Example. Continuing with the previous example $f(x) = \frac{1}{20} \quad 20 < x \leq 40$

$$E[X^2] = \int_{20}^{40} x^2 \frac{1}{20} dx = \frac{1}{20} \frac{x^3}{3} \Big|_{20}^{40} = 933.3$$

$$\Rightarrow \sigma^2 = E[X^2] - E[X]^2 = 933.3 - 30^2 = 33.3$$

If this sawtooth voltage were measured with a dc voltmeter the reading would be 30V. If it were measured with a rms-reading ac voltmeter the reading would be $\sqrt{33.3}$ V

Observation. σ^2 and σ are spread measures, the main difference is that σ has the same scale (and measure units) as the original random variable

Theorem. Let X be a random variable and $Var(X)$ exist $\Rightarrow$ for $Y = a + bX \quad Var(Y) = b^2 Var(X)$

Proof Since $E[Y] = = a + bE[X] \Rightarrow$

$$Var(Y) = E[(Y - E[Y])] = E\{[a + bX - a - bE[X]]^2\}$$
$$= E\{b^2[X - E[X]]^2\} = b^2 E[(X - E[X])^2 = b^2 Var(X)$$

Definition. Higher order moments. Let X be a random variable, then the nth moment is defined as $E[X^n]$

Continuous r.v $\quad E[X^n] = \int_{-\infty}^{\infty} x^n f(x) dx$

Discrete r.v. $\quad E[X^n] = \sum_{i=\omega}^{\omega} x_i f(x_i)$

Some important discrete distributions

Bernoulli: An experiment that can result only in "success" or "failure" is performe The probability of success is p and the probability of failure $1-p$. Define the r.v. X: succes or failure $\Rightarrow X \sim Ber(p)$. $X \in \{0,1\}$

$$f(x) = Pr(X=x) = p^x(1-p)^{1-x} \quad \text{if} \quad x=0,1$$

$$E[X] = \sum_{x \in S} x f(x) = 0 f(0) + 1 f(1) = p$$

$$E[X^2] = \sum_{x \in S} x^2 f(x) = 0^2 f(0) + 1^2 f(1) = p$$

$$Var(X) = E[X^2] - E[X]^2 = p - p^2 = p(1-p)$$

Binomial n independent trials of a "success" or "failure" (Bernoulli) experiment are performed. The probability of success is p and failure $1-p$.
X: # of successes $\Rightarrow X \sim Bin(n,p)$
$X \in \{0,1,2,\dots n\}$

$$Pr(X=x) = f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$E[X] = np \qquad Var(X) = np(1-p)$$

Example. Error correction coding. Digital communication systems transmit sequences of 0 and 1. For practical reasons the sequences are separated in words of the same length. Any transmitted word is received correctly if and only if all the bits of the word are read correctly. Because of noise, interference, there is a probability that a word is detected incorrectly (if one of its bits was detected incorrectly).

Consider the following option. Double the length of each word by adding bits (known as check-digits) that are uniquely related to the actual message bits. A double-error-correcting code will produce the correct message word if no more than two bits are received in error in each word. Is this approach actually effective? This is not clear since the check digits can also be incorrectly read. How much more effective, if at all, is?

To illustrate assume words are 5-bit long and consider a system without error-correction coding where the probability of a bit being incorrectly read is 0.01. Define X: number of bits received in error, then $X \sim Bin(5, 0.01)$

$$Pr(\text{Correct word}) = Pr(\text{no bits received in error}) \underset{\uparrow\, Pr(X=0)}{=} \binom{5}{0} 0.01^0 (1-0.01)^5 = 0.951$$

Now consider a double-error-correction system, each word is then 10 bits long and a word will be correctly read if no bits are received in error, one bit is received in error or two bits are received in error. This time $X \sim Bin(10, 0.01)$

$$Pr(\text{Correct word}) \overset{Pr(X=0)+Pr(X=1)+Pr(X=2)}{=} \binom{10}{0} 0.01^0 (1-0.01)^{10} + \binom{10}{1} 0.01^1 (1-0.01)^9 + \binom{10}{2} 0.01^2 (1-0.01)^8$$
$$= 0.9999$$

Geometric. An infinite number of independent Bernoulli trials is considered. On each trial, success occurs again with probability p and failure with probability $1-p$.

X: # of trials up to, and including, the first success $\Rightarrow X \sim Geo(p)$

$X = \{1, 2, \ldots\}$

$$Pr(X=x) = f(x) = (1-p)^{x-1} p$$

$$E[X] = \sum_{x \in S} x f(x) = \sum_{x=1}^{\infty} x(1-p)^{x-1} p = p \sum_{x=1}^{\infty} x(1-p)^{x-1}$$

Since $x(1-p)^{x-1} = -\frac{d}{dp}(1-p)^x$

$$= p\left(-\frac{d}{dp}\sum_{x=1}^{\infty}(1-p)^x\right) = -p\frac{d}{dp}\left[\frac{1-p}{1-(1-p)}\right] = -p\left[\frac{(-1)(p)-(1-p)}{p^2}\right]$$

$$= \frac{1}{p}$$

$$Var(X) = \frac{1-p}{p^2}$$

In this case is possible to find a closed form for the distribution function $P_r(X=x) = F(x) = 1 - q^x$ where $q = 1-p$

<u>Poisson</u> Consider again the Binomial setting: n independent repetitions of Bernoulli trials with probability of success p. Let the number of trials to approach infinity and the probability of success aproach zero in each trial in such a way that $np = \lambda$ constant (distribution of rare events or Poisson law of small numbers).

X: # number of successes $\Rightarrow$ $X \sim Poisson(\lambda)$, $x \in \{0,1,2,\dots$

$$P(X=x) = f(x) = \frac{\lambda^k e^{-\lambda}}{k!} \qquad E[X] = \lambda \quad Var(X) = \lambda$$

X describes then the number of events that occur during a certain amount of time, when this events occur with a given average frequency (λ)

Example. The average number of cars that pass a certain traffic light each hour is 10. Let X: # of cars that pass the traffic light $\Rightarrow$ $X \sim Poisson(10)$

$$\Pr(X \geq 3) = 1 - \Pr(X < 3) = 1 - \Pr(X \leq 2) = 1 - \sum_{x=0}^{2} f(x) = 1 - \sum_{x=0}^{2} \frac{10^x e^{-10}}{x!}$$

$$= 0.9972$$

Example. An office receives telephone calls as a Poisson proces with $\lambda = 0.5$ per minute. Then the number of calls in a 5-min interval is $5\lambda = 2.5$

$\Pr(\text{no calls in a 5-min interval}) = \Pr(X = 0) = e^{-2.5} = 0.082$