Example. Consider a resistor carrousel with six bins

| Ohms \ Bins | 1 | 2 | 3 | 4 | 5 | 6 | Totals |
|---|---|---|---|---|---|---|---|
| 10 | 500 | 0 | 200 | 800 | 1200 | 1000 | 3700 |
| 100 | 300 | 400 | 600 | 200 | 800 | 0 | 2300 |
| 1000 | 200 | 600 | 200 | 600 | 0 | 1000 | 2600 |
| Totals | 1000 | 1000 | 1000 | 1600 | 2000 | 2000 | 8600 |

What's the probability that a resistor selected at random will be $10\,\Omega$?

Solution. Let $A_i = \{\text{the resistor is selected from bin } i\}$ $i = 1,2,3,4,5,6$

$\Rightarrow Pr(A_i) = \frac{1}{6}$. Let $B = \{\text{the resistor is } 10\,\Omega\}$

$$Pr(B|A_1) = \frac{500}{1000} = \frac{1}{2} \quad Pr(B|A_2) = \frac{0}{1000} = 0 \quad \dots \quad Pr(B|A_6) = \frac{1000}{2000} = \frac{1}{2}$$

$$\Rightarrow Pr(B) = \sum_{i=1}^{6} Pr(B|A_i)\, P(A_i) = \frac{1}{2} \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + \dots + \frac{1}{2} \cdot \frac{1}{6} = 0.3833$$

Bayes' theorem. Let $A_i$, $i = 1, 2, \dots, 6$ and $B$ be events as before and $Pr(B) > 0$ then

a priori probability

$$Pr(A_i|B) = \frac{Pr(B|A_i)\, Pr(A_i)}{Pr(B)} = \frac{Pr(B|A_i)\, Pr(A_i)}{\sum_{j=1}^{n} P(B|A_j)\, P(A_j)}$$

a posteriori probability

Proof. By the multiplication law $Pr(A_i|B)\,Pr(B) = Pr(B|A_i)\,Pr(A_i)$

(if $Pr(B) > 0$ and $Pr(A_i) > 0$ $\forall i = 1, 2, \dots 6$)

then $Pr(A_i|B) = \dfrac{Pr(B|A_i)\, Pr(A_i)}{Pr(B)}$

Example. Continuing with the previous one, what's the probability that a 10Ω resistor comes from bin 3?

$$Pr(A_3|B) = \frac{P(B|A_3)P_r(A_3)}{P_r(B)} = \frac{\left(\frac{200}{1000}\right)\left(\frac{1}{6}\right)}{0.3833} = 0.0869$$

## Independence

Definition. Two events A and B are said to be independent is

$P_r(A \cap B) = P_r(A) P_r(B)$

Note. To stablish independence for more than two events all the independency condition of size $2, 3, \dots, n$ must be satisfied.

Example. Suppose you throw two dice. let $A = \{7\}$ and $B = \{11\}$, are this events independent?

$P_r(A \cap B) = P_r(\emptyset) = 0 \neq P_r(A) p_r(B)$

Mutually exclusive events can't be independent: if one occurs, the other one can't.

Example. $A = \{\text{odd number}\}$, $B = \{11\}$ as $B \subset A \Rightarrow A \cap B = B \Rightarrow$

$P_r(A \cap B) = P_r(B) = \frac{2}{36} = \frac{1}{18}$

On the other hand $P_r(A) = \frac{1}{2}$ and $P_r(B) = \frac{1}{18} \Rightarrow$

$P_r(A \cap B) = \frac{1}{18} \neq \frac{1}{2} \cdot \frac{1}{18} = P_r(A) \cdot P_r(B)$ $\Rightarrow$ not independent

Example. $A = \{1, 2, 3\}$, $B = \{3, 4\}$

$P_r(A \cap B) = P_r(\{3\}) = \frac{1}{6}$

$P_r(A) = \frac{1}{2}$ $P_r(B) = \frac{1}{3}$

$P_r(A) P_r(B) = \frac{1}{6} \Rightarrow$ independent

Recall combined experiments. If a combined experiment is composed of 2 independent experiments, $A_1$ and $A_2$ are events in experiments 1 and 2 respectively and $A = A_1 \times A_2$ then $Pr(A) = Pr(A_1 \times A_2) = Pr(A_1) Pr(A_2)$

# Random variables

Definition. A random variable is a realvalued function defined over the sample space of an experiment.

