9.11 Problems

1. A coin is thrown independently 10 times to test the hypothesis that the probability of heads is $\frac{1}{2}$ versus the alternative that the probability is not $\frac{1}{2}$. The test rejects if either 0 or 10 heads are observed.
   a. What is the significance level of the test?
   b. If in fact the probability of heads is .1, what is the power of the test?

2. Which of the following hypotheses are simple, and which are composite?
   a. X follows a uniform distribution on [0, 1].
   b. A die is unbiased.
   c. X follows a normal distribution with mean 0 and variance $\sigma^2 > 10$.
   d. X follows a normal distribution with mean $\mu = 0$.

3. Suppose that $X \sim \text{bin}(100, p)$. Consider the test that rejects H_0: $p = .5$ in favor of H_A: $p \neq .5$ for $|X - 50| > 10$. Use the normal approximation to the binomial distribution to answer the following:
   a. What is α?
   b. Graph the power as a function of p.

4. Let X have one of the following distributions:

X	H_0	H_A
x_1	.2	.1
x_2	.3	.4
x_3	.3	.1
x_4	.2	.4

   a. Compare the likelihood ratio, Λ, for each possible value X and order the x_i according to Λ.
   b. What is the likelihood ratio test of H_0 versus H_A at level $\alpha = .2$? What is the test at level $\alpha = .5$?
   c. If the prior probabilities are $P(H_0) = P(H_A)$, which outcomes favor H_0?
   d. What prior probabilities correspond to the decision rules with $\alpha = .2$ and $\alpha = .5$?

5. True or false, and state why:
   a. The significance level of a statistical test is equal to the probability that the null hypothesis is true.
   b. If the significance level of a test is decreased, the power would be expected to increase.
   c. If a test is rejected at the significance level α, the probability that the null hypothesis is true equals α.
   d. The probability that the null hypothesis is falsely rejected is equal to the power of the test.
   e. A type I error occurs when the test statistic falls in the rejection region of the test.

f. A type II error is more serious than a type I error.
g. The power of a test is determined by the null distribution of the test statistic.
h. The likelihood ratio is a random variable.

6. Consider the coin tossing example of Section 9.1. Suppose that instead of tossing the coin 10 times, the coin was tossed until a head came up and the total number of tosses, X, was recorded.

a. If the prior probabilities are equal, which outcomes favor H_0 and which favor H_1?
b. Suppose $P(H_0)/P(H_1) = 10$. What outcomes favor H_0?
c. What is the significance level of a test that rejects H_0 if $X \geq 8$?
d. What is the power of this test?

7. Let $X_1, \ldots, X_n$ be a sample from a Poisson distribution. Find the likelihood ratio for testing $H_0\colon \lambda = \lambda_0$ versus $H_A\colon \lambda = \lambda_1$, where $\lambda_1 > \lambda_0$. Use the fact that the sum of independent Poisson random variables follows a Poisson distribution to explain how to determine a rejection region for a test at level α.

8. Show that the test of Problem 7 is uniformly most powerful for testing $H_0\colon \lambda = \lambda_0$ versus $H_A\colon \lambda > \lambda_0$.

9. Let $X_1, \ldots, X_{25}$ be a sample from a normal distribution having a variance of 100. Find the rejection region for a test at level $\alpha = .10$ of $H_0\colon \mu = 0$ versus $H_A\colon \mu = 1.5$. What is the power of the test? Repeat for $\alpha = .01$.

10. Suppose that $X_1, \ldots, X_n$ form a random sample from a density function, $f(x|\theta)$, for which T is a sufficient statistic for θ. Show that the likelihood ratio test of $H_0\colon \theta = \theta_0$ versus $H_A\colon \theta = \theta_1$ is a function of T. Explain how, if the distribution of T is known under H_0, the rejection region of the test may be chosen so that the test has the level α.

