribution from ψ_0 since the limit of functions in $\mathrm{Dom}\big(\overline{\mathfrak{D}}_{\min}\big)$ tends to zero at the origin. Let us introduce the notation $\sigma(\alpha) = \Gamma(\alpha)2^{\alpha}$. Using the asymptotics for the Bessel functions we get

$$c^{+}_{-\alpha}(\psi) = 0, \quad c^{+}_{\alpha-1}(\psi) = \frac{\mu}{2}(1+e^{i\tau})\sigma(1-\alpha),$$
$$c^{-}_{\alpha-1}(\psi) = 0, \quad c^{-}_{-\alpha}(\psi) = \frac{\mu}{2}(e^{i\tau}-1)\sigma(\alpha)$$

for such functions $\psi \in \mathrm{Dom}(\mathfrak{D}_\tau)$. Here μ is the same constant as in (2.2). An equivalent description of all self-adjoint Dirac extensions is

$$\begin{aligned}\mathrm{Dom}(\mathfrak{D}_\tau) = \Big\{\psi \in \mathcal{H} \;\Big|\; \eth\psi \in \mathcal{H};\\ \frac{c^{+}_{\alpha-1}(\psi)}{c^{-}_{-\alpha}(\psi)} = -i\cot(\tau/2)\frac{\sigma(1-\alpha)}{\sigma(\alpha)},\\ c^{+}_{-\alpha}(\psi) = c^{-}_{\alpha-1}(\psi) = 0\Big\};\end{aligned}$$
$$\mathfrak{D}_\tau\psi = \eth\psi, \quad \psi \in \mathrm{Dom}(\mathfrak{D}_\tau).$$

2.2 The Dirac operator with several AB solenoids together with a regular field

In this subsection we are going to study the Dirac operator for a magnetic field consisting of a finite number of AB solenoids together with a regular background field. We will use the same method as in [AR04] to glue together the different self-adjoint Dirac operators corresponding to only one AB solenoid and the self-adjoint Dirac operator corresponding to the regular magnetic field.

Note here that we do not study all self-adjoint extensions but only the ones that are subject to the natural locality principle.

We start by defining the Dirac operator with two AB solenoids together with a smooth field. The general case does not give any extra difficulties. Let the magnetic field B consist of a smooth field B_0 with compact support and two AB solenoids located at z_1 and z_2 with intensities α_1 and α_2,

$$B(z) = B_0(z) + 2\pi\alpha_1\delta_{z_1} + 2\pi\alpha_2\delta_{z_2}. \tag{2.3}$$

In this case our scalar potential W can be written as

$$\begin{aligned} W(z) &= W_0(z) + W_1(z) + W_2(z) \\ &= \frac{1}{2\pi}(\log|\cdot| * B_0)(z) + \alpha_1 \log|z - z_1| + \alpha_2 \log|z - z_2|. \end{aligned}$$

From the previous section we have self-adjoint Dirac operators $\mathfrak{D}^{W_1}_{\tau_1}$ and $\mathfrak{D}^{W_2}_{\tau_2}$ corresponding to each of the AB solenoids separately. We will often drop the parameters τ_1 and τ_2 from the subscripts. So, for example, when we write $\mathfrak{D}^{W_1}$ we mean some self-adjoint extension with one AB solenoid located at z_1.

Let $\varphi_j \in C_0^\infty(\mathbb{R}^2)$, $j = 1, 2$, be equal to 1 in a neighborhood of z_j and have small support not touching a neighborhood of z_k, $k \neq j$ and $0 \le \varphi_j \le 1$. Let $\varphi_0 = 1 - \varphi_1 - \varphi_2$. We denote by E_{jk} the set $\operatorname{supp}\varphi_j \cap \operatorname{supp}\varphi_k$.

Let us introduce the multiplication operators V^{W_j} as

$$V^{W_j} = 2i \begin{pmatrix} 0 & -\frac{\partial W_j}{\partial z} \\ \frac{\partial W_j}{\partial z} & 0 \end{pmatrix}.$$

Note that V^{W_0} is bounded in $\mathcal{H}$. For $j \neq 0$ we will be sure to apply the operators V^{W_j} only on functions being zero in a neighborhood of the singular points z_j.

Definition 2.1 The Dirac operator $\mathfrak{D}^W$ corresponding to the magnetic field B in (2.3) is defined as

$$\operatorname{Dom}\big(\mathfrak{D}^W\big) = \left\{ \psi \in \mathcal{H} \;\middle|\; \varphi_j\psi \in \operatorname{Dom}(\mathfrak{D}^{W_j}),\ j = 0, 1, 2 \right\}$$

and

$$\begin{aligned}\mathfrak{D}^W\psi &= (\mathfrak{D}^{W_0} + V^{W_1} + V^{W_2})(\varphi_0\psi)\\ &\quad + (\mathfrak{D}^{W_1} + V^{W_0} + V^{W_2})(\varphi_1\psi)\\ &\quad + (\mathfrak{D}^{W_2} + V^{W_0} + V^{W_1})(\varphi_2\psi)\end{aligned}$$

for $\psi \in \mathrm{Dom}(\mathfrak{D}^W)$.

It is easily verified that the definition is independent of the partition of unity $1 = \varphi_0 + \varphi_1 + \varphi_2$.

Theorem 2.1 *The Dirac operator $\mathfrak{D}^W$ is self-adjoint.*

For the proof of this theorem, we need some lemmas.

Lemma 2.2 *The Dirac operator $\mathfrak{D}\colon \mathcal{H} \to \mathcal{H}$ without any magnetic field is a self-adjoint operator with the Sobolev space $\mathcal{H}^1$ as domain.*

Proof See [Tha92]. □

Lemma 2.3 *The Dirac operator $\mathfrak{D}^{W_0}$ corresponding to the magnetic field B_0 is self-adjoint in $\mathcal{H}$ with domain $\mathcal{H}^1$.*

Proof The operator $\mathfrak{D}^{W_0}$ can be written as $\mathfrak{D}^{W_0} = \mathfrak{D} + V^{W_0}$ and the multiplication operator V^{W_0} is relatively bounded with respect to $\mathfrak{D}$ with relative bound zero, so the lemma follows from the Kato-Rellich theorem. □

Lemma 2.4 *Let T be a bounded operator from $\mathcal{H}$ to $\mathcal{H}^1$ and let V be a function, $V(z) \to 0$ as $|z| \to \infty$. Then the composition VT is compact in $\mathcal{H}$.*

Proof For $n = 1, 2, \ldots$ we write V as $V = V_n + \tilde{V}_n$, where

$$V_n(z) = \begin{cases} V(z) & |V(z)| > \frac{1}{n} \\ 0 & |V(z)| \le \frac{1}{n}. \end{cases}$$

The functions V_n all have compact support, so the operators V_nT are compact. But $\|V_nT - VT\| \le \frac{1}{n}\|T\|$ for all $n = 1, 2, \ldots$, so VT is also compact. □

Remark 2.1 Lemma 2.4 is also true for 2×2 matrix valued functions V where all components tend to zero at infinity. It also holds if T is bounded from $L_2(\mathbb{R}^2)$ to the Sobolev space $H^1(\mathbb{R}^2)$.

Lemma 2.5 *Let* $0 \neq s \in \mathbb{R}$ *and let* $\varphi \in C_0^\infty(\mathbb{R}^2)$ *with zero in its support. Then the operator* φR *is compact, where* $R = (\mathfrak{D}_\tau + is)^{-1}$ *and* $\mathfrak{D}_\tau$ *is any self-adjoint extension of the Dirac operator corresponding to one AB solenoid (which is assumed to be located at the origin).*

Proof First, φR is compact if and only if $\varphi R(\varphi R)^* = \varphi RR^*\varphi$ is compact. To show that $\varphi RR^*\varphi$ is compact, it is sufficient to show that φRR^* is compact.

The operator RR^* is equal to $\left(\mathfrak{D}_\tau^2 + s^2\right)^{-1}$. Note that $\mathfrak{D}_\tau^2$ is a self-adjoint Pauli operator corresponding to the same magnetic field (see Section 3.1 for a discussion of the Pauli operators that are the square of some Dirac operator). If we denote by $\mathfrak{P}$ any other self-adjoint Pauli operator corresponding to this magnetic field, then by the Kreĭn resolvent formula (see [AG93]) the resolvents of $\mathfrak{D}_\tau^2$ and $\mathfrak{P}$ differ by a finite rank operator. Thus, it is enough to show that $\varphi(\mathfrak{P} + s^2)^{-1}$ is compact for a convenient choice of self-adjoint Pauli extension $\mathfrak{P}$. Let us choose $\mathfrak{P}$ to be the Friedrich extension. The functions in the domain of this extension $\mathfrak{P}$ vanish at the origin so

$$\mathfrak{P} = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix},$$

where H is the Friedrich extension of the Schrödinger operator corresponding to the same magnetic field (see [GŠ04a] for a discussion of this). Hence it is enough to show that $\varphi(H + s^2)^{-1}$ is compact.

Let $H_0 = -\Delta$ be the Schrödinger operator corresponding to no magnetic field. Then, by the diamagnetic inequality (see [MOR04]) it follows that $|\varphi(H+s^2)^{-1}u| \leq \varphi(H_0+s^2)^{-1}|u|$ (pointwise) for all $u \in L_2(\mathbb{R}^2)$. This inequality implies that $\varphi(H+s^2)^{-1}$ is compact if $\varphi(H_0+s^2)^{-1}$ is compact (see [DF79, Pit79]).

The compactness of $\varphi(H_0 + s^2)^{-1}$ follows from Lemma 2.4 since the operator $(H_0 + s^2)^{-1}$ is bounded from $L_2(\mathbb{R}^2)$ to $H^1(\mathbb{R}^2)$.

□

Lemma 2.6 *The operator* $\mathfrak{D}^W$ *is symmetric.*

Proof This follows easily from an integration by parts. □

In the following lemma we look at our operator as acting from its domain $\mathrm{Dom}(\mathfrak{D}^W)$ considered as a Hilbert space equipped with graph norm

$$\begin{aligned}\|\psi\|^2_{\mathfrak{D}^W} &= \|(\mathfrak{D}^{W_0} + V^{W_1} + V^{W_2})(\varphi_0\psi)\|^2 + \|(\mathfrak{D}^{W_1} + V^{W_0} + V^{W_2})(\varphi_1\psi)\|^2 \\ &\quad + \|(\mathfrak{D}^{W_2} + V^{W_0} + V^{W_1})(\varphi_2\psi)\|^2 + \|\psi\|^2.\end{aligned}$$

Lemma 2.7 *Let $0 \neq s \in \mathbb{R}$ be fixed. The operator*

$$\mathfrak{D}^W + is\colon \big(\mathrm{Dom}(\mathfrak{D}^W), \|\cdot\|_{\mathfrak{D}^W}\big) \to \mathcal{H}$$

is a bounded Fredholm operator with index zero.

Proof First, it is clear that $\mathfrak{D}^W + is$ is bounded from the domain space with graph norm. To show that $\mathfrak{D}^W + is$ is a Fredholm operator, it is enough to find a left and a right parametrix (see [Agr90]). We start by finding a right parametrix. Let R_j denote the resolvent $R_j = (\mathfrak{D}^{W_j} + is)^{-1}$, $j = 0, 1, 2$, and define the operator $R\colon \mathcal{H} \to \mathcal{H}$ as

$$Ru = \varphi_0 R_0 u + \varphi_1 R_1 u + \varphi_2 R_2 u, \quad \text{for } u \in \mathcal{H}.$$

For $u \in \mathcal{H}$ we have $\varphi_j \varphi_k R_j u \in \mathcal{H}^1$ and being zero in a neighborhood of the singular point(s) if $j \neq k$. Thus

$$(\mathfrak{D}^{W_k} + V^{W_j})(\varphi_j \varphi_k R_j u) = (\mathfrak{D}^{W_j} + V^{W_k})(\varphi_j \varphi_k R_j u), \quad j \neq k.$$

From this it follows that

$$(\mathfrak{D}^W + is)Ru = u + K_R u$$

where $K_R\colon \mathcal{H} \to \mathcal{H}$ is the operator

$$\begin{aligned} K_R u = {} & \big((V^{W_1} + V^{W_2})\varphi_0 + \mathfrak{D}(\varphi_0)\big) R_0 u \\ & + \big((V^{W_0} + V^{W_2})\varphi_1 + \mathfrak{D}(\varphi_1)\big) R_1 u \\ & + \big((V^{W_0} + V^{W_1})\varphi_2 + \mathfrak{D}(\varphi_2)\big) R_2 u. \end{aligned}$$

K_R is compact. Indeed, the first term is compact according to Lemma 2.4 since the operator R_0 is bounded from $\mathcal{H}$ to $\mathcal{H}^1$ and the matrix-valued function $(V^{W_1} + V^{W_2})\varphi_0 + \mathfrak{D}(\varphi_0)$ tends to zero at infinity. The other two terms are compact by Lemma 2.5. Hence K_R is compact, so R is a right parametrix.

In the same way it is easily checked that the operator

$$L = R_0 \varphi_0 + R_1 \varphi_1 + R_2 \varphi_2$$

is a left parametrix. Thus any of R and L works as a parametrix and hence $\mathfrak{D}^W + is$ is a Fredholm operator.

To see that $\mathfrak{D}^W + is$ has index zero, we note that since $\mathfrak{D}$ with domain $\mathcal{H}^1$ is self-adjoint, the operator $\mathfrak{D} + is$ has index zero and $R_s := (\mathfrak{D} + is)^{-1}$ is a parametrix for $\mathfrak{D} + is$. The operator

$$R - R_s = \varphi_0 R_0 + \varphi_1 R_1 + \varphi_2 R_2 - R_s$$

is compact. To see this, we write $R - R_s$ as

$$R - R_s = (\varphi_0 - 1)R_0 + \varphi_1 R_1 + \varphi_2 R_2 + (R_0 - R_s).$$

The first term is compact according to Lemma 2.4, the second and third according to Lemma 2.5. For the last term we note that $R_0 - R_s = -R_0 V^{W_0} R_s$. The compactness of $V^{W_0} R_s$ follows from Lemma 2.4. Composition with the bounded operator R_0 preserves compactness. Thus $R - R_s$ is compact. It follows that $\operatorname{ind}(R) = \operatorname{ind}(R_s)$.

Since R and R_s are parametrices for $\mathfrak{D}^W + is$ and $\mathfrak{D} + is$ respectively, it holds that

$$\operatorname{ind}(\mathfrak{D}^W + is) = -\operatorname{ind}(R) = -\operatorname{ind}(R_s) = \operatorname{ind}(\mathfrak{D} + is) = 0,$$

so we are done. □

Proof (of Theorem 2.1). We know from Lemma 2.6 that $\mathfrak{D}^W$ is symmetric, so for $0 \neq s \in \mathbb{R}$ we have

$$\|(\mathfrak{D}^W + is)\psi\|^2 = \|\mathfrak{D}^W \psi\|^2 + s^2\|\psi\|^2 \geq s^2\|\psi\|^2.$$

It follows that $\dim\ker(\mathfrak{D}^W + is) = 0$. From Lemma 2.7 we have that $\mathfrak{D}^W + is$ has index zero, so it follows that $\dim\ker((\mathfrak{D}^W)^* - is) = 0$. Choosing s positive and negative respectively gives that the deficiency indices for $\mathfrak{D}^W$ is $(0,0)$, so $\mathfrak{D}^W$ is self-adjoint. □

2.3 Spin flip invariance

Since the particle we are studying moves only in a plane, and the magnetic field is orthogonal to this plane, physically it should be no difference if the sign of the magnetic field is changed. This transformation has to come together with a flip of the spin-up and spin-down components and a normalization of the AB intensities. We say that a self-adjoint extension is spin flip invariant if, after applying these transformations, we end up with a (anti)-unitarily equivalent operator. We will show that there are only two values of the parameter that give spin flip invariant Dirac extensions. Let $\tilde{\tau} = (\tau_1, \ldots, \tau_n)$ and denote the Dirac operator by $\mathfrak{D}^W_{\tilde{\tau}}$. We will use the linear functionals

$$c^{\pm}_{-\alpha_j}(\psi) = \lim_{r_j \to 0} r_j^{\alpha_j} \frac{1}{2\pi} \int_0^{2\pi} \psi_{\pm}(r_j e^{i\theta_j})\, d\theta_j, \text{ and} \tag{2.4}$$

$$c^{\pm}_{\alpha_j - 1}(\psi) = \lim_{r_j \to 0} r_j^{1-\alpha_j} \frac{1}{2\pi} \int_0^{2\pi} \psi_{\pm}(r_j e^{i\theta_j}) e^{i\theta_j}\, d\theta_j. \tag{2.5}$$

We define anti-unitarily operator $S_1 \colon \mathcal{H} \to \mathcal{H}$ as the spin-flip operator that maps $(\psi_+, \psi_-)^t$ to $(\overline{\psi_-}, \overline{\psi_+})^t$.

