Covariance and Correlation

Definition. Covariance If X and Y are jointly distributed random variables with expectation μ_X and μ_Y respectively, then the covariance of X and Y is

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

which can be simplified as follows:

$$\begin{aligned} Cov(X,Y) &= E[(X-\mu_X)(Y-\mu_Y)] = E[XY - X\mu_Y - Y\mu_X + \mu_X\mu_Y] \\ &= E[XY] - E[X]\mu_Y - E[Y]\mu_X + \mu_X\mu_Y \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

In particular $Cov(X,Y)=0$ if X and Y are independent (the converse is not always true, normality is needed).

Properties

$Cov(aW+bX, cY+dZ) = ac\,Cov(W,Y) + bc\,Cov(X,Y) + ad\,Cov(W,Z) + bd\,Cov(X,Z)$

$Var(X) = Cov(X,X) \implies Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$

$Var(X+Y) = Var(X) + Var(Y)$ if X and Y are independent

Definition Correlation If X and Y are jointly distributed and $Var(X), Var(Y) > 0$ exist, the correlation ρ of X and Y is

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$$

Note $\rho = 0$ if X and Y are independent

Theorem. $-1 \le \rho \le 1$. Furthermore $\rho = \pm 1$ if and only if $Pr(Y = a + bX) = 1$ for some constant a, b

Example: X and Y have joint density function $f(x,y) = x+y \quad 0 \le x,y \le 1$

$$f_X(x) = \int_0^1 x+y \, dy = x + \frac{1}{2} \qquad f_Y(y) = \int_0^1 x+y \, dx = y + \frac{1}{2}$$

$$E[X] = \int_0^1 x\left(x+\frac{1}{2}\right)dx = \frac{7}{12} \qquad E[Y] = \int_0^1 y\left(y+\frac{1}{2}\right)dx = \frac{7}{12}$$

$$Var(X) = E[(X-\mu_X)^2] = \int_0^1 \left(x-\frac{7}{12}\right)^2\left(x+\frac{1}{2}\right)dx = \frac{11}{144}$$

$$Var(Y) = \int_0^1 \left(y-\frac{7}{12}\right)^2\left(y+\frac{1}{2}\right)dx = \frac{11}{144}$$

$$E[XY] = \int_0^1\int_0^1 xy(x+y)\,dx\,dy = \frac{1}{3}$$

$$Cov(X,Y) = E[XY] - E[X]E[Y] = \frac{1}{3} - \left(\frac{7}{12}\right)^2 = \frac{97}{144}$$

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{\frac{97}{144}}{\frac{11}{144}} = -\frac{1}{11}$$

Example: X and Y have joint density function $f(x,y) = \lambda^2 e^{-\lambda y} \quad 0 \le x \le y$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^{\infty} \lambda^2 e^{-\lambda y}\,dy = \lambda e^{-\lambda x} \qquad x \ge 0 \qquad X \sim Exp(\lambda)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^y \lambda^2 e^{-\lambda y}\,dx = \lambda^2 y e^{-\lambda y} \qquad y \ge 0 \qquad Y \sim Gamma(2,\lambda)$$

$$f_{Y|X}(y|x) = \frac{\lambda^2 e^{-\lambda y}}{\lambda e^{-\lambda x}} = \lambda e^{-\lambda(y-x)} \qquad y \ge 0 \qquad Y|X \sim Exp(\lambda)$$

$$f_{X|Y}(x|y) = \frac{\lambda^2 e^{-\lambda y}}{\lambda^2 y e^{-\lambda y}} = \frac{1}{y} \qquad 0 \le x \le y \qquad X|Y \sim U[0,y]$$

Proposition. If $\rho = 0$ and X, Y are jointly Gaussian $\Rightarrow$ X and Y are independent.

Proof. $$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2(x-\mu_x)(y-\mu_y)\rho}{\sigma_x\sigma_y}\right]\right\}$$

when $\rho = 0 \Rightarrow$ $$= \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{-\frac{1}{2}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right\}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left\{-\frac{1}{2}\frac{(x-\mu_x)^2}{\sigma_x^2}\right\} \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left\{-\frac{1}{2}\frac{(y-\mu_y)^2}{\sigma_y^2}\right\}$$

$$= f_X(x) f_Y(y)$$

<u>Density function of sums (Convolution)</u>

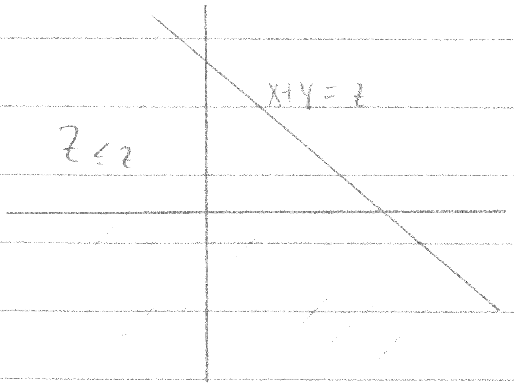
Let X and Y be jointly distributed random variables and let $Z = X+Y$. It is desired to obtain the probability density function of Z.

$$F_Z(z) = P_r(Z \le z) = P_r(X+Y \le z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f(x,y)\,dx\,dy$$

$$= \left\{\begin{array}{l} x = v - y \\ dx = dv \end{array}\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{z} f(v-y,y)\,dv\,dy = \int_{-\infty}^{z}\int_{-\infty}^{\infty} f(v-y,y)\,dy\,dv$$

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \int_{-\infty}^{\infty} f(z-y,y)\,dy$$

And if X and Y are independent $f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\,dy$



Similarly is possible to obtain

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dy$$

Example: The lifetime of a component is $T_1 \sim Exp(\lambda)$ and an identical component, $T_2 \sim Exp(\lambda)$, is available as backup. If a system works as long as one of the components work, then it's lifetime has the distribution of $S = T_1 + T_2$

$$f_{T_1}(t) = \lambda e^{-\lambda t} \quad f_{T_2}(t) = \lambda e^{-\lambda t} \quad t \geq 0$$

$$f_S(s) = \int_{-\infty}^{\infty} f_{T_1}(t) f_{T_2}(s-t)\,dt = \int_0^s \lambda e^{-\lambda t} \lambda e^{-\lambda(s-t)}\,dt$$

$$= \lambda^2 \int_0^s e^{-\lambda s}\,dt = \lambda^2 s e^{-\lambda s} = \text{Gamma}(2, \lambda)$$

If $X \sim \text{Gamma}(n, \lambda)$ $\quad f(x) = \frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x}$ $\quad$ sum of n $Exp(\lambda)$

In general $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\,dt$

Example: If $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ then

$Z = X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$

In general, the sum of any number of normal random variables is still normal.