11. Suppose that $X_1, \ldots, X_{25}$ form a random sample from a normal distribution having a variance of 100. Graph the power of the likelihood ratio test of $H_0\colon \mu = 0$ versus $H_A\colon \mu \neq 0$ as a function of μ, at significance levels .10 and .05. Do the same for a sample size of 100. Compare the graphs and explain what you see.

12. Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with the density function $f(x|\theta) = \theta \exp[-\theta x]$. Derive a likelihood ratio test of $H_0\colon \theta = \theta_0$ versus $H_A\colon \theta \neq \theta_0$, and show that the rejection region is of the form $\{\overline{X} \exp[-\theta_0 \overline{X}] \leq c\}$.

13. Suppose, to be specific, that in Problem 12, $\theta_0 = 1$, $n = 10$, and that $\alpha = .05$. In order to use the test, we must find the appropriate value of c.

a. Show that the rejection region is of the form $\{\overline{X} \leq x_0\} \cup \{\overline{X} \geq x_1\}$, where x_0 and x_1 are determined by c.
b. Explain why c should be chosen so that $P(\overline{X} \exp(-\overline{X}) \leq c) = .05$ when $\theta_0 = 1$.
c. Explain why $\sum_{i=1}^{10} X_i$ and hence $\overline{X}$ follow gamma distributions when $\theta_0 = 1$. How could this knowledge be used to choose c?

d. Suppose that you hadn't thought of the preceding fact. Explain how you could determine a good approximation to c by generating random numbers on a computer (simulation).

14. Suppose that under H_0, a measurement X is $N(0, \sigma^2)$, and that under H_1, X is $N(1, \sigma^2)$ and that the prior probability $P(H_0) = 2 \times P(H_1)$. As in Section 9.1, the hypothesis H_0 will be chosen if $P(H_0|x) > P(H_1|x)$. For $\sigma^2 = 0.1, 0.5, 1.0, 5.0$:

a. For what values of X will H_0 be chosen?

b. In the long run, what proportion of the time will H_0 be chosen if H_0 is true $\frac{2}{3}$ of the time?

15. Suppose that under H_0, a measurement X is $N(0, \sigma^2)$, and that under H_1, X is $N(1, \sigma^2)$ and that the prior probability $P(H_0) = P(H_1)$. For $\sigma = 1$ and $x \in [0, 3]$, plot and compare (1) the p-value for the test of H_0 and (2) $P(H_0|x)$. Can the p-value be interpreted as the probability that H_0 is true? Choose another value of σ and repeat.

16. In the previous problem, with $\sigma = 1$, what is the probability that the p-value is less than 0.05 if H_0 is true? What is the probability if H_1 is true?

17. Let $X \sim N(0, \sigma^2)$, and consider testing $H_0 : \sigma_1 = \sigma_0$ versus $H_A : \sigma = \sigma_1$, where $\sigma_1 > \sigma_0$. The values σ_0 and σ_1 are fixed.

a. What is the likelihood ratio as a function of x? What values favor H_0? What is the rejection region of a level α test?

b. For a sample, $X_1, X_2, \ldots, X_n$ distributed as above, repeat the previous question.

c. Is the test in the previous question uniformly most powerful for testing $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma > \sigma_0$?

18. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables from a double exponential distribution with density $f(x) = \frac{1}{2}\lambda \exp(-\lambda|x|)$. Derive a likelihood ratio test of the hypothesis $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda = \lambda_1$, where λ_0 and $\lambda_1 > \lambda_0$ are specified numbers. Is the test uniformly most powerful against the alternative $H_1 : \lambda > \lambda_0$?

19. Under H_0, a random variable has the cumulative distribution function $F_0(x) = x^2$, $0 \le x \le 1$; and under H_1, it has the cumulative distribution function $F_1(x) = x^3$, $0 \le x \le 1$.

a. If the two hypotheses have equal prior probability, for what values of x is the posterior probability of H_0 greater than that of H_1?

b. What is the form of the likelihood ratio test of H_0 versus H_1?

c. What is the rejection region of a level α test?

d. What is the power of the test?