Proposition 2.8 *The operators $\mathfrak{D}^{-W}_{\tilde{\tau}'}$ and $\mathfrak{D}^{W}_{\tilde{\tau}}$ are anti-unitarily equivalent via the operator S_1 if and only if for all $j = 1, \ldots, n$ we have $\tau'_j + \tau_j = \pi$ or $\tau'_j + \tau_j = 3\pi$.*

Proof Let $\beta_j = 1 - \alpha_j$ be the normalized AB intensities for the magnetic field $-B$ that corresponds to $\mathfrak{D}^{-W}_{\tilde{\tau}'}$. A function ψ in the domain of $\mathfrak{D}^{-W}_{\tilde{\tau}'}$ has the asymptotics

$$\psi \sim \frac{\mu_j}{2} \begin{pmatrix} (1 + e^{i\tau'_j})\sigma(1-\beta_j) r_j^{\beta_j - 1} + O(r_j^{1-\beta_j}) \\ (e^{i\tau'_j} - 1)\sigma(\beta_j) r^{-\beta_j} e^{i\theta_j} + O(r_j^{\beta_j}) \end{pmatrix}$$

as $z \to z_j$ for some constant $\mu_j \in \mathbb{C}$. We see that $S_1\psi$ has the asymptotics

$$S_1 \psi \sim \frac{\bar{\mu}_j}{2} \begin{pmatrix} (e^{-i\tau'_j} - 1)\sigma(1-\alpha_j) r_j^{\alpha_j - 1} e^{-i\theta_j} + O(r_j^{1-\alpha_j}) \\ (1 + e^{-i\tau'_j})\sigma(\alpha_j) r_j^{-\alpha_j} + O(r_j^{\alpha_j}) \end{pmatrix}$$

Applying the functionals (2.4) and (2.5) we see that $S_1\psi$ satisfies

$$\frac{c^{+}_{\alpha_j - 1}(S_1\psi)}{c^{-}_{-\alpha_j}(S_1\psi)} = -i \tan(\tau'_j/2) \frac{\sigma(1-\alpha_j)}{\sigma(\alpha_j)}$$

and $c^{-}_{\alpha_j - 1}(S_1\psi) = c^{+}_{-\alpha_j}(S_1\psi) = 0$, so the requirements that the domain change properly is that

$$\tan(\tau'_j/2) = \cot(\tau_j/2), \quad \text{for } j = 1, \ldots, n.$$

We see that $\tau_j/2$ and $\pi/2 - \tau'_j/2$ must differ by a integer multiple of π. Both τ_j and τ'_j belong to the interval $[0, 2\pi)$, so the only possibilities are $\tau'_j + \tau_j = \pi$ or $\tau'_j + \tau_j = 3\pi$. □

Corollary 2.9 *The operators $\mathfrak{D}_{\tilde{\tau}}^{-W}$ and $\mathfrak{D}_{\tilde{\tau}}^{W}$ are anti-unitarily equivalent via the operator S_1 if and only if for all $j = 1, \dots, n$ we have $\tau_j = \pi/2$ or $\tau_j = 3\pi/2$.*

Proof Take $\tau'_j = \tau_j$ in the previous Proposition. □

If we let $S_2 \colon \mathcal{H} \to \mathcal{H}$ be the operator that takes $(\psi_+, \psi_-)^t$ to $(\psi_-, \psi_+)^t$ we get some other symmetries if we compose it with the gauge transform that only act on the spin-up component.

Proposition 2.10 *The operators $\mathfrak{D}_{\tilde{\tau}'}^{-W}$ and $\mathfrak{D}_{\tilde{\tau}}^{W}$ are unitarily equivalent via the operator S_2 composed with a gauge multiplication of $\exp\left(-2i \sum_{j=1}^{n} \theta_j\right)$ of the spin-up component if and only if $|\tau'_j - \tau_j| = \pi$ for all $j = 1, \dots, n$.*

Proof The proof goes on as in the proof of Proposition 2.8. This time the requirement on τ_j and τ'_j becomes

$$-\tan(\tau'_j/2) = \cot(\tau_j/2), \quad \text{for } j = 1, \dots, n$$

which gives $|\tau'_j - \tau_j| = \pi$ for all $j = 1, \dots, n$. □

2.4 Zero-modes

Let us calculate the dimension of the kernel of $\mathfrak{D}^W$ under the assumption that $\tau_j = \tau$ for all $j = 1, \dots, n$, which means that we assume that we have the same physical conditions of the behavior of the particle close to all solenoids. Denote by Φ the total flux of B divided by 2π, that is

$$\Phi = \frac{1}{2\pi} \int_{\mathbb{C}} B(z) \, \mathrm{d}m(z) = \frac{1}{2\pi} \int_{\mathbb{C}} B_0(z) \, \mathrm{d}m(z) + \sum_{j=1}^{n} \alpha_j.$$

As usual, the definition of the total flux is a matter of agreement, due to the arbitrariness in the choice of normalization for AB intensities. The asymptotics of e^W at infinity and at the singular points Λ are given by

$$e^W \sim \begin{cases} |z|^{\Phi}, & |z| \to \infty; \\ |z - z_j|^{\alpha_j}, & z \to z_j. \end{cases} \tag{2.6}$$

We recall that the functions in the domain of $\mathfrak{D}^W$ satisfies

$$\frac{c^+_{\alpha_j-1}(\psi)}{c^-_{-\alpha_j}(\psi)} = -i\cot(\tau_j/2)\frac{\sigma(1-\alpha_j)}{\sigma(\alpha_j)}, \quad j = 1,\dots,n. \tag{2.7}$$

Let $\{x\}$ denote the lower integer part, that is

$$\{x\} = \begin{cases} \lfloor x \rfloor, & x > 1 \text{ and } x \notin \mathbb{N}; \\ x-1, & x > 1 \text{ and } x \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.11 *If $\tau_j = \tau$, $j = 1,\dots,n$ then the dimension of the kernel of $\mathfrak{D}^W$ is given by*

$$\dim\ker\mathfrak{D}^W_{\tau} = \begin{cases} \{|n-\Phi|\}, & \textit{if } \tau = 0; \\ \{|\Phi|\}, & \textit{if } \tau = \pi; \\ 0, & \textit{otherwise.} \end{cases}$$

The proof follows the same idea as the original proof by Aharonov-Casher with the same changes as in [Per05] and using the fact that the spin-up and spin-down components are coupled if $\tau \notin \{0,\pi\}$.

Proof We start by calculating the zero-modes as if the spin-up and spin-down components were not coupled; so these components are studied separately.

Let us start with the spin-up component, that is, we consider the solutions to $\mathfrak{Q}\psi_+ = 0$. This is equivalent to $\frac{\partial}{\partial\bar{z}}(e^{-W}\psi_+) = 0$, and thus the function $f_+ = e^{-W}\psi_+$ must be analytic in $\mathbb{C}\setminus\Lambda$. The behavior of f_+ at the singular points Λ is different for different values of the parameter τ, but a pole of order at most $\{-\Phi\}-1$ at infinity is allowed independently of the value of τ.

Case I, $\tau = \pi$: For square integrable ψ_+, as we see from (2.6), the function f_+ is not allowed to have any poles at the singular points Λ. Thus, if $\tau = \pi$ then f_+ may be a polynomial of order at most $\{-\Phi\}-1$. There are as many as $\{-\Phi\}$ many linearly independent such polynomials.

Case II, $\tau \neq \pi$: From (2.7) we see that a pole of order at most one is allowed at each $z_j \in \Lambda$. The calculation in [Per05] then yields that the dimension is $\{n-\Phi\}$.

Let us now turn to the spin-down component. We look for solutions to the equation $\mathfrak{Q}^*\psi_- = 0$, which is equivalent to finding solutions to $\frac{\partial}{\partial z}(e^{W}\psi_-) = 0$. If we now let $f_- = e^{W}\psi_-$, then f_- must be anti-analytic in $\mathbb{C}\setminus\Lambda$, and from the asymptotics (2.6) we see that f_- may have a polynomial part of degree at most $\{\Phi\}-1$ independent of the value of the parameter

τ. Again we get two different cases for the behavior of the functions at the singular points Λ.

Case I, $\tau = 0$: In this case we see from (2.7) that no singular parts for ψ_- are allowed at Λ, and hence f_- must have a zero of order at least 1 at each point in Λ. That is we have a polynomial in $\bar{z}$ of degree $\{\Phi\} - 1$ with n predicted zeroes. There are $\{\Phi - n\}$ linearly independent polynomials of this type.

Case II, $\tau \neq 0$: Now f_- must be a polynomial in $\bar{z}$ of degree at most $\{\Phi\} - 1$, but without any forced zeroes. Thus the dimension of the kernel is $\{\Phi\}$.

Since the spin-up and spin-down components are not coupled in the cases $\tau = 0$ and $\tau = \pi$ the calculations above yield

$$\dim\ker \mathfrak{D}^W_{\vec{\tau}} = \begin{cases} \{|n - \Phi|\}, & \text{if } \tau = 0; \\ \{|\Phi|\}, & \text{if } \tau = \pi. \end{cases}$$

Let us now assume that $\tau \not\in \{0, \pi\}$. We should evaluate how the spin-up zero-modes match the spin-down zero-modes to satisfy the conditions at the singularities. First we note that to be able to have zero-modes both $\{n - \Phi\}$ and $\{\Phi\}$ must be positive. From the calculations in the last two paragraphs of the proof of Theorem 3.3 in [Per05] it follows that f_+ must be of the form

$$f_+(z) = \sum_{j=1}^{n} \frac{\eta_j}{z - z_j}$$

where $\eta_j \in \mathbb{C}$ satisfy

$$\sum_{j=1}^{n} \eta_j z_j^k = 0, \quad \text{for } k = 0, 1, \ldots, n - \{n - \Phi\} - 1 \tag{2.8}$$

and $f_-(z)$ must be a polynomial in $\bar{z}$ of degree at most $\{\Phi\} - 1$. Actually, we will show that even if the degree of the polynomial f_- is $\{\Phi\}$ or in some cases $\{\Phi\} + 1$, all coefficients of the polynomial must be zero. Let us define the natural number m as $m = n - \{n - \Phi\} - 1$ and note that $m = \lfloor \Phi \rfloor$. Let

$$f_-(z) = \sum_{k=0}^{m} s_k \bar{z}^k. \tag{2.9}$$

From the asymptotics (2.7) we see that

$$\frac{\eta_j}{f_-(z_j)} = -e^{-2h_0(z_j)} \prod_{l \neq j} \left(|z_j - z_l|^{-2\alpha_l} \right) i \cot(\tau/2) \frac{\sigma(1 - \alpha_j)}{\sigma(\alpha_j)}, \quad j = 1, \ldots, n.$$

From the requirements (2.8) of the coefficients η_j we get

$$0 = \sum_{j=1}^{n} \eta_j z_j^k = -i \cot(\tau/2) \sum_{j=1}^{n} t_j f_-(z_j) z_j^k, \quad k = 0, 1, \dots, m, \tag{2.10}$$

where

$$t_j = e^{-2h_0(z_j)} \prod_{l \neq j} \left(|z_j - z_l|^{-2\alpha_l} \right) \frac{\sigma(1-\alpha_j)}{\sigma(\alpha_j)} > 0.$$

We introduce the vector $s = (s_0, \dots, s_m)^t$ where s_k, $k = 0, \dots, m$ are the coefficients in (2.9). Let us also introduce the matrix

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & \ddots & \vdots \\ z_1^m & z_2^m & \cdots & z_n^m \end{pmatrix},$$

and the diagonal matrix T having the positive number t_j at the jth diagonal position. Then (2.10) can be written as

$$-i \cot(\tau/2) VBV^* s = 0.$$

The matrix VTV^* is clearly Hermitian and since T is positive, we can write VTV^* as $(V\sqrt{T})(V\sqrt{T})^*$. Hence the null space of VTV^* is the same as that of the matrix $(V\sqrt{T})^* = \sqrt{T}V^*$. Since V^* is (a part of) a Vandermonde matrix it has full rank, so the dimension of the null space of $\sqrt{T}V^*$ is zero. Hence the polynomial f_-, and thus also ψ_-, must be zero. Since the spin-up and spin-down components are coupled, it follows that ψ_+ is also zero. Consequently, $\dim \ker \mathfrak{D}_{\tilde{\tau}}^W = 0$, and the proof is complete. □

3 The Pauli operator

In this section we will study the Pauli operator corresponding to the magnetic field (2.1), obtained as the square of a self-adjoint Dirac operator.

3.1 The Pauli operators with several AB solenoids

Since there are more self-adjoint Pauli extensions than Dirac extensions corresponding to our singular magnetic field, it is clear that not all Pauli operators can be obtained as the square of a self-adjoint Dirac operator. Here we will study the Pauli operators that can be obtained in this way.

Definition 3.1 We define the Pauli operator $\mathfrak{P}^W_{\tilde{\tau}}$ as $(\mathfrak{D}^W_{\tilde{\tau}})^2$ where $\mathfrak{D}^W_{\tilde{\tau}}$ is a self-adjoint Dirac operator defined in Definition 2.1. This means that

$$\mathrm{Dom}\big(\mathfrak{P}^W_{\tilde{\tau}}\big) = \big\{\psi \in \mathcal{H} \,\big|\, \mathfrak{d}\psi \in \mathrm{Dom}(\mathfrak{D}^W_{\tilde{\tau}})\big\};$$
$$\mathfrak{P}^W_{\tilde{\tau}}\psi = (\mathfrak{d})^2\psi, \quad \psi \in \mathrm{Dom}(\mathfrak{P}^W_{\tilde{\tau}}).$$

Let us again introduce the boundary value linear functionals acting on $\mathrm{Dom}(\mathfrak{P}^W)$, but this time for all singular points Λ. For $j = 1, \dots, n$, let

$$c^{\pm}_{-\alpha_j}(\psi) = \lim_{r_j \to 0} r_j^{\alpha_j} \frac{1}{2\pi}\int_0^{2\pi} \psi_{\pm}(r_j e^{i\theta_j})\,d\theta_j,$$
$$c^{\pm}_{\alpha_j}(\psi) = \lim_{r_j \to 0} r_j^{-\alpha_j} \left(\frac{1}{2\pi}\int_0^{2\pi} \psi_{\pm}(r_j e^{i\theta_j})\,d\theta_j - r_j^{-\alpha_j} c^{\pm}_{-\alpha_j}(\psi)\right),$$
$$c^{\pm}_{\alpha_j-1}(\psi) = \lim_{r_j \to 0} r_j^{1-\alpha_j} \frac{1}{2\pi}\int_0^{2\pi} \psi_{\pm}(r_j e^{i\theta_j}) e^{i\theta_j}\,d\theta_j, \text{ and}$$
$$c^{\pm}_{1-\alpha_j}(\psi) = \lim_{r_j \to 0} r_j^{\alpha_j-1} \left(\frac{1}{2\pi}\int_0^{2\pi} \psi_{\pm}(r_j e^{i\theta_j}) e^{i\theta_j}\,d\theta_j - r_j^{\alpha_j-1} c^{\pm}_{\alpha_j-1}(\psi)\right).$$

Proposition 3.1 *For an arbitrary self-adjoint Pauli extension $\mathfrak{P}$, it is the square of some self-adjoint Dirac extension $\mathfrak{D}^W_{\tilde{\tau}}$ if and only if the following equations are satisfied for all $\psi \in \mathrm{Dom}(\mathfrak{P})$*

$$\frac{c^{+}_{\alpha_j-1}(\psi)}{c^{-}_{-\alpha_j}(\psi)} = -i\cot(\tau_j/2)\frac{\sigma(1-\alpha_j)}{\sigma(\alpha_j)}, \tag{3.1}$$
$$\frac{c^{-}_{\alpha_j}(\psi)}{c^{+}_{1-\alpha_j}(\psi)} = -i\cot(\tau_j/2)\frac{\sigma(-\alpha_j)}{\sigma(\alpha_j-1)}, \tag{3.2}$$
$$c^{+}_{-\alpha_j}(\psi) = 0, \text{ and}$$
$$c^{-}_{\alpha_j-1}(\psi) = 0. \tag{3.3}$$

Proof Given $\psi \in \mathrm{Dom}(\mathfrak{D}^W_{\tilde{\tau}})$, a calculation of the asymptotics of ψ at the singular points shows that the requirements on $\mathfrak{D}^W_{\tilde{\tau}}\psi$ to belong to $\mathrm{Dom}(\mathfrak{D}^W_{\tilde{\tau}})$ are exactly that it should fulfill equations (3.1)–(3.3). □

Remark 3.1 The domain of $\mathfrak{P}^W$ can be written as

$$\mathrm{Dom}\big(\mathfrak{P}^W_{\tilde{\tau}}\big) = \big\{\psi \in \mathcal{H} \,\big|\, \mathfrak{d}^2\psi \in \mathcal{H}, \quad \text{(3.1)–(3.3) hold for all } \psi\big\}.$$

We see also that $\mathrm{Dom}\big(\mathfrak{P}_{\tilde{\tau}}^{W}\big)$ is exactly the subset of $\mathrm{Dom}\big(\mathfrak{D}_{\tilde{\tau}}^{W}\big)$ for which also the conditions (3.2) hold.

