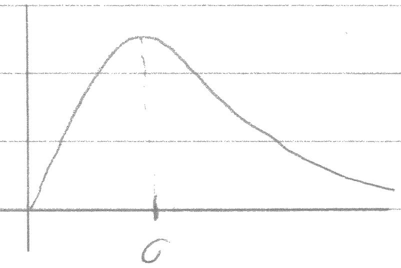
__Rayleigh__ Peak values (envelope) of Gaussian random voltage or current are Rayleigh distributed. Peak values of the sum of sine waves with different frequencies. Error associated with aiming: suppose the origin of a rectangular coordinate system is the aim and the errors along the axes is $X \sim N(0, \sigma^2)$ and $Y \sim N(0, \sigma^2)$ respectively, then the miss distance is $R = \sqrt{X^2 + Y^2}$ and $R \sim \text{Rayleigh}(\sigma^2)$

$$f(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right), \quad r \geq 0$$



$$E[R] = \int_0^\infty r f(r)\, dr = \int_0^\infty \frac{r^2}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) dr$$

$$= \left| x = \frac{r}{\sigma} \quad dx = \frac{dr}{\sigma} \right| = \sigma \int_0^\infty x^2 e^{-\frac{x^2}{2}}\, dx = \sigma\sqrt{2\pi} \int_0^\infty \frac{1}{\sqrt{2\pi}} x^2 e^{-\frac{x^2}{2}}\, dx = \sigma\sqrt{2\pi}\, \frac{E[Z^2]}{2}$$

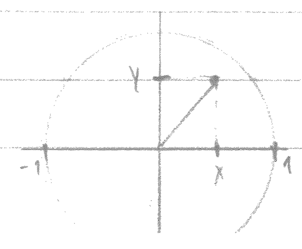
Now $\operatorname{var}(Z) = 1 \iff E[Z^2] - E[Z]^2 = 1 \iff E[Z^2] = 1 \implies$

$$E[R] = \sigma\sqrt{2\pi}\, \frac{1}{2} = \sqrt{\frac{\pi}{2}}\, \sigma$$

$$\operatorname{Var}(R) = \left(2 - \frac{\pi}{2}\right)\sigma^2$$

$$F(r) = \int_0^r \frac{u}{\sigma^2} \exp\left(\frac{-u^2}{2\sigma^2}\right) du = 1 - \exp\left(\frac{-r^2}{2\sigma^2}\right), \quad r \geq 0$$

Example.



$X \sim N(0, \frac{1}{4}) \quad Y \sim N(0, \frac{1}{4})$

$\implies R \sim \text{Rayleig}(\frac{1}{4}) \implies f(r) = 16 r \exp(-8r^2)$

The mean value of the miss distance is thus

$$E[R] = \sqrt{\frac{\pi}{2}} \cdot \frac{1}{4} = 0.313$$

The standard deviation is

$$\sqrt{Var(R)} = \sqrt{\left(2 - \frac{\pi}{2}\right)\sigma^2} = \sqrt{0.429 \cdot \frac{1}{4}} = 0.164$$

The probability of missing completely is

$$Pr(R > 1) = 1 - Pr(R \le 1) = 1 - F(1) = 1 - \left[1 - \exp\left(-\frac{1^2}{2(0.25)^2}\right)\right] = e^{-8} = 3.35 \cdot 10^{-4}$$

If the bulls-eye is 2 in in diameter (1 ft = 12 in) then

$$Pr(\text{bulls-eye}) = Pr\left(R \le \frac{1}{12}\right) = F\left(\frac{1}{12}\right) = 1 - \exp\left(-\frac{8}{144}\right) = 0.0540$$

check again example in Expected values section

Maxwell Suppose we want to determine the probability that a given gas molecule will have more than twice the mean kinetic energy for all molecules. The kinetic energy is given by $e = \frac{1}{2} m V^2 \Rightarrow V = \sqrt{\frac{2e}{m}}$. The mean kinetic energy is $\frac{3}{2} m \sigma^2$, then the velocity of a molecule with twice the mean kinetic energy is $\sqrt{2 \cdot \left(2 \cdot \frac{3}{2} m \sigma^2\right) / m} = \sqrt{6}\,\sigma$

$$Pr(V > \sqrt{6}\sigma) = \int_{\sqrt{6}\sigma}^{\infty} \sqrt{\frac{2}{\pi}} \frac{v^2}{\sigma^3} \exp\left(-\frac{v^2}{2\sigma^2}\right) dv = 0.1116$$

numerically

Chi-square Suppose $X \sim N(0,1) \Rightarrow Y = X^2 \sim \chi^2_{(1)}$

$$f(y) = \frac{y^{-\frac{1}{2}}}{\sqrt{2\pi}} \exp\left(-\frac{y}{2}\right), \quad y \ge 0$$