20. Consider two probability density functions on [0, 1]: $f_0(x) = 1$, and $f_1(x) = 2x$. Among all tests of the null hypothesis $H_0 : X \sim f_0(x)$ versus the alternative $X \sim f_1(x)$, with significance level $\alpha = 0.10$, how large can the power possibly be?

21. Suppose that a single observation X is taken from a uniform density on $[0, \theta]$, and consider testing $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.

a. Find a test that has significance level $\alpha = 0$. What is its power?

b. For $0 < \alpha < 1$, consider the test that rejects when $X \in [0, \alpha]$. What is its significance level and power?

c. What is the significance level and power of the test that rejects when $X \in [1 - \alpha, 1]$?

d. Find another test that has the same significance level and power as the previous one.

e. Does the likelihood ratio test determine a unique rejection region?

f. What happens if the null and alternative hypotheses are interchanged—$H_0: \theta = 2$ versus $H_1: \theta = 1$?

22. In Example A of Section 8.5.3 a confidence interval for the variance of a normal distribution was derived. Use Theorem B of Section 9.3 to derive an acceptance region for testing the hypothesis H_0: $\sigma^2 = \sigma_0^2$ at the significance level α based on a sample $X_1, X_2, \dots, X_n$. Precisely describe the rejection region if $\sigma_0 = 1, n = 15, \alpha = .05$.

23. Suppose that a 99% confidence interval for the mean μ of a normal distribution is found to be $(-2.0, 3.0)$. Would a test of H_0: $\mu = -3$ versus H_A: $\mu \neq -3$ be rejected at the .01 significance level?

24. Let X be a binomial random variable with n trials and probability p of success.

a. What is the generalized likelihood ratio for testing H_0: $p = .5$ versus H_A: $p \neq .5$?

b. Show that the test rejects for large values of $|X - n/2|$.

c. Using the null distribution of X, show how the significance level corresponding to a rejection region $|X - n/2| > k$ can be determined.

d. If $n = 10$ and $k = 2$, what is the significance level of the test?

e. Use the normal approximation to the binomial distribution to find the significance level if $n = 100$ and $k = 10$.

This analysis is the basis of the **sign test,** a typical application of which would be something like this: An experimental drug is to be evaluated on laboratory rats. In n pairs of litter mates, one animal is given the drug and the other is given a placebo. A physiological measure of benefit is made after some time has passed. Let X be the number of pairs for which the animal receiving the drug benefited more than its litter mate. A simple model for the distribution of X if there is no drug effect is binomial with $p = .5$. This is then the null hypothesis that must be made untenable by the data before one could conclude that the drug had an effect.

25. Calculate the likelihood ratio for Example B of Section 9.5 and compare the results of a test based on the likelihood ratio to those of one based on Pearson's chi-square statistic.

26. True or false:

a. The generalized likelihood ratio statistic Λ is always less than or equal to 1.

b. If the p-value is .03, the corresponding test will reject at the significance level .02.

c. If a test rejects at significance level .06, then the p-value is less than or equal to .06.

d. The p-value of a test is the probability that the null hypothesis is correct.

e. In testing a simple versus simple hypothesis via the likelihood ratio, the p-value equals the likelihood ratio.

f. If a chi-square test statistic with 4 degrees of freedom has a value of 8.5, the p-value is less than .05.

27. What values of a chi-square test statistic with 7 degrees of freedom yield a p-value less than or equal to .10?

28. Suppose that a test statistic T has a standard normal null distribution.

a. If the test rejects for large values of $|T|$, what is the p-value corresponding to $T = 1.50$?

b. Answer the same question if the test rejects for large T.

29. Suppose that a level α test based on a test statistic T rejects if $T > t_0$. Suppose that g is a monotone-increasing function and let $S = g(T)$. Is the test that rejects if $S > g(t_0)$ a level α test?