3.2 Spin-flip invariance and Zero-modes

Proposition 3.2 *The only self-adjoint Pauli extensions* $\mathfrak{P}_{\tilde{\tau}}^{W} = (\mathfrak{D}_{\tilde{\tau}}^{W})^2$ *that are spin-flip invariant under the transform* S_1 *are these where for all* $j = 1,\dots,n$ *we have* $\tau_j = \pi/2$ *or* $\tau_j = 3\pi/2$.

Proof The proof is the same as for the Dirac operators, see Proposition 2.8. □

Theorem 3.3 *If* $\tau_j = \tau$, $j = 1,\dots,n$ *then the dimension of the kernel of* $\mathfrak{P}_{\tilde{\tau}}^{W}$ *is given by*

$$\dim\ker\mathfrak{P}_{\tilde{\tau}}^{W} = \begin{cases} \{|n-\Phi|\}, & \text{if } \tau = 0; \\ \{|\Phi|\}, & \text{if } \tau = \pi; \\ 0, & \text{otherwise.} \end{cases}$$

Proof This follows from Theorem 2.11 since $\ker\mathfrak{P}_{\tilde{\tau}}^{W} = \ker\mathfrak{D}_{\tilde{\tau}}^{W}$. □

3.3 Discussion

Let us compare the different self-adjoint Pauli operators from [EV02] (which we will denote by $\mathfrak{P}_{\mathrm{EV}}$) and [Per05] (which we will denote by $\mathfrak{P}_{\max}$) with the ones obtained above as the square of a self-adjoint Dirac operator. It is easier to do this comparison if we have the same AB flux normalization for all operators. Thus, we let all AB intensities α_j belong to the interval $(0,1)$. In the case of the Pauli operator $\mathfrak{P}_{\mathrm{EV}}$, where the AB intensities were normalized to $[-1/2,1/2)$, we have to do a gauge transformation if there are intensities α_j belonging to $[-1/2,0)$. This is not a problem, since $\mathfrak{P}_{\mathrm{EV}}$ is gauge invariant.

In Table 3.1 we see a comparison of the boundary conditions of the Pauli operators obtained above that are the square of a Dirac operator and the Maximal and EV Pauli operators (see [Per05, EV02]). We see that $\mathfrak{P}_{\max}$ is not the square of a Dirac operator. However, if we let

$$\tau_j = \begin{cases} \pi, & \text{if} \quad 0 < \alpha_j < 1/2 \\ 0, & \text{if} \quad 1/2 \le \alpha_j < 1 \end{cases}, \quad j = 1,\dots,n,$$

and $\tilde{\tau} = (\tau_1,\dots,\tau_n)$, then $\mathfrak{P}_{\mathrm{EV}}$ is the square of the self-adjoint Dirac operator corresponding to $\tilde{\tau}$. Note that it is possible to have different physical situations at the singular points Λ. Indeed, if not all intensities α_j belong to either $(0,1/2)$ or $[1/2,1)$ then this is the case.

Table 3.1 The boundary value conditions for the squared Dirac operators compared with the ones for the Maximal and EV Pauli operators. The constants μ_j depend on the functions in the domain.

	$\mathfrak{P}_{\tilde{\tau}}^{W} = (\mathfrak{D}_{\tilde{\tau}}^{W})^2$	$\mathfrak{P}_{\mathrm{EV}}$	$\mathfrak{P}_{\max}$
$\frac{c^+_{\alpha_j}}{c^+_{-\alpha_j}}$	∞	∞	∞
$\frac{c^+_{1-\alpha_j}}{c^+_{\alpha_j-1}}$	$-\mu_j \frac{\sigma(\alpha_j-1)}{\sigma(1-\alpha_j)} \tan(\tau_j/2)$	$\begin{cases} \infty, & \text{if } 0<\alpha_j<1/2 \\ 0, & \text{if } 1/2 \le \alpha_j < 1 \end{cases}$	0
$\frac{c^-_{\alpha_j}}{c^-_{-\alpha_j}}$	$\mu_j \frac{\sigma(-\alpha_j)}{\sigma(\alpha_j)} \cot(\tau_j/2)$	$\begin{cases} 0, & \text{if } 0<\alpha_j<1/2 \\ \infty, & \text{if } 1/2 \le \alpha_j < 1 \end{cases}$	0
$\frac{c^-_{1-\alpha_j}}{c^-_{\alpha_j-1}}$	∞	∞	∞

Remark 3.2 If the AB intensities in [EV02] would have been normalized to $(0,1)$ instead of $[-1/2,1/2)$, then the operator $\mathfrak{P}_{\mathrm{EV}}$ would have become the square of the Dirac operator where $\tau_j = \pi$ for all $j = 1,\dots,n$. If the AB intensities would have been normalized to $(-1,0)$ then $\mathfrak{P}_{\mathrm{EV}}$ would have been the square of the Dirac operator where $\tau_j = 0$ for all $j = 1,\dots,n$.

Among the Pauli operators studied in this article, the ones for $\tau = \pi/2$ (which is (anti)-unitarily equivalent to the one for $\tau = 3\pi/2$), $\tau = 0$ and $\tau = \pi$ seems to be the most interesting ones. For $\tau = \pi/2$ we get a very symmetric domain of the operator, which implies that the operator is spin-flip invariant. Lacking zero-modes, it does not satisfy the original Aharonov-Casher formula, but it can be approximated component-wise according to Table 3.1 and the result in [BP03]. See the end of [Per05] for a discussion of this.

The Pauli operators corresponding to $\tau = 0$ and $\tau = \pi$ have very asymmetric domains. Only one of the components contain singular terms at the points Λ. This lack of symmetry implies that these extensions are not spin-flip invariant. On the other hand, the Pauli operator corresponding to $\tau = \pi$ does satisfy the original Aharonov-Casher formula and there is no doubt that both of these Pauli operators can be approximated as in [BP03], even as Pauli Hamiltonians.

The Maximal Pauli operator studied [Per05] is spin-flip invariant and has zero-modes, even more than is present in the original Aharonov-Casher formula. It can be approximated component-wise as in [BP03]. However, it is not the square of a self-adjoint Dirac operator. It is still not clear which Pauli extension that describes the physics in the best way.

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Paper III

Zero modes for the magnetic Pauli operator in even-dimensional Euclidean space

Mikael Persson

Abstract: We study the ground state of the Pauli Hamiltonian with a magnetic field in $\mathbb{R}^{2d}$, $d > 1$. We consider the case where a scalar potential W is present and the magnetic field B is given by $B = 2i\bar{\partial}\partial W$. The main result is that there are no zero modes if the magnetic field decays faster than quadratically at infinity. If the magnetic field decays quadratically then zero modes may appear, and we give a lower bound for the number of them. The results in this paper partly correct a mistake in a paper from 1993.

1 Introduction and main result

The Pauli operator $\mathfrak{P}$ in $\mathbb{R}^n$ describes a charged spin-$\frac{1}{2}$ particle in a magnetic field. Along with the Dirac operator, it lies in the base of numerous models in quantum physics. The problem about zero modes, the bound states with zero energy, is one of many questions to be asked about the spectral properties of these operators.

Zero modes were discovered in [AC79] in dimension $n = 2$. Unlike the purely electric interaction, a compactly supported magnetic field can generate zero modes, as soon as the total flux of the field is sufficiently large. Quantitatively, this is expressed by the famous Aharonov-Casher formula. The two-dimensional case is by now quite well studied; the AC formula is extended to rather singular magnetic field, moreover, if the total flux is infinite (and the field has constant sign), there are infinitely many zero modes.

On the other hand, in the three-dimensional case the presence of zero modes is a rather exceptional feature, and the conditions for them to appear are not yet found, see the discussion in [MR03] and references therein.

Even less clear is the situation in the higher dimensions. In [Shi91], for *even* n some sufficient conditions for the infiniteness of the number of zero modes were found, requiring, in particular, that the field decays rather slowly (more slowly than r^{-2}) at infinity. On the other hand, in [Ogu93], again for even n, the case where a finite number of zero modes should appear was considered. Under the assumption of a rather regular behavior of the *scalar potential* of the magnetic field at infinity the number of zero modes was calculated. In particular, for a field with compact support or decaying faster than quadratically at infinity the formula in [Ogu93] implies the absence of zero modes, thus making a difference with the two-dimensional situation.

Unfortunately, it turned out that the reasoning in [Ogu93] contains an error. A miscalculation in an important integral leads to an erroneous conclusion, thus destroying the final results. This is the reason for us to return to the question on zero modes in the even higher-dimensional case. We try to revive the results in [Ogu93] and succeed partially.

We use the representation of the Pauli and Dirac operators in the terms of multi-variable complex analysis proposed in [Shi91] and used further in [Ogu93]. This approach puts a certain restriction on the class of magnetic field considered, equivalent to the existence of a scalar potential. At the moment it is unclear how to treat the general case.

Under the above condition, the operators are represented as acting on the complex forms, the action expressed via the $\bar{\partial}$ operator. The mistake in [Ogu93] occurs in calculating the L_2 norm of the form one gets after applying the $\bar{\partial}$ operator. We present the detailed analysis of this miscalculation in Section 3.

The strategy of our treatment of zero modes differs from the one in [AC79] and other previous papers including [Ogu93]. Usually, when studying zero modes, one shows first that they, after having been multiplied by some known factor, are holomorphic function in the whole space; then one easily counts the number of such functions. This strategy fails in our case, so we use another one, involving more advanced machinery of complex and real analysis. The main ingredient of the proofs is a combination of the techniques of using the Bochner-Martinelli-Koppelman kernel to solve a $\bar{\partial}$ equation and the use of a weighted Hardy-Littlewood-Sobolev inequality to estimate that solution.

As a result, we establish some of the properties presented in [Ogu93]. We show that there are no zero modes if the magnetic field decays faster than quadratically at infinity (in particular, if it is compactly supported). Another result is that zero modes may exist if the magnetic field decays exactly quadratically, and the formula in [Ogu93] gives a lower bound for their number.

1.1 The Pauli operator

Let $x = (x^1, \dots, x^{2d})$, denote the usual Euclidean coordinates in $\mathbb{R}^{2d}$. From now on it is always assumed that $d > 1$. According to the Maxwell equations, a magnetic field B in $\mathbb{R}^{2d}$ is a real closed two-form

$$B(x) = \sum_{j<k} b_{j,k}(x)\,\mathrm{d}x^j \wedge \mathrm{d}x^k. \tag{1.1}$$

Throughout this paper we assume that all the coefficient functions $b_{j,k}$ belong to $C^\infty(\mathbb{R}^{2d})$. The condition that the magnetic field B is closed is given by

$$0 = \mathrm{d}B = \sum_{j<k<l} \left(\frac{\partial b_{j,k}}{\partial x^l} - \frac{\partial b_{j,l}}{\partial x^k} + \frac{\partial b_{k,l}}{\partial x^j} \right) \mathrm{d}x^j \wedge \mathrm{d}x^k \wedge \mathrm{d}x^l,$$

where d is the usual exterior differential operator. Since B is closed there exists a one-form

$$a(x) = \sum_{j=1}^{n} a_j(x)\,\mathrm{d}x^j$$

satisfying

$$B = \mathrm{d}a = \sum_{j<k} \left(\frac{\partial a_k}{\partial x^j} - \frac{\partial a_j}{\partial x^k} \right) \mathrm{d}x^j \wedge \mathrm{d}x^k$$

Any such one-form a is called a magnetic one-form or magnetic vector potential. It is not unique. In fact, given one magnetic one-form, another one is obtained by adding $\mathrm{d}f$ for some regular function f. The choice of magnetic one-form a is usually referred to as the choice of gauge.

The analysis of the Pauli operator was successful in [Shi91] using complex analysis under a condition that the magnetic field is a complex $(1,1)$-type form. It is not clear what this condition means physically, but to be able to use the theory of complex analysis in several variables, we will throughout use the same assumption. Thus, the coefficient functions in (1.1) of the closed 2-form B must satisfy the $d(d-1)$ equations

$$\begin{cases} b_{2j-1,2k-1} = b_{2j,2k}, \\ b_{2j-1,2k} = -b_{2j,2k-1}, \end{cases} \qquad \text{for } j+1 \le k \le d, \quad 1 \le j \le d-1. \tag{1.2}$$

The spinless Schrödinger operator H in $\mathbb{R}^{2d}$ corresponding to the magnetic field B is defined in $L_2(\mathbb{R}^{2d})$ as

$$H = \sum_{j=1}^{2d} \Big(-i \frac{\partial}{\partial x^j} - a_j \Big)^2.$$

We are interested in spin-$\frac{1}{2}$ particles (including the electron). Such systems are described by the Pauli operator $\mathfrak{P}$, acting in $L_2(\mathbb{R}^{2d}) \otimes \mathbb{C}^{2^d}$. Let $\{\gamma^j\}_{j=0}^{2d}$ be Hermitian $2^d \times 2^d$ matrices satisfying

$$\gamma^j \gamma^k + \gamma^k \gamma^j = 2\delta^{jk} I_{2^d}, \tag{1.3}$$

where I_{2^d} denotes the $2^d \times 2^d$ identity matrix. These matrices $\{\gamma^j\}_{j=0}^{2d}$ generate a Clifford algebra, and are usually called the Dirac matrices. The Pauli operator $\mathfrak{P}$ is defined by

$$\mathfrak{P} = H I_{2^d} + \sum_{0<j<k} i b_{jk}(x) \gamma^j \gamma^k.$$

To be more precise, $\mathfrak{P}$ is first defined on $C_0^\infty \otimes \mathbb{C}^{2^d}$, where it is essential self-adjoint (see [Che73]). We denote the self-adjoint closure by $\mathfrak{P}$. The Pauli operator $\mathfrak{P}$ can also be written as $\mathfrak{P} = \mathfrak{D}^2$, where $\mathfrak{D}$ is the self-adjoint Dirac operator

$$\mathfrak{D} = \sum_{j=1}^{2d} \gamma^j \Big(-i \frac{\partial}{\partial x^j} - a_j \Big).$$

From this it follows that the Pauli operator is non-negative.

1.2 The main result

Theorem 1.1 *Assume that the equations in (1.2) are satisfied, and that there exist constants $C > 0$ and $\rho > 2$ such that*

$$|B(x)| \le \frac{C}{(1+|x|)^\rho} \quad \text{for all } x \in \mathbb{R}^{2d}.$$

Then

$$\dim \ker \mathfrak{P} = 0.$$

We will prove this theorem in Section 2. The case

$$|B(x)| \sim 1/|x|^2, \quad \text{as } |x| \to \infty$$

is more complicated and is discussed in Section 3. In Remark 3.3 we give an example of a magnetic field B satisfying

$$|B(x)| = \frac{\Phi(d-1)}{2|x|^2}, \quad \text{for large values of } |x|,$$

such that $\dim\ker\mathfrak{P} = 0$ if $|\Phi| < d$ and $\dim\ker\mathfrak{P} > 0$ otherwise. This result is somehow strange and suggests that the situation for magnetic fields with a quadratic decay is quite complicated and unstable.

1.3 Complex analysis and Differential forms

Let us now switch to the complex analysis viewpoint. We identify the point $x = (x^1, \dots, x^{2d})$ in $\mathbb{R}^{2d}$ with $z = (z^1, \dots, z^d)$ in $\mathbb{C}^d$, where $z^j = x^{2j-1} + i x^{2j}$. We define tangent and cotangent vectors by

$$\begin{aligned} \frac{\partial}{\partial z^j} &= \frac{1}{2}\left(\frac{\partial}{\partial x^{2j-1}} - i\frac{\partial}{\partial x^{2j}}\right), \\ \frac{\partial}{\partial \bar{z}^j} &= \frac{1}{2}\left(\frac{\partial}{\partial x^{2j-1}} + i\frac{\partial}{\partial x^{2j}}\right), \\ \mathrm{d}z^j &= \mathrm{d}x^{2j-1} + i\,\mathrm{d}x^{2j}, \quad \text{and} \\ \mathrm{d}\bar{z}^j &= \mathrm{d}x^{2j-1} - i\,\mathrm{d}x^{2j}. \end{aligned}$$

Written in complex terms, the magnetic field B can be written as a sum of $(1,1)$, $(2,0)$, and $(0,2)$ type forms as

$$B(z) = \sum_{j,k=1}^{d} \mathfrak{b}_{j,k}(z)\,\mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k + \sum_{j,k=1}^{d} \mathfrak{b}'_{j,k}(z)\,\mathrm{d}z^j \wedge \mathrm{d}z^k + \sum_{j,k=1}^{d} \mathfrak{b}''_{j,k}(z)\,\mathrm{d}\bar{z}^j \wedge \mathrm{d}\bar{z}^k.$$

The equations in (1.2) state that B is of type $(1,1)$ which means that all coefficient functions $\mathfrak{b}'_{j,k}$ and $\mathfrak{b}''_{j,k}$ in the representation above vanish, so the magnetic field B has the form

$$B(z) = \sum_{j,k=1}^{d} \mathfrak{b}_{j,k}(z)\,\mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k \tag{1.4}$$

To magnetic fields that are $(1,1)$-type forms there exist scalar potentials $W \in C^\infty(\mathbb{C}^d \to \mathbb{R})$ satisfying

$$B = 2i\bar{\partial}\partial W, \tag{1.5}$$

see [Wel80]. In [Shi91] it was shown that the Dirac and Pauli operators can be defined in terms of W and operators acting on differential forms in a very nice way. For the sake of completeness we show how this is done.

