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Applied Mathematics and Statistics
Chalmers and GU

MVE055 / MSG810 Matematisk statistik och diskret matematik

Exam 6 April 2018, 8:30 - 12:30

Allowed aids: Chalmers-approved calculator
and one (two-sided) A4 sheet of paper with your own notes.
Total number of points: 30. To pass, at least 12 points are needed.
Note: All answers should be motivated.

1 Solutions

1. (a) Looking at the scatterplot and the value of the sample correlation coefficient, there seems to be a dependence between the two variables considered. However, this is not enough to conclude a causal relationship between them, thus the first two possible explanations are not enough supported. The third explanation is the most reasonable, and the one factor influencing both variables could be the time.

   (b) Linear regression is based on the following assumptions:
   - The residuals are independent and normally distributed;
   - the mean of the residuals is zero;
   - the variance of the residuals is σ^2 (not dependent on the independent variable);

   In the graph in the left the variance of the residuals increases as the value of the independent variable increses. Thus the variance of the data is not constant. In the graph in the center, the residuals only assume three possible values $-1, 0, 1$ and hence the data cannot be normally distributed. Finally, in the last graph the residuals shows that a linear relationship is not a good model to predict the dependent variable as a function of the independent one.

2. (a) The minimum and maximum values for $E[X]$ are -2 and 2. The expected value of X is given by

$$E[X] = \sum_{x\in\{-2,-1,0,1,2\}} xP(X = x).$$

   By using the fact that the expected value is a convex combination of the values taken by X

$$E[X] = \sum_{x\in\{-2,-1,0,1,2\}} xP(X = x) \leq \max_{x\in\{-2,-1,0,1,2\}} x \sum_{x\in\{-2,-1,0,1,2\}} P(X = x) \leq 2$$

$$E[X] = \sum_{x\in\{-2,-1,0,1,2\}} xP(X = x) \geq \min_{x\in\{-2,-1,0,1,2\}} x \sum_{x\in\{-2,-1,0,1,2\}} P(X = x) \geq -2.$$

The example such that $E[X] = 2$ is given by the constant random variable $X = 2$, that is $P(X = 2) = 1$. Similarly, the constant random variable$X = -2$ provide the other example.

(b) $0 \leq Var(X) = E[X^2] - E[X]^2 \leq E[X^2]$. The lower bound of the variance is 0, as the variance is always non negative. An example of random variables with variance 0 is a constant. The upper bound for the variance is given by $E[X^2] = 4$, as $E[X]^2 \geq 0$. An example attaining a variance of 4 is the one characterized by $P(X = -2) = P(X = 2) = 0.5$. For this random variable, $E[X] = 0$ and $E[X^2] = 4 = Var(X)$.

3. We will use the following notation: E= denote the event that the student passes the exam, PC = denote the event that the student will partecipate in the Pub crawl. The information given in the text can be written as

$$P(E|PC) = 0.99, P(E^c|PC) = 0.01$$

$$P(E|PC^c) = 0.5, P(PC) = P(PC^c) = 0.5.$$

Using Bayes' theorem

$$P(PC|E) = \frac{P(E|PC)P(PC)}{P(E)} = \frac{P(E|PC)P(PC)}{P(E|PC)P(PC) + P(E|PC^c)P(PC^c)} = \frac{0.99 \cdot 0.5}{0.99 \cdot 0.5 + 0.5 \cdot 0.5} = 0.6644$$

4. (a) The sequence $\{Y_n\}$ is a Markov chain. This is due to the fact that the random variables $X_1, ..., X_n$ are independent. Y_n can assume only the two values 0 and 1. We can prove that the sequence is a Markov chain by cases and note that $Y_{n+1} \geq Y_n$ as the maximum can only increase:

$$P(Y_n = 0|Y_{n-1} = 0, ...Y_1 = 0) = P(X_n = 0|X_{n-1} = 0, ...X_1 = 0) = P(X_n = 0|Y_{n-1} = 0) = P(Y_n = 0|Y_{n-1} = 0) = 1-p$$

$$P(Y_n = 1|Y_{n-1} = 0, ...Y_1 = 0) = P(X_n = 1|X_{n-1} = 0, ...X_1 = 0) = P(X_n = 1|Y_{n-1} = 0) = P(Y_n = 1|Y_{n-1} = 0) = p$$

$$P(Y_n = 1|Y_{n-1} = 1, ..., Y_j = 1, Y_{j-1} = 0, ...Y_1 = 0) = 1 = P(Y_n = 1|Y_{n-1} = 1)$$

and the transition matrix is given by (first column and first row correspond to state 0, second colum and row to state 1)

$$\begin{bmatrix} 1-p & p \\ 0 & 1 \end{bmatrix}$$

(b)

$$P(Y_n = 0) = P(\max(X_1, ..., X_n) = 0) = P(X_1 = 0, ..., X_n = 0) = P(X_1 = 0)P(X_2 = 0) \cdots P(X_n = 0) = (1-p)^n$$

$$P(Y_n = 1) = 1 - P(Y_n = 0) = 1 - (1 - p)^n$$

5. (a) We want the condition to ensure that $E[S_n] = \mu$.

$$E[S_n] = E[\sum_{i=1}^{n} w_i X_i] = \sum_{i=1}^{n} w_i E[X_i] = \mu \sum_{i=1}^{n} w_i.$$

Hence, the condition that ensures S_n is an unbiased estimator is $\sum_{i=1}^{n} w_i = 1$.

(b) $S_2 = w_1X_1 + w_2X_2$ and we know from a) that $w_1 + w_2 = 1$ for unbiasedness. Thus, we can rewrite $S_2 = w_1X_1 + (1 - w_1)X_2$. Using the properties of the variance we have

$$Var(S_2) = w_1^2 Var(X_1) + (1 - w_1)^2 Var(X_2) = [w_1^2 + (1 - w_1)^2]\sigma^2.$$

We then need to find the minimum of the function $f(w_1) = w_1^2 + (1 - w_1)^2$.

$$f'(w_1) = 2w_1 - 2(1 - w_1) = 0 \iff w_1 = \frac{1}{2}.$$

The estimator with minimum variance is $S_2 = \frac{1}{2}X_1 + \frac{1}{2}X_2$.

6. (a) The simplest assumptions are: 1) the calls received over two non overlapping time intervals are independent of each other; 2) the number of calls in a unit of time is constant; 3) the number of calls in a time interval follows a Poisson distribution.

(b) Using the above assumptions and the informations in the statement, Petter receives on average $\frac{113}{50}$ calls per hour. Then,

$$P(\text{Peter will receive exactly 2 calls in the next hour}) = P(Poisson(\frac{113}{50}) = 2) = 0.2665.$$

(c)

$$P(Poisson(113) > 120) \approx P(N(113, 113) > 120) = P(N(0, 1) > \frac{120 - 113}{\sqrt{113}}) = 0.2551.$$