30. Suppose that the null hypothesis is true, that the distribution of the test statistic, T say, is continuous with cdf F and that the test rejects for large values of T. Let V denote the p-value of the test.

a. Show that $V = 1 - F(T)$.

b. Conclude that the null distribution of V is uniform. (*Hint:* See Proposition C of Section 2.3.)

c. If the null hypothesis is true, what is the probability that the p-value is greater than .1?

d. Show that the test that rejects if $V < \alpha$ has significance level α.

31. What values of the generalized likelihood ratio Λ are necessary to reject the null hypothesis at the significance level $\alpha = .1$ if the degrees of freedom are 1, 5, 10, and 20?

32. The intensity of light reflected by an object is measured. Suppose there are two types of possible objects, A and B. If the object is of type A, the measurement is normally distributed with mean 100 and standard deviation 25; if it is of type B, the measurement is normally distributed with mean 125 and standard deviation 25. A single measurement is taken with the value $X = 120$.

a. What is the likelihood ratio?

b. If the prior probabilities of A and B are equal ($\frac{1}{2}$ each), what is the posterior probability that the item is of type B?

c. Suppose that a decision rule has been formulated that declares the object to be of type B if $X > 125$. What is the significance level associated with this rule?

d. What is the power of this test?

e. What is the p-value when $X = 120$?

33. It has been suggested that dying people may be able to postpone their death until after an important occasion, such as a wedding or birthday. Phillips and King

47. **a.** $\hat{\theta} = \overline{X}/(\overline{X} - x_0)$
b. $\tilde{\theta} = n/(\Sigma \log X_i - n \log x_0)$
c. $\mathrm{Var}(\tilde{\theta}) \approx \theta^2/n$

49. **a.** Let $\hat{p}$ be the proportion of the n events that go forward. Then $\hat{\alpha} = 4\hat{p} - 2$.
b. $\mathrm{Var}(\hat{\alpha}) = (2 - \alpha)(2 + \alpha)/n$

53. **a.** $\hat{\theta} = 2\overline{X};\ E(\hat{\theta}) = \theta;\ \mathrm{Var}(\hat{\theta}) = \theta^2/3n$
b. $\tilde{\theta} = \max(X_1, X_2, \ldots, X_n)$
c. $E(\tilde{\theta}) = n\theta/(n+1);$ bias $= -\theta/(n+1);$ $\mathrm{Var}(\hat{\theta}) = n\theta^2/(n+2)(n+1)^2;$ $\mathrm{MSE} = 2\theta^2/(n+1)(n+2)$
d. $\theta^* = (n+1)\tilde{\theta}/n$

55. **a.** Let n_1, n_2, n_3, n_4 denote the counts. The mle of θ is the positive root of the equation

$$(n_1 + n_2 + n_3 + n_4)\theta^2 - (n_1 - 2n_2 - 2n_3 - n_4)\theta - 2n_4 = 0$$

The asymptotic variance is $\mathrm{Var}(\hat{\theta}) = 2(2+\theta)(1-\theta)\theta/(n_1 + n_2 + n_3 + n_4)(1+\theta)$. For these data, $\hat{\theta} = .0357$ and $s_{\hat{\theta}} = .0057$.
b. An approximate 95% confidence interval is $.0357 \pm .0112$.

57. **a.** s^2 is unbiased. **b.** $\hat{\sigma}^2$ has smaller MSE. **c.** $\rho = 1/(n+1)$

59. **b.** $\hat{\alpha} = (n_1 + n_2 - n_3)/(n_1 + n_2 + n_3)$ if this quantity is positive and 0 otherwise.

63. In case (1) the posterior is $\beta(4, 98)$ and the posterior mean is 0.039. In case (2) the posterior is $\beta(3.5, 102)$ and the posterior mean is 0.033. The posterior for case (2) rises more steeply and falls off more rapidly than that of case (1).