Let $\bigwedge^{0,q}(\mathbb{C}^d)^*$ denote the space of $(0,q)$-type differential forms and let $\bigwedge^{0,*}(\mathbb{C}^d)^* = \bigoplus_{q=0}^d \bigwedge^{0,q}(\mathbb{C}^d)^*$. The Dirac operator $\mathfrak{D}$ is realized as an operator in the Hilbert space $\mathcal{H} := L_2(\mathbb{C}^d;\, \mathrm{d}m) \otimes \bigwedge^{0,*}(\mathbb{C}^d)^*$ in the way

$$\mathfrak{D} = 2\big(\overline{\partial}_W + \overline{\partial}_W^*\big). \tag{1.6}$$

Here

$$\overline{\partial}_W = \overline{\partial} - \mathrm{ext}(\overline{\partial}W) = \sum_{j=1}^{d} \mathrm{ext}(\,\mathrm{d}\bar{z}^j)\left(\frac{\partial}{\partial \bar{z}^j} - \frac{\partial W}{\partial \bar{z}^j}\right),$$

$$\overline{\partial}_W^* = \overline{\partial}^* - \mathrm{int}(\partial W) = -\sum_{j=1}^{d} \mathrm{int}(\,\mathrm{d}z^j)\left(\frac{\partial}{\partial z^j} + \frac{\partial W}{\partial z^j}\right),$$

$\mathrm{ext}(\,\mathrm{d}\bar{z}^j)$ is the operator on $\bigwedge^{0,*}(\mathbb{C}^d)^*$ acting as

$$\mathrm{ext}(\,\mathrm{d}\bar{z}^j)\eta = \mathrm{d}\bar{z}^j \wedge \eta, \quad \text{for } \eta \in \bigwedge\nolimits^{0,*}(\mathbb{C}^d)^*,$$

and $\mathrm{int}(\,\mathrm{d}z^j)$ is the adjoint operator of $\mathrm{ext}(\,\mathrm{d}\bar{z}^j)$ in $\mathcal{H}$.

To see that (1.6) is true we use the anti-commutation relations

$$\begin{aligned} \big[\,\mathrm{ext}(\,\mathrm{d}\bar{z}^j), \mathrm{ext}(\,\mathrm{d}\bar{z}^k)\big]_+ &= 0;\\ \big[\,\mathrm{int}(\,\mathrm{d}z^j), \mathrm{int}(\,\mathrm{d}z^k)\big]_+ &= 0;\\ \big[\,\mathrm{ext}(\,\mathrm{d}\bar{z}^j), \mathrm{int}(\,\mathrm{d}z^k)\big]_+ &= \delta^{jk}. \end{aligned}$$

By defining

$$\begin{aligned} \gamma^{2j-1} &= i(\mathrm{ext}(\,\mathrm{d}\bar{z}^j) - \mathrm{int}(\,\mathrm{d}z^j));\\ \gamma^{2j} &= -(\mathrm{ext}(\,\mathrm{d}\bar{z}^j) + \mathrm{int}(\,\mathrm{d}z^j)) \end{aligned}$$

one can easily check that

$$\big[\gamma^j, \gamma^k\big]_+ = 2\delta^{jk}.$$

Hence $\{\gamma^j\}$ so defined satisfies the relation (1.3) of a Clifford algebra. Now it is easy to see that (1.6) holds:

$$
\begin{aligned}
2\left(\overline{\partial}_W + \overline{\partial}_W^*\right) &= \sum_{j=1}^{d} \left(-i\gamma^{2j-1} - \gamma^{2j}\right)\left(\frac{\partial}{\partial \bar{z}^j} - \frac{\partial W}{\partial \bar{z}^j}\right) \\
&\qquad - \left(i\gamma^{2j-1} - \gamma^{2j}\right)\left(\frac{\partial}{\partial z^j} + \frac{\partial W}{\partial z^j}\right) \\
&= \sum_{j=1}^{d} \gamma^{2j-1}\left(-i\frac{\partial}{\partial x^{2j-1}} - \frac{\partial W}{\partial x^{2j}}\right) + \gamma^{2j}\left(-i\frac{\partial}{\partial x^{2j}} + \frac{\partial W}{\partial x^{2j-1}}\right) \\
&= \sum_{j=1}^{2d} \gamma^j \left(-i\frac{\partial}{\partial x^j} - a_j(x)\right) = \mathfrak{D}
\end{aligned}
$$

where $a_{2j-1}(x) = \frac{\partial W}{\partial x^{2j}}$ and $a_{2j}(x) = -\frac{\partial W}{\partial x^{2j-1}}$, so $a = i(\partial - \overline{\partial})W$, which fits well with (1.5), since $B = \mathrm{d}a = (\partial + \overline{\partial})a = (\partial + \overline{\partial})i(\partial - \overline{\partial})W = 2i\overline{\partial}\partial W$.

For a form $\alpha \in \mathcal{H}$ to belong to the kernel of $\mathfrak{P}$ it is necessary and sufficient that α belongs to the kernel of the quadratic form

$$
\mathfrak{p}[\alpha] = 4\int_{\mathbb{C}^d} \left(\left|\overline{\partial}_W \alpha\right|^2 + \left|\overline{\partial}_W^* \alpha\right|^2\right) \mathrm{d}m(z), \quad \alpha \in \mathcal{H}.
$$

Let $U\colon \mathcal{H} \to \mathcal{H}_W := L_2(\mathbb{C}^d; e^{2W}\,\mathrm{d}m) \otimes \bigwedge^{0,*}(\mathbb{C}^d)^*$ be the unitary operator $U\colon \alpha \mapsto e^{-W}\alpha$. Then $\mathfrak{P}$ and $\widetilde{\mathfrak{P}} = U\mathfrak{P}U^*$ are unitarily equivalent. The quadratic form $\tilde{\mathfrak{p}}$ on $\mathcal{H}_W$ corresponding to $\widetilde{\mathfrak{P}}$ is given by

$$
\tilde{\mathfrak{p}}[\alpha] = 4\int_{\mathbb{C}^d} \left(\left|\overline{\partial}\alpha\right|^2 + \left|\overline{\partial}^*\alpha\right|^2\right) e^{2W}\,\mathrm{d}m(z), \quad \alpha \in \mathcal{H}_W. \tag{1.7}
$$

Here $\overline{\partial}^*$ is the adjoint operator to $\overline{\partial}$ in $\mathcal{H}_W$.

2 Proof of Theorem 1.1

Let $K_q(\zeta, z)$ be the Bochner-Martinelli-Koppelman kernel

$$
K_q(\zeta, z) = \frac{(d-1)!}{2^{q+1}\pi^d} \frac{1}{|\zeta - z|^{2d}} \sum_{\substack{j,J \\ |L|=q+1}} \varepsilon^L_{jJ}(\overline{\zeta}^j - \bar{z}^j)(*\,\mathrm{d}\zeta^L) \wedge \mathrm{d}\bar{z}^J. \tag{2.1}
$$

Here J is a multiindex of length q and if A and B are ordered subsets of $\{1, 2, \dots, d\}$ then ε^A_B denotes the sign of the permutation which takes A into B if $|A| = |B|$ and zero if $|A| \neq |B|$. If $A \subset \{1, 2, \dots, d\}$ and $|A| = q$ then

$$* \, \mathrm{d}\zeta_A = \frac{(-1)^{q(q-1)/2}}{2^{d-q} i^d} \, \mathrm{d}\zeta^A \wedge \Big(\bigwedge_{\nu \in A'} \mathrm{d}\bar{\zeta}^\nu \wedge \mathrm{d}\zeta^\nu \Big),$$

where A' is the complementary multiindex of A. We see that $K_q(\zeta, z)$ is of type $(d, d-q-1)$ in ζ and $(0, q)$ in z.

Let f be a smooth $(0, q)$-type form with compact support in $\mathbb{C}^d$. Then f satisfies the Bochner-Martinelli-Koppelman formula (see [Ran86])

$$f(z) = -\int_{\zeta \in \mathbb{C}^d} \bar{\partial} f(\zeta) \wedge K_q(\zeta, z) - \bar{\partial}_z \int_{\zeta \in \mathbb{C}^d} f(\zeta) \wedge K_{q-1}(\zeta, z). \tag{2.2}$$

Lemma 2.1 *Under the same conditions as in Theorem 1.1 there exists a scalar potential $W \in L^\infty(\mathbb{C}^d \to \mathbb{R})$ such that $2i\bar{\partial}\partial W = B$.*

We know from [Wel80] that solutions W exist, the essential part of this Lemma is that there exist a bounded solution to (1.5).

Proof In the proof below, to avoid logarithms to appear, assume that ρ is not an integer. If it is, then let $\rho' = \rho - 1/2$ and do the following proof with ρ' instead.

Denote by $b_j = \sum_{k=1}^d \mathfrak{b}_{j,k} \, \mathrm{d}\bar{z}^k$. Then $B = \sum_{j=1}^d \mathrm{d}z^j \wedge b_j$ and $\bar{\partial} b_j = 0$ for all $j = 1, \dots, d$. Denote by u_j the function

$$u_j(z) = -\int_{\zeta \in \mathbb{C}^d} b_j(\zeta) \wedge K_0(\zeta, z). \tag{2.3}$$

Step 1: $\bar{\partial} u_j = b_j$:

It is enough to show that the equation holds in the sense of distributions. Let η_k be a family of cut-off functions, such that $\eta_k(\zeta) = 1$ if $|\zeta| < k$, $\eta_k(\zeta) = 0$ if $|\zeta| > k+1$ and $|\bar{\partial}\eta_k| \le 2$ for all $k = 1, 2, \dots$. The $(0,1)$-type forms $\eta_k b_j$ are smooth and have compact support and thus satisfy (2.2). Let Φ be a test form with support in $|z| < M$. Fix $\varepsilon > 0$. Then, using (2.2) and the triangle inequality we have

$$
\begin{aligned}
\left|\langle u_j, \overline{\partial}^* \Phi\rangle - \langle b_j, \Phi\rangle\right| &= \Big|\left\langle u_j, \overline{\partial}^* \Phi\right\rangle + \left\langle \int_{\zeta\in\mathbb{C}^d} (\eta_k b_j) \wedge K_0(\zeta,z), \overline{\partial}^* \Phi \right\rangle \\
&\quad + \left\langle \int_{\zeta\in\mathbb{C}^d} \overline{\partial}(\eta_k b_j) \wedge K_1(\zeta,z), \Phi \right\rangle + \langle \eta_k b_j, \Phi\rangle - \langle b_j, \Phi\rangle \Big| \\
&\leq \left|\left\langle \int_{\zeta\in\mathbb{C}^d} (\eta_k - 1) b_j \wedge K_0(\zeta,z), \overline{\partial}^* \Phi \right\rangle\right| \\
&\quad + \left|\left\langle \int_{\zeta\in\mathbb{C}^d} \overline{\partial}\eta_k \wedge b_j \wedge K_1(\zeta,z), \Phi \right\rangle\right| \\
&\quad + \left|\langle (\eta_k - 1) b_j, \Phi\rangle\right| \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

We will let k tend to infinity. For $k > 2M$ we have $|K_0(\zeta,z)| \leq C|\zeta|^{1-2d}$. We get

$$
\begin{aligned}
I_1 &\leq \int_{|z|<M} \int_{|\zeta|>k} |\eta_k(\zeta) - 1| \cdot |b_j(\zeta)| \cdot |K_0(\zeta,z)| \,\mathrm{d}m(\zeta) \left|\overline{\partial}^* \Phi(z)\right| \,\mathrm{d}m(z) \\
&\leq C \sup\left|\overline{\partial}^* \Phi\right| \cdot \int_{|z|<M} \mathrm{d}m(z) \int_{|\zeta|>k} \frac{1}{|z|^{2d-1+\rho}} \,\mathrm{d}m(\zeta) \\
&\leq C \sup\left|\overline{\partial}^* \Phi\right| \cdot \int_k^\infty \frac{1}{r^\rho} \,\mathrm{d}m(r) \\
&\leq C \sup\left|\overline{\partial}^* \Phi\right| \cdot k^{1-\rho}
\end{aligned}
$$

so $I_1 < \varepsilon$ if k is large enough. Similarly, for I_2, we have

$$
\begin{aligned}
I_2 &\leq \int_{|z|<M} \int_{k<|\zeta|<k+1} \left|\overline{\partial}\eta_k\right| \cdot |b_j| \cdot |K_1(\zeta,z)| \,\mathrm{d}m(\zeta) \, |\Phi(z)| \,\mathrm{d}m(z) \\
&\leq C \sup|\Phi| \cdot \int_{|z|<M} \mathrm{d}m(z) \int_{k<|\zeta|<k+1} \frac{1}{|z|^{2d-1+\rho}} \,\mathrm{d}m(\zeta) \\
&\leq C \sup|\Phi| \cdot \int_k^{k+1} \frac{1}{r^\rho} \,\mathrm{d}m(r) \\
&\leq C \sup|\Phi| \cdot k^{-\rho}
\end{aligned}
$$

so $I_2 < \varepsilon$ if k is large enough. I_3 is equal to zero if k is large enough, since then $(1 - \eta_k)$ and Φ has disjoint support. We conclude that $\overline{\partial} u_j = b_j$.

Step 2: There exists a constant C such that $|u_j(z)| \leq C\frac{1}{(1+|z|)^{\rho-1}}$ for all $z \in \mathbb{C}^d$:

This is standard, but let us prove it for the sake of completeness. Using the triangle inequality and the estimate for B and $|K_0(\zeta,z)| \leq C/|\zeta - z|^{2d-1}$, we have

$$\begin{aligned} |u_j(z)| &\leq C\int_{\mathbb{C}^d} \frac{1}{(1+|\zeta|)^\rho}\frac{1}{|\zeta - z|^{2d-1}}\,\mathrm{d}m(\zeta) \\ &= C\int_{\mathbb{C}^d} \frac{1}{(1+|\zeta - z|)^\rho}\frac{1}{|\zeta|^{2d-1}}\,\mathrm{d}m(\zeta). \end{aligned}$$

For a fixed $z \neq 0$ divide $\mathbb{C}^d$ into three domains:

$$E_1 = \left\{\zeta \;\middle|\; |\zeta| < \frac{1}{2}|z|\right\}, \quad E_2 = \left\{\zeta \;\middle|\; \frac{1}{2}|z| < |\zeta| < 2|z|\right\}, \quad E_3 = \left\{\zeta \;\middle|\; |\zeta| > 2|z|\right\}.$$

On E_1 we have $|\zeta - z| \geq \frac{1}{2}|z|$ and hence

$$\begin{aligned} \int_{E_1} \frac{1}{(1+|\zeta - z|)^\rho}\frac{1}{|\zeta|^{2d-1}}\,\mathrm{d}m(\zeta) &\leq \frac{C}{|z|^\rho}\int_{|\zeta|<\frac{1}{2}|z|}\frac{1}{|\zeta|^{2d-1}}\,\mathrm{d}m(\zeta) \\ &\leq C\frac{1}{|z|^{\rho-1}}. \end{aligned}$$

On E_2 we have $|\zeta - z| \leq \frac{5}{2}|z|$, so we get

$$\begin{aligned} \int_{E_2} \frac{1}{(1+|\zeta - z|)^\rho}\frac{1}{|\zeta|^{2d-1}}\,\mathrm{d}m(\zeta) &\leq \frac{C}{|z|^{2d-1}}\int_{\frac{1}{2}|z|<|\zeta|<2|z|}\frac{1}{|\zeta - z|^\rho}\,\mathrm{d}m(\zeta) \\ &\leq \frac{C}{|z|^{2d-1}}\int_{|\zeta - z|<\frac{5}{2}|x|}\frac{1}{|\zeta - z|^\rho}\,\mathrm{d}m(\zeta) \\ &\leq C\frac{1}{|z|^{\rho-1}}. \end{aligned}$$

On E_3 we have $|\zeta - z| > \frac{1}{2}|\zeta|$, so we get

$$\begin{aligned} \int_{E_3} \frac{1}{(1+|\zeta - z|)^\rho}\frac{1}{|\zeta|^{2d-1}}\,\mathrm{d}m(\zeta) &\leq \int_{|\zeta|>2|z|}\frac{1}{|\zeta|^{2d-1+\rho}}\,\mathrm{d}m(\zeta) \\ &\leq C\frac{1}{|z|^{\rho-1}}. \end{aligned}$$

u_j is clearly bounded for $z = 0$, so the claimed estimate follows.

