Example: Let $N_1 \sim Poisson(\lambda_1)$ and $N_2 \sim Poisson(\lambda_2)$ and independent
Show that $N = N_1 + N_2 \sim Poisson(\lambda_1 + \lambda_2)$

For discrete independent random variables X and Y, the convolution (density of $Z = X+Y$) is computed as $f_Z(z) = \sum_{x=-\infty}^{\infty} f_X(x) f_Y(z-x)$

In this case $f_{N_1}(n) = \frac{\lambda_1^n}{n!} e^{-\lambda_1}$ $f_{N_2}(n) = \frac{\lambda_2^n}{n!} e^{-\lambda_2}$ thus

$$f_N(n) = \sum_{s=-\infty}^{\infty} f_{N_1}(s) f_{N_2}(n-s) = \sum_{s=0}^{n} \frac{\lambda_1^s}{s!} e^{-\lambda_1} \frac{\lambda_2^{n-s}}{(n-s)!} e^{-\lambda_2}$$

$$= e^{-(\lambda_1+\lambda_2)} \sum_{s=0}^{n} \frac{1}{s!(n-s)!} \lambda_1^s \lambda_2^{n-s} = \frac{1}{n!} e^{-(\lambda_1+\lambda_2)} \sum_{s=0}^{n} \frac{n!}{s!(n-s)!} \lambda_1^s \lambda_2^{n-s}$$

$$= \frac{1}{n!} e^{-(\lambda_1+\lambda_2)} (\lambda_1+\lambda_2)^n = \frac{(\lambda_1+\lambda_2)^n}{n!} e^{-(\lambda_1+\lambda_2)} \Rightarrow N \sim Poisson(\lambda_1+\lambda_2)$$

The Characteristic Function

Definition. The charateristic function of a random variable X is defined as
$\varphi(u) = E[e^{jux}]$ where $j = \sqrt{-1}$

$$= \int_{-\infty}^{\infty} f(x) e^{jux} dx$$

which, except for the minus sign is the Fourier transform of $f(x)$, and preserves all the properties of the Fourier transform.

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(u) e^{-jux} du$$ (distribution uniquely determined by the characteristic)

If $Z = X+Y$ and X and Y are independent then $\varphi_Z(u) = \varphi_X(u) \varphi_Y(u)$

$$f_Z(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_X(x) \varphi_Y(y) e^{-juz} du$$

Finally $E[X^n] = \frac{1}{j^n} \frac{d^n \varphi(u)}{du^n}\Big|_{u=0}$

which is easily proved for the case $n=1$ as follows:

$$\frac{d\varphi(u)}{du} = \int_{-\infty}^{\infty} f(x)(jx) e^{jux} dx \Rightarrow \frac{d}{du}\varphi(u)\Big|_{u=0} = j\int_{-\infty}^{\infty} x f(x)\, dx = j\, E[X]$$

and using induction for general n.

Example: $X \sim Exp(\lambda)$, $Y \sim Exp(\lambda)$, $Z = X+Y$

$$\varphi_X(u) = \int_{-\infty}^{\infty} f(x) e^{jux} dx = \int_0^{\infty} \lambda e^{-\lambda x} e^{jux} dx = \lambda \int_0^{\infty} e^{-(\lambda - ju)x} dx$$

$$= \left(1 - \frac{ju}{\lambda}\right)^{-1}$$

Similarly $\varphi_Y(u) = \left(1 - \frac{ju}{\lambda}\right)^{-1}$

$$\varphi_Z(u) = \varphi_X(u)\varphi_Y(u) = \left(1 - \frac{ju}{\lambda}\right)^{-2}$$

For a Gamma(n, λ)

$$\int_0^{\infty} \frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x} e^{jux} dx = \frac{\lambda^n}{\Gamma(n)} \int_0^{\infty} x^{n-1} e^{-(\lambda - ju)x} dx = \left(1 - \frac{ju}{\lambda}\right)^{-n}$$

The characteristic function can be extended to joint density functions as follows

$$\varphi_{X,Y}(u,v) = E\left[e^{j(uX+vY)}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{j(ux+vy)} dx dy$$

and then can be used to find the covariance of X and Y by using

$$E[XY] = -\frac{\partial^2}{\partial u \partial v} \varphi_{X,Y}(u,v) \Big|_{u=v=0}$$

Limit theorems

Law of large numbers

Let $X_1, X_2, \ldots$ be a sequence of independent random variables with $E[X_i] = \mu$ and $Var(X_i) = \sigma^2$. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ then,

for any $\varepsilon > 0$ $\quad P_r(|\bar{X}_n - \mu| > \varepsilon) \longrightarrow 0$ as $n \to \infty$

This type of convergence is called convergence in probability or weak convergence.

Central Limit Theorem

Let $X_1, X_2, \ldots$ be a sequence of independent random variables with $E[X_i] = \mu$ and $Var(X_i) = \sigma^2$ and common distribution function. Let $S_n = \sum_{i=1}^{n} X_i$, then $\lim_{n\to\infty} P_r\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right) = \Phi(x)$

Example. Suppose $X_1, \ldots, X_n$ are repeated measurements of a quantity μ and that $E[X_i] = \mu$ and $Var(X_i) = \sigma^2$ obviously $\bar{X}$ is used to estimate μ and because of $Var(\bar{X}) = \frac{\sigma^2}{n}$ we hope that $\bar{X}$ is close to μ if n is large, but how close?

$$P_r(|\bar{X} - \mu| < c) = P_r(-c < \bar{X} - \mu < c) = P_r\left(\frac{-c}{\sigma/\sqrt{n}} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \frac{c}{\sigma/\sqrt{n}}\right)$$

$$\simeq \Phi\left(\frac{c\sqrt{n}}{\sigma}\right) - \Phi - \left(\frac{c\sqrt{n}}{\sigma}\right)$$

For example, if 16 measurements are taken with $\sigma = 1$, the probability that $\bar{X}$ deviates from μ by less than 0.5 is approximately

$$P_r(|\bar{X} - \mu| < 0.5) = \Phi(0.5\ 4) - \Phi(-0.5\ 4) = 0.954$$

Example. Let $X_n \sim Bin(n,p)$, then $X_n = \sum_{j=1}^{n} Y_j$ where $Y_j \sim Bernoulli(p)$. Then, by TLC

$$\lim_{n\to\infty} P_r\left(\frac{X_n - np}{\sqrt{p(1-p)}\sqrt{n}} < x \right) = \Phi(x)$$

Or equivalently $\frac{X_n - np}{\sqrt{np(1-p)}} \overset{d}{\sim} N(0,1)$ or $X_n \overset{d}{\sim} N(np, np(1-p))$

or $f(x) = \binom{n}{x} p^x (1-p)^{n-x} \approx \frac{1}{\sqrt{2\pi np(1-p)}} e^{-\frac{1}{2}\frac{(x-np)^2}{np(1-p)}}$

Example. Suppose a coin is tossed 100 times and we obtain 60 heads. Should we be surprised and doubt that the coin is fair?

X: number of heads. then $X \sim Bin(100, 0.5)$

Then $E[X] = np = 50$ and $Var(X) = np(1-p) = 25$

$P_r(X=60)$ and even $P_r(X=50)$ are small $(0.07$ and $0.01)$

$P_r(X \geq 60) = P_r\left(\frac{X-50}{5} \geq \frac{60-50}{5}\right) \approx 1 - \Phi(2) = 0.0228$