Solutions to ESS011 2011-08-23

① Let C_i be the event "instrument i is chosen" $i=1,2,3$

and let F_i be the event "instrument i works" $i=1,2,3$

a) $P((C_1 \cap F_1) \cup (C_2 \cap F_2) \cup (C_3 \cap F_3))$

$= P(C_1 \cap F_1) + P(C_2 \cap F_2) + P(C_3 \cap F_3)$

$= P(C_1)P(F_1) + P(C_2)P(F_2) + P(C_3)P(F_3)$

$= \frac{1}{3} \cdot 0.9 + \frac{1}{3} \cdot 0.8 + \frac{1}{3} \cdot 0.4 = 0.7$

b) $$P(C_i \mid F_i) = \frac{P(C_i)P(F_i \mid C_i)}{\sum_{i=1}^{3} P(C_i)P(F_i \mid C_i)}$$

$$i=1) \quad = \frac{\frac{1}{3} \cdot 0.9}{\frac{1}{3} \cdot 0.9 + \frac{1}{3} \cdot 0.8 + \frac{1}{3} 0.4} = 0.4286$$

$$i=2) \quad = \frac{\frac{1}{3} \cdot 0.8}{\frac{1}{3} \cdot 0.9 + \frac{1}{3} \cdot 0.8 + \frac{1}{3} \cdot 0.4} = 0.3810$$

$$i=3) \quad = \frac{\frac{1}{3} \cdot 0.4}{\frac{1}{3} \cdot 0.9 + \frac{1}{3} 0.8 + \frac{1}{3} 0.4} = 0.1905$$

② a) $0 < X < 10$

b) $$f_X(x) = \begin{cases} \frac{1}{10} & 0 < x < 10 \\ 0 & \text{otherwise} \end{cases}$$

$$P(3.5 < X \leq 7) = \int_{3.5}^{7} f_X(x)\,dx = \int_{3.5}^{7} \frac{1}{10}\,dx = \frac{1}{10} x \Big|_{3.5}^{7} = \frac{7}{10} - \frac{3.5}{10} = 0.35$$

c) $F_X(x) = \int_0^x \frac{1}{10}\,dx = \frac{x}{10}$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{10} & 0 \le x \le 10 \\ 1 & x \geq 10 \end{cases}$$

③ a) $L(\theta) = \prod_{i=1}^{2} f(x_i) = [\theta(1+0.2)^{-\theta-1}][\theta(1+0.8)^{-\theta-1}]$

$= \theta^2 \cdot 2.16^{-\theta-1}$

$L(2) = 2^2 \cdot 2.16^{-2-1} = 0.3969$

$L(3) = 3^2 \cdot 2.16^{-3-1} = 0.4135$

$L(4) = 4^2 \cdot 2.16^{-4-1} = 0.3403$

b) $L(\theta)$ is max for $\theta = 3$ so $\theta^* = 3$ is the maximum likelihood estimate

c) $E(X) = \int_0^\infty x\,\theta(1+x)^{-\theta-1}\,dx = \left\{ \begin{matrix} u = x & dv = \theta(1+x)^{-\theta-1}\,dx \\ du = dx & v = -(1+x)^{-\theta} \end{matrix} \right\}$

$= -x(1+x)^{-\theta}\Big|_{x=0}^{\infty} - \int_0^\infty -(1+x)^{-\theta}\,dx = 0 + \int_0^\infty (1+x)^{-\theta}\,dx$

$= \frac{(1+x)^{-\theta+1}}{-\theta+1}\Big|_{x=0}^{\infty} = \frac{1}{\theta-1}$

$\theta > 1$

d) $\bar{X} = \frac{0.2+0.8}{2} = 0.5 \Rightarrow \frac{1}{\theta-1} = 0.5 \Rightarrow \theta^* = 3$

(4) $f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx = *$

$0 \leq z \leq 1)\quad * = \int_0^z e^{-x} \cdot 1\, dx = -e^{-x} \Big|_{x=0}^{z} = -e^{-z} + 1 = 1 - e^{-z}$