Step 3: It holds that $|\partial u_j| \to 0$ as $|z| \to \infty$:

We have $\partial u_j = \sum_{k=1}^{d} \frac{\partial u_j}{\partial z^k} \, \mathrm{d}z^k$ so it is enough to show that $\frac{\partial u_j}{\partial z^k}$ tends to zero as $|z| \to \infty$. To do this, we differentiate inside the integral formula in (2.3), and use the fact that

$$\left| \frac{\partial K_0(\zeta, z)}{\partial z^k} \right| \le C \frac{1}{|\zeta - z|^{2d}}, \quad k = 1, 2, \dots, d; \; \zeta \neq z,$$

which follows by differentiating formula (2.1). We get that

$$\left| \frac{\partial u_j}{\partial z^k} \right| \le C \int_{\mathbb{C}^d} \frac{1}{(1 + |\zeta|)^{\rho}} \frac{1}{|\zeta - z|^{2d}} \, \mathrm{d}m(\zeta) = C \int_{\mathbb{C}^d} \frac{1}{(1 + |\zeta - z|)^{\rho}} \frac{1}{|\zeta|^{2d}} \, \mathrm{d}m(\zeta).$$

With the same sets E_1, E_2 and E_3 as above, a similar calculation as in Step 2 gives

$$\left| \frac{\partial u_j}{\partial z^k} \right| \le C \frac{\log |z|}{|z|^{\rho}}.$$

It follows that $|\partial u_j| \to 0$ as $|z| \to \infty$.

Step 4: Let $u(z) = -\sum_{j=1}^{d} u_j \, \mathrm{d}z^j$. Then $\bar{\partial} u = B$, $\partial u = 0$ and u satisfies the bound $|u(z)| \le C \frac{1}{(1+|z|)^{\rho - 1}}$:

First

$$\bar{\partial} u = -\sum_{j=1}^{d} \bar{\partial} u_j \wedge \mathrm{d}z^j = -\sum_{j=1}^{d} b_j \wedge \mathrm{d}z^j = B.$$

Next

$$\bar{\partial} \partial u = -\partial \bar{\partial} u = -\partial B = 0.$$

Hence, all coefficient functions of ∂u are holomorphic functions. On the other hand we have from Step 3 that

$$\partial u = \sum_{j=1}^{d} \mathrm{d}z^j \wedge \partial u_j,$$

so it follows that $|\partial u| \to 0$ as $|z| \to \infty$. But then a Liouville type argument implies that $\partial u = 0$.

Finally, the inequality $|u(z)| \le C \frac{1}{(1+|z|)^{\rho - 1}}$ follows directly from Step 2.

Step 5: Solving the ∂-equation $2i \partial W = u$ in a same, up to complex adjoint sign, way as the $\bar{\partial}$-equation was solved in Step 1, we get a function W satisfying

$$2i\bar\partial\partial W = \bar\partial u = B,$$

and we also get the existence of a positive constant C such that

$$|W(z)| \le C\frac{1}{(1+|z|)^{\rho-2}}.$$

Since $\rho > 2$ this implies that W is bounded. Thus this function W satisfies the conditions of the Lemma. □

To prove Theorem 1.1 it is clearly enough to prove the following theorem.

Theorem 2.2 *Assume that* $W \in L^\infty(\mathbb{C}^d \to \mathbb{R})$*. Then*

$$\dim\ker\mathfrak{P} = 0.$$

Since $\mathfrak{P}$ and $\widetilde{\mathfrak{P}}$ are unitarily equivalent we will show instead that $\dim\ker\widetilde{\mathfrak{P}} = 0$. We need some Lemmas.

Lemma 2.3 *Let* $\Omega\colon \mathbb{C}^d \to \mathbb{C}$ *be a homogeneous function of degree zero, and let* Ω *be bounded on the unit sphere* $|z| = 1$*. Define the operator* T *as*

$$(Tf)(z) = \int_{\mathbb{C}^d} \frac{\Omega(z-\zeta)}{|z-\zeta|^{2d-1}} f(\zeta)\,\mathrm{d}m(\zeta).$$

Then T *is bounded as an operator from* $L_2(\mathbb{C}^d)$ *to* $L_{2d/(d-1)}(\mathbb{C}^d)$*.*

Proof This is a special case of the Hardy-Littlewood-Sobolev theorem, see Theorem V.1 in [Ste70]. □

Lemma 2.4 *Let* $\alpha \in \mathcal{H}_W$ *be a* $(0,q)$*-type form,* $1 \le q \le d-1$*, satisfying* $\bar\partial\alpha = \bar\partial^*\alpha = 0$*. Then the* $(0,q-1)$*-type form*

$$\beta(z) = -\int_{\mathbb{C}^d} \alpha(\zeta) \wedge K_{q-1}(\zeta, z)$$

satisfies $\bar\partial\beta = \alpha$*. Moreover, there exists a constant* $C > 0$ *such that*

$$\int_{R<|z|<2R} \frac{|\beta(z)|^2}{|z|^2}\,\mathrm{d}m(z) \le C\|\alpha\|^2_{\mathcal{H}_W} \tag{2.4}$$

for all $R > 0$*, where the constant* C *does not depend on* α *or* R*.*

Proof To show that $\bar\partial\alpha = \beta$ it is enough to show that this equality holds in the sense of distributions.

Let η_k be a family of cut-off functions, such that $\eta_k(\zeta) = 1$ if $|\zeta| < k$, $\eta_k(\zeta) = 0$ if $|\zeta| > k + 1$ and $|\bar{\partial}\eta_k| \leq 2$ for all $k = 1, 2, \ldots$. Since the form α belongs to the kernel of the elliptic Pauli operator (with smooth coefficient functions), it must itself be smooth. The $(0, q)$-type forms $\eta_k \alpha$ are smooth and have compact support and thus satisfy (2.2). Let Φ be a test form with support in $|z| < M$. Fix $\varepsilon > 0$. Then by (2.2) and the triangle inequality we get

$$\begin{aligned}
\left|\left\langle \beta, \bar{\partial}^* \Phi \right\rangle - \langle \alpha, \Phi \rangle\right| &= \Bigg| \langle \beta, \bar{\partial}^* \Phi \rangle + \left\langle \int_{\zeta \in \mathbb{C}^d} (\eta_k \alpha) \wedge K_{q-1}(\zeta, z), \bar{\partial}^* \Phi \right\rangle \\
&\qquad + \left\langle \int_{\zeta \in \mathbb{C}^d} \bar{\partial}(\eta_k \alpha) \wedge K_q(\zeta, z), \Phi \right\rangle + \langle \eta_k \alpha, \Phi \rangle - \langle \alpha, \Phi \rangle \Bigg| \\
&\leq \left| \left\langle \int_{\zeta \in \mathbb{C}^d} (\eta_k - 1) \alpha \wedge K_{q-1}(\zeta, z), \bar{\partial}^* \Phi \right\rangle \right| \\
&\qquad + \left| \left\langle \int_{\zeta \in \mathbb{C}^d} \bar{\partial} \eta_k \wedge \alpha \wedge K_q(\zeta, z), \Phi \right\rangle \right| \\
&\qquad + \left| \langle (\eta_k - 1) \alpha, \Phi \rangle \right| \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

We will let k tend to infinity. For $k > 2M$ we have $|K_{q-1}(\zeta, z)| \leq C|\zeta|^{1-2d}$. We get

$$\begin{aligned}
I_1 &\leq \int_{|z|<M} \int_{|\zeta|>k} |\eta_k(\zeta) - 1| \cdot |\alpha(\zeta)| \cdot |K_{q-1}(\zeta, z)| \, \mathrm{d}m(\zeta) \left| \bar{\partial}^* \Phi(z) \right| \, \mathrm{d}m(z) \\
&\leq C \sup \left| \bar{\partial}^* \Phi \right| \cdot \|\alpha\|_{\mathcal{H}} \int_{|z|<M} \left(\int_{|\zeta|>k} |K_{q-1}(\zeta, z)|^2 \, \mathrm{d}m(\zeta) \right)^{1/2} \mathrm{d}m(z) \\
&\leq C \sup \left| \bar{\partial}^* \Phi \right| \cdot \|\alpha\|_{\mathcal{H}} \left(\int_k^\infty r^{2-4d+2d-1} \, \mathrm{d}m(r) \right)^{1/2} \\
&\leq C \sup \left| \bar{\partial}^* \Phi \right| \cdot \|\alpha\|_{\mathcal{H}_W} \cdot k^{1-d}
\end{aligned}$$

so $I_1 < \varepsilon$ if k is large enough. Similarly, for I_2, we have

$$
\begin{aligned}
I_2 &\le \int_{|z|<M}\int_{k<|\zeta|<k+1} \left|\bar\partial\eta_k\right| \cdot |\alpha| \cdot |K_q(\zeta,z)|\, \mathrm{d}m(\zeta)\ |\Phi(z)|\ \mathrm{d}m(z) \\
&\le C\sup|\Phi|\cdot\|\alpha\|_{\mathcal{H}} \int_{|z|<M}\left(\int_{k<|\zeta|<k+1} |K_q(\zeta,z)|^2 e^{2W(\zeta)}\,\mathrm{d}m(\zeta)\right)^{1/2} \mathrm{d}m(z) \\
&\le C\sup|\Phi|\cdot\|\alpha\|_{\mathcal{H}} \left(\int_k^{k+1} r^{2-4d+2d-1}\,\mathrm{d}m(r)\right)^{1/2} \\
&\le C\sup|\Phi|\cdot\|\alpha\|_{\mathcal{H}_W}\cdot k^{\frac{1}{2}-d},
\end{aligned}
$$

so $I_2 < \varepsilon$ if k is large enough. I_3 is equal to zero if k is large enough, since then $(1-\eta_k)$ and Φ has disjoint support. We conclude that $\bar\partial\beta = \alpha$ in the sense of distributions.

To show the estimate (2.4), we use Lemma 2.3. Indeed, note that β can be written as

$$\beta = \sum_J T_J\alpha_J$$

where $\alpha = \sum_J \alpha_J d\bar z^J$, $|J| = q$, and all operators T_J are of the kind in Lemma 2.3. Denote by E_R the set $\{z \in \mathbb{C}^d : R < |z| \le 2R\}$. Using the Hölder inequality and Lemma 2.3, we have

$$
\begin{aligned}
\int_{E_R}\frac{|\beta|^2}{|z|^2}\,\mathrm{d}m(z) &\le \left(\int_{E_R}\frac{1}{|z|^{2d}}\,\mathrm{d}m(z)\right)^{1/d}\left(\int_{E_R}|\beta|^{2d/(d-1)}\,\mathrm{d}m(z)\right)^{(d-1)/d} \\
&\le C\|\alpha\|^2_{\mathcal{H}} \le C\|\alpha\|^2_{\mathcal{H}_W}.
\end{aligned}
$$

Note that the integral

$$\int_{R<|z|<2R}\frac{1}{|z|^{2d}}\,\mathrm{d}m(z) = c_d\int_R^{2R} r^{-2d+2d-1}\,\mathrm{d}m(r) = c_d\log(2)$$

is independent of R, so the constant C above is also independent of R. □

Proof (of Theorem 2.2) Let $1 \le q \le d-1$. Assume that $\alpha \in \mathcal{H}_W$ is a $(0,q)$-type form in the kernel of $\widetilde{\mathfrak{P}}$. Then $\bar\partial\alpha = \bar\partial^*\alpha = 0$, and so we get the form β from Lemma 2.4. We don't know, a priori, that β belongs to the domain of the $\bar\partial$ operator. We introduce a family of cut-off functions to be able to integrate by parts.

Let $\varphi_k(r)$, $k = 1, 2, \ldots$, be a C^∞ family of cut-off functions, such that $\varphi_k(r) = 1$ if $0 < r \le 2^k$, $\varphi_k(r) = 0$ if $r \ge 2^{k+1}$ and such that $0 \le \varphi_k$ and $|\varphi_k'(r)| \le 2^{1-k}$. Let $\chi_k(z) = \varphi_k(|z|)$. We have

$$\begin{aligned}
0 &= \langle \bar{\partial}^* \alpha, \chi_k \beta \rangle_{\mathcal{H}_W} \\
&= \langle \alpha, \bar{\partial}(\chi_k \beta) \rangle_{\mathcal{H}_W} \\
&= \int_{\mathbb{C}^d} |\alpha|^2 \chi_k e^{2W} \, \mathrm{d}m(z) + \int_{\mathbb{C}^d} \alpha \cdot \overline{\bar{\partial}\chi_k \wedge \beta} e^{2W} \, \mathrm{d}m(z) \\
&= I_k + II_k.
\end{aligned}$$

The integration by parts above is permitted thanks to the cut-off function χ_k. It is clear that $I_k \to \|\alpha\|^2_{\mathcal{H}_W}$ as $k \to \infty$. We shall prove that $II_k \to 0$ as $k \to \infty$.

Let $m_k^2 = \int_{E_k} |\alpha|^2 e^{2W} \, \mathrm{d}m$. Then it holds that $\sum_k m_k^2 = \|\alpha\|^2_{\mathcal{H}_W} < \infty$ so $m_k \to 0$ as $k \to \infty$. Since $\bar{\partial}\chi_k$ has support in E_k and $|\bar{\partial}\chi_k| \le C2^{-k}$ we have

$$\begin{aligned}
|II_k| &\le \int_{E_k} |\alpha| \cdot |\beta| \cdot \left|\bar{\partial}\chi_k\right| e^{2W} \, \mathrm{d}m \\
&\le C2^{-k} \left(\int_{E_k} |\alpha|^2 \, \mathrm{d}m \right)^{1/2} \left(\int_{E_k} |\beta|^2 \, \mathrm{d}m \right)^{1/2} \\
&\le C m_k \left(\int_{E_k} \frac{|\beta|^2}{|z|^2} \, \mathrm{d}m \right)^{1/2} \\
&\le C m_k \|\alpha\|_{\mathcal{H}_W} \to 0, \quad \text{as } k \to \infty.
\end{aligned}$$

The first inequality is just the triangle inequality. The second one is the inequality for χ_k and the Cauchy-Schwarz inequality. In the third inequality we use the fact that $|z| \approx 2^k$, and in the fourth we use Lemma 2.4.

Next let $q = 0$, and assume that α is a $(0, 0)$-type form in the kernel of $\widetilde{\mathfrak{P}}$. According to (1.7) α has to be an entire function in $z^1, \ldots, z^d$. Since the function α also belongs to $L_2(\mathbb{C}^d, e^{2W} \, \mathrm{d}m)$ a Liouville-type argument gives that it must be zero.

Finally let $q = d$. Then (1.7) implies that $\bar{\partial}^* \alpha = 0$. If $\alpha = \hat{\alpha} \, \mathrm{d}\bar{z}^1 \wedge \cdots \wedge \mathrm{d}\bar{z}^d$, then this means that

$$\frac{\partial \hat{\alpha}}{\partial z^j} + 2 \frac{\partial W}{\partial z^j} \hat{\alpha} = 0, \qquad j = 1, \ldots, d.$$

If we put $f(z) = e^{2W(z)} \hat{\alpha}(z)$ we obtain

$$\frac{\partial f}{\partial z^j} = 0, \qquad j = 1, \dots, d,$$

that is the function f is an entire function in $\bar{z}^1, \dots, \bar{z}^d$. Moreover the function f belongs to $L_2(\mathbb{C}^d, e^{-2W}\,\mathrm{d}m)$ so it must be zero.

□

3 Quadratically decaying magnetic fields

The case of determining the kernel of the Pauli operator for potentials with a logarithmic growth, which includes quadratically decaying magnetic fields, is more complicated. Given a real number Φ, denote by $N_d(\Phi)$ the number of all monomials in d variables of degree strictly less than $|\Phi| - d$. The following Theorem was proposed in [Ogu93].

