Density functions

Although a complete description, the distribution function is not necessarily the most convenient form.

Definition. Probability density function For continuous random variables

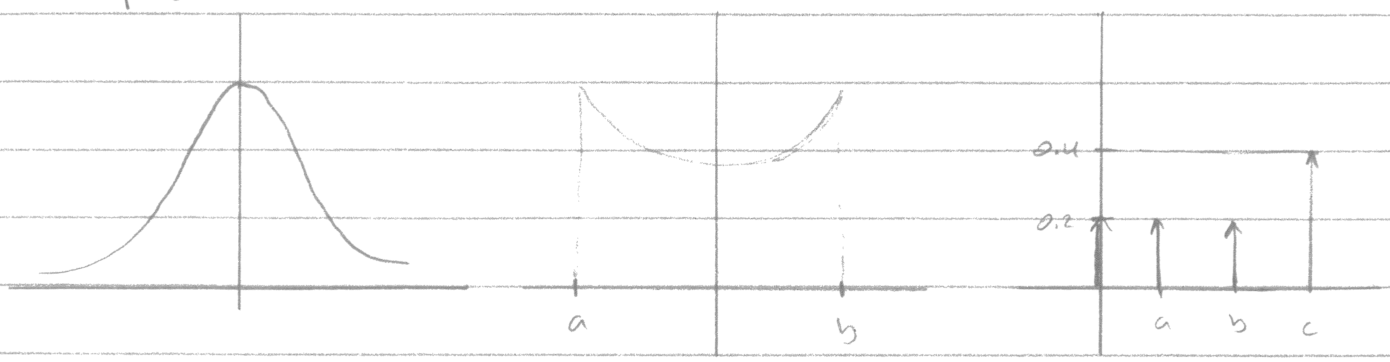
$$f(x) = \frac{d}{dx} F(x)$$

For discrete variables: $f(x) = P_r(X = x)$

Properties

	Continuous	Discrete	
Required	1) $f(x) \geq 0$	$f(x) \geq 0$	$-\infty < x < \infty$
Required	2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$	$\sum_{x_i \in S} f(x_i) = 1$	
	3) $F(x) = \int_{-\infty}^{x} f(u)\,du$	$F(x) = \sum_{u \leq x} f(u)$	
	4) $P_r(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f(x)\,dx$		

Example



Mean values and moments

Integration over an interval familiar concept because is used to find length of interval, the dc component or average power of the time function

Definition. Expected value. The expected value of a continuous random variable X is defined as

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

For a discrete random variable $E[X] = \sum_{x_i \in S} x_i f(x_i)$

For a function g of a random variable X, the expected value can be found in a similar way

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

For a discrete random variable $E[g(X)] = \sum_{x_i \in S} g(x_i) f(x_i)$

(Finding the mean value of a random voltage or current is equivalent to finding its dc component)

Example. A gambler bets \$1 at trial 0; if he loses he bets \$2 at trial 1; if he hasn't won by the kth trial, he bets \$$2^k$. This way, when he finally wins, he will be \$1 ahead:

$W_0 = 1$

$$W_k = 2^k - \sum_{i=0}^{k-1} 2^i = 2^k - \left[\frac{1-2^k}{1-2}\right] = 2^k + 1 - 2^k = \underline{1}$$

X: amount of money he bets on the last game (the one he wins)

The probability of k losses followed by one win (in a fair game) is $\frac{1}{2^{k+1}}$

$\Rightarrow f(2^k) = Pr(X = 2^k) = \frac{1}{2^{k+1}}$ (which is a density function and sums up to 1) and

$$E[X] = \sum_{x_i \in S} x_i f(x_i) = \sum_{k=0}^{\infty} 2^k \frac{1}{2^{k+1}} = \sum_{k=0}^{\infty} \frac{1}{2} = \underbrace{\frac{1}{2} + \frac{1}{2} + \dots}_{k=\infty \text{ times}} = \infty$$

Example. Consider a sawtooth voltage waveform that varied linearly from 20 to 40 V. The representation for this density function is:

$$f(x) = \begin{cases} 0 & -\infty < x \leq 20 \\ \frac{1}{20} & 20 < x \leq 40 \\ 0 & 40 < x < \infty \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{20}^{40} x \frac{1}{20}\, dx = \frac{1}{20} \cdot \frac{x^2}{2} \Big|_{20}^{40} = \frac{1}{40}(1600 - 400) = 30$$

Example The velocity V (magnitude) of a gas molecule is random an follows Maxwell's distribution: $f(x) = \frac{\sqrt{2/\pi}}{\sigma^3} x^2 \exp\left\{-\frac{1}{2}\frac{x^2}{\sigma^2}\right\}$ *

where the parameter σ depends on the temperature of the gas.

