Continuous distributions

Uniform. It arises in situations where there's not a prefered value for the random variable: times at which a radioactive element emits particles, phase angle of an unknown sinusoidal signal.

$X \sim U[x_1, x_2]$, $X \in [x_1, x_2]$

$$f(x) = \frac{1}{x_2 - x_1}, \quad x_1 \leq x \leq x_2$$

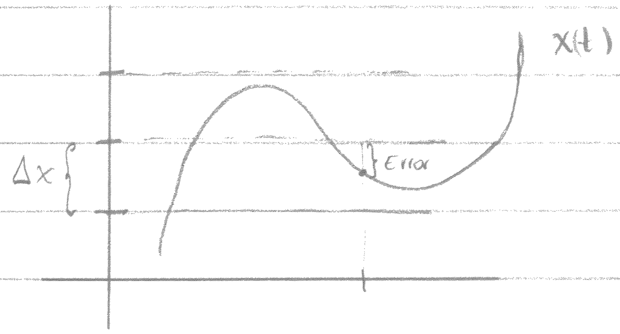
$$E[X] = \int_{x_1}^{x_2} x \frac{1}{x_2 - x_1} dx = \frac{1}{x_2 - x_1} \frac{x^2}{2} \Big|_{x_1}^{x_2} = \frac{1}{2} \frac{1}{x_2 - x_1} (x_2^2 - x_1^2) =$$

$$\frac{1}{2} \frac{1}{x_2 - x_1} (x_2 - x_1)(x_1 + x_2) = \frac{1}{2}(x_1 + x_2)$$

$$Var(X) = \frac{1}{12}(x_2 - x_1)^2$$

$$P(X \leq x) = F(x) = \int_{x_1}^{x} t \frac{1}{x_2 - x_1} dt = \frac{x - x_1}{x_2 - x_1} \quad x_1 \leq x \leq x_2$$

Example. Analog to digital conversion. The uniform distribution describes the errors associated with this conversion.

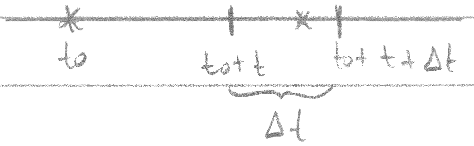


$\text{Error} \sim U[-\frac{\Delta x}{2}, \frac{\Delta x}{2}]$

mean error $= 0$

$\text{var}(\text{error}) = \frac{1}{2}(\Delta x)^2$

<u>Exponential</u>. This distribution model the length of time intervals between independent events that occur at times that are equally probable. Suppose that an event has occured at time t_0 and we want to know the probability that the next one will occur between t_0+t and $t_0+t+\Delta t$ (the probability that it occurs at a fixed time t_1 is 0)



That is, we want to know the distribution of the random variable T: time for next event

If the distribution for T is F then $P_r(t_0+t \le T \le t_0+t+\Delta t) = P_r(t \le T \le t+\Delta t) = F(t+\Delta t) - F(t)$

This probability has to be equal to the product of the probabilities:

The event didn't occur between t_0 and t_0+t:

$1 - P_r(t_0 \le T \le t_0+t) = 1 - P_r(0 \le T \le t) = 1 - F(t)$

The event occur between t_0+t and $t_0+t+\Delta t$: $\Delta t/\lambda$

$F(t+\Delta t) - F(t) = [1-F(t)]\dfrac{\Delta t}{\lambda}$ where λ is the average length of time intervals between events.

$$\Rightarrow \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{1-F(t)}{\lambda}$$

$$\Rightarrow \frac{d}{dt}F(t) = \frac{1-F(t)}{\lambda}$$

$T \sim Exp(\lambda)$, $T \in [0, \infty)$

$$\Rightarrow F(t) = 1 - e^{-\frac{t}{\lambda}}, \quad t \geq 0$$

$$\Rightarrow f(t) = \frac{1}{\lambda} e^{-\frac{t}{\lambda}}, \quad t \geq 0$$

$$E[T] = \int_0^\infty t \frac{1}{\lambda} e^{-\frac{t}{\lambda}} dt = \frac{1}{\lambda}\int_0^\infty t e^{-\frac{t}{\lambda}} dt = \left| \begin{matrix} u = t & v = -\lambda e^{-t/\lambda} \\ dv = e^{-t/\lambda} dt & du = dt \end{matrix} \right|$$