Theorem 3.1 *Assume that* $W \in C^\infty(\mathbb{C}^d \to \mathbb{R})$ *and that there exists a real constant* Φ *such that the limit*

$$\lim_{|z|\to\infty} \frac{e^{W(z)}}{|z|^\Phi}$$

exists and is greater than zero. Then

$$\dim \ker \mathfrak{P} = N_d(\Phi).$$

Let us sketch the idea of the proof in the case $d = 2$. First, assume that $\Phi > 0$, and that

$$\alpha = \alpha_{00} + \alpha_{10}\,\mathrm{d}\bar{z}^1 + \alpha_{01}\,\mathrm{d}\bar{z}^2 + \alpha_{11}\,\mathrm{d}\bar{z}^1 \wedge \mathrm{d}\bar{z}^2$$

is an element of $\ker \widetilde{\mathfrak{P}}$. Then

$$\bar{\partial}\alpha = \frac{\partial \alpha_{00}}{\partial \bar{z}^1}\,\mathrm{d}\bar{z}^1 + \frac{\partial \alpha_{00}}{\partial \bar{z}^2}\,\mathrm{d}\bar{z}^2 + \left(\frac{\partial \alpha_{01}}{\partial \bar{z}^1} - \frac{\partial \alpha_{10}}{\partial \bar{z}^2}\right)\mathrm{d}\bar{z}^1 \wedge \mathrm{d}\bar{z}^2$$

and thus

$$\begin{aligned} 0 &= \int_{\mathbb{C}^2} \left|\bar{\partial}\alpha\right|^2 e^{2W}\,\mathrm{d}m(z) \\ &= \int_{\mathbb{C}^2} \left(\left|\frac{\partial \alpha_{00}}{\partial \bar{z}^1}\right|^2 + \left|\frac{\partial \alpha_{00}}{\partial \bar{z}^2}\right|^2 + \left|\frac{\partial \alpha_{01}}{\partial \bar{z}^1} - \frac{\partial \alpha_{10}}{\partial \bar{z}^2}\right|^2\right) e^{2W}\,\mathrm{d}m(z). \end{aligned}$$

However, in [Ogu93] this is written as

$$
\begin{aligned}
0 &= \int_{\mathbb{C}^2} \left|\bar{\partial}\alpha\right|^2 e^{2W}\,\mathrm{d}m(z) \\
&= \int_{\mathbb{C}^2} \left(\left|\frac{\partial \alpha_{00}}{\partial \bar{z}^1}\right|^2 + \left|\frac{\partial \alpha_{00}}{\partial \bar{z}^2}\right|^2 + \left|\frac{\partial \alpha_{01}}{\partial \bar{z}^1}\right|^2 + \left|\frac{\partial \alpha_{10}}{\partial \bar{z}^2}\right|^2 \right) e^{2W}\,\mathrm{d}m(z), \qquad (3.1)
\end{aligned}
$$

which is not correct. The rest of the proof uses (3.1) and some arguments to show that α_{00}, α_{10} and α_{01} must vanish. Then it is shown, correctly, that the term $\alpha_{11}\,\mathrm{d}\bar{z}^1 \wedge \mathrm{d}\bar{z}^2$ contains elements in the kernel if Φ is big enough. It is similar if $\Phi < 0$.

So, we know from [Ogu93] that if the potential W satisfies $W(z) \sim \Phi \log|z|$, as $|z| \to \infty$, for $|\Phi| > d$, then the kernel is non-empty, and the dimension of the kernel is at least $N_d(\Phi)$. We are not able to prove the Theorem proposed in [Ogu93], but we can show the following Theorem.

Theorem 3.2 *Assume that the limit*

$$
\lim_{|z|\to\infty} \frac{e^{W(z)}}{|z|^{\Phi}}
$$

exists and is positive. If $|\Phi| < d$ *then* $\dim\ker\mathfrak{P} = 0$. *If* $|\Phi| \geq d$ *then* $\dim\ker\mathfrak{P} \geq N_d(\Phi)$.

Remark 3.3 If $W(z) = \Phi\log|z|$ for large $|z|$ then an easy calculation shows that $|B| = \frac{\Phi(d-1)}{2|z|^2}$ for large $|z|$. According to the above Theorem the number of zero modes is 0 if $|\Phi| < d$ and at least $N_d(\Phi) > 0$ otherwise.

The proof of Theorem 3.2 goes in the same way as the proof of Theorem 2.2 so we will just point out the main differences. First of all we can assume that $\Phi \geq 0$. If Φ is negative we can apply a unitary transform that changes the sign of W.

We need a replacement of Lemma 2.4 where weights of polynomial growth are allowed. To prepare for this we introduce the Muckenhoupt weight class.

Definition 3.2 A non-negative function ψ is said to belong to the Muckenhoupt class $A(p,q)$, $1 < p, q < \infty$, if there exists a constant $C > 0$ such that

$$
\sup_{B\subset\mathbb{C}^d} \left(\frac{1}{|B|}\int_B \psi^q\,\mathrm{d}m(z)\right)^{1/q} \left(\frac{1}{|B|}\int_B \psi^{-p/(p-1)}\,\mathrm{d}m(z)\right)^{(p-1)/p} \leq C.
$$

Here the supremum is taken over all balls in $\mathbb{C}^d$ and $|B|$ denotes the Lebesgue measure of the ball B.

Lemma 3.3 *Let $0 \le \Phi < d$. Assume that the limit*

$$\lim_{|z|\to\infty} \frac{e^{W(z)}}{|z|^{\Phi}}$$

exists and is positive. Then the weight function e^{-W} belongs to the Muckenhoupt class $A(2, 2d/(d-1))$.

Proof Let $\gamma = 2d/(d-1)$. We should show that

$$I := \left(\frac{1}{|B|}\int_B e^{\gamma W}\,\mathrm{d}m(z)\right)^{1/\gamma}\left(\frac{1}{|B|}\int_B e^{-2W}\,\mathrm{d}m(z)\right)^{1/2} \le C,$$

where C does not depend on the ball B. From the assumptions on e^W we know that there exist positive constants c_1, c_2, c_3 and c_4 such that

$$c_1|z|^{\Phi} \le e^{W(z)} \le c_2|z|^{\Phi}, \qquad \text{if } |z| \ge 1 \tag{3.2}$$

and

$$c_3 \le e^{W(z)} \le c_4, \qquad \text{if } |z| < 5. \tag{3.3}$$

We divide the balls into different classes. Say that a ball $B = B(z_0, R)$ is of Type 1 if $|z_0| > 3/2R$ and otherwise of Type 2.

First, assume that B is of Type 1. Then for $z \in B$ we have $|z| \le |z_0| + R \le 5/3|z_0|$ and $|z| \ge |z_0| - R \ge 1/3|z_0|$. If $|z_0| \ge 3$ we can use (3.2) and get

$$\begin{aligned}
I &\le C\left(\frac{1}{|B|}\int_B |z|^{\gamma\Phi}\,\mathrm{d}m(z)\right)^{1/\gamma}\left(\frac{1}{|B|}\int_B \frac{1}{|z|^{2\Phi}}\,\mathrm{d}m(z)\right)^{1/2}\\
&\le C\left(\frac{1}{|B|}\int_B |z_0|^{\gamma\Phi}\,\mathrm{d}m(z)\right)^{1/\gamma}\left(\frac{1}{|B|}\int_B \frac{1}{|z_0|^{2\Phi}}\,\mathrm{d}m(z)\right)^{1/2}\\
&= C\left(|z_0|^{\gamma\Phi}\right)^{1/\gamma}\left(\frac{1}{|z_0|^{2\Phi}}\right)^{1/2}\\
&= C.
\end{aligned}$$

If $|z_0| \le 3$ then $|z| \le 5$, so we can easily use (3.3) to get that $I \le C$ independent of R.

Now assume that B is of Type 2. Let $B' := B(0,3R)$. Then $B \subset B'$ and $|B'| = 3^{2d}|B|$, so we have

$$I \leq C\left(\frac{1}{|B'|}\int_{B'} e^{\gamma W}\,\mathrm{d}m(z)\right)^{1/\gamma}\left(\frac{1}{|B'|}\int_{B'} e^{-2W}\,\mathrm{d}m(z)\right)^{1/2} =: J.$$

If $R \leq 5/3$ we can use (3.3) to get that $J \leq C$ independent of R. If $R > 5/3$ we have

$$J \leq C\left(\frac{1}{R^{2d}}\left(\int_{|z|<5}(1/c_4)^{\gamma}\,\mathrm{d}m(z) + \int_{5<|z|<3R}\frac{|z|^{\gamma\Phi}}{c_2^{\gamma}}\,\mathrm{d}m(z)\right)\right)^{1/\gamma} \times$$
$$\left(\frac{1}{R^{2d}}\left(\int_{|z|<5} c_3^{\gamma}\,\mathrm{d}m(z) + \int_{5<|z|<3R}\frac{c_1^2}{|z|^{2\Phi}}\,\mathrm{d}m(z)\right)\right)^{1/2}$$

In this product the first factor is of order $O(R^{\Phi})$ while the second factor is of order $O(R^{-\min(d,\Phi)})$ as $R \to \infty$. Since the expression clearly is bounded for bounded values of R there exists a constant C such that $J \leq C$ independent of R.

We conclude that $e^W \in A(2, 2d/(d-1))$. □

The following Lemma replaces Lemma 2.3.

Lemma 3.4 *Let $\Omega\colon \mathbb{C}^d \to \mathbb{C}$ be a homogeneous function of degree zero, and let Ω be bounded on the unit sphere $|z| = 1$. Define the operator T as*

$$(Tf)(z) = \int_{\mathbb{C}^d}\frac{\Omega(z-\zeta)}{|z-\zeta|^{2d-1}}f(\zeta)\,\mathrm{d}m(\zeta).$$

If the weight ψ belongs to the Muckenhoupt class $A\big(2, 2d/(d-1)\big)$ then there exists a constant $C > 0$, independent of f, such that

$$\left(\int_{\mathbb{C}^d}\big((Tf)(z)\cdot\psi(z)\big)^{2d/(d-1)}\right)^{(d-1)/(2d)} \leq C\left(\int_{\mathbb{C}^d}\big|f(z)\psi(z)\big|^2\right)^{1/2}.$$

Proof This is a special case of Theorem 1 in [DL98]. □

Finally we get the result that replaces Lemma 2.4.

Lemma 3.5 *Let $\alpha \in \mathcal{H}_W$ be a $(0,q)$-type form, $1 \leq q \leq d-1$, satisfying $\bar\partial\alpha = \bar\partial^*\alpha = 0$. Then the $(0,q-1)$-type form*

$$\beta(z) = -\int_{\mathbb{C}^d} \alpha(\zeta) \wedge K_{q-1}(\zeta, z) \tag{3.4}$$

satisfies $\bar{\partial}\beta = \alpha$ *in the sense of distributions. Moreover, there exists a constant* $C > 0$ *such that*

$$\int_{R<|z|<2R} \frac{|\beta(z)|^2}{|z|^2} e^{2W(z)}\, \mathrm{d}m(z) \le C\|\alpha\|^2_{\mathcal{H}_W} \tag{3.5}$$

for all $R > 0$*, where the constant* C *does not depend on* α *or* R*.*

Proof The part that β solves $\bar{\partial}\beta = \alpha$ is just the same as in the proof of Lemma 2.4. Using Lemma 3.4 we get

$$\begin{aligned}\int_{E_R} |\beta|^2 e^{2W}\, \mathrm{d}m(z) &\le \left(\int_{E_R} |z|^{-2d}\, \mathrm{d}m(z)\right)^{1/d} \times \\ &\quad \times \left(\int_{E_R} \left(|\beta|e^W\right)^{2d/(d-1)}\, \mathrm{d}m(z)\right)^{(d-1)/d} \\ &\le C\|\alpha\|^2_{\mathcal{H}_W},\end{aligned}$$

and the estimate (3.5) is proved. □

Proof (of Theorem 3.2) First, let $1 \le q \le d-1$. The proof runs in the same way as in the proof of Theorem 2.2, but with the use of Lemmas with weights.

Next, let $q = 0$, and assume that α is a $(0,0)$-type form in the kernel of $\widetilde{\mathfrak{P}}$. According to (1.7) α has to be an entire function in $z^1, \dots, z^d$. Also belonging to $L_2(\mathbb{C}^d, e^{2W}\, \mathrm{d}m)$, it must tend to zero at infinity. Hence it must be constant equal to zero by a Liouville type argument.

Finally, let $q = d$. Then (1.7) implies that $\bar{\partial}^* \alpha = 0$. If $\alpha = \hat{\alpha} d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^d$, then this means that the function $f(z) = e^{2W(z)}\hat{\alpha}(z)$ is an entire function in $\bar{z}^1, \dots, \bar{z}^d$. Moreover there exist constants c_1 and c_2 such that

$$\frac{c_1}{|z|^{\Phi}} \le e^{-W(z)} \le \frac{c_2}{|z|^{\Phi}}$$

if $|z|$ is large enough. The condition $\alpha \in \mathcal{H}_W$ means $e^{-W} f \in L_2(\mathbb{C}^d)$. This is the case if and only if f is a polynomial in $\bar{z}^1, \dots, \bar{z}^d$ of degree strictly less than $\Phi - d$. The dimension of the space of such polynomials is exactly $N_d(\Phi)$. □

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Paper IV

Eigenvalue asymptotics of the even-dimensional exterior Landau-Neumann Hamiltonian

Mikael Persson

Abstract: We study the Schrödinger operator with a constant magnetic field in the exterior of a compact domain in $\mathbb{R}^{2d}$, $d \geq 1$. The spectrum of this operator consists of clusters of eigenvalues around the Landau levels. We give asymptotic formulas for the rate of accumulation of eigenvalues in these clusters. When the compact is a Reinhart domain we are able to show a more precise asymptotic formula.

1 Introduction

The Landau Hamiltonian describes a charged particle moving in a plane, influenced by a constant magnetic field of strength $B > 0$ orthogonal to the plane. It is a classical result, see [Foc28, Lan30], that the spectrum of the Landau Hamiltonian consists of infinitely degenerate eigenvalues $B(2q+1)$, $q = 0, 1, 2, \ldots$, called Landau levels.

In this paper we will study the even-dimensional Landau Hamiltonian outside a compact obstacle, imposing magnetic Neumann conditions at the boundary. Our motivation to study this operator comes mainly from the papers [HS02, PR07]. Spectral properties of the exterior Landau Hamiltonian in the plane are discussed in [HS02], under both Dirichlet and Neumann conditions at the boundary. A more qualitative study of the spectrum is done in [PR07], where the authors fix an interval around a Landau level and describe how fast the eigenvalues in that cluster converges to that Landau level. They work in the plane and with Dirichlet boundary conditions only. The goal of this paper is to perform the same qualitative description when we impose magnetic Neumann conditions at the boundary. Moreover, we do not limit ourself to the plane, but work in arbitrary evendimensional Euclidean space.

The result is that the eigenvalues do accumulate with the same rate to the Landau levels for both types of boundary conditions, see Theorem 3.2 for the details. However, the eigenvalues can only accumulate to a Landau level from below in the Neumann setting. In the Dirichlet case they accumulate only from above.

It should be mentioned that we suppose that the compact set removed has no holes and that its boundary is smooth. This is far more restrictive than the conditions imposed on the compact set in [PR07].

Several different perturbations of the Landau Hamiltonian have been studied in the last years, see [MR03, FP06, RS08, PR07]. They all share the common idea of making a reduction to a certain Toeplitz-type operator whose spectral asymptotics is known. We also do this kind of reduction. The method we use is based on the theory for pseudodifferential operators and boundary PDE methods, which we have not seen in any of the mentioned papers.

In Section 2 we define the Landau Hamiltonian and mention some auxiliary results about its spectrum, eigenspaces and Green function.

We begin Section 3 by defining the exterior Landau Hamiltonian with magnetic Neumann boundary condition and formulate and prove the main theorems (Theorem 3.1 and 3.2) about the spectral asymptotics of the operator. The main part of the proof, the reduction step, is quite technical and therefore moved to Section 4. When the reduction step is done we use the asymptotic formulas of the spectrum of the Toeplitz-type operators, given in [FP06, MR03], to obtain the asymptotic formulas in Theorem 3.2.

In the higher dimensional case ($\mathbb{R}^{2d}$, $d > 1$) we also consider the case when the compact obstacle is a Reinhart domain. We use some ideas from [Par94] to prove a more precise asymptotic formula for the eigenvalues. This is done in Section 5.

2 The Landau Hamiltonian in $\mathbb{R}^{2d}$

We denote by $x = (x^1, \dots, x^{2d})$ a point in $\mathbb{R}^{2d}$. Let $B > 0$ and denote by $\vec{a}$ the magnetic vector potential

$$\vec{a}(x) = (a_1(x), \dots, a_{2d}(x)) = \frac{B}{2}\left(-x^2, x^1, -x^4, x^3, \dots, -x^{2d}, x^{2d-1}\right).$$

It corresponds to an isotropic magnetic field of constant strength B. The Landau Hamiltonian L in $\mathbb{R}^{2d}$ describes a charged, spinless particle in this homogeneous magnetic field. It is given by

$$L = (-i\nabla - \vec{a})^2$$

and is essentially self-adjoint on the set $C_0^\infty(\mathbb{R}^{2d})$ in the usual Hilbert space $\mathcal{H} = L_2(\mathbb{R}^{2d})$. For $j = 1, \dots, d$ we also introduce the self-adjoint operators

$$L_j = \left(-i\left(\frac{\partial}{\partial x^{2j-1}}, \frac{\partial}{\partial x^{2j}}\right) - (a_{2j-1}, a_{2j})\right)^2,$$

in the Hilbert spaces $\mathcal{H}_j = L_2(\mathbb{R}^2)$. Note that $\mathcal{H} = \bigotimes_{j=1}^d \mathcal{H}_j$, and

$$L = L_1 \otimes I^{\otimes(d-1)} + I \otimes L_2 \otimes I^{\otimes(d-2)} + \ldots + I^{\otimes(d-1)} \otimes L_d. \tag{2.1}$$

2.1 Landau levels

The spectrum of each twodimensional Landau Hamiltonian L_j consist of so called Landau levels, eigenvalues $B(2q+1)$, $q \in \mathbb{N}$, each of infinite multiplicity. Let $\hat{\kappa} = (\kappa_1, \dots, \kappa_d) \in \mathbb{N}^d$ be a multiindex. We denote by $|\hat{\kappa}| = \kappa_1 + \ldots + \kappa_d$ the length of the multiindex $\hat{\kappa}$ and also set $\hat{\kappa}! = \kappa_1! \cdot \ldots \cdot \kappa_d!$. From (2.1) it follows that the spectrum of L consists of the infinitely degenerate eigenvalues

$$\Lambda_{\hat{\kappa}} = B \sum_{j=1}^d (2\kappa_j + 1), \quad \kappa_j \in \mathbb{N}.$$

Note that $\Lambda_{\hat{\kappa}} = \Lambda_{\hat{\kappa}'}$ if $|\hat{\kappa}| = |\hat{\kappa}'|$. Hence the spectrum of L consists of eigenvalues of the form $\Lambda_\mu = B(2\mu + d)$, $\mu \in \mathbb{N}$.