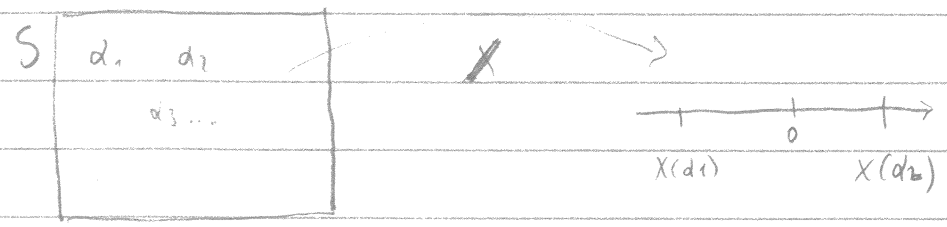
Suppose $S = \{\alpha_i \mid i \geq 0\}$, then, if the outcome of one trial of the experiment is $\alpha_j$, the random variable $X$ has the value $X(\alpha_j)$

From an engineering viewpoint, a random variable is simply a numerical description of the outcome of a random experiment. For engineering applications is usually not necessary to list the underlying sampling space, but only to be able to assign probabilities to the events associated with the random variables of interest.

Example. Consider tossing a coin 3 times, then
$S = \{HHH, HHT, HTT, HTH, TTT, TTH, THH, THT\}$
Some random variables are
a) Total number of heads: $X_1(HHH) = 3$, $X_1(THT) = 1$
b) Number of heads − number of tails: $X_2(HHT) = 1$, $X_2(TTT) = -3$

Definition. Discrete random variable. Is a random variable that can take a countable set of values.

Definition. Continuous random variable. Is a random variable that can take any value within a specified range.




$P_r(X = k) =$
$P_r(\underbrace{\{\alpha \in S : X(\alpha) = k\}}_{\text{event}})$

Example. $P_r(X_1 = 3) = P_r(\{HHH\}) = \frac{1}{8}$

$P_r(X_2 = 1) = P_r(\{HHT, HTH, THH\}) = \frac{3}{8}$

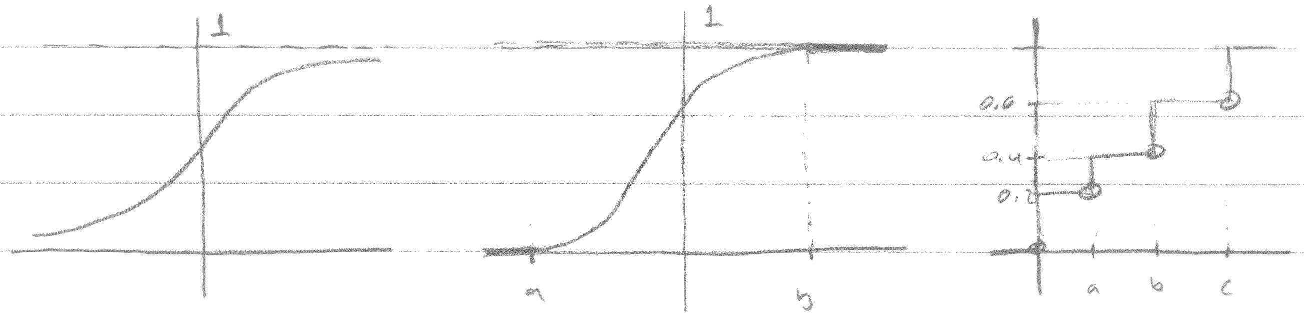
## Distribution functions

Definition Probability distribution function. It's defined as the probability of the event that the observed random variable $X$ is less than or equal the corresponding value $x$, that is $F(x) = P_r(X \leq x)$

Properties

1) $0 \leq F(x) \leq 1 \quad -\infty < x < \infty$
2) $\lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to \infty} F(x) = 1$
3) $F(x)$ is non-decreasing for increasing $x$
4) $P_r(x_1 < X \leq x_2) = F(x_2) - F(x_1)$

Examples




$X \in (-\infty, \infty) \qquad X \in [a, b] \qquad X \in \{0, a, b, c\}$

Note: $F(a) = 0.4$

5) $P_r(\underbrace{X > x}) = 1 - F(x)$

↑

complement of $X \leq x$ which has probability $F(x)$