$$\frac{1}{\lambda}\left[-t\lambda e^{-t/\lambda} \Big|_0^\infty + \int_0^\infty \lambda e^{-t/\lambda} dt \right] = \int_0^\infty e^{-t/\lambda} dt = -\lambda e^{-t/\lambda} \Big|_0^\infty =$$

$$\lambda$$

$$Var(T) = \lambda^2$$

Example. Lifetimes of some electronic components are modeled as exponential. Assume that the mean-time-to-failure of a certain component in a communication system is of 4 years. Then, if T is the (random) lifetime $\Rightarrow T \sim Exp(4)$

$Pr(\text{component lasts more than 4 years}) = Pr(T > 4) = 1 - F(4) = 1 - (1 - e^{-4/4}) = 0.368$

$Pr(\text{component fails the first year}) = Pr(T \leq 1) = F(1) = 0.221$

$Pr(\text{component fails between years 4 and 6}) = Pr(4 \leq T \leq 6) = F(6) - F(4)$

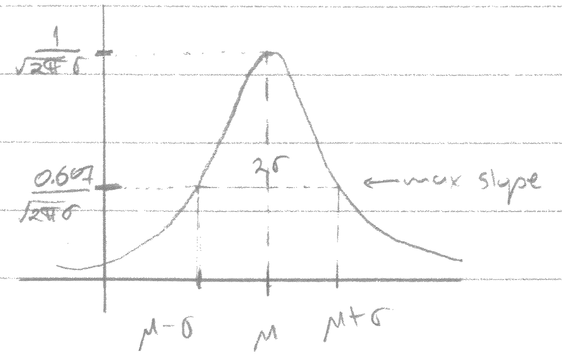
$= (1 - e^{-6/4}) - (1 - e^{-4/4}) = 0.1447$

Normal (Gaussian). Most important distribution we will study

Originally proposed by Gauss as a model for measurement errors. The Central Limit Theorem justifies its use in many applications.

$X \sim N(\mu, \sigma^2)$, $X \in (-\infty; \infty)$, $E(X) = \mu$ $Var(X) = \sigma^2$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{1}{2\sigma^2}(x-\mu)^2 \right\} \quad -\infty < x < \infty$$



1. Max occurs at μ
2. Symmetrical about μ
3. Width is proportional to σ
4. Can be use to represent δ function

$$\delta(x-\mu) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{1}{2\sigma^2} (x-\mu)^2 \right\}$$

This representation has the advantage of being infinitely differentiable.

By definition, the distribution function is

$$F(x) = \int_{-\infty}^{x} f(u)\, du = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{1}{2\sigma^2} (u-\mu)^2 \right\} du$$

The special case for $\mu = 0$ and $\sigma = 1$ is designated by $\Phi(x)$, that is:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{u^2}{2} \right) du$$

And the general case can be expressed in terms of $\Phi(x)$ by a change of variable: If $X \sim N(\mu, \sigma^2) \Rightarrow Z = \frac{X-\mu}{\sigma} \sim N(0,1)$, thus:

$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$. Note that $\Phi(-x) = 1 - \Phi(x)$

Another function closely related to $\Phi(x)$ is the Q function

$Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{u^2}{2} \right) du$. Also $Q(-x) = 1 - Q(x)$

and so $Q(x) = 1 - \Phi(x)$. Finally we have that $F(x) = 1 - Q\left(\frac{x-\mu}{\sigma}\right)$

Example. A circuit receives an input voltage of 0.5V, but it has Gaussian random noise superimposed with standard deviation $\sqrt{0.2}$ V. The circuit incorrectly changes state when the voltage exceeds 2.5V, what's the probability that this happens?

$V \sim N(0.5,\ 0.2) \Rightarrow$ we want to compute

$$P_r(V > 2.5) = \int_{2.5}^{\infty} \frac{1}{\sqrt{2\pi(0.2)}} \exp\left\{-\frac{1}{2(0.2)}(u-0.5)^2\right\} du = Q\left(\frac{2.5-0.5}{\sqrt{0.2}}\right) = Q(4.472)$$

from tables

$$\approx 0.39 \times 10^{-5}$$