2.2 Creation and annihilation operators

The structure of the eigenspaces of L has been described before in [MR03]. We give the results without proofs. It is convenient to introduce complex notation. Let $z = (z^1, \dots, z^d) \in \mathbb{C}^d$, where $z^j = x^{2j-1} + ix^{2j}$. Also, we use the scalar potential $W(z) = -\frac{B}{4}|z|^2$ and the complex derivatives

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^{2j-1}} - i\frac{\partial}{\partial x^{2j}}\right), \quad \frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^{2j-1}} + i\frac{\partial}{\partial x^{2j}}\right).$$

We define creation and annihilation operators $\mathfrak{Q}_j^*, \mathfrak{Q}_j$ as

$$\mathfrak{Q}_j^* = -2ie^{-W}\frac{\partial}{\partial z^j}e^W, \qquad \mathfrak{Q}_j = -2ie^W\frac{\partial}{\partial \bar{z}^j}e^{-W},$$

and note that

$$\left[\mathfrak{Q}_j^*, \mathfrak{Q}_k^*\right] = \left[\mathfrak{Q}_j, \mathfrak{Q}_k\right] = \left[\mathfrak{Q}_j^*, \mathfrak{Q}_k\right] = 0, \quad \text{if } j \neq k. \tag{2.2}$$

The notation $\mathfrak{Q}_j^*$ for the creation operators is motivated by the fact that it is the formal adjoint of $\mathfrak{Q}_j$ in $\mathcal{H}$.

A function u belongs to the lowest Landau level Λ_0 if and only if $\mathfrak{Q}_j u = 0$ for $j = 1, \dots, d$. This means that the function $f = e^{-W} u$ is an entire function, so via multiplication by e^{-W} the eigenspace $\mathfrak{L}_{\Lambda_0}$ corresponding to Λ_0 is equivalent to the Fock space

$$\mathfrak{F}_B^2 = \left\{ f \;\middle|\; f \text{ is entire and } \int_{\mathbb{C}^d} |f|^2 e^{-\frac{B}{2}|z|^2} \, \mathrm{d}m(z) < \infty \right\}.$$

Here, and elsewhere, dm denotes the Lebesgue measure. A function u belongs to the eigenspace $\mathfrak{L}_{\Lambda_\mu}$ of the Landau level Λ_μ if and only if it can be written in the form

$$u = \sum_{|\hat{\kappa}|=\mu} c_{\hat{\kappa}} (\mathfrak{Q}^*)^{\hat{\kappa}} (e^W f_{\hat{\kappa}}),$$

where $(\mathfrak{Q}^*)^{\hat{\kappa}} = (\mathfrak{Q}_1^*)^{\kappa_1} \cdots (\mathfrak{Q}_d^*)^{\kappa_d}$ and $f_{\hat{\kappa}}$ all belong to $\mathfrak{F}_B^2$. The multiplicity of the eigenvalue Λ_μ is equal to $\binom{\mu+d-1}{d-1}$. We denote by $\mathcal{P}_{\Lambda_{\hat{\kappa}}}$ and $\mathcal{P}_{\Lambda_\mu}$ the projection onto the eigenspaces $\mathfrak{L}_{\Lambda_{\hat{\kappa}}}$ and $\mathfrak{L}_{\Lambda_\mu}$ respectively, and note by (2.2) that the orthogonal decompositions

$$\mathfrak{L}_{\Lambda_\mu} = \bigoplus_{|\hat{\kappa}|=\mu} \mathfrak{L}_{\Lambda_{\hat{\kappa}}}, \quad \mathcal{P}_{\Lambda_\mu} = \bigoplus_{|\hat{\kappa}|=\mu} \mathcal{P}_{\Lambda_{\hat{\kappa}}} \tag{2.3}$$

hold in $\mathcal{H}$.

2.3 The resolvent

Let $R_\rho = (L + \rho I)^{-1}$ be the resolvent of L, $\rho \geq 0$. An explicit formula of the kernel $G_\rho(x, y)$ of R_ρ was given in [HS02] for $d = 1$. In Section 4.2 we will use the behavior of $G_\rho(x, y)$ near the diagonal $x = y$, given in the following Lemma.

Lemma 2.1 *R_ρ is an integral operator with kernel $G_\rho(x, y)$ that has the following singularity at the diagonal,*

$$G_\rho(x,y) \sim \begin{cases} \frac{1}{\pi}\log(1/|x-y|) + O(1), & d=1; \\ \frac{1}{2\pi^2}|x-y|^{-2} + O(\log(1/|x-y|)), & d=2; \\ \frac{\Gamma(d-1)}{2\pi^d}|x-y|^{2-2d} + O(|x-y|^{4-2d}), & d>2; \end{cases} \tag{2.4}$$

as $|x-y| \to 0$.

Proof The kernel $G_\rho(x,y)$ of R_ρ can be written as

$$G_\rho(x,y) = \int_0^\infty e^{-\rho t} e^{-Lt}(x,y)\,\mathrm{d}m(t).$$

Now, since the variables separate pairwise, we have

$$e^{-Lt}(x,y) = \prod_{j=1}^{d} e^{-L_j t}(x^{2j-1}, x^{2j}, y^{2j-1}, y^{2j}).$$

The formula for $e^{-L_j t}$ is given in [Sim79a]. It reads

$$e^{-L_j t} = \frac{B}{4\pi}\exp\left(-\frac{iB}{2}(x^{2j-1}y^{2j} - x^{2j}y^{2j-1})\right)\frac{1}{\sinh(Bt/2)} \times$$
$$\times \exp\left(-\frac{B}{4}\coth(Bt/2)\left((x^{2j-1}-y^{2j-1})^2 + (x^{2j}-y^{2j})^2\right)\right)$$

Hence the formula for $G_\rho(x,y)$ becomes

$$G_\rho(x,y) = \left(\frac{B}{4\pi}\right)^d \exp\Big(-\frac{iB}{2}\sum_{j=1}^{d}(x^{2j-1}y^{2j} - x^{2j}y^{2j-1})\Big) I(|x-y|^2) \tag{2.5}$$

where

$$I(s) = \int_0^\infty e^{-\rho t}\frac{1}{\sinh^d(Bt/2)}\exp\left(-\frac{B}{4}\coth(Bt/2)s\right)\mathrm{d}m(t).$$

An expansion of $I(s)$ shows that

$$I(s) \sim \begin{cases} \left(\frac{2}{B}\right)\log(1/s) + O(1), & d=1; \\ \frac{8}{B^2}s^{-1} + O(\log(1/s)), & d=2; \\ \left(\frac{4}{B}\right)^d \frac{\Gamma(d-1)}{2}s^{1-d} + O(s^{2-d}); & d>2, \end{cases} \qquad \text{as } s \to 0,$$

from which (2.4) follows. □

3 The exterior Landau-Neumann Hamiltonian in $\mathbb{R}^{2d}$

Let $K \subset \mathbb{R}^{2d}$ be a simply connected compact domain with smooth boundary Γ and let $\Omega = \mathbb{R}^{2d} \setminus K$. We define the exterior Landau-Neumann Hamiltonian L_Ω in $\mathcal{H}_\Omega = L_2(\Omega)$ by

$$L_\Omega = (-i\nabla - \vec{a})^2, \quad \text{in } \Omega; \tag{3.1}$$

with Neumann boundary conditions

$$\partial_N u := (-i\nabla - \vec{a})u \cdot \nu = 0 \quad \text{on } \Gamma. \tag{3.2}$$

Here ν denotes the exterior normal to Γ. Our aim is to study how much the spectrum of L_Ω differs from the Landau levels discussed in the previous section. The first Theorem below states that the eigenvalues of L_Ω can accumulate to each Landau level only from below. The second Theorem says that the eigenvalues do accumulate to the Landau levels from below, and the rate of convergence is given.

Theorem 3.1 *For every $\mu \in \mathbb{N}$ and each ε, $0 < \varepsilon < dB$, the number of eigenvalues of L_Ω in the interval $(\Lambda_\mu, \Lambda_\mu + \varepsilon)$ is finite.*

Denote by $l_1^{(\mu)} \le l_2^{(\mu)} \le \cdots$ the eigenvalues of L_Ω in the interval $(\Lambda_{\mu-1}, \Lambda_\mu)$ and $N(a, b, T)$ the number of eigenvalues of the operator T in the interval (a, b), counting multiplicities. Also, let $\mathrm{Cap}(K)$ denote the logarithmic capacity of K, see [Lan72].

Theorem 3.2 *Let $\mu \in \mathbb{N}$.*

(*a*) *If $d = 1$ then* $\lim_{j\to\infty} \left(j!\left(\Lambda_\mu - l_j^{(\mu)}\right)\right)^{1/j} = \frac{B}{2}\left(\mathrm{Cap}(K)\right)^2.$

(*b*) *If $d > 1$ then* $N(\Lambda_{\mu-1}, \Lambda_\mu - \lambda, L_\Omega) \sim \binom{\mu+d-1}{d-1}\frac{1}{d!}\left(\frac{|\ln\lambda|}{\ln|\ln\lambda|}\right)^d$ *as* $\lambda \searrow 0$.

3.1 Proof of the Theorems

We want to compare the spectrum of the operators L and L_Ω. However, the expression $L - L_\Omega$ has no meaning since L and L_Ω acts in different Hilbert spaces. We introduce the Hilbert space $\mathcal{H}_K = L_2(K)$ and define the interior Landau-Neumann Hamiltonian L_K in $\mathcal{H}_K$ by the same formulas as in (3.1) and (3.2) but with Ω replaced by K. Note that $\mathcal{H} = \mathcal{H}_K \oplus \mathcal{H}_\Omega$. Define $\tilde{L}$ as

$$\tilde{L} = L_K \oplus L_\Omega, \quad \text{in } \mathcal{H}_K \oplus \mathcal{H}_\Omega.$$

The inverse of L_K is compact, so L_K has at most a finite number of eigenvalues in each interval $(\Lambda_{\mu-1}, \Lambda_\mu)$. The operators L_K and L_Ω act in orthogonal subspaces of $\mathcal{H}$, so $\sigma(\tilde{L}) = \sigma(L_K) \cup \sigma(L_\Omega)$. This means that $\tilde{L}$ has the same spectral asymptotics as L_Ω in each interval $(\Lambda_{\mu-1}, \Lambda_\mu)$, so it is enough to prove the statements in Theorem 3.1 and 3.2 for the operator $\tilde{L}$ instead of L_Ω.

Since the unbounded operators L and $\tilde{L}$ have different domains, we cannot compare them directly. However, they act in the same Hilbert space, so we can compare their inverses. Let

$$R = R_0 = L^{-1}, \quad \text{and} \quad \tilde{R} = \tilde{L}^{-1} = L_K^{-1} \oplus L_\Omega^{-1},$$

and set

$$V = \tilde{R} - R, \quad \text{and} \quad T_\mu = \mathcal{P}_{\Lambda_\mu} V \mathcal{P}_{\Lambda_\mu}, \quad \text{for } \mu \in \mathbb{N}.$$

Lemma 3.3 *V is non-negative and compact.*

Proof See Section 4.1. □

By Weyl's theorem the essential spectrum of R and $\tilde{R}$ coincides. Since $\tilde{R} = R + V$ and $V \geq 0$, Theorem 3.1 follows immediately from Theorem 9.4.7 in [BS87] and the fact that $\sigma(R) = \sigma_{\mathrm{ess}}(R) = \{\Lambda_\mu^{-1}\}$. We continue with the proof of Theorem 3.2.

Let $\tau > 0$ be such that $\big((\Lambda_\mu^{-1} - 2\tau, \Lambda_\mu^{-1} + 2\tau) \setminus \{\Lambda_\mu^{-1}\}\big) \cap \sigma_{\mathrm{ess}}(R) = \varnothing$. Denote the eigenvalues of T_μ by

$$t_1^{(\mu)} \geq t_2^{(\mu)} \geq \cdots,$$

and the eigenvalues of $\tilde{R}$ in the interval $(\Lambda_\mu^{-1}, \Lambda_\mu^{-1} + \tau)$ by

$$r_1^{(\mu)} \geq r_2^{(\mu)} \geq \cdots.$$

Lemma 3.4 *Given $\varepsilon > 0$ there exists an integer l such that*

$$(1-\varepsilon) t_{j+l}^{(\mu)} \leq r_j^{(\mu)} - \Lambda_\mu^{-1} \leq (1+\varepsilon) t_{j-l}^{(\mu)}, \quad \textit{for all sufficiently large } j.$$

Proof See Proposition 2.2 in [PR07]. □

Hence the study of the asymptotics of the eigenvalues of $\tilde{R}$ is reduced to the study of the eigenvalues of the Toeplitz-type operator T_μ. For a bounded simply connected set U in $\mathbb{R}^{2d}$ we define the Toeplitz operator S_μ^U as

$$S_\mu^U = \mathcal{P}_{\Lambda_\mu}\chi_U\mathcal{P}_{\Lambda_\mu},$$

where χ_U denotes the characteristic function of U. The following lemma reduces our problem to the study of these Toeplitz operators, which are easier to study than T_μ.

Lemma 3.5 *Let $K_0 \Subset K \Subset K_1$ be compact domains such that $\partial K_i \cap \Gamma = \varnothing$. There exist a constant $C > 0$ and a subspace $\mathcal{S} \subset \mathcal{H}$ of finite codimension such that*

$$\frac{1}{C}\langle f, S_\mu^{K_0} f\rangle \le \langle f, T_\mu f\rangle \le C\langle f, S_\mu^{K_1} f\rangle \tag{3.3}$$

for all $f \in \mathcal{S}$.

Proof See Section 4.2. □

The asymptotic expansion of the spectrum of S_μ^U is given in the following lemma.

Lemma 3.6 *Denote by $s_1^{(\mu)} \ge s_2^{(\mu)} \ge \dots$ the eigenvalues of S_μ^U and by $n(\lambda, S_\mu^U)$ the number of eigenvalues of S_μ^U greater than λ (counting multiplicity). Then*

(*a*) *if $d = 1$ we have $\lim_{j\to\infty}\left(j!s_j^{(\mu)}\right)^{1/j} = \frac{B}{2}\left(\mathrm{Cap}(U)\right)^2$,*

(*b*) *if $d > 1$ we have $n(\lambda, S_\mu^U) \sim \binom{\mu+d-1}{d-1}\frac{1}{d!}\left(\frac{|\ln\lambda|}{\ln|\ln\lambda|}\right)^d$ as $\lambda \searrow 0$.*

Proof See Lemma 3.2 in [FP06] for part (*a*) and Proposition 7.1 in [MR03] for part (*b*). □

We are now able to finish the proof of Theorem 3.2. By letting K_0 and K_1 in Lemma 3.5 get closer and closer to our compact K we see that the eigenvalues $\left\{t_j^{(\mu)}\right\}$ of T_μ satisfy

$$\lim_{n\to\infty}\left(j!t_j^{(\mu)}\right)^{1/j} = \frac{B}{2}\left(\mathrm{Cap}(K)\right)^2 \tag{3.4}$$

if $d = 1$, and

$$n(\lambda, T_\mu) \sim \binom{\mu+d-1}{d-1}\frac{1}{d!}\left(\frac{|\ln\lambda|}{\ln|\ln\lambda|}\right)^d, \quad \text{as } \lambda \searrow 0 \tag{3.5}$$

if $d > 1$. Since neither of the formulas (3.4) nor (3.5) are sensitive for finite shifts in the indices it follows from Lemma 3.4 that the eigenvalues of $\{r_j^{(\mu)}\}$ $\tilde{R}$ satisfies

$$\lim_{j\to\infty}\left(j!(r_j^{(\mu)} - \Lambda_\mu^{-1})\right)^{1/j} = \frac{B}{2}\left(\mathrm{Cap}(K)\right)^2$$

if $d = 1$, and

$$N(\Lambda_\mu^{-1} + \lambda, \Lambda_{\mu-1}^{-1}, \tilde{R}) \sim \binom{\mu+d-1}{d-1}\frac{1}{d!}\left(\frac{|\ln\lambda|}{\ln|\ln\lambda|}\right)^d, \quad \text{as } \lambda \searrow 0$$

If we translate this in terms of $\tilde{L}$ we get

$$\lim_{j\to\infty}\left(j!(\Lambda_\mu - l_j^{(\mu)})\right)^{1/j} = \frac{B}{2}\left(\mathrm{Cap}(K)\right)^2$$

for $d = 1$, and

$$\begin{aligned} N(\Lambda_{\mu-1}, \Lambda_\mu - \lambda, \tilde{L}) &\sim \binom{\mu+d-1}{d-1}\frac{1}{d!}\left(\frac{\left|\ln\frac{\lambda}{\Lambda_\mu(\Lambda_\mu-\lambda)}\right|}{\ln\left|\ln\frac{\lambda}{\Lambda_\mu(\Lambda_\mu-\lambda)}\right|}\right)^d \\ &\sim \binom{\mu+d-1}{d-1}\frac{1}{d!}\left(\frac{|\ln\lambda|}{\ln|\ln\lambda|}\right)^d, \quad \text{as } \lambda \searrow 0, \end{aligned}$$

for $d > 1$. This completes the proof of Theorem 3.2. □

4 Proof of the Lemmas

In this section we prove Lemma 3.3 and 3.5.