65. $\mu_0 = 16.25, \xi_0 = 80$

71. $\prod_{i=1}^{n} (1 + X_i)$

73. $\sum_{i=1}^{n} X_i^2$

Chapter 9

1. **a.** $\alpha = .002$ **b.** power $= .349$

3. **a.** $\alpha = .046$

5. F, F, F, F, F, F, F, T

7. Reject when $\sum X_i > c$. Since under H_0, $\sum X_i$ follows a Poisson distribution with parameter $n\lambda$, c can be chosen so that $P(\sum X_i > c | H_0) = \alpha$.

9. For $\alpha = .10$, the test rejects for $\overline{X} > 2.56$, and the power is .2981. For $\alpha = .01$, the test rejects for $\overline{X} > 4.66$, and the power is .0571.

17. **a.** $LR = \frac{\sigma_1}{\sigma_0} \exp\left[\frac{1}{2}x^2\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)\right]$. A level α test rejects for $X^2 > \sigma_0^2 \chi_1^2(\alpha)$.
b. Reject for $\sum_{i=1}^{n} X_i^2 > \sigma_0^2 \chi_n^2(\alpha)$ **c.** Yes

19. **a.** $X < 2/3$ **b.** Reject for large values of X
c. Reject for $X > \sqrt{1 - \alpha}$ **d.** $1 - (1 - \alpha)^3/2$

21. **a.** Reject for $X > 1$; power $= 1/2$
b. Significance level $= \alpha$, power $= 1 - \alpha/2$
c. Significance level $= \alpha$, power $= 1 - \alpha/2$
d. Reject when $(1-\alpha)/2 \le X \le (1+\alpha)/2$
e. For $\alpha > 0$, the rejection region is not uniquely determined.
f. The rejection region is not uniquely determined.

23. yes **25.** $-2\log\Lambda = 54.6$. Strongly rejects **27.** ≥ 12.02

29. yes **31.** 2.6×10^{-1}, 9.8×10^{-3}, 3×10^{-4}, 7×10^{-7}

33. $-2\log\Lambda$ and X^2 are both approximately 2.93. $.05 < p < .10$; not significant for Chinese and Japanese; both $\approx .3$.

35. $X^2 = .0067$ with 1 df and $p \approx .90$. The model fits well.

37. $X^2 = 79$ with 11 df and $p \approx 0$. The accidents are not uniformly distributed, apparently varying seasonally with the greatest number in November–January and the fewest in March–June. There is also an increased incidence in the summer months, July–August.

39. $\chi^2 = 85.5$ with 9 df, and thus provides overwhelming evidence against the null hypothesis of constant rate.

41. Let $\hat{p}_i = X_i/n_i$ and $\hat{p} = \sum X_i / \sum n_i$. Then

$$\Lambda = \frac{\hat{p}^{\Sigma n_i \hat{p}_i}(1-\hat{p})^{\Sigma n_i(1-\hat{p}_i)}}{\prod \hat{p}_i^{n_i\hat{p}_i}(1-\hat{p}_i)^{n_i(1-\hat{p}_i)}}$$

and

$$-2\log\Lambda \approx \sum \frac{(X_i - n_i\hat{p})^2}{n_i\hat{p}(1-\hat{p})}$$

is approximately distributed as χ^2_{m-1} under H_0.

43. **a.** 9207 heads out of 17950 tosses is not consistent with the null hypothesis of 17950 independent Bernoulli trials with probability .5 of heads. ($X^2 = 11.99$ with 1 df).
b. The data are not consistent with the model ($X^2 = 21.57$ with 5 df, $p \approx .001$).
c. A chi-square test gives $X^2 = 8.74$ with 4 df and $p \approx .07$. Again, the model looks doubtful.

45. The binomial model does not fit the data ($X^2 = 110.5$ with 11 df). Relative to the binomial model, there are too many families with very small and very large numbers of boys. The model might fail because the probability of a male child differs from family to family.

51. The horizontal bands are due to identical data values.

57. The tails decrease less rapidly than do those of a normal probability distribution, causing the normal probability plot to deviate from a straight line at the ends by curving below the line on the left and above the line on the right.

59. The rootogram shows no systematic deviation.