Using the characteristic function of the Binomial show that $Bin(n,p) \longrightarrow Poisson(\lambda)$ as $n \to \infty$, $p \to 0$ and $np \to \infty$

Let $X \sim Bin(n,p)$ then $X = \sum_{i=1}^{n} Y_i$ where $Y_i \sim Ber(p)$ then

$$\varphi_X(u) = \prod_{i=1}^{n} \varphi_{Y_i}(u)$$

$$\varphi_{Y_i}(u) = \sum_{y=-\infty}^{\infty} f_{Y_i}(y) e^{juy} = \sum_{y=0}^{1} p^y (1-p)^{1-y} e^{juy}$$

$$= 1-p + pe^{ju}$$

Then $\varphi_X(u) = (1-p+pe^{ju})^n$

Taking logarithm on both sides we get

$$\ln[\varphi_X(u)] = n \ln(1-p+pe^{ju}) = n \ln[1+p(e^{ju}-1)]$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = x + O(x^2)$$

$$= np(e^{ju}-1) + n\,O\{[p(e^{ju}-1)]^2\}$$

Letting $n \to \infty$, $p \to 0$ and $np \to \lambda$ on both sides we get

$$\ln[\varphi_X(u)] = \lambda(e^{ju}-1) \quad (\text{because } n\,O[p^2(e^{ju}-1)^2] \to 0) \Rightarrow \varphi_X(u) = e^{\lambda(e^{ju}-1)}$$

On the other hand, for $Z \sim Poisson(\lambda)$

$$\varphi_Z(u) = \sum_{z=-\infty}^{\infty} f_Z(z) e^{juz} = \sum_{z=0}^{\infty} \frac{\lambda^z}{z!} e^{-\lambda} e^{juz} = e^{-\lambda} \sum_{z=0}^{\infty} \frac{(\lambda e^{ju})^z}{z!}$$

$$= e^{-\lambda} e^{\lambda e^{ju}} = e^{\lambda e^{ju} - \lambda} = e^{\lambda(e^{ju}-1)}$$ as required

A drunk guy does a "random walk" as follows: each minute he takes a step south with probability $\frac{2}{3}$ and north with probability $\frac{1}{3}$. Each successive step is independent of the previous one. His steps are 50 cm long. Use the Central Limit Theorem to approximate the distribution of his location after 1 h where is he most likely to be?

Let X_i be the ith step. Then $X_i = \begin{cases} 50 & \text{(north)} \\ -50 & \text{(south)} \end{cases}$

and $P_r(X_i = 50) = \frac{1}{3}$ $P_r(X_i = -50) = \frac{2}{3}$

This way, his location after 1 h is $S = \sum_{i=1}^{60} X_i$

$$E[X_i] = 50\left(\tfrac{1}{3}\right) + (-50)\left(\tfrac{2}{3}\right) = -\frac{50}{3}$$

$$E[X_i^2] = 50^2\left(\tfrac{1}{3}\right) + (-50)^2\left(\tfrac{2}{3}\right) = \frac{2500}{3}$$

$$Var(X_i) = E[X_i^2] - E[X_i]^2 = \frac{2500}{3} - \left(-\frac{50}{3}\right)^2 = \frac{5000}{9}$$

$$E[S] = \sum_{i=1}^{60} E[X_i] = \sum_{i=1}^{60} -\frac{50}{3} = -1000$$

$$Var(S) = \sum_{i=1}^{60} Var(X_i) = \sum_{i=1}^{60} \frac{5000}{9} = \frac{100\,000}{3}$$

According to the CLT $S \overset{d}{\approx} N\left(-1000, \frac{100\,000}{3}\right)$

As the normal distribution is symmetric about its mean then the most probable value is actually its mean so the drunk guy is most likely to be at -1000 cm of where he started, that is 10 m south.