* Consider two experiments with sample spaces $S_1=\{\alpha_1,\alpha_2,\dots,\alpha_n\}$ and $S_2=\{\beta_1,\beta_2,\dots,\beta_m\}$ respectively, then
the combined experiment consisting of these two experiments has sample space $S=S_1 \times S_2$ (the cartesian product of S_1 and S_2)

Example. Experiment 1 throwing a die $\Rightarrow S_1=\{1,2,3,4,5,6\}$

Experiment 2 tossing a coin $\Rightarrow S_2=\{H,T\}$

Combined experiment $S=\{(1,H),(2,H),\dots,(6,H),(1,T),(2,T),\dots,(6,T)\}$

If A_1 is an event in experiment 1 and A_2 is an event in experiment 2 then $A=A_1 \times A_2$ is an event in the combined experiment

Example. If $A_1=\{1,2,3\}$ and $A_2=\{H\} \Rightarrow$

$A=A_1 \times A_2=\{(1,H),(3,H),(5,H)\}$

In the following we will consider combined experiments that are repetitions of the same experiment.

Combined experiments and counting methods *

The multiplication principle. Suppose we have p experiments and the first has n_1 possible outcomes,
" second " n_2 " " , ..., and
" pth " n_p " "
Then there are $\prod_{i=1}^{p} n_i = n_1 \cdot n_2 \cdots n_p$ possible outcomes for the p experiments.

Definition. Permutation. A permutation is an ordered arrangement of objects. Suppose that from the set $C = \{c_1, c_2, \ldots, c_n\}$ we choose r elements and list them in order. In how many ways can we do this?

Number of permutations without replacement of a set of size n and a sample of size r is: $_nP_r = n(n-1)(n-2)\ldots(n-r+1)$ by the multiplication principle

$$= \frac{n!}{(n-r)!}$$

The number of orderings of n elements O_n equals $_nP_n$, thus

$$O_n = n(n-1)(n-2)\ldots(n-n+1) = n!$$

Example. Suppose a room with r people. what is the probability that at least two of them have a common birthday?
Solution. Event $A = \{$at least 2 people have a common birthday$\}$
Assume every day of the year is equally likely to be the birthday of someone and disregard leap years. In this case is easier to compute $\bar{A} = \{$no two people have a common birthday$\}$.
There are 365^r possible outcomes and $\bar{A}$ can happen in $_{365}P_r$ ways:
1st person can have birthday in any of the 365 days

2^{nd} person can have birthday in 364 days

$\vdots$

rth person can have birthday in $364 - r + 1$ days

$$\Rightarrow P_r(\bar{A}) = \frac{\frac{365!}{(365-r)!}}{365^r} \quad \Rightarrow \quad P_r(A) = 1 - \frac{\frac{365!}{(365-r)!}}{365^r}$$

r	$P(A)$
4	0.016
16	0.284
23	0.507
56	0.988

Definition. Combinations. A combination is an arrangement of objects regardless of the order in which they were obtained. In how many ways can we choose r elements from a set with n elements without replacement and disregarding order?

From the multiplication principle

\# of ordered samples = \# of unordered samples X \# ways to order the sample

$$\Rightarrow \quad {}_nP_r = {}_nC_r \cdot O_r$$

$$\Rightarrow \quad \frac{n!}{(n-r)!} = {}_nC_r \cdot r!$$

$$\Rightarrow \quad {}_nC_r = \frac{n!}{(n-r)!\,r!} \qquad \text{By convention } 0! = 1$$

This numbers, sometimes denoted as $\binom{n}{r}$ are called the binomial coefficients since they appear in the binomial expansion $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$. In particular $\sum_{r=0}^{n} \binom{n}{r} = 2^n$

Conditional probability

Definition. Let A and B be two events such that $P_r(B) > 0$ then the conditional probability of A given B is

$$P_r(A|B) = \frac{P_r(A \cap B)}{P_r(B)}$$

As the conditional probability has been defined this way, it's necessary to verify it's indeed a probability, that is, that it satisfies the axioms

1) $P_r(A|B) \geq 0$ is obvious from the definition.

2) $P_r(S|B) = 1$

Proof. $P_r(S|B) = \dfrac{P_r(S \cap B)}{P_r(B)} = \dfrac{P_r(B)}{P_r(B)} = 1$

3) $P_r(A \cup C|B) = P_r(A|B) + P_r(C|B)$ exercise

Example. Consider the experiment of throwing a die, then $S = \{1,2,3,4,5,6\}$. Let $A = \{1,2\} \Rightarrow P_r(A) = P_r(\{1\} \text{ or } \{2\}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

Define now $B = \{\text{even number}\} = \{2,4,6\} \Rightarrow P_r(B) = \frac{1}{2}$

$$P_r(A|B) = \frac{P_r(A \cap B)}{P_r(B)} = \frac{P_r(\{2\})}{P_r(B)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

on the other hand

$$P_r(B|A) = \frac{P_r(B \cap A)}{P_r(A)} = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2}$$

*

Proposition. Law of total probability. Let $A_1, A_2, \ldots, A_n$ be mutually exclusive events such that $\bigcup_{i=1}^{n} A_i = S$. Then, for an arbitrary event B

$$P_r(B) = \sum_{i=1}^{n} P_r(B|A_i) P_r(A_i)$$

Proof. For $i \neq j$ $B \cap A_i$ and $B \cap A_j$ are mutually exclusive. Also

$$B = B \cap S = B \cap \left(\bigcup_{i=1}^{n} A_i \right) = (B \cap A_1) \cup (B \cap A_2) \cup \ldots \cup (B \cap A_n) \Rightarrow$$

$$P_r(B) = \sum_{i=1}^{n} P_r(B \cap A_i) = \sum_{i=1}^{n} P_r(B | A_i) P_r(A_i)$$

* Definition. Multiplication law. Let A and B be two events such that $P_r(B) > 0$ then

$$P_r(A \cap B) = P_r(A | B) P_r(B)$$
$$= P_r(B | A) P_r(A) \text{ if } P_r(A) > 0$$