4.1 Proof of Lemma 3.3

The operators L and $\tilde{L}$ are defined by the same expression, but the domain of $\tilde{L}$ is contained in the domain of L. It follows from Proposition 2.1 in [PR07] that $L - \tilde{L} \geq 0$. This means that $V = \tilde{R} - R \geq 0$.

Next we prove the compactness of V. Let f and g belong to $\mathcal{H}$. Also, let $u = Rf$ and $v = \tilde{R}g$. Then u belongs to the domain of L and v belongs to the domain of $\tilde{L}$, so $v = v_K \oplus v_\Omega$, and $L_K v_K \oplus L_\Omega v_\Omega = g$. Integrating by parts and using (3.2) for v_K and v_Ω, we get

$$\begin{aligned}\langle f, Vg\rangle &= \langle f, \tilde{R}g\rangle - \langle Rf, g\rangle \\ &= \int_K Lu \cdot \overline{v_K}\, \mathrm{d}m(x) + \int_\Omega Lu \cdot \overline{v_\Omega}\, \mathrm{d}m(x) \\ &\quad - \int_K u \cdot \overline{L_K v_K}\, \mathrm{d}m(x) - \int_\Omega u \cdot \overline{L_\Omega v_\Omega}\, \mathrm{d}m(x) \\ &= \int_\Gamma \partial_N u \cdot \overline{(v_\Omega - v_K)}\, \mathrm{d}S. \end{aligned} \tag{4.1}$$

Here $\mathrm{d}S$ denotes the surface measure on Γ.

Take a smooth cut-off function $\chi \in C_0^\infty(\mathbb{R}^{2d})$ such that $\chi(x) = 1$ in a neighborhood of K. Then we can replace u and v by $\tilde{u} = \chi u$ and $\tilde{v} = \chi v$ in the right hand side of (4.1). By local elliptic regularity we have that $\tilde{u} \in H^2(\mathbb{R}^{2d})$ and $\tilde{v} \in H^2(\mathbb{R}^{2d} \setminus \Gamma)$. However, the operator $\tilde{u} \mapsto \partial_N \tilde{u}|_\Gamma$ is compact as considered from $H^2(\mathbb{R}^{2d})$ to $L_2(\Gamma)$ and both $\tilde{v} \mapsto \tilde{v}_\Omega|_\Gamma$ and $\tilde{v} \mapsto \tilde{v}_K|_\Gamma$ are compact as considered from $H^2(\mathbb{R}^{2d} \setminus \Gamma)$ to $L_2(\Gamma)$, so it follows that V is compact. □

4.2 Proof of Lemma 3.5

We start by showing that T_μ can be considered as an elliptic Pseudodifferential operator of order 1 on some subspace of $L_2(\Gamma)$ of finite codimension, and hence that there exists a constant $C > 0$ such that

$$\frac{1}{C}\|f\|_{L_2(\Gamma)}\|f\|_{H^1(\Gamma)} \le \langle f, T_\mu f\rangle \le C\|f\|_{L_2(\Gamma)}\|f\|_{H^1(\Gamma)} \tag{4.2}$$

for all f in that subspace.

Let f and g belong to $\mathcal{H}$. Also, let $u = Rf$, $v = \tilde{R}g$ and $w = Rg$. We saw in (4.1) that

$$\langle f, Vg\rangle = \int_\Gamma \partial_N u \cdot \overline{(v_\Omega - v_K)}\, \mathrm{d}S.$$

To go further we will introduce the Neumann to Dirichlet and Dirichlet to Neumann operators. Let $G_\rho(x, y)$ be as in (2.5) We start with the single and double layer integral operators, defined by

$$\mathcal{A}\alpha(x) = \int_\Gamma G_0(x-y)\alpha(y)\,\mathrm{d}S(y), \quad x \in \mathbb{R}^{2d},$$

$$\mathcal{B}\alpha(x) = \int_\Gamma \partial_{N_y} G_0(x-y)\alpha(y)\,\mathrm{d}S(y), \quad x \in \mathbb{R}^{2d} \setminus \Gamma,$$

$$A\alpha(x) = \int_\Gamma G_0(x-y)\alpha(y)\,\mathrm{d}S(y), \quad x \in \Gamma, \text{ and}$$

$$B\alpha(x) = \int_\Gamma \partial_{N_y} G_0(x-y)\alpha(y)\,\mathrm{d}S(y), \quad x \in \Gamma.$$

The last two operators are compact on $L_2(\Gamma)$, since, by Lemma 2.1, their kernels have weak singularities. Moreover, since the kernel G_0 has the same singularity as the Green kernel for the Laplace operator in $\mathbb{R}^{2d}$ (see [Tay96b]), we have the following limit relations on Γ

$$\begin{aligned} \mathcal{A}\alpha_K &= A\alpha_K, & \mathcal{B}\alpha_K &= \frac{1}{2}\alpha + B\alpha, \\ \mathcal{A}\alpha_\Omega &= A\alpha_\Omega, & \mathcal{B}\alpha_\Omega &= -\frac{1}{2}\alpha + B\alpha. \end{aligned} \tag{4.3}$$

Using a Green-type formula for L in K we see that

$$\beta = \mathcal{B}\beta_K - \mathcal{A}(\partial_N \beta_K).$$

If we combine this with the limit relations (4.3) we get

$$\Big(B - \frac{1}{2}I\Big)\beta_K = A(\partial_N \beta_K), \quad \text{on } \Gamma.$$

A similar calculation for Ω gives

$$\Big(B + \frac{1}{2}I\Big)\beta_\Omega = A(\partial_N \beta_\Omega), \quad \text{on } \Gamma.$$

It seems natural to do the following definitions.

Definition 4.1 We define the Dirichlet-to-Neumann and Neumann-to-Dirichlet operators in K and Ω as

$$(DN)_K = A^{-1}\Big(B - \frac{1}{2}I\Big), \quad (ND)_K = \Big(B - \frac{1}{2}I\Big)^{-1} A,$$
$$(DN)_\Omega = A^{-1}\Big(B + \frac{1}{2}I\Big), \quad (ND)_\Omega = \Big(B + \frac{1}{2}I\Big)^{-1} A.$$

Remark 4.1 The inverses above exist at least on a space of finite codimension. This follows from the fact that A is elliptic and B is compact.

Lemma 4.1 *The operator* $(ND)_K - (ND)_\Omega$ *is an elliptic pseudodifferential operator of order* -1.

Proof Using a resolvent identity, we see that

$$(ND)_K - (ND)_\Omega = \left(B + \frac{1}{2}I\right)^{-1}\left(B - \frac{1}{2}I\right)^{-1} A.$$

It follows from the asymptotic expansion of $G_0(x,y)$ in Lemma 2.1 that A is an elliptic pseudodifferential operator of order -1. Moreover the operator B is compact, so the other two factors are pseudodifferential operators of order 0 which do not change the principal symbol noticeably. □

Let us now return to the expression of V. We have

$$\begin{aligned}
\langle f, Vg\rangle &= \int_\Gamma \partial_N u \cdot \overline{(v_\Omega - v_K)}\, \mathrm{d}S \\
&= \int_\Gamma \partial_N u \cdot \overline{(v_\Omega - w + w - v_K)}\, \mathrm{d}S \\
&= \int_\Gamma \partial_N u \cdot \overline{\big((ND)_\Omega(\partial_N(v_\Omega - w) + (ND)_K(\partial_N(w - v_K)))\big)}\, \mathrm{d}S \\
&= \int_\Gamma \partial_N u \cdot \overline{\big(((ND)_K - (ND)_\Omega)(\partial_N w)\big)}\, \mathrm{d}S.
\end{aligned}$$

Since we are interested in T_μ and not V, we may assume that f and g belong to $\mathfrak{L}_{\Lambda_\mu}$. Then $u = Rf = \Lambda_\mu^{-1} f$ and $w = Rg = \Lambda_\mu^{-1} g$. For such f and g we get

$$\langle f, Vg\rangle = (\Lambda_\mu)^{-2} \int_\Gamma \partial_N f \cdot \overline{\big(((ND)_K - (ND)_\Omega)(\partial_N g)\big)}\, \mathrm{d}S$$

or, with the introduced operators above

$$\langle f, Vg\rangle = (\Lambda_\mu)^{-2} \int_\Gamma f \cdot \overline{\big((DN)_K^*((ND)_K - (ND)_\Omega)((DN)_K g)\big)}\, \mathrm{d}S. \tag{4.4}$$

Moreover, $(DN)_K$ is an elliptic pseudodifferential operator of order 1. This follows from the identity $A(DN)_K = B - \frac{1}{2}I$ and the fact that A is an elliptic Pseudodifferential operator of order -1. It follows from (4.4) that T_μ is an elliptic pseudodifferential operator or order 1.

Next, we prove the inequality (3.3). Because of the projections, it is enough to show it for functions f in $\mathfrak{L}_{\Lambda_\mu}$.

The lower bound: We prove that there exists a subspace $\tilde{\mathcal{S}} \subset \mathfrak{L}_{\Lambda_\mu}$ of finite codimension such that the lower bound in (3.3) is valid for all $f \in \tilde{\mathcal{S}}$. Since $f \in \mathfrak{L}_{\Lambda_\mu}$ we have $L_\mu f := (L - \Lambda_\mu) f = 0$ so f belongs to the kernel of the second order elliptic operator L_μ. Let $\varphi = f|_\Gamma$. We study the problem

$$\begin{cases} L_\mu f = 0 & \text{in } K^\circ \\ f = \varphi & \text{on } \Gamma. \end{cases} \tag{4.5}$$

Let $E(x,y)$ be the Schwarz-kernel for L_μ. It is smooth away from the diagonal $x = y$. One can repeat the theory with the single and double layer potentials for L_μ and write the solution f in the case it the solution exists.

Let B_μ be the double layer operator evaluated at the boundary,

$$B_\mu \alpha(x) = \int_\Gamma \partial_{N_y} E(x,y)\alpha(y)\, \mathrm{d}S(y), \quad x \in \Gamma.$$

The operator B_μ is compact, since the kernel $\partial_{N_y} E(x,y)$ has a weak singularity at the diagonal $x = y$. Thus there exists a subspace $\mathcal{S}_1 \subset L_2(\Gamma)$ of finite codimension such that the operator $\frac{1}{2}I + B_\mu$ is invertible on $\mathcal{S}_1$. Hence, there exists a subspace $\tilde{\mathcal{S}} \subset \mathfrak{L}_{\Lambda_\mu}$ of finite codimension where we have the representation formula

$$f(x) = \int_\Gamma \frac{\partial E(x,y)}{\partial \nu_y} \Big(\Big(\frac{1}{2} I + B_\mu \Big)^{-1} \varphi \Big)(y)\, \mathrm{d}S(y), \quad x \in K^\circ \tag{4.6}$$

for all $f \in \tilde{\mathcal{S}}$. The inequality $\|f\|_{L_2(K_0)} \le C\|f\|_{L_2(\Gamma)}$ follows easily from 4.6 for all such functions f.

Since we also have $\|f\|_{L_2(\Gamma)} \le C\|f\|_{H^1(\Gamma)}$ the lower bound in (3.3) follows via the lower bound in (4.2).

The upper bound: By the upper bound in (4.2) it is enough to show the following inequalities

$$\|f\|_{L_2(\Gamma)}\|f\|_{H^1(\Gamma)} \le C\|f\|_{H^{1/2}(K)}\|f\|_{H^{3/2}(K)} \le C\|f\|^2_{H^2(K)} \le C\|f\|^2_{L_2(K_1)}.$$

However, the first inequality is just the Trace theorem, the second is the Sobolev-Rellich embedding theorem. We note that $L_\mu f = 0$, so the third inequality is a standard estimate for elliptic operators. □

5 Spectrum of Toeplitz operators in a Reinhart domain

In the case when K is a Reinhart domain one can strengthen part (b) of Lemma 3.6. Assume that K°, the interior of K, is a Reinhart domain. This means that $0 \in K^\circ$ and if $z \in K^\circ$, then the set

$$\{(w^1, \dots, w^d),\ w^j = tz^j,\ t \in \mathbb{C}, |t| < 1\}$$

is a subset of K°. If the set

$$\log|K| = \left\{(y^1, \dots, y^d),\ y^j = \log|z^j|, z \in K^\circ\right\}$$

is convex in the usual sense, then K° is said to be logarithmically convex, and K° is a domain of holomorphy. Denote by $V_K\colon \mathbb{R}^d \to \mathbb{R}$ the function defined by

$$V_K(x) = \sup_{y \in \log|K|} \langle x, y\rangle.$$

We denote by $J\colon \mathfrak{F}^2_B \to \widetilde{\mathcal{H}} := L_2\big(K, e^{-\frac{B}{2}|z|^2}\,\mathrm{d}m(z)\big)$ the embedding operator. The s-values $s_{\hat\kappa}$, $\hat\kappa \in \mathbb{N}^d$, of J coincides with the numbers

$$\left\{\|z^{\hat\kappa}\|^2_{\widetilde{\mathcal{H}}} \Big/ \|z^{\hat\kappa}\|^2_{\mathfrak{F}^2_B}\right\}_{\hat\kappa \geq 0} \tag{5.1}$$

Unlike the case $d = 1$, see [FP06], it is natural to numerate the eigenvalues by the d-tuples $\hat\kappa = (\kappa_1, \dots, \kappa_d)$, just as for the eigenvalues of the Laplace operator in the unit cube $[0,1]^d$, where the eigenvalues are given by $(2\pi)^{-d}|\hat\kappa|_2^2 = (2\pi)^{-d}\left(\kappa_1^2 + \cdots + \kappa_d^2\right)$.

Lemma 5.1 *Let $d > 1$ and $\omega = \hat\kappa/|\hat\kappa|$. Then*

$$(\hat\kappa! s_{\hat\kappa})^{1/|\hat\kappa|} \sim \frac{B}{2}\exp\big(2V_K(\omega)\big)\big(1 + o(1)\big), \quad \text{as } |\hat\kappa| \to \infty. \tag{5.2}$$

Proof The denominator in (5.1) is easily calculated to be

$$\|z^{\hat\kappa}\|^2_{\mathfrak{F}^2_B} = \left(\frac{2\pi}{B}\right)^d \left(\frac{2}{B}\right)^{|\hat\kappa|} \hat\kappa!.$$

For the numerator, we do estimations from above and below, as in [Par94]. First, note that

$$I_{\hat\kappa} = \|z^{\hat\kappa}\|^2_{\widetilde{\mathcal{H}}} = \int_{\log|K|} \exp(2\langle\hat\kappa, x\rangle)\,\mathrm{d}\widetilde{m}(x),$$

where $\mathrm{d}\widetilde{m}(x)$ is the transformed measure. It is clear that

$$I_{\hat{\kappa}} \leq \exp(2|\hat{\kappa}|V_K(\omega))m(K).$$

For the inequality in the other direction, fix $\delta > 0$. The hyperplane

$$\langle \hat{\kappa}, x\rangle = (1-\delta)V_K(\hat{\kappa})$$

cuts $\log|K|$ in two components. Let P_δ be the component for which the inequality $\langle \hat{\kappa}, x\rangle \geq (1-\delta)V_K(\hat{\kappa})$ holds. Then we have

$$I_{\hat{\kappa}} \geq \int_{P_\delta} \exp\left(2|\hat{\kappa}|(1-\delta)V_K(\omega)\right) \mathrm{d}\widetilde{m}(x) \geq C_\delta \exp\left(2|\hat{\kappa}|(1-\delta)V_K(\omega)\right),$$

where $C_\delta = \int_{P_\delta} \mathrm{d}\widetilde{m}(x) > 0$. It follows that

$$(\hat{\kappa}!s_{\hat{\kappa}})^{1/|\hat{\kappa}|} \leq \left(m(K)\left(\frac{B}{2\pi}\right)^d\right)^{1/|\hat{\kappa}|}\frac{B}{2}\exp\left(2V_K(\omega)\right)$$

and

$$(\hat{\kappa}!s_{\hat{\kappa}})^{1/|\hat{\kappa}|} \geq \left(C_\delta\left(\frac{B}{2\pi}\right)^d\right)^{1/|\hat{\kappa}|}\frac{B}{2}\exp\left(2(1-\delta)V_K(\omega)\right),$$

from which (5.2) follows. □

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