The kinetic energy Y of the molecule is given by $Y = \frac{1}{2} m V^2$ where m is the mass of the molecule, then the average kinetic energy is

$$E[Y] = \int_0^{\infty} \frac{1}{2} m x^2 f(x)\, dx = \int_0^{\infty} \frac{1}{2} m x^2 \frac{\sqrt{2/\pi}}{\sigma^3} x^2 \exp\left\{-\frac{1}{2}\frac{x^2}{\sigma^2}\right\} dx$$

$$= \frac{m}{2} \frac{\sqrt{2/\pi}}{\sigma^3} \int_0^{\infty} x^4 \exp\left\{-\frac{1}{2}\frac{x^2}{\sigma^2}\right\} dx = \left\{ u = \frac{x^2}{2\sigma^2},\ du = \frac{x\, dx}{\sigma^2} \right\}$$

* The assumption is that each component of the velocity $V = (V_x, V_y, V_z)$ is $N \sim (0, \sigma^2)$ where $\sigma^2 = \frac{kT}{m}$ where $k = 1.38 \cdot 10^{23} \frac{Ws}{K}$ (Boltzmann's constant), T is the temperature in Kelvin degrees and m is the mass of the molecule in kilograms. The magnitude of the velocity is therefore

$$V = \sqrt{V_x^2 + V_y^2 + V_z^2} \sim \text{Maxwell}(\sigma)$$

$$= \frac{m}{2} \frac{\sqrt{2/\pi}}{\sigma^3} \int_0^\infty (2\sigma^2 u)^{\frac{3}{2}} e^{-u} \sigma^2 du = \frac{m}{2} \frac{\sqrt{2/\pi}}{\sigma^3} \int_0^\infty (\sqrt{2})^3 \sigma^5 u^{\frac{3}{2}} e^{-u} du$$

$$= \frac{m}{2} \frac{(\sqrt{2})^4 \sigma^2}{\sqrt{\pi}} \int_0^\infty u^{3/2} e^{-u} du = \frac{2m\sigma^2}{\sqrt{\pi}} \int_0^\infty u^{\frac{5}{2}-1} e^{-u} du$$

$$= \frac{2m\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{5}{2}\right) \underset{\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)}{=} \frac{2m\sigma^2}{\sqrt{\pi}} \left[\frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \right] \underset{\Gamma(\frac{1}{2}) = \sqrt{\pi}}{=} \frac{3}{2} \frac{m\sigma^2}{\sqrt{\pi}} \sqrt{\pi} = \frac{3}{2} m\sigma^2 = \frac{3}{2} kT$$

Observation: In general $E[g(X)] \neq g(E[X])$

Theorem Let $X_1, X_2, \ldots, X_n$ be independent random variables with expectation $E[X_i]$ and let $Y = a + \sum_{i=1}^{n} b_i X_i$ for some a and b_i, then

$$E[Y] = a + \sum_{i=1}^{n} b_i E[X_i]$$

Proof later on, we need to define joint distributions

Definition Variance: For a random variable X with mean value $E[X] = \mu$ the variance σ^2 is defined as: $\sigma^2 = E[(X-\mu)^2]$

Proposition For X as above we have that $\sigma^2 = E[X^2] - E[X]^2$

Proof. $\sigma^2 = E[(X-\mu)^2] = E[X^2 - 2X\mu + \mu^2] = E[X^2] - 2\mu E[X] + \mu^2$
$= E[X^2] - 2\mu^2 + \mu^2 = E[X^2] - \mu^2 = E[X^2] - E[X]^2$

Definition Standard deviation For a random variable X, the standard deviation is defined as the square root of its variance