$z \geq 1)\quad * = \int_{z-1}^{z} e^{-x} \cdot 1\, dx = -e^{-x} \Big|_{x=z-1}^{z} = -e^{-z} + e^{-(z-1)} = e^{-z}(e-1)$

$$f_Z(z) = \begin{cases} 1 - e^{-z} & 0 \leq z \leq 1 \\ e^{-z}(e-1) & z \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

(5) a) $(\bar{x} - \sigma_{\bar{x}} z_{(\alpha/2)} - \Delta,\ \bar{x} + \sigma_{\bar{x}} z_{(\alpha/2)} + \Delta)$

$= (\bar{x} - \frac{\sigma}{\sqrt{n}} z_{(\alpha/2)} - \Delta,\ \bar{x} + \frac{\sigma}{\sqrt{n}} z_{(\alpha/2)} + \Delta)$

$= \left(8.2 - \frac{0.05}{\sqrt{4}} \cdot 2.58 - 0.1,\ 8.2 + \frac{0.05}{\sqrt{4}} \cdot 2.58 - 0.1\right)$

$= (8.0355, 8.1645)$

b) $(\bar{x} - s_{\bar{x}} t_{n-1}(\alpha/2) - \Delta,\ \bar{x} + s_{\bar{x}} t_{n-1}(\alpha/2) + \Delta)$

$= (\bar{x} - \frac{s_x}{\sqrt{n}} t_{n-1}(\alpha/2) - \Delta,\ \bar{x} + \frac{s_x}{\sqrt{n}} t_{(n-1)}(\alpha/2) + \Delta)$

$= \left(8.2 - \frac{0.0424}{\sqrt{4}} \cdot 5.84 - 0.1,\ 8.2 - \frac{0.0424}{\sqrt{4}} \cdot 5.84 - 0.1\right)$

$= (7.9761, 8.2239)$

⑥ a) $H_0: \mu = 2.5$

$H_1: \mu < 2.5$

b) Test statistic $T = \dfrac{\bar{X} - \mu}{s/\sqrt{n}} \sim t_{(n-1)}$

$T^* = \dfrac{1.8 - 2.5}{0.8/\sqrt{16}} = -3.5$

$t_{(n-1)}(\alpha) = t_{(15)}(0.99) = -2.6$ since $T^* < -2.6$

we reject H_0

c) X is normally (or approximately) distributed

d) The mean noise level is below 2.5 db. Type I would be to conclude that the new transistor reduces noise when, in fact, it doesn't.

⑦ a) $y = 25.44x - 0.0032$

b) $\hat{y} = 25.44x - 0.0032 =$ | $r = \hat{y} - y =$ | sum $= -0.002$

$\hat{y}$	r
3.8128	0.0028
2.5408	0.0008
2.4899	−0.0101
2.0066	0.0066
1.2688	−0.0012
1.0144	−0.0056
0.9890	−0.0110
0.8109	0.0109
0.5056	0.0056
0.4038	0.0038
0.2512	0.0012
0.1240	−0.0060

c) $0.1 \cdot \hat{\beta}_1 = 0.1 \cdot 25.44 = 2.544$

d) In general prediction is less trustworthy the farter away from the mean but in this case the fit is very good and 0.175 is close to our data.

e) Test statistic $T = \dfrac{\hat{\beta}_1 - \beta_1}{s/\sqrt{S_{xx}}} \sim t_{(n-2)}$

$S_{xx} = 0.0217$

$S_{yy} = 14.0764$

$S_{xy} = 0.5532$

$s^2 = \dfrac{1}{n-2}\left(S_{yy} - \dfrac{S_{xy}^2}{S_{xx}}\right) = 0.00005 \qquad s = 0.0071$

$T^* = \dfrac{25.44}{0.0071/\sqrt{0.0217}} = 527.8234$

$t_{(n-2)}\left(\frac{\alpha}{2}\right) = t_{(10)}\left(\frac{0.01}{2}\right) = 3.17$ since $T^* > 3.17$ we reject H_0