The power W dissipated in a resistor is given by $W = RI^2$. If the current I is assumed to be Gaussian with mean zero and variance $\sigma^2 \Rightarrow RI \sim N(0, R\sigma^2)$
$\Rightarrow W \sim \chi^2_{(1)}$ (non-centered)

$$f(w) = \frac{1}{\sqrt{2\pi R w}\,\sigma} \exp\left(-\frac{w}{2R\sigma^2}\right) \quad w \ge 0$$

In general, if $X_1, X_2, \ldots, X_n$ are independent Gaussian random variables with mean 0 and variance 1 then $Y = \sum_{i=1}^{n} X_i^2 \sim \chi^2_{(n)}$

$$f(y) = \frac{y^{\frac{n}{2}-1}}{2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} \exp\left(-\frac{y}{2}\right), \quad y \geq 0$$

$E[Y] = n$ and $Var(Y) = 2n$

This distribution arises statistics problems like hypothesis testing (for example, to determine if a sampled voltage is just noise or contains a signal)

Example Back to the distribution of the power in a resistor

$$P_r(W \leq w) = P_r(RI^2 \leq w) = P_r\left(I^2 \leq \frac{w}{R}\right) = P_r\left(|I| \leq \sqrt{\frac{w}{R}}\right)$$

$$= P_r\left(-\sqrt{\frac{w}{R}} \leq I \leq \sqrt{\frac{w}{R}}\right) = P_r\left(\frac{-\sqrt{w/R}}{\sigma} \leq \frac{I}{\sigma} \leq \frac{\sqrt{w/R}}{\sigma}\right) = \Phi\left(\frac{\sqrt{w/R}}{\sigma}\right) - \Phi\left(\frac{-\sqrt{w/R}}{\sigma}\right)$$

$$= \Phi\left(\frac{\sqrt{w/R}}{\sigma}\right) - \left[1 - \Phi\left(\frac{\sqrt{w/R}}{\sigma}\right)\right] = 2\Phi\left(\frac{\sqrt{w/R}}{R}\right) - 1$$

Consider a speaker with resistance of 4Ω rated for a max power of 25 W. If we assume a Gaussian input current with mean 4W, what's the probability that the maximum power level of the speaker will be exceeded? Since 4W dissipated in 4Ω implies $\sigma^2 = 1$ it follows that:

$$P_r(W > 25) = 1 - P_r(W \leq 25) = 1 - \left[2\Phi\left(\frac{\sqrt{25/4}}{1}\right) - 1\right] = 2 - 2\Phi(2.5) = 2 - 2(0.9938)$$

$$= 0.0124$$

Log-normal

Let $Y \sim N(\mu, \sigma^2)$ and $X \sim e^Y \Rightarrow X \sim \text{LogNormal}(\mu, \sigma^2)$

In communication systems the atenuation A of a signal power is computed as $A = \ln\left(\frac{W_{out}}{W_{in}}\right)$ where W_{out} and W_{in} are the output and input signal powers respectively. A is very often quite close to a Gaussian random variable $\Rightarrow$ $X = \frac{W_{out}}{W_{in}} \sim \text{Log-Normal}(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left\{ -\frac{(\ln(x) - \mu)^2}{2\sigma^2} \right\}, \quad x > 0$$

$$E[X] = \exp\left\{\mu + \frac{1}{2}\sigma^2\right\} \qquad Var(X) = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2}$$

Functions of a random variable

Suppose that a random variable X has density function $f(x)$. We often need to find the density function of a new random variable defined as $Y = g(X)$. If g is differentiable, strictly monotonic in some interval I and $f(x) = 0$ if $x \notin I$ then

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

Proof. $F_Y(y) = P_r(Y \leq y) = P_r(g(X) \leq y) \overset{g \text{ monotonically increasing}}{=} P_r(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$

$$f_Y(y) = \frac{d}{dx} F_Y(y) = \frac{d}{dx} F_X(g^{-1}(y)) = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$$

By considering g monotonically decreasing and putting the results together we conclude the proof.

Example $X \sim N(0,1)$, $Y = X^2 \Rightarrow g(x) = x^2 \Rightarrow g^{-1}(y) = \pm\sqrt{y}$, $y > 0$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = \frac{1}{\sqrt{2\pi}} \left[e^{-\frac{(-\sqrt{y})^2}{2}} + e^{-\frac{(\sqrt{y})^2}{2}} \right] \left| \frac{1}{2\sqrt{y}} \right| = \frac{y^{-\frac{1}{2}}}{\sqrt{2\pi}} e^{-\frac{y}{2}}$$

Alternatively: $F_Y(y) = P_r(Y \leq y) = P_r(X^2 \leq y) = P_r(|X| \leq \sqrt{y}) = P_r(-\sqrt{y} \leq X \leq \sqrt{y}) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y})$

$$f_Y(y) = \frac{1}{2} y^{-\frac{1}{2}} \phi(\sqrt{y}) + \frac{1}{2} y^{-\frac{1}{2}} \phi(-\sqrt{y}) = y^{-\frac{1}{2}} \phi(\sqrt{y}) = \frac{y^{-\frac{1}{2}}}{\sqrt{2\pi}} e^{-\frac{